Local stability of Dirac masses in a kinetic model of alignment on the sphere

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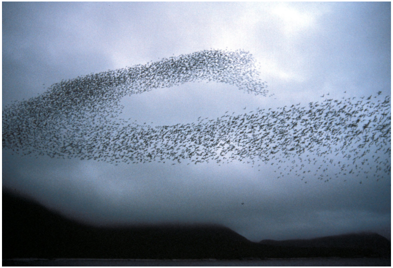
Alignment models for oriented particles

Framework: Spatially homogeneous version of the “Bertin, Droz and Grégoire” model [2006].

General Collision Model

- Metric space $\mathcal{M}$, particles interacting at a constant rate.
- Probability density that a particle at position x_* interacting with another one at x'_* is found at x after collision: $K(x, x_*, x'_*)$.

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Evolution of the probability measure ρ_t

$$\partial_t \rho(t,x) = \int_{\mathcal{M}\times\mathcal{M}} K(x, x_*, x'_*)\, \mathrm{d}\rho_t(x_*)\mathrm{d}\rho_t(x'_*) - \rho(t,x).$$

- Existence and uniqueness of solutions in $\mathcal{P}_2(\mathcal{M})$ if $x_* \mapsto K(\cdot, x_*, x'_*)$ is Lipschitz for W_2.

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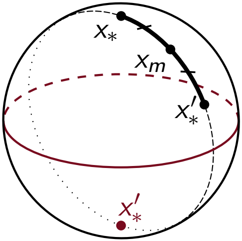
Example: $\mathcal{M} = \mathbb{S}$ (unit sphere of $\mathbb{R}^n$), with $K(\cdot, x_*, x'_*) = \delta_{x_m}$, where $x_m = \frac{x_*+x'_*}{\|x_*+x'_*\|}$.



Case $\mathcal{M} = \mathbb{R}^n$, midpoint collision

- Case $K(\cdot, x_*, x_*') = \delta_{\frac{x_*+x_*'}{2}}$, linked to models in economy (Pareschi, Toscani [2006]).
- Mass conservation $\bar{x} = \int_{\mathbb{R}^n} x \,\mathrm{d}\rho(x)$.
- 2nd moment $m_2 = \int_{\mathbb{R}^n} |x - \bar{x}|^2 \,\mathrm{d}\rho(x)$: $\frac{\mathrm{d}}{\mathrm{d}t} m_2 = -\frac{m_2}{2}$.

Exponential convergence towards a Dirac mass

$$W_2(\rho, \delta_{\bar{x}}) = W_2(\rho_0, \delta_{\bar{x}}) e^{-\frac{t}{4}}.$$

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- Decreasing energy: $E(\rho) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y)$ (equal to $2m_2$), no need for $\bar{x}$ in this definition.

Link "Energy – Wasserstein"

Energy:

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Useful Lemma – Markov inequalities

For $\rho \in \mathcal{P}(\mathbb{S})$, there exists $\bar{x} \in \mathbb{S}$ and for all $x \in \mathbb{S}$:

$$W_2(\rho, \delta_{\bar{x}})^2 \leqslant E(\rho) \leqslant 4\, W_2(\rho, \delta_x)^2,$$

For such a $\bar{x}$ and for all $\kappa > 0$, we have

$$\int_{\{x\in\mathbb{S};\, d(x,\bar{x})\geqslant\kappa\}} \mathrm{d}\rho(x) \leqslant \frac{1}{\kappa^2} E(\rho),$$

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Evolution of the energy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(\rho) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(x_*, x_*', y)\,\mathrm{d}\rho(x_*)\,\mathrm{d}\rho(x_*')\,\mathrm{d}\rho(y).$$

Local contribution to the variation of energy

$$\alpha(x_*, x_*', y) = \int_{\mathbb{S}} d(x,y)^2 K(x, x_*, x_*')\,\mathrm{d}x - \frac{d(x_*, y)^2 + d(x_*', y)^2}{2}.$$

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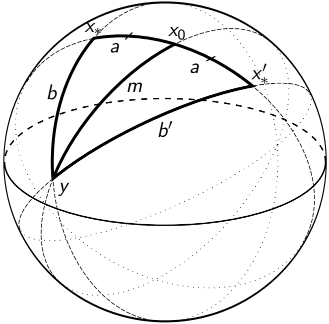

Configuration of Apollonius:

$$\alpha(x_*, x_*', y) = m^2 - \frac{b^2 + b'^2}{2}.$$

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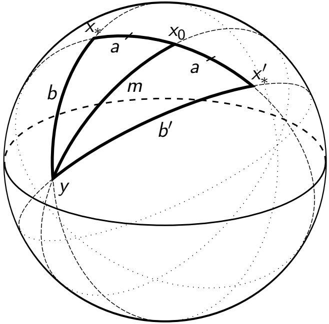

