Different types of phase transitions for a simple model of alignment of oriented particles

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Kinetic Description of Multiscale Phenomena
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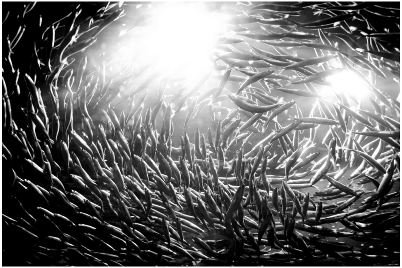
Goal: macroscopic description of some animal societies



- Local interactions without leader
- Emergence of macroscopic structures, phase transitions

Example : Vicsek model [1995], following more realistic similar models (Aoki [1982], Reynolds [1987], Huth-Wissel [1992]).

Kinetic mean-field model

Ingredients : moving at unit speed, alignment with neighbors, orientational noise.

Theorem (following Bolley, Cañizo, Carrillo, 2012)

Probability density function $f(x, v, t)$ of finding an individual at $x \in \mathbb{R}^n$, with speed $v \in \mathbb{S}$ (unit sphere of $\mathbb{R}^n$):

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{transport}} + \underbrace{\nu(|\bar{J}_f|)\nabla_v \cdot (\nabla_v(v \cdot \bar{\Omega}_f)\, f)}_{\text{alignment}} = \underbrace{\tau(|\bar{J}_f|)\Delta_v f}_{\text{diffusion}}$$

Local momentum: $J_f = \int_{v \in \mathbb{S}} v\, f(x, v, t)\, \mathrm{d}v.$

Target orientation: $\bar{\Omega}_f = \frac{\bar{J}_f}{|\bar{J}_f|}$, with $\bar{J}_f = K *_x J_f$.

Assumptions : K with finite second moment, and K, $|J| \mapsto \frac{\nu(|J|)}{|J|}$ and τ bounded Lipschitz.

Space-homogeneous version

Smoluchowski equation on the sphere

Reduced equation, for a function $f(v,t)$:

$$\partial_t f = Q(f),$$
$$Q(f) = \tau(|J_f|)\nabla_v \cdot [\nabla_v f - k(|J_f|)\nabla_v(v \cdot \Omega_f)\, f]$$
$$\Omega_f = \frac{J_f}{|J_f|}, \quad J_f(t) = \int_{\mathbb{S}} f(v,t)\, v \,\mathrm{d}v.$$

Key parameter: the conserved quantity $\rho = \int_{\mathbb{S}} f$.

Key function: $k(|J|) = \frac{\nu(|J|)}{\tau(|J|)}$ (competition between alignment and noise). More precisely, its inverse $\kappa \mapsto j(\kappa)$:

Main assumption: $|J| \mapsto k(|J|)$ increasing.

$$\kappa = k(|J|) \Leftrightarrow |J| = j(\kappa)$$

Equilibria

Definitions: von Mises–Fisher distribution

$$M_{\kappa\Omega}(v) = \frac{e^{\kappa\, v\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\, w\cdot\Omega}\, \mathrm{d}w}.$$

Orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geqslant 0$.

Order parameter: $c(\kappa) = |J_{M_{\kappa\Omega}}| = \frac{\int_0^\pi \cos\theta\, e^{\kappa\cos\theta} \sin^{n-2}\theta\, \mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta\, \mathrm{d}\theta}.$

We rewrite Q:

$$Q(f) = \tau(|J_f|)\nabla_\omega \cdot \left[M_{k(|J_f|)\Omega_f} \nabla_\omega \left(\frac{f}{M_{k(|J_f|)\Omega_f}} \right) \right].$$

Compatibility relation for steady-states

$$Q(f) = 0 \Leftrightarrow f = \rho M_{\kappa\Omega}, \text{ with } \Omega \in \mathbb{S} \text{ and } j(\kappa) = \rho c(\kappa).$$

Local analysis near uniform equilibria

Critical value: $\rho_c = \lim_{\kappa \to 0} \frac{j(\kappa)}{c(\kappa)} \in (0, +\infty]$.

Theorem: Strong instability – Exponential stability

- $\rho > \rho_c$: if $J_{f_0} \neq 0$, then f cannot converge to the uniform equilibrium.
- $\rho < \rho_c$: There exists a constant δ such that if $\|f_0 - \rho\|_{H^s} < \delta$, then we have for all $t \geqslant 0$

$$\|f(t) - \rho\|_{H^s} \leqslant \frac{\|f_0 - \rho\|_{H^s} e^{-\lambda t}}{1 - \frac{1}{\delta}\|f_0 - \rho\|_{H^s}}, \text{ with } \lambda = (n-1)\tau_0(1 - \frac{\rho}{\rho_c}).$$

Tools: linearization for the evolution of J_f, and then energy estimates for the whole equation.

