Mixing time of the lazy and simple random walk on a randomly twisted hypercube

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Abstract

We consider a randomly twisted hypercube on N vertices and study a simple or lazy random walk on this graph. We show that for both walks the mixing time is of order $\log N$, and the model does not exhibit cutoff. We also establish that both walks have cover time of order $N \log N$ and we show that the chromatic number of the graph diverges.

1 Introduction

In this paper we consider the mixing properties of a random walk on a twisted hypercube, which is a random generalisation of a Boolean hypercube, defined as follows.

Definition 1.1. We recursively define the random graphs $G^{(n)}$ for $n \geq 0$ as follows. Let $G^{(0)}$ be a graph consisting of a single vertex, and for $n \geq 1$ let $G^{(n)}$ be obtained by considering two independent copies $G^{(n-1,1)}$ and $G^{(n-1,2)}$ of $G^{(n-1)}$ and adding the edges corresponding to a uniform random matching between their vertices. Then we say that $G^{(n)}$ is a *twisted hypercube* on 2^n vertices.

We call the copies of $G^{(k)}$ appearing in the construction of $G^{(n)}$ the *type k subgraphs* of $G^{(n)}$, and we label them $G^{(k,1)}$, ..., $G^{(k,2^{n-k})}$ in such a way that for each $k \in \{1, 2, ..., n\}$ and $\ell \in \{1, 2, ..., 2^{n-k}\}$ the type $(k-1)$ subgraphs of $G^{(k,\ell)}$ are labelled $G^{(k-1,2\ell-1)}$ and $G^{(k-1,2\ell)}$. We say that the edges running between $G_{k-1}^{(2\ell-1)}$ and $G_{k-1}^{(2\ell)}$ are *type k edges.* (See Figure 1 for illustration; disregard the colouring for now.) //

Since the 1990s various modifications of a Boolean hypercube have been considered as models of networks that retain some of the structure of a hypercube, but have smaller diameter and better connectivity properties.

The above random model was first introduced by Dudek et al [6] in 2018, and they showed that with high probability between any pair of vertices there are at least n internally disjoint paths of length at most $\frac{n}{\log_2 n} + O\left(\frac{n}{(\log_2 n)^2}\right)$. Among n-regular graphs with 2^n vertices this number of paths is optimal and the length is asymptotically optimal.

In 2023 Benjamini et al [5] considered a more general random model, where the two copies of $G^{(n-1)}$ in the construction of $G^{(n)}$ do not need to be independent, and proved that with high probability it has a small diameter, large vertex expansion, its eigenvalues satisfy a semicircle law, and it has no non-trivial automorphisms.

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The paper [5] also included a list of remarks and open questions (in Section 3). In particular, they asked about the mixing time of a lazy random walk on a twisted hypercube, noting that by simple estimates it is $O(n \log n)$ and $\Omega(n)$, and wondering whether it is $o(n \log n)$. They also asked about the chromatic number of a twisted hypercube, noting that with high probability it is at least 3.

In this work we study the mixing properties of a randomly twisted hypercube and we establish that both a lazy and a simple random walk have mixing time of order n. Using this we also show that the cover time is of order $n2^n$ for both walks.

Although it is not the focus of the paper, we also present a quick argument showing that the chromatic number is $\omega(1)$.

We recall the definitions of mixing time, cutoff, cover time and chromatic number.

Definition 1.2. Let X be a Markov chain on a finite state space V, with a unique invariant distribution π. Then for $\varepsilon \in (0,1)$ we define the ε-mixing time of X as

$$t_{\text{mix}}(\varepsilon) := \max_{x \in V} \inf \{t \geq 0 : d_{\text{TV}}(\mathbb{P}_x(X_t = \cdot), \pi(\cdot)) \leq \varepsilon\}.$$

where d_{TV} denotes the total variation distance between two distributions, defined as $d_{\text{TV}}(\mu, \nu) = \frac{1}{2}\sum_x |\mu(x) - \nu(x)|$.

We say that a sequence $\left(X^{(n)}\right)$ of Markov chains exhibits cutoff at time t_n, with window s_n if $\frac{s_n}{t_n} \to 0$ as $n \to \infty$, and for any $\varepsilon \in (0,1)$ there exists a constant $c(\varepsilon)$ such that for all n the ε-mixing time of $X^{(n)}$ satisfies

$$t_n - c(\varepsilon)s_n \quad \leq \quad t_{\text{mix}}^{(n)}(\varepsilon) \quad \leq \quad t_n + c(\varepsilon)s_n. \tag{1}$$

In case the Markov chains $X^{(n)}$ are random, we say that the sequence exhibits cutoff with high probability if (1) holds with a probability $1 - o(1)$ as $n \to \infty$. //

Definition 1.3. Let X be a Markov chain on a finite state space V. Then the cover time of X is defined as

$$t_{\text{cov}} := \max_{x \in V} \mathbb{E}_x \left[\max_{y \in V} \tau_y\right],$$

where τ_y denotes the first time the chain visits y. //

Definition 1.4. Given a graph G, a colouring of its vertices with k colours is a function $c : V(G) \to \mathcal{C}$ where $\mathcal{C}$ is a set of size k. We say that c is a proper colouring if for any pair $\{x, y\}$ of neighbouring vertices, we have $c(x) \neq c(y)$. The chromatic number $\chi(G)$ of G is the smallest positive integer k such that G has a proper colouring with k colours. //

Our main results are the following.

