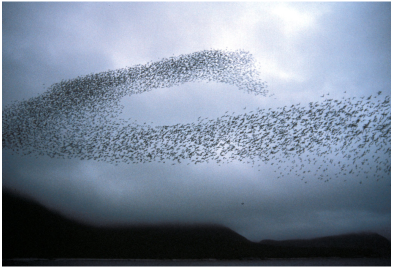
Local stability of Dirac masses in a kinetic model of alignment on the sphere

Amic Frouvelle

Collaboration with Pierre Degond (Imperial College, London)
and Gaël Raoul (CMAP, École Polytechnique)

Cambridge Geometric Analysis and PDE Seminar
March 21st, 2016

Alignment models for oriented particles



Framework: Spatially homogeneous version of the "Bertin, Droz and Grégoire" model [2006].

General Collision Model

- Metric space $\mathcal{M}$, particles interacting at a constant rate.
- Probability density that a particle at position x_* interacting with another one at x'_* is found at x after collision: $K(x, x_*, x'_*)$.

General Collision Model

- Metric space $\mathcal{M}$, particles interacting at a constant rate.
- Probability density that a particle at position x_* interacting with another one at x_*' is found at x after collision: $K(x, x_*, x_*')$.

Evolution of the probability measure ρ_t

$$\partial_t \rho(t,x) = \int_{\mathcal{M}\times\mathcal{M}} K(x, x_*, x_*')\,\mathrm{d}\rho_t(x_*)\mathrm{d}\rho_t(x_*') - \rho(t,x).$$

- Existence and uniqueness of solutions in $\mathcal{P}_2(\mathcal{M})$ if $x_* \mapsto K(\cdot, x_*, x_*')$ is Lipschitz for W_2.

General Collision Model

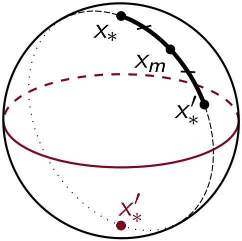
- Metric space $\mathcal{M}$, particles interacting at a constant rate.
- Probability density that a particle at position x_* interacting with another one at x'_* is found at x after collision: $K(x, x_*, x'_*)$.

Evolution of the probability measure ρ_t

$$\partial_t \rho(t,x) = \int_{\mathcal{M}\times\mathcal{M}} K(x, x_*, x'_*)\,\mathrm{d}\rho_t(x_*)\mathrm{d}\rho_t(x'_*) - \rho(t,x).$$

- Existence and uniqueness of solutions in $\mathcal{P}_2(\mathcal{M})$ if $x_* \mapsto K(\cdot, x_*, x'_*)$ is Lipschitz for W_2.

Example: $\mathcal{M} = \mathbb{S}$ (unit sphere of $\mathbb{R}^n$), with $K(\cdot, x_*, x'_*) = \delta_{x_m}$, where $x_m = \frac{x_*+x'_*}{\|x_*+x'_*\|}$.



Case $\mathcal{M} = \mathbb{R}^n$, midpoint collision

- Case $K(\cdot, x_*, x_*') = \delta_{\frac{x_* + x_*'}{2}}$, linked to models in economy (Pareschi, Toscani [2006]).
- Conservation of center of mass $\bar{x} = \int_{\mathbb{R}^n} x \,\mathrm{d}\rho(x)$.
- 2nd moment $m_2 = \int_{\mathbb{R}^n} |x - \bar{x}|^2 \,\mathrm{d}\rho(x)$: $\frac{\mathrm{d}}{\mathrm{d}t} m_2 = -\frac{m_2}{2}$.

Exponential convergence towards a Dirac mass

$$W_2(\rho, \delta_{\bar{x}}) = W_2(\rho_0, \delta_{\bar{x}}) e^{-\frac{t}{4}}.$$

Case $\mathcal{M} = \mathbb{R}^n$, midpoint collision

- Case $K(\cdot, x_*, x_*') = \delta_{\frac{x_*+x_*'}{2}}$, linked to models in economy (Pareschi, Toscani [2006]).
- Conservation of center of mass $\bar{x} = \int_{\mathbb{R}^n} x \, \mathrm{d}\rho(x)$.
- 2nd moment $m_2 = \int_{\mathbb{R}^n} |x - \bar{x}|^2 \, \mathrm{d}\rho(x)$: $\frac{\mathrm{d}}{\mathrm{d}t} m_2 = -\frac{m_2}{2}$.

Exponential convergence towards a Dirac mass

$$W_2(\rho, \delta_{\bar{x}}) = W_2(\rho_0, \delta_{\bar{x}}) e^{-\frac{t}{4}}.$$

- Decreasing energy: $E(\rho) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y)$ (equal to $2m_2$), no need for $\bar{x}$ in this definition.

