Passage from micro to macro, part II: hydrodynamics limits of the mean-field model

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The kinetic mean-field model

Self-propulsion, alignment with neighbors, orientational noise:

$$\begin{cases} \mathrm{d}X_i = V_i \, \mathrm{d}t \\ \mathrm{d}V_i = P_{V_i^\perp}\left[\frac{1}{N}\sum_j K(X_j - X_i)V_j\right] + \sqrt{2\tau}\, P_{V_i^\perp} \circ \mathrm{d}B_t^i \end{cases}$$

Theorem (following Bolley, Cañizo, Carrillo, 2012)

Probability density function $f(x, v, t)$ of finding an individual at $x \in \mathbb{R}^n$, with speed $v \in \mathbb{S}$ (unit sphere of $\mathbb{R}^n$):

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{transport}} + \underbrace{\nabla_v \cdot (\nabla_v (v \cdot \bar{J}_f)\, f)}_{\text{alignment}} = \underbrace{\tau(|\bar{J}_f|)\Delta_v f}_{\text{diffusion}}$$

Local momentum: $J_f = \int_{v \in \mathbb{S}} v\, f(x, v, t)\, \mathrm{d}v.$

Target orientation: $\bar{\Omega}_f = \frac{\bar{J}_f}{|\bar{J}_f|}$, with $\bar{J}_f = K *_x J_f$.

Hydrodynamic scaling

Scaling, with $\varepsilon \ll 1$ (and $K_0 = \int_{\mathbb{R}^n} K(x)\mathrm{d}x$):

$$f^\varepsilon(x, v, t) = K_0 f(\tfrac{1}{\tau\varepsilon}x, v, \tfrac{1}{\tau\varepsilon}t).$$

Mean-field reduced and rescaled equation, for the measure f^ε:

$$\varepsilon(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon) + O(\varepsilon^2),$$

with an effect of localization in space:

$$\begin{aligned} Q(f) &= -\nabla_v \cdot [P_{v^\perp} J_f\, f] + \Delta_v f, \\ &= \nabla_v \cdot (e^{v\cdot J_f} \nabla_v (e^{-v\cdot J_f} f)). \end{aligned}$$

Local equilibria

Definitions: Fisher–von Mises distribution

$$M_{\kappa\Omega}(v) = \frac{e^{\kappa\, v\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\, w\cdot\Omega}\,\mathrm{d}w}.$$

Orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geqslant 0$.

Order parameter: $c(\kappa) = |J_{M_{\kappa\Omega}}| = \dfrac{\int_0^\pi \cos\theta\, e^{\kappa\cos\theta}\sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{n-2}\theta\,\mathrm{d}\theta}$.

For $\kappa_f = |J_f|$ and $\Omega_f = \frac{J_f}{|J_f|}$, we get:

$$Q(f) = \nabla_v \cdot \left[M_{\kappa_f\Omega_f} \nabla_v \left(\frac{f}{M_{\kappa_f\Omega_f}} \right) \right].$$

Local equilibria: $f_{eq} = \rho M_{\kappa\Omega}$, for some $\Omega \in \mathbb{S}$.

Compatibility condition: $\kappa = \kappa_{f_{eq}} = |J_{f_{eq}}| = \rho |J_{\kappa\Omega}| = \rho c(\kappa)$.

Solutions to the compatibility condition $\rho c(\kappa) = \kappa$

Proposition

The function $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing, its limit is $\frac{1}{n}$ when $\kappa \to 0$.

- $\rho \leqslant n$, only one solution: $\kappa = 0$. Uniform equilibrium.
- $\rho > n$, uniform equilibrium for $\kappa = 0$.
  Unique solution $\kappa(\rho) > 0$. Manifold of equilibria:

$$\{\rho M_{\kappa(\rho)\Omega},\ \Omega \in \mathbb{S}\}.$$

Homogeneous case: convergence to the equilibrium

Spatial homogeneous case: the equation becomes

$$\varepsilon \partial_t f = -\nabla_v \cdot (P_{v^\perp} J_f \, f) + \Delta_v f,$$

also called Smoluchowski equation (with dipolar potential).

Theorem (AF, J.-G. Liu)

- If $\rho_{f_0} < n$, exponential convergence to the uniform distribution $f \to \rho_{f_0}$.
- If $\rho_{f_0} > n$ and $J_{f_0} \neq 0$, there exists $\Omega_\infty \in \mathbb{S}$ such that f converges exponentially to $\rho_{f_0} M_{\kappa(\rho)\Omega_\infty}$.