Theorem 1. *Let $\left(G^{(n)}\right)$ be a sequence of twisted hypercubes as in Definition 1.1 and let $X^{(n)}$ be a simple random walk on $G^{(n)}$. Then with high probability the graphs $G^{(n)}$ have the following property.*

For any $\varepsilon \in (0,1)$ there exist positive constants $c(\varepsilon)$ and $C(\varepsilon)$ such that for all sufficiently large n the ε-mixing time of $X^{(n)}$ satisfies

$$c(\varepsilon)n \quad \leq \quad t_{\text{mix}}^{(n)}(\varepsilon) \quad \leq \quad C(\varepsilon)n.$$

Also, the sequence $\left(X^{(n)}\right)$ does not exhibit cutoff.

Moreover, the same statement also holds for a lazy random walk instead of a simple one.

Theorem 2. *There exist positive constants c and C with the following property. With high probability the twisted hypercube $G^{(n)}$ is such that the cover time of a simple random walk on it satisfies*

$$cn2^n \quad \leq \quad t_{\mathrm{cov}}^{(n)} \quad \leq \quad Cn2^n.$$

Moreover, the same statement holds for a lazy random walk instead of a simple one.

Theorem 3. *For any positive constant c, with high probability the chromatic number of $G^{(n)}$ satisfies $\chi\left(G^{(n)}\right) > c$.*

1.1 Some notation

We let $N = 2^n$, and we write $\mathcal{U}$ to denote the uniform distribution on the vertices of $G^{(n)}$.

We write SRW to mean a simple random walk.

For functions $a, b : \mathbb{Z}_{\geq 0} \to [0, \infty)$ we write $a(n) \lesssim b(n)$ if there exists a constant $C > 0$ such that for all sufficiently large n we have $a(n) \leq Cb(n)$, and write $a(n) \ll b(n)$ if for any constant $c > 0$, for all sufficiently large n (in terms of c) we have $a(n) \leq cb(n)$. We define $\gtrsim$ and $\gg$ analogously. We write $a(n) \asymp b(n)$ if we have $a(n) \lesssim b(n) \lesssim a(n)$. Unless specified otherwise, these relations are considered as $n \to \infty$.

We often consider quantities like $0.9n$ or $C \log n$ and treat them as integers. In these cases one can take $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$, but we omit this from the notation.

1.2 Overview

In this section we give an overview of the proof ideas of Theorem 1, which is the most involved one of our three results. Instead of trying to estimate the mixing times of a walk directly, we bound the hit times of the walk and then compare the mixing times to these. The hit times are defined as follows.

Definition 1.5. Let X be a Markov chain on a finite state space V with a unique invariant distribution π. For each $\alpha, \theta \in (0, 1)$ we define the corresponding *hit time* of X as

$$\mathrm{hit}_\alpha(\theta) := \quad \inf\{t : \forall x, \forall A \subseteq V \text{ with } \pi(A) \geq \alpha \text{ we have } \mathbb{P}_x(\tau_A > t) \leq \theta\},$$

where $\tau_A = \inf\{t : X_t \in A\}$. //

We will prove the following two results.

Proposition 1.6. *With high probability there exist $\alpha, \theta \in (0, 1)$ such that the corresponding hit time of a simple random walk on $G^{(n)}$ satisfies $\mathrm{hit}_\alpha(\theta) \lesssim n$.*

Proposition 1.7. *With high probability the absolute relaxation time of a simple random walk on $G^{(n)}$ satisfies $t_{\mathrm{rel}}^{\mathrm{abs}} \asymp n$.*

Once we have these, the proof of Theorem 1 can be concluded as follows.

Proof of Theorem 1. From Proposition 1.7 we know that $\mathrm{hit}_\alpha(\theta) \lesssim n$ for some $\alpha, \theta \in (0, 1)$ and from [5, Proposition 1.5] we know that $t_{\mathrm{rel}} \asymp n$. Then [3, Proposition 3.3 and Corollary 3.4] allows us to bound the hit time with all parameters, and we get that $\mathrm{hit}_\beta(\varphi) \lesssim n$ for all $\beta, \varphi \in (0, 1)$.

Using [2, Lemma A.3] we can compare the hit times of a lazy random walk and a SRW, and get that $\mathrm{hit}_{\beta}^{\mathrm{lazy}}(\varphi) \lesssim n$ for all $\beta, \varphi \in (0,1)$.

Then using [3, Proposition 1.8], the monotonicity of mixing time in the parameter, and that $t_{\mathrm{mix}}^{\mathrm{lazy}}(\varepsilon) \gtrsim t_{\mathrm{rel}}$ for all $\varepsilon \in (0,1)$ (by [10, Theorem 12.4] for the lazy walk) we get that mixing time of the lazy walk satisfies $t_{\mathrm{mix}}^{\mathrm{lazy}}(\varepsilon) \asymp n \asymp t_{\mathrm{rel}}$ for all $\varepsilon \in (0,1)$, hence by [3, Theorem 3] the lazy walk does not exhibit cutoff.

Since the lazy walk does not exhibit cutoff, neither does the simple random walk. (See [2, Lemma 6.9].)

Using the above bound on the hit times, [3, Remark 1.9], and that $t_{\mathrm{mix}}(\varepsilon) \gtrsim t_{\mathrm{rel}}^{\mathrm{abs}}$ for all $\varepsilon \in (0,1)$ (see [10, Theorem 12.4]), we also get that $t_{\mathrm{mix}}(\varepsilon) \asymp n$ for all $\varepsilon \in (0,1)$. $\square$

To prove Proposition 1.6, we define a random time τ and show that with positive probability it satisfies $\tau \lesssim n$, and X_τ is comparable to the uniform distribution.