Link "Energy – Wasserstein"

Energy:

$$E(\rho) = \int_{\mathbb{S}\times\mathbb{S}} d(x,y)^2 \,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y).$$

Link "Energy – Wasserstein"

Energy:

$$E(\rho) = \int_{\mathbb{S}\times\mathbb{S}} d(x,y)^2 \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y).$$

Useful Lemma – Markov inequalities

For $\rho \in \mathcal{P}(\mathbb{S})$, there exists $\bar{x} \in \mathbb{S}$ and for all $x \in \mathbb{S}$:

$$W_2(\rho, \delta_{\bar{x}})^2 \leqslant E(\rho) \leqslant 4 \, W_2(\rho, \delta_x)^2,$$

For such a $\bar{x}$ and for all $\kappa > 0$, we have

$$\int_{\{x\in\mathbb{S};\, d(x,\bar{x})\geqslant\kappa\}} \mathrm{d}\rho(x) \leqslant \frac{1}{\kappa^2} E(\rho),$$

$$\int_{\{x\in\mathbb{S};\, d(x,\bar{x})\geqslant\kappa\}} d(x,\bar{x}) \, \mathrm{d}\rho(x) \leqslant \frac{1}{\kappa} E(\rho).$$

Evolution of the energy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(\rho) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(x_*, x_*', y)\,\mathrm{d}\rho(x_*)\,\mathrm{d}\rho(x_*')\,\mathrm{d}\rho(y).$$

Local contribution to the variation of energy

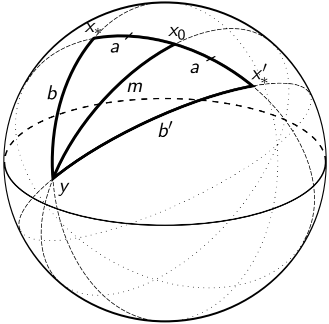
$$\alpha(x_*, x_*', y) = \int_{\mathbb{S}} d(x,y)^2 K(x, x_*, x_*')\,\mathrm{d}x - \frac{d(x_*,y)^2 + d(x_*',y)^2}{2}.$$

Evolution of the energy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(\rho) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(x_*, x_*', y)\,\mathrm{d}\rho(x_*)\,\mathrm{d}\rho(x_*')\,\mathrm{d}\rho(y).$$

Local contribution to the variation of energy

$$\alpha(x_*, x_*', y) = \int_{\mathbb{S}} d(x,y)^2 K(x, x_*, x_*')\,\mathrm{d}x - \frac{d(x_*,y)^2 + d(x_*',y)^2}{2}.$$



Configuration of Apollonius:

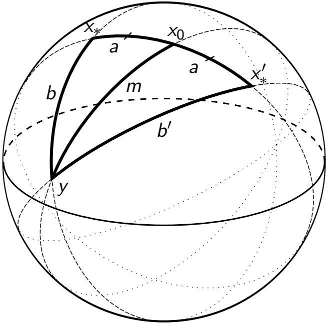
$$\alpha(x_*, x_*', y) = m^2 - \frac{b^2 + b'^2}{2}.$$

Evolution of the energy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(\rho) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(x_*, x'_*, y)\,\mathrm{d}\rho(x_*)\,\mathrm{d}\rho(x'_*)\,\mathrm{d}\rho(y).$$

Local contribution to the variation of energy

$$\alpha(x_*, x'_*, y) = \int_{\mathbb{S}} d(x, y)^2 K(x, x_*, x'_*)\,\mathrm{d}x - \frac{d(x_*, y)^2 + d(x'_*, y)^2}{2}.$$



Configuration of Apollonius:

$$\alpha(x_*, x'_*, y) = m^2 - \frac{b^2 + b'^2}{2}.$$

Flat case (we get $-a^2$):

$$\alpha(x_*, x'_*, y) = -\frac{1}{4}d(x_*, x'_*)^2.$$

Error estimates in Apollonius' formula

Lemma: global estimate (only triangular inequalities)

For all x_*, x'_*, $y \in \mathbb{S}$, we have

$$\alpha(x_*, x'_*, y) \leqslant -\frac{1}{4} d(x_*, x'_*)^2 + 2\, d(x_*, x'_*) \min\big(d(x_*, y),\, d(x'_*, y)\big).$$

Error estimates in Apollonius' formula

Lemma: global estimate (only triangular inequalities)

For all x_*, x_*', $y \in \mathbb{S}$, we have

$$\alpha(x_*, x_*', y) \leqslant -\frac{1}{4} d(x_*, x_*')^2 + 2\, d(x_*, x_*') \min\big(d(x_*, y),\, d(x_*', y)\big).$$

Lemma: local estimate, more precise

For all $\kappa_1 < \frac{2\pi}{3}$, there exists $C_1 > 0$ such that for all $\kappa \leqslant \kappa_1$, and all x_*, x_*', $y \in \mathbb{S}$ with $\max(d(x_*, y), d(x_*', y), d(x_*, x_*')) \leqslant \kappa$, we have

$$\alpha(x_*, x_*', y) \leqslant -\frac{1}{4} d(x_*, x_*')^2 + C_1\, \kappa^2\, d(x_*, x_*')^2.$$

Spherical Apollonius: $\frac{1}{2}(\cos b + \cos b') = \cos a \cos m$.