Region where $\rho^\varepsilon(x,t) - n \gg \varepsilon$

Starting point: when $\varepsilon \to 0$, f^ε converges (formally) to $\rho M_{\kappa(\rho)\Omega}$.
Equation on ρ: conservation of mass (integration of the kinetic equation against a constant).

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \bar{J}^\varepsilon = 0$$

In the limit $\varepsilon \to 0$, we get

$$\partial_t \rho + \nabla_x \cdot (\rho c(\kappa(\rho))\Omega) = 0$$

Evolution of Ω? No more conservation relation...

$$\int_{\mathbb{S}} Q(f^\varepsilon)\,\psi(v)\mathrm{d}v \neq 0 \text{ in general } (\psi \text{ non constant}).$$

Idea: integrate against $\psi_{\rho^\varepsilon,\Omega^\varepsilon}(v)$ instead.

Generalized collisional invariants

Linearized operator: $Q(f) = L_{\kappa_f \Omega_f}(f)$, with

$$L_{\kappa\Omega}(f) = -\Delta_v f + \kappa \nabla_v \cdot (P_{v^\perp} \Omega f) = -\nabla_v \cdot \left[M_{\kappa\Omega} \nabla_v \left(\frac{f}{M_{\kappa\Omega}} \right) \right],$$

Definition: GCIs associated to κ and Ω

$$\mathcal{C}_{\kappa\Omega} = \left\{ \psi | \int_{v\in\mathbb{S}} L_{\kappa\Omega}(f)\, \psi \, \mathrm{d}v = 0, \ \forall f \text{ such that } J_f \parallel \Omega \right\}.$$

In particular, for any generalized collisional invariant $\psi \in \mathcal{C}_{\kappa\Omega}$:

$$\forall f \text{ such that } \Omega_f = \Omega \text{ and } \kappa_f = \kappa, \int_{v\in\mathbb{S}} Q(f)\, \psi \, \mathrm{d}v = 0.$$

Proposition

$$\psi \in \mathcal{C}_{\kappa\Omega} \Leftrightarrow \psi = C + h_\kappa(v \cdot \Omega)\, A \cdot v, A \in \mathbb{R}^n, A \perp \Omega.$$

The macroscopic model

$$A \cdot \int_{v \in \mathbb{S}} Q(f^\varepsilon)\, h_{\kappa_{f^\varepsilon}}(v \cdot \Omega_{f^\varepsilon})\, v \, \mathrm{d}v = 0 \text{ for all } A \in \mathbb{R}^n \text{ s.t. } A \cdot \Omega_{f^\varepsilon} = 0$$

Equivalently, defining $\vec{\psi}_{\kappa,\Omega} = h_\kappa(v \cdot \Omega)\, P_{\Omega^\perp} v$, we get

$$\int_{v \in \mathbb{S}} Q(f^\varepsilon) \vec{\psi}_{\kappa_{f^\varepsilon}, \Omega_{f^\varepsilon}} \mathrm{d}v = 0$$

Theorem (P. Degond, AF, J.-G. Liu)

When $\varepsilon \to 0$, the (formal) limit of f^ε is $f^0 = \rho(x,t) M_{\kappa(\rho)\Omega(x,t)}$ and the functions ρ, Ω satisfy the system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, c\, \Omega) = 0, \\ \rho\, (\partial_t \Omega + \widetilde{c}(\Omega \cdot \nabla_x)\Omega) + \lambda\, P_{\Omega^\perp} \nabla_x \rho = 0, \end{cases}$$

with $\widetilde{c} = \langle \cos\theta \rangle_{\widetilde{M}_\kappa}$, and $\lambda = \frac{\rho - n - \kappa \widetilde{c}}{\kappa(\rho - n - \kappa c)}$.

Region where $n - \rho^\varepsilon(x,t) \gg \varepsilon$

Chapman–Enskog expansion.

Theorem (P. Degond, AF, J.-G. Liu)

When $\varepsilon \to 0$, a first order correction is (formally) given by

$$f^\varepsilon(x,v,t) = \rho^\varepsilon(x,t) - \varepsilon \frac{n\, v \cdot \nabla_x \rho^\varepsilon(x,t)}{(n-1)(n-\rho^\varepsilon(x,t))},$$

And the density $\rho^\varepsilon(x,t)$ satisfies the following (nonlinear) diffusion equation:

$$\partial_t \rho^\varepsilon = \frac{\varepsilon}{n-1} \nabla_x \cdot \left(\frac{1}{n-\rho^\varepsilon} \nabla_x \rho^\varepsilon \right).$$

Open questions:

- Convergence to a uniform steady state (inhomogeneous perturbation)
- Understanding the behavior at the boundary…