Inspired by [1], we colour the 'short', 'medium length' and 'long' edges of $G^{(n)}$ green, red, and blue respectively. We consider a random time τ so that the first τ steps of a random walk X on $G^{(n)}$ roughly speaking correspond to taking a geometric number of steps as a random walk on the green and red edges, then walking on the green and red edges until $K+L$ further red edges are crossed (where K and L are well-chosen parameters), then crossing a blue edge, then walking on the green and red edges again until $L+K$ more red edges are crossed.

If after the initial geometric number of steps the walk is at a given vertex q, we approximate the probability of $X_\tau = y$ as

$$\sum_{u,v,w,z} \mathbb{P}_q\left(Y_{\tau_{\mathrm{red}}^K} = u \;\middle|\; G^{(n)}\right)\mathbb{P}_u\left(Y_{\tau_{\mathrm{red}}^L - 1} = w \;\middle|\; G^{(n)}\right) \\ \cdot \mathbb{P}_y\left(Y_{\tau_{\mathrm{red}}^K} = v \;\middle|\; G^{(n)}\right)\mathbb{P}_v\left(Y_{\tau_{\mathrm{red}}^L - 1} = z \;\middle|\; G^{(n)}\right)\mathbf{1}_{\eta(w)=z}, \tag{2}$$

where Y is a simple random walk on the green and red edges of $G^{(n)}$, τ_{red}^i denotes the ith time that a red edge is crossed, and $\eta(w)$ is a uniformly chosen blue neighbour of w.

To approximate the neighbourhoods of u and v formed by green and red edges of $G^{(n)}$, we consider random 'quasi-trees' T_u and T_v that consist of 'balls' that are distributed like balls of a given radius R in the green distance in a random copy of $G^{(n)}$, and 'long-range edges' between them that correspond to the red edges of $G^{(n)}$ and form a tree-like structure.

Using the iid structure of a quasi-tree T, and the fact that due to the high degrees of vertices, a random walk $\widetilde{Y}$ on it is unlikely to return to a ball it already left, we can get a good understanding of the behaviour $\widetilde{Y}$. We then prove that for a sufficiently large proportion of pairs (u,v), their neighbourhood in the green and red edges of $G^{(n)}$, and a simple random walk on it is well approximated by a random quasi-tree and simple random walk on it. This allows us to get a good control over the probabilities $\mathbb{P}_u\left(Y_{\tau_{\mathrm{red}}^L - 1} = w \;\middle|\; G^{(n)}\right)$ and $\mathbb{P}_v\left(Y_{\tau_{\mathrm{red}}^L - 1} = z \;\middle|\; G^{(n)}\right)$. After this we use the randomness of the blue edges and a concentration result similar to [4, Lemma 5.1] to show that the sum (2) is $\gtrsim \frac{1}{N}$.

To prove Proposition 1.7, we use a result that roughly speaking states that if a graph is 'far from being bipartite', then its relaxation time and absolute relaxation time are of the same order. We prove that whp $G^{(n)}$ satisfies the above property, and use that $t_{\mathrm{rel}} \asymp n$ from [5, Proposition 1.5].

1.3 Organisation

In Sections 2 to 5 we work towards the proof of Proposition 1.6. In Section 2 we introduce the colouring of the edges of $G^{(n)}$ and the exact definition of the random time τ. In Section 3 we give the definition of a quasi-tree and explain how to couple $G^{(n)}$ to it. In Section 4 we estimate the success probability of this coupling. In Section 5 we conclude the proof of Proposition 1.6.

In Section 6 we prove Proposition 1.7, hence concluding the proof of Theorem 1. In Section 7 we prove Theorem 2, and finally in Section 8 we prove Theorem 3.

2 The random time τ

We colour the edges of $G^{(n)}$ with three different colours as follows. We colour the type n edges of $G^{(n)}$ blue, the type k edges for $k \in \{0.9n, ..., n-1\}$ red, and all remaining edges green. (See Figure 1 for an illustration.)