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Flat case (we get $-a^2$):

$$\alpha(x_*, x_*', y) = -\frac{1}{4}d(x_*, x_*')^2.$$

Error estimates in Apollonius' formula

Lemma: global estimate (only triangular inequalities)

For all x_*, x_*', $y \in \mathbb{S}$, we have

$$\alpha(x_*, x_*', y) \leqslant -\frac{1}{4} d(x_*, x_*')^2 + 2\, d(x_*, x_*') \min\big(d(x_*, y),\ d(x_*', y)\big).$$

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Lemma: local estimate, more precise

For all $\kappa_1 < \frac{2\pi}{3}$, there exists $C_1 > 0$ such that for all $\kappa \leqslant \kappa_1$, and all x_*, x'_*, $y \in \mathbb{S}$ with $\max(d(x_*, y), d(x'_*, y), d(x_*, x'_*)) \leqslant \kappa$, we have

$$\alpha(x_*, x'_*, y) \leqslant -\frac{1}{4} d(x_*, x'_*)^2 + C_1\, \kappa^2\, d(x_*, x'_*)^2.$$

Spherical Apollonius: $\frac{1}{2}(\cos b + \cos b') = \cos a \cos m$.

Decreasing energy – Control on displacement

We set $\bar{\omega} := \{x \in \mathbb{S};\ d(x,\bar{x}) \leqslant \frac{1}{2}\kappa\}$, and we cut the triple integral in four parts following if x_*, x_*', y is in $\bar{\omega}$ or not.

$$\frac{1}{2}\frac{d}{dt}E(\rho) + \frac{1}{4}E(\rho) \leqslant \underbrace{C\,\kappa^2\,E(\rho)}_{\text{Local lemma}} + \underbrace{12\frac{E(\rho)^{\frac{3}{2}}}{\kappa} + 24\frac{E(\rho)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

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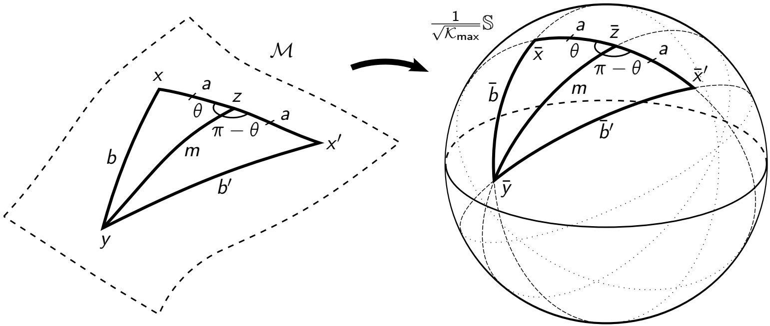
Theorem: local stability of Dirac masses

There exists $C_1 > 0$ and $\eta > 0$ such that for all solution $\rho \in C(\mathbb{R}_+, \mathcal{P}(\mathbb{S}))$ with initial condition ρ_0 satisfying $W_2(\rho_0, \delta_{x_0}) < \eta$ for a $x_0 \in \mathbb{S}$, there exists $x_\infty \in \mathbb{S}$ such that

$$W_2(\rho_t, \delta_{x_\infty}) \leqslant C_1 W_2(\rho_0, \delta_{x_0})\, e^{-\frac{1}{4}t}.$$

Comparison lemmas



$$\left.\begin{array}{c}\text{Sectional curvature } \leqslant \mathcal{K}_{\text{max}} \\ \text{Positive injectivity radius}\end{array}\right\} \Rightarrow \begin{array}{l}\text{Same result, same rate} \\ \text{of convergence (sharp)}\end{array}$$

Larger class of models

(H1) Contraction property: there exists $\beta \in [0,1)$ such that for all $x_*, x_*' \in \mathcal{M}$, we have

$$\int_{\mathcal{M}} d(x,x_*)^2 K(x,x_*,x_*')\mathrm{d}x \leqslant \tfrac{1}{4}(1+\beta)d(x_*,x_*')^2,$$

(H2) Midpoint symmetry for small distances,

(H3) Control of a moment > 2.

Theorem: local stability of Dirac masses

Under (H1), (H2) and (H3), sectional curvature bounded below and above, and positive injectivity radius, same result with

$$W_2(\rho_t,\delta_{x_\infty}) \leqslant C_1 W_2(\rho_0,\delta_{x_0})\, e^{-\frac{1}{4}(1-\beta)t}.$$

Thanks !