Decreasing energy – Control on displacement

We set $\bar{\omega} := \{x \in \mathbb{S};\ d(x,\bar{x}) \leqslant \frac{1}{2}\kappa\}$, and we cut the triple integral in four parts following if x_*, x'_*, y is in $\bar{\omega}$ or not.

$$\frac{1}{2}\frac{d}{dt}E(\rho) + \frac{1}{4}E(\rho) \leqslant \underbrace{C\,\kappa^2\,E(\rho)}_{\text{Local lemma}} + \underbrace{12\frac{E(\rho)^{\frac{3}{2}}}{\kappa} + 24\frac{E(\rho)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

Decreasing energy – Control on displacement

We set $\bar{\omega} := \{x \in \mathbb{S};\ d(x,\bar{x}) \leqslant \frac{1}{2}\kappa\}$, and we cut the triple integral in four parts following if x_*, x'_*, y is in $\bar{\omega}$ or not.

$$\frac{1}{2}\frac{d}{dt}E(\rho) + \frac{1}{4}E(\rho) \leqslant \underbrace{C\,\kappa^2\,E(\rho)}_{\text{Local lemma}} + \underbrace{12\frac{E(\rho)^{\frac{3}{2}}}{\kappa} + 24\frac{E(\rho)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

Same kind of cut to show that $\bar{x}(t)$ satisfies Cauchy's criteria (for $t \to \infty$) and converges with the same rate as $\sqrt{E(\rho)}$.

Decreasing energy – Control on displacement

We set $\bar{\omega} := \{x \in \mathbb{S};\ d(x,\bar{x}) \leqslant \frac{1}{2}\kappa\}$, and we cut the triple integral in four parts following if x_*, x'_*, y is in $\bar{\omega}$ or not.

$$\frac{1}{2}\frac{d}{dt}E(\rho) + \frac{1}{4}E(\rho) \leqslant \underbrace{C\,\kappa^2\,E(\rho)}_{\text{Local lemma}} + \underbrace{12\frac{E(\rho)^{\frac{3}{2}}}{\kappa} + 24\frac{E(\rho)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

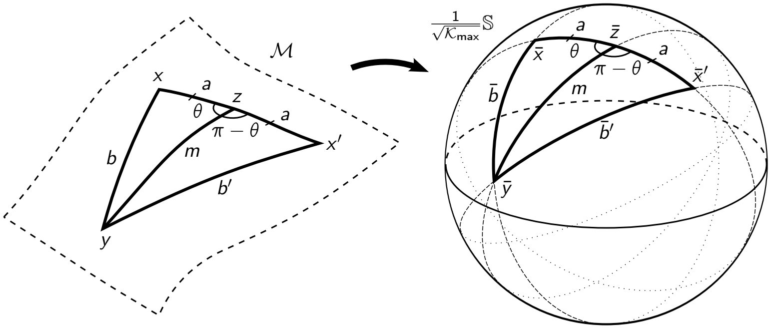
Same kind of cut to show that $\bar{x}(t)$ satisfies Cauchy's criteria (for $t \to \infty$) and converges with the same rate as $\sqrt{E(\rho)}$.

Theorem: local stability of Dirac masses

There exists $C_1 > 0$ and $\eta > 0$ such that for all solution $\rho \in C(\mathbb{R}_+, \mathcal{P}(\mathbb{S}))$ with initial condition ρ_0 satisfying $W_2(\rho_0, \delta_{x_0}) < \eta$ for a $x_0 \in \mathbb{S}$, there exists $x_\infty \in \mathbb{S}$ such that

$$W_2(\rho_t, \delta_{x_\infty}) \leqslant C_1 W_2(\rho_0, \delta_{x_0})\, e^{-\frac{1}{4}t}.$$

Comparison lemmas



$$\left.\begin{array}{c}\text{Sectional curvature } \leqslant \mathcal{K}_{\max} \\ \text{Positive injectivity radius}\end{array}\right\} \Rightarrow \begin{array}{l}\text{Same result, same rate} \\ \text{of convergence (sharp)}\end{array}$$

Larger class of models

(H1) Contraction property: there exists $\beta \in [0,1)$ such that for all $x_*, x_*' \in \mathcal{M}$, we have

$$\int_{\mathcal{M}} d(x,x_*)^2 K(x,x_*,x_*')\mathrm{d}x \leqslant \tfrac{1}{4}(1+\beta)d(x_*,x_*')^2,$$

(H2) Midpoint symmetry for small distances,

(H3) Control of a moment > 2.

Theorem: local stability of Dirac masses

Under (H1), (H2) and (H3), sectional curvature bounded below and above, and positive injectivity radius, same result with

$$W_2(\rho_t, \delta_{x_\infty}) \leqslant C_1 W_2(\rho_0, \delta_{x_0})\, e^{-\frac{1}{4}(1-\beta)t}.$$

Thanks !