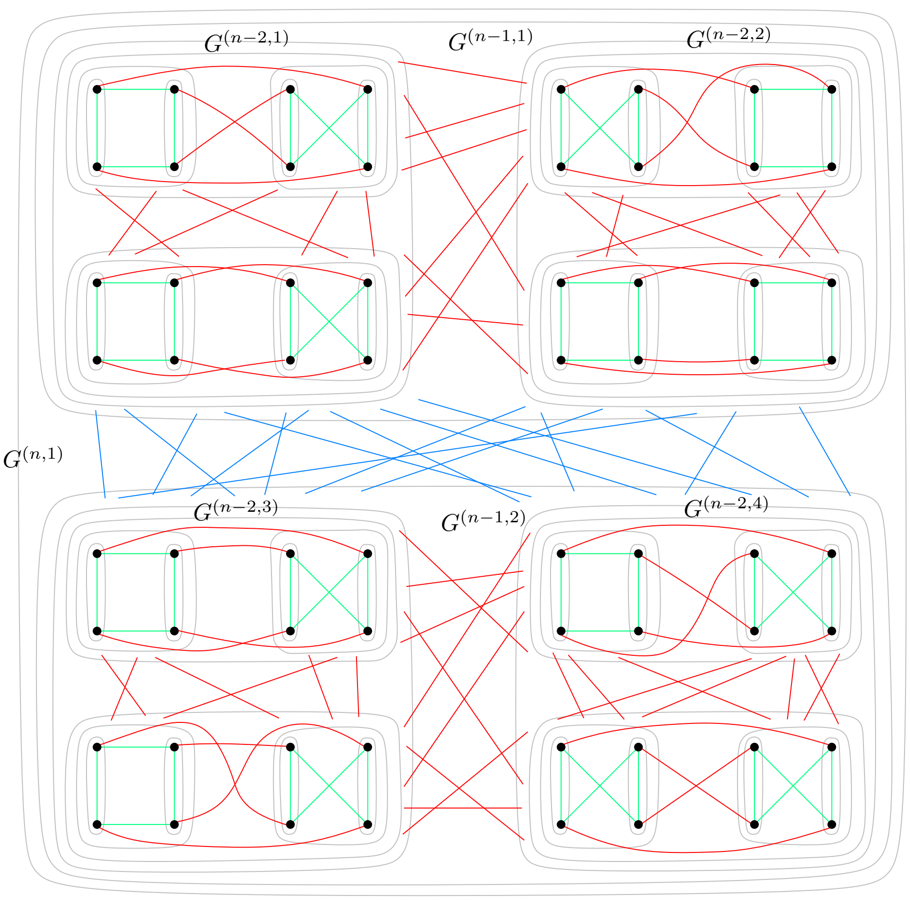

Figure 1: An illustration of $G^{(n)}$, the naming of its subgraphs, and the colouring of its edges.

Given $G^{(n)}$, we define a random time τ and an auxiliary walk $\widetilde{X}$ as follows.

Definition 2.1. Let

$$L := \quad C_L \frac{n}{\log n} + C_{L,2}\sqrt{\frac{n}{\log n}}, \qquad K := \quad C_K, \tag{3}$$

where C_L is an appropriately chosen constant (see choice after Lemma 3.2) and $C_{L,2}$ and C_K are sufficiently large constants, to be specified later.

Let $\tau_{\text{red}}^{(L+K,j)} = g^{(1,j)} + ... + g^{(L+K,j)} + (L+K-1)$ for $j = 1, 2$ where $g^{(i,j)}$ are independent $\text{Geom}_{\geq 0}\left(\frac{0.1n}{n-1}\right)$ random variables and let $\tau_{\text{geom}} \sim \text{Geom}_{\geq 0}\left(\frac{1}{n}\right)$, independently of these. Let

$$\tau := \quad \tau_{\text{geom}} + 1 + \tau_{\text{red}}^{(L+K,1)} + 1 + \tau_{\text{red}}^{(L+K,2)}.$$

Let us fix a vertex x of $G^{(n)}$, and let $\widetilde{X}$ be a walk on $G^{(n)}$ defined up to time τ, taking steps as follows. The first τ_{geom} steps are a SRW from x on the green and red edges of $G^{(n)}$ (i.e. on the edges of the type $(n-1)$ subgraph containing x). Then for $i = 1, 2, ..., L+K$ it crosses a uniformly chosen red edge emenating from its current position, then it takes $g^{(i,1)}$ steps as a SRW on the green edges. Then it crosses the blue edge emenating from its current position. Then for $i = L+K, ..., 2$ it takes $g^{(i,2)}$ steps as a SRW on the green edges and crosses a uniformly chosen red edge from its location. Finally, it takes $g^{(1,2)}$ steps as a SRW on the green edges. See Figure 2 for an illustration. //

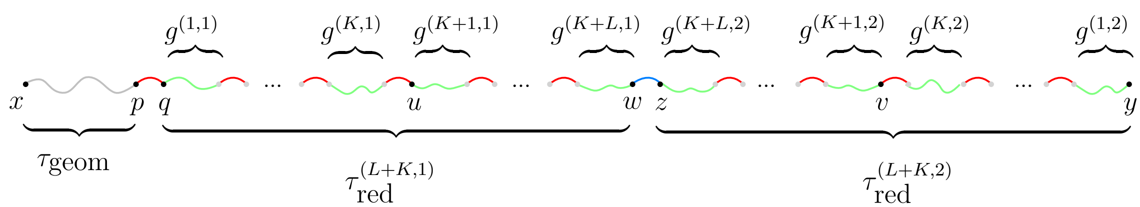

Figure 2: An illustration of the walk $\widetilde{X}$ up to time τ.

The proof of Proposition 1.6 relies on the following three results.

The distribution of $\widetilde{X}_\tau$ is comparable to the uniform distribution $\mathcal{U}$.

Proposition 2.2. *There exists a constant θ such that with high probability $G^{(n)}$ has the following property. For any vertices x and y of $G^{(n)}$ that are in different type $(n-1)$ subgraphs, we have*

$$\mathbb{P}_x\left(\widetilde{X}_\tau = y \,\middle|\, G^{(n)}\right) \quad \geq \quad \frac{\theta}{N}.$$

The time τ is likely of order at most n.

Lemma 2.3. *For any $\theta \in (0, 1)$, for a sufficiently large constant C, for any realisation of the graph $G^{(n)}$ and any vertex x, we have $\mathbb{P}_x\left(\tau > Cn \,\middle|\, G^{(n)}\right) \leq \theta$.*

The walk $\widetilde{X}$ can be coupled closely with a simple random walk on $G^{(n)}$.

Lemma 2.4. *For any choice of C_L, $C_{L,2}$ and C_K, any realisation of the graph $G^{(n)}$ and any vertex x, there exists a coupling between the pair $\left(\tau, \widetilde{X}\right)$ defined as above, and a simple random walk X on $G^{(n)}$ starting from x, such that with probability $1 - o(1)$ the walks X and $\widetilde{X}$ agree up to time τ. Here the $o(1)$ can depend on the parameters C_L, $C_{L,2}$ and C_K, but it does not depend on the realisation of $G^{(n)}$ and the choice of the vertex x.*

The proof of Lemma 2.3 is very simple and omitted. The proof of Lemma 2.4 is also quick, and it is presented below.