Close to a nonisotropic equilibria $\rho M_{\kappa\Omega}$

Theorem: Exponential stability in case $(\frac{j}{c})'(\kappa) > 0$

For all $s > \frac{n-1}{2}$, there exist constants $\delta > 0$ and $C > 0$, such that if $\|f_0 - \rho M_{\kappa\Omega_0}\|_{H^s} < \delta$ for some $\Omega_0 \in \mathbb{S}$, there exists $\Omega_\infty \in \mathbb{S}$ such that

$$\|f - \rho M_{\kappa\Omega_\infty}\|_{H^s} \leqslant C\|f_0 - \rho M_{\kappa\Omega_0}\|_{H^s} e^{-\lambda t},$$

with $\lambda = \frac{c\tau(j)}{j'}\Lambda_\kappa(\frac{j}{c})'$, where Λ_κ is the best constant for the following weighted Poincaré inequality:

$$\langle|\nabla g|^2\rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa\langle(g - \langle g\rangle_{M_{\kappa\Omega}})^2\rangle_{M_{\kappa\Omega}}$$

Tools: Free energy $\mathcal{F}(f) = \int_{\mathbb{S}} f \ln f - \Phi(|J_f|)$, with $\frac{\mathrm{d}\Phi}{\mathrm{d}|J|} = k(|J|)$, and its dissipation $\mathcal{D}(f) = \tau(|J_f|)\int_{\mathbb{S}} f|\nabla_\omega(\ln f - h(|J_f|)\omega\cdot\Omega_f)|^2$.

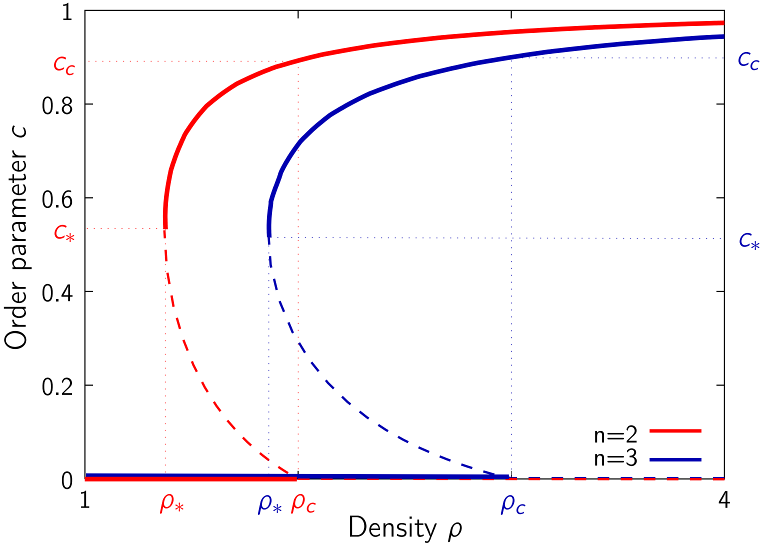
- Instability of $\rho M_{\kappa\Omega}$ if $(\frac{j}{c})'(\kappa) < 0$.

General statements in the case $(\frac{j}{c})' > 0$ for all κ

- If $\rho < \rho_c$, then the solution converges exponentially fast towards the uniform distribution $f_\infty = \rho$.
- If $\rho = \rho_c$, the solution converges to the uniform distribution.
- If $\rho > \rho_c$ and $J_{f_0} \neq 0$, then there exists Ω_∞ such that f converges exponentialy fast to the von Mises distribution $f_\infty = \rho M_{\kappa\Omega_\infty}$, where $\kappa > 0$ is the unique positive solution to the equation $\rho c(\kappa) = j(\kappa)$.

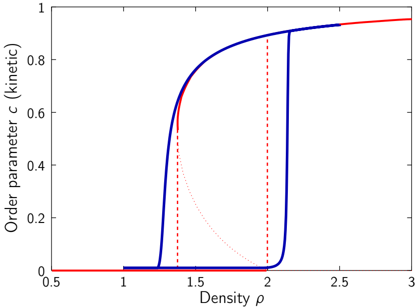
We can then define c (order parameter) as a function of ρ, and this function is continuous.
Critical exponent β: when $c(\rho) \asymp (\rho - \rho_c)^\beta$. Can be any number in $(0, 1]$, as one can artificially choose $j(\kappa) = c(\kappa)(1 + \kappa^{\frac{1}{\beta}})$.