Proof of Lemma 2.4. Note that a simple random walk X on $G^{(n)}$ at each step crosses a blue edge with probability $\frac{1}{n}$, independently for different steps, and conditional on which steps are blue, the remaining steps are green with probability $\frac{0.9n-1}{n-1}$ and red otherwise, independently of each other. Because of this we can sample a simple random walk X on $G^{(n)}$ up to the second time it crosses a blue edge as follows.

Let $\tau_{\text{blue}}^{(1)}, \tau_{\text{blue}}^{(2)} \sim \text{Geom}_{\geq 1}\left(\frac{1}{n}\right)$ and let $g^{(i,j)} \sim \text{Geom}_{\geq 0}\left(\frac{0.1n}{n-1}\right)$, all independently of each other.

Let $B^{(j)} = \tau_{\text{blue}}^{(j)}$, let $G_1^{(j)} = g^{(1,j)} \wedge (B^{(j)}-1)$, $R_1^{(j)} = 1 \wedge (B^{(j)}-1-G_1^{(j)})$, $G_2^{(j)} = g^{(2,j)} \wedge (B^{(j)}-1-G_1^{(j)}-R_1^{(j)})$, ... $R_{L+K}^{(j)} = 1 \wedge (B^{(j)}-1-G_1^{(j)}-R_1^{(j)} ... - G_{L+K}^{(j)})$, $S^{(j)} = B^{(j)}-1-G_1^{(j)} - ... - R_{L+K}^{(j)}$.

Let X take $S^{(1)}$ steps as a SRW on the green and red edges, then for $i = L+K, L+K-1, ..., 1$ let it take $R_i^{(1)}$ steps as a SRW on the red edges, then $G_i^{(1)}$ steps as a SRW on the green edges. Then let it cross a blue edge. Then for $i = 1, 2, ..., L+K$ let it take $G_i^{(2)}$ steps as a SRW on the green edges, then $R_i^{(2)}$ steps as a SRW on the red edges. Finally, let it take $S^{(2)}$ steps as a SRW on the green and red edges, and then let it cross a blue edge.

Having this representation of X, let us couple it to $\widetilde{X}$ as follows. Let us sample $g^{(i,j)}$, and conditional on these, let us couple $\tau_{\text{blue}}^{(1)} \sim \text{Geom}_{\geq 0}\left(\frac{1}{n}\right)$ and $\tau_{\text{geom}} + \tau_{\text{red}}^{(L+K,1)} + 1 \sim \text{Geom}_{\geq 1}\left(\frac{1}{n}\right) + g^{(1,1)} + ... + g^{(L+K,1)} + L + K + 1$ using their optimal coupling, so that they agree on the event $\left\{\tau_{\text{blue}}^{(1)} \geq g^{(1,1)} + ... + g^{(L+K,1)} + L + K + 1\right\}$.

Then on the event $\left\{\tau_{\text{blue}}^{(j)} \geq g^{(1,j)} + ... + g^{(L+K,j)} + L + K + 1 \text{ for } j = 1, 2\right\}$, the walks X and $\widetilde{X}$ agree up to time τ. It is easy to see that this event has probability $1 - o(1)$. $\square$

In the rest of this section and the following sections we will be working towards the proof of Proposition 2.2. The proof of Proposition 1.6 given Proposition 2.2 and Lemmas 2.3 and 2.4 is presented in Section 5.

For any two vertices x and y that are not in the same type $(n-1)$ subgraph of $G^{(n)}$, we can express the transition probability $\mathbb{P}_x\left(\widetilde{X}_\tau = y \,\middle|\, G^{(n)}\right)$ as follows.

Let Y be a SRW on the green and red edges of $G^{(n)}$, independently of τ_{geom}, and let τ_{red}^i denote the ith time that Y crosses a red edge. For a vertex w of $G^{(n)}$ let $\eta(w)$ denote its blue neighbour. For vertices p, q of $G^{(n)}$ let $\{p \sim_r q\}$ denote the event that p and q are connected via a red edge. Then we have

$$\begin{aligned} &\mathbb{P}_x\left(\widetilde{X}_\tau = y \,\middle|\, G^{(n)}\right) \qquad (4) \\ &= \sum_{p,q,u,w,z,v} \mathbb{P}_x\left(Y_{\tau_{\text{geom}}} = p \,\middle|\, G^{(n)}\right) \frac{\mathbf{1}_{p \sim_r q}}{0.1n} \mathbb{P}_q\left(Y_{\tau_{\text{red}}^K} = u \,\middle|\, G^{(n)}\right) \mathbb{P}_y\left(Y_{\tau_{\text{red}}^K} = v \,\middle|\, G^{(n)}\right) \\ &\qquad \cdot \mathbb{P}_u\left(Y_{\tau_{\text{red}}^L - 1} = w \,\middle|\, G^{(n)}\right) \mathbb{P}_v\left(Y_{\tau_{\text{red}}^L - 1} = z \,\middle|\, G^{(n)}\right) \mathbf{1}_{\eta(w) = z}, \end{aligned}$$

where the sums are taken over all vertices of $G^{(n)}$.

In what follows we will show that for most u and v in different type $(n-1)$ subgraphs, their neighbourhoods according to the green and red edges, and the walks on these neighbourhoods can be approximated well by some random tree-like structures (quasi-trees) and walks on these. We use this to get a good control over $\mathbb{P}_u\left(Y_{\tau_{\text{red}}^L - 1} = w \,\middle|\, G^{(n)}\right)$ and $\mathbb{P}_v\left(Y_{\tau_{\text{red}}^L - 1} = z \,\middle|\, G^{(n)}\right)$. Then we use the randomness of the blue edges to show that $\sum_{w,z} \mathbb{P}_v\left(Y_{\tau_{\text{red}}^L - 1} = w \,\middle|\, G^{(n)}\right) \mathbb{P}_y\left(Y_{\tau_{\text{red}}^L - 1} = z \,\middle|\, G^{(n)}\right) \mathbf{1}_{\eta(w) = z}$ is concentrated.

To extend the result to all x and y in different type $(n-1)$ subgraphs, we show that for any starting point, with high probability $Y_{\tau^K_{\mathrm{red}}}$ is such that the above approximation can be applied for its neighbourhood.

3 The random quasi-tree and its coupling to $G^{(n)}$

In this section we define random quasi-trees and explain how to we use these to approximate the green and red neighbourhoods of two vertices in $G^{(n)}$.

3.1 Definition of a quasi-tree

Analogously to [1] we consider a random quasi-tree with the green edges forming the balls, and the red edges serving as long-range edges.

Let

$$R := \quad C_R \log n, \tag{5}$$

where C_R is a sufficiently large constant, to be specified later.

Definition 3.1. A *random quasi-tree* T is a random graph together with a map ι from its vertices to the vertex set $V^{(n)}$ of $G^{(n)}$, obtained as follows. Let us start from a root vertex ρ, with a given $\iota(\rho)$. Then consider a ball of radius R around $\iota(\rho)$ in a random copy of the type $(0.9n-1)$ subgraph containing $\iota(\rho)$, and attach it to ρ. We call this an *R-ball*. For each vertex v in the ball let $\iota(v)$ be the corresponding vertex in $G^{(n)}$. Then for each vertex v in the ball, except for ρ we add $0.1n$ edges leading to new vertices $w_{v,0.9n}, ..., w_{v,n-1}$, while for $v = \rho$ we add $0.1n - 1$ new edges leading to new vertices $w_{v,0.9n+1}, ..., w_{v,n-1}$. We call these *long-range edges*, and in particular call $(v, w_{v,k})$ a type k long-range edge. For each $w_{v,k}$, we sample $\iota(w_{v,k})$ by choosing uniformly from the 2^k vertices of $G^{(n)}$ with the property that they are in the same type $(k+1)$ subgraph as $\iota(v)$ but in different type k subgraphs. Analogously to the above, we consider a ball of radius R in an independent random copy of the type $(0.9n-1)$ subgraph containing $\iota(w)$, attach it to w and let ι map each of its vertices to the corresponding vertex of $G^{(n)}$. We continue analogously, but if v is the new endpoint of a newly revealed long-range edge of type k, then the further long-range edges we add from v should be of types $\{0.9n, ..., n-1\} \setminus \{k\}$. This way we obtain an infinite random graph consisting of R-balls and long-range edges between them, where apart from ρ, each vertex is the endpoint of exactly $0.1n$ long-range edges which are of types $0.9n, ..., n-1$. //

Note that the edges of the R-balls correspond to green edges, and the long-range edges correspond to red edges, but the R-balls and the long-range edges are always sampled from a new copy of $G^{(n)}$.

For a vertex v in the quasi-tree, we define its level as the minimal number of long-range edges a path from ρ to v needs to cross. For a long-range edge e, we define its level as the level of its endpoint that is further from ρ, and we denote it by $\ell(e)$.

For a vertex v of the quasi-tree T, we let $T(v)$ denote the subgraph of T induced by the R-ball of v and the R-balls descending from this.

Let Z be a walk on the quasi-tree that takes steps as a SRW, except that it is not allowed to cross a long-range in the direction towards to root.[1] Let ξ_k be the kth long-range edge crossed by Z.

If Z starts from ρ, then the entropy of ξ_k is concentrated in the following sense.

[1]If Z is at ρ or at a vertex that is not the centre of an R-ball, it takes a step to a uniform neighbour. Otherwise it disregards the one neighbour that has a smaller level than the current vertex, chooses uniformly from the others, and takes a step there.

Lemma 3.2. *For any $\theta > 0$, there exists a constant $C > 0$ with the following property. Let T be a random quasi-tree and let $\xi^{(1)}$ and $\xi^{(2)}$ be two independent copies of ξ on T. Then for any k we have*

$$\mathbb{P}\Big(\Big|-\log \mathbb{P}_\rho\Big(\xi_k^{(1)} = \xi_k^{(2)} \;\Big|\; \xi^{(1)}, T\Big) - k\mathfrak{h}\Big| > C\sqrt{k\mathfrak{V}}\Big) \quad \leq \quad \theta,$$

where $\mathfrak{h} = c_1 \log n$, $\mathfrak{V} = c_2(\log n)^2$ and $c_1, c_2 \asymp 1$ as $n \to \infty$.

Proof. Let us use the notation $\mathrm{H}_b(\cdot)$ and $\mathrm{H}_b(\cdot|\cdot)$ like in [2, Definitions 2.23 and 2.24], and their properties from Section B of the same article.

Then we can note that for any realisation t of T, we have $\mathrm{H}_1(\xi_1|T = t) \asymp \log n$ and $\mathrm{H}_2(\xi_1|T = t) \asymp (\log n)^2$, and the values of these only depend on the R-ball of ρ. [2]