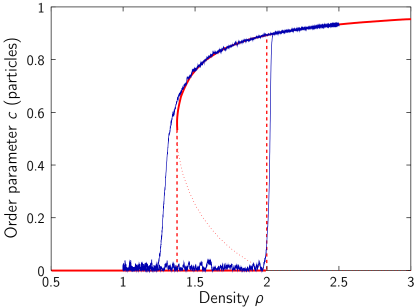
A phase diagram: $k(|J|) = |J| + |J|^2$



Numerical illustration of the hysteresis phenomena

Change of scale $\widetilde{f} = \frac{f}{\rho}$. The parameter ρ can now be considered as a free parameter that we let evolve in time.





Scalings for the kinetic equation

2 scaling parameters : ε (hydrodynamic scaling) and η (characteristic length for the observation kernel K).

Reduced kinetic equation

$$\varepsilon(\partial_t f + v\cdot\nabla_x f) + K_2\eta^2\left[\nabla_v\cdot(P_{v^\perp}\ell_f f) - m_f\Delta_v f\right] = Q(f) + \mathcal{O}(\eta^4),$$

with

$$\ell_f = \frac{\nu(|J_f|)}{|J_f|}P_{\Omega_f^\perp}\Delta_x J_f + (\Omega_f\cdot\Delta_x J_f)\,\nu'(|J_f|)\Omega_f\,,$$
$$m_f = (\Omega_f\cdot\Delta_x J_f)\,\tau'(|J_f|),$$

Limit as $\varepsilon\to 0$, in the cases where $\eta=\mathcal{O}(\varepsilon)$, or $\eta=\mathcal{O}(\sqrt{\varepsilon})$?

Branch of a stable nonisotropic equilibrium

Stable branch of von Mises distributions given by $\rho \mapsto \kappa(\rho)$.

Theorem: formal hydrodynamic limit

When $\varepsilon \to 0$, in a region where $f^\varepsilon \to f^0 = \rho(x,t) M_{\kappa(\rho)\Omega(x,t)}$, the functions ρ, Ω satisfy the following system:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, c\, \Omega) = 0, \\ \rho\, (\partial_t \Omega + \widetilde{c}(\Omega \cdot \nabla_x)\Omega) + \Theta\, P_{\Omega^\perp} \nabla_x \rho = \mathcal{K}_2 \delta P_{\Omega^\perp} \Delta_x(\rho c \Omega). \end{cases}$$

$$\widetilde{c} = \langle \cos\theta \rangle_{\widetilde{M}_\kappa}, \quad \Theta = \tfrac{1}{\kappa} + \tfrac{\rho}{\kappa} \tfrac{\mathrm{d}\kappa}{\mathrm{d}\rho} (\widetilde{c} - c), \quad \delta = \tfrac{\nu(\sigma)}{c} \left(\tfrac{n-1}{\kappa} + \widetilde{c}\right).$$

Scaling parameter $\mathcal{K}_2 = \lim_{\varepsilon\to 0} K_2 \frac{\eta^2}{\varepsilon}$.
Hyperbolicity linked to the critical exponent β (second order), or non hyperbolicity in the neighborhood of ρ_* (first order).

Region where $\rho_c - \rho^\varepsilon(x,t) \gg \varepsilon$

Chapman–Enskog expansion.

Theorem: formal diffusion correction

As $\varepsilon \to 0$, at first order, in a region where $f^\varepsilon \to f^0 = \rho(x,t)$, f^ε is (formally) given by

$$f^\varepsilon(x,v,t) = \rho^\varepsilon(x,t) - \varepsilon \frac{n\,\rho_c\, v\cdot\nabla_x \rho^\varepsilon(x,t)}{(n-1)n\tau_0(\rho_c - \rho^\varepsilon(x,t))},$$

And the density ρ^ε satisfies the following diffusion equation:

$$\partial_t \rho^\varepsilon = \frac{\varepsilon\,\rho_c}{(n-1)n\tau_0}\nabla_x\cdot\Big(\frac{1}{\rho_c-\rho^\varepsilon}\,\nabla_x\rho^\varepsilon\Big).$$

Second order phase transition : “boundary” region where $\rho^\varepsilon(x,t) - \rho_c = O(\varepsilon)$? How to connect the two models?