Since the R-balls are iid and the walk Z does not backtrack long-range edges, we get that $-\log \mathbb{P}_\rho\Big(\xi_k^{(1)} = \xi_k^{(2)} \;\Big|\; \xi^{(1)}, T\Big)$ decomposes as a sum of iid copies of $-\log \mathbb{P}_\rho\Big(\xi_1^{(1)} = \xi_1^{(2)} \;\Big|\; \xi^{(1)}, T\Big)$. We can also note that $\mathrm{Var}\Big(-\log \mathbb{P}_\rho\Big(\xi_1^{(1)} = \xi_1^{(2)} \;\Big|\; \xi^{(1)}, T\Big)\Big) \leq \mathrm{H}_2(\xi_1|T)$, hence for any θ, for sufficiently large C we have

$$\mathbb{P}\Big(\Big|-\log \mathbb{P}_\rho\Big(\xi_k^{(1)} = \xi_k^{(2)} \;\Big|\; \xi^{(1)}, T\Big) - k\mathfrak{h}\Big| > C\sqrt{k\mathfrak{V}}\Big) \quad \leq \quad \theta$$

where $\mathfrak{h} = \mathrm{H}_1(\xi_1|T) = c_1 \log n$, $\mathfrak{V} = \mathrm{H}_2(\xi_1|T) = c_2(\log n)^2$ with $c_1, c_2 \asymp 1$ as $n \to \infty$. □

Let us choose the constant C_L in (3) as $C_L = \frac{\log 2}{2c_1}$.

3.2 Coupling to $G^{(n)}$

In this section we explain how we couple the (green and red) neighbourhoods of two vertices of $G^{(n)}$ and the walks on them with random quasi-trees and walks on them.

Similarly to previous works, we are not able to closely couple the neighbourhood of a vertex v in $G^{(n)}$ to the entire first L levels of a quasi-tree T, since the chance that some vertices appear multiple times in the ι of the first L levels of T is too high. Instead, we focus on coupling the parts that are likely to be visited by a walk. If a long-range edge of T is unlikely to be crossed by Z, then we disregard the part of T descending from it (we 'truncate' that edge). Below we define 'truncation criteria' that specify when exactly to do this. Also, if we already have a disagreement between some parts of T and $G^{(n)}$, then we do not continue trying to couple the parts of T descending from it.

We also restrict our attention to vertices v of $G^{(n)}$ that are 'K-roots', i.e. where we already know that their neighbourhood is tree-like in the first $K < L$ levels, and we only couple with T from then onwards.

Once we coupled the neighbourhood of v with a tree T, we also couple the walks Y and Z on these. We say that the coupling is successful if Y and Z reach level L without visiting any parts of $G^{(n)}$ and T that disagree or got truncated.

[2] We drop the conditioning on $T = t$ from the notation. Let τ_{red} be the first time that Z crosses a long-range edge. Then $\tau_{\mathrm{red}} \leq_{\mathrm{st}} \mathrm{Geom}_{\geq 1}\Big(\frac{0.1n}{n-1}\Big)$, hence $\mathrm{H}_1(\tau_{\mathrm{red}}) \asymp \mathrm{H}_2(\tau_{\mathrm{red}}) \asymp 1$. Also, Z_k can take $\leq n^k$ different values, hence $\mathrm{H}_1(Z_k|\tau_{\mathrm{red}} = k) \leq k \log n$ and $\mathrm{H}_2(Z_k|\tau_{\mathrm{red}} = k) \leq (k \log n)^2$. Conditional on the value of τ_{red} and $Z_{\tau_{\mathrm{red}}-1}$, the value of ξ_1 is uniform over $0.1n$ or $0.1n - 1$ different edges, hence we have $\mathrm{H}_1(\xi_1|Z_{\tau_{\mathrm{red}}} = v, \tau_{\mathrm{red}} = k) \asymp \log n$ and $\mathrm{H}_2(\xi_1|Z_{\tau_{\mathrm{red}}} = v, \tau_{\mathrm{red}} = k) \asymp (\log n)^2$ for any possible v and k. Using these and that $\mathrm{H}_b(\xi_1|Z_{\tau_{\mathrm{red}}}, \tau_{\mathrm{red}}) - O(1) \lesssim \mathrm{H}_b(\xi_1) \lesssim \mathrm{H}_b(\xi_1|Z_{\tau_{\mathrm{red}}}, \tau_{\mathrm{red}}) + \mathrm{H}_b(Z_{\tau_{\mathrm{red}}}|\tau_{\mathrm{red}}) + \mathrm{H}_b(\tau_{\mathrm{red}})$ we get the result.

Let $K = C_K$ as defined in (3), where C_K is a sufficiently large constant, to be specified later.

For a long-range edge e of a given quasi-tree T and a given constant A we define truncation criteria as follows.

$$\mathrm{Tr}(e,A) := \quad \left\{ -\log \mathbb{P}\big(\xi_{\ell(e)} = e \mid \xi, T\big) > \frac{1}{2}\log N + A\sqrt{(\log\log N)(\log N)} \right\},$$

$$\mathrm{Tr}'(e) := \quad \left\{ -\log \mathbb{P}(\xi_{L-K} = e) < \frac{1}{2}\log N + \sqrt{(\log\log N)(\log N)} \right\}.$$

Then we can bound the probability of Z crossing a truncated edge before hitting level $L-K$ as follows.

Lemma 3.3. *For any $\theta > 0$, for sufficiently large values of A (in terms of θ and C_L), and sufficiently large n we have*

$$\mathbb{P}\left(\bigcup_{k=1}^{L-K} \mathrm{Tr}(\xi_k, A) \right) < \theta.$$

Also, for sufficiently large values of $C_{L,2}$ (in terms of θ and C_L), and sufficiently large n we have

$$\mathbb{P}\big(\mathrm{Tr}'(\xi_{L-K})\big) < \theta.$$

Sketch proof. We use that $\mathrm{Tr}(e,A)$ implies $\mathrm{Tr}(e',A)$ for all descendants e' of e, and Lemma 3.2 with $k = L-K$. $\square$

Definition 3.4. Let u be a vertex of $G^{(n)}$ and let e_u be the type $0.9n$ red edge from it. Then we say that u is a K-root in $G^{(n)}$ if a neighbourhood of u of green and red edges in $G^{(n)} \setminus \{e_u\}$ is a possible realisation of (ι of) the first K levels of a quasi-tree. We denote the set of K-roots by $V_{K-\mathrm{root}}$. //

Let us consider any two vertices x and y that are in different type $(n-1)$ subgraphs of $G^{(n)}$ and work on the event that both of them are K-roots, with corresponding neighbourhoods $T_{x,0}$ and $T_{y,0}$ respectively. Then we couple the neighbourhoods of x and y with the first L levels of two independent quasi-trees T_x and T_y conditioned to have the first K levels as $T_{x,0}$ and $T_{y,0}$ respectively, as follows.

We denote the level K vertices of T_x by z_1, ..., z_{L_x}, and the level K vertices of T_y by z_{L_x+1}, ..., $z_{L_x+L_y}$. Then we explore from z_1 up to level L as follows.

Firstly, we pick $k \in \{0.9n, ..., n-1\}$ such that the type k long-range edge of v is not yet explored, we reveal the other endpoint v of the type k long-range edge from z_1 in T_x, and couple $\iota(v)$ with the other endpoint of the type k red edge from $\iota(z_1)$ in $G^{(n)}$ according to the optimal coupling. We also couple the R-ball of v with the distance R green neighbourhood of $\iota(v)$ in $G^{(n)}$ according to the optimal coupling. If either of these optimal couplings fails, we say that the long-range edge (z_1, v) is truncated due to the optimal coupling failing. Otherwise, if the distance R green neighbourhood of $\iota(v)$ contains any of the already explored vertices, we say that (z_1, v) is truncated due to an overlap. Otherwise, if $\mathrm{Tr}((z_1, v), A)$ or $\mathrm{Tr}'((z_1, v))$ holds with respect to $T_x(z_1)$, we say that (z_1, v) is truncated according to a truncation criterion. We proceed similarly with the other yet unexplored long-range edges from z_1. If a long-range edge (z_1, v) is truncated for any of the above reasons, then we will not explore further into this direction. Otherwise we continue exploring analogously from each vertex in the R-ball of v. We continue the exploration up to level L. (In each level, the truncation criteria are considered with respect to $T_x(z_1)$.)

Then we explore similarly from z_2, z_3, ..., $z_{L_x+L_y}$. (When exploring from z_i we consider the truncation criteria with respect to $T_x(z_i)$ or $T_y(z_i)$.)

We denote by $\mathcal{F}_i$ the σ-algebra generated by the explorations from z_1, ..., z_i. We say that z_i is *good* if it is not contained in any R-ball revealed during the explorations from z_1, ..., z_{i-1}. Otherwise we call z_i *bad*.

After coupling the neighbourhoods of x and y with the trees, we couple independent walks $Y^{(1)}$ and $Y^{(2)}$ on the green and red edges of $G^{(n)}$ from x and y respectively, with independent walks $Z^{(1)}$ and $Z^{(2)}$ on T_x and T_y from x and y respectively such that the following are satisfied. Let $\tau_{\text{red}}^{(L,i)}$ denote the Lth time that $Y^{(i)}$ crosses a red edge. We say that a vertex v of T is at the boundary of an R-ball if it is at graph distance R from the centre of the R-ball it is contained in.

We couple $Y^{(1)}$ and $Y^{(2)}$ with $Z^{(1)}$ and $Z^{(2)}$ up to $\tau_{\text{red}}^{(K,1)}$ and $\tau_{\text{red}}^{(K,2)}$ respectively such that on the event that for $i = 1, 2$

(i) up to $\tau_{\text{red}}^{(K,i)}$ the walk $Z^{(i)}$ does not hit the boundary of an R-ball,

(ii) the first red edge that $Y^{(i)}$ crosses is not the type $(0.9n - 1)$ red edge from $Y_0^{(i)}$, and

(iii) up to the Kth time that $Y^{(i)}$ crosses a red edge, it never backtracks the most recently crossed red edge,

we have $Y_j^{(i)} = \iota\left(Z_j^{(i)}\right)$ for $i = 1, 2$, $j \leq \tau_{\text{red}}^{(K,i)}$.

We say that the coupling is successful in the first K levels if (i), (ii) and (iii) hold. In this case if $Z^{(i)}$ reached level K at z_{k_i}, then we couple the rest of the walks $Y^{(1)}$ and $Y^{(2)}$ with $Z^{(1)}$ and $Z^{(2)}$ from z_{k_1} and z_{k_2} until $\tau_{\text{red}}^{(L,1)}$ and $\tau_{\text{red}}^{(L,2)}$ respectively such that on the event that for $i = 1, 2$

(vi) between $\tau_{\text{red}}^{(K,i)}$ and $\tau_{\text{red}}^{(L,i)}$ the walk $Z^{(i)}$ does not cross a truncated long-range edge and does not hit the boundary of an R-ball,

(v) between the Kth and the Lth time that $Y^{(i)}$ crosses a red edge, it never backtracks the most recently crossed red edge,

we have $Y_j^{(i)} = \iota\left(Z_j^{(i)}\right)$ for $i = 1, 2$, $\tau_{\text{red}}^{(K,i)} \leq j \leq \tau_{\text{red}}^{(L,i)}$.

We say that the coupling is successful between levels K and L if (iv) and (v) hold.

We say that the coupling is successful if it is successful in the first K levels, and also between levels K and L.

4 Estimates regarding K-roots and the success of coupling

In this section we prove an estimate about $V_{K-\text{root}}$ that will make it sufficient to consider the case when u and v in (4) are both K-roots, and then we estimate the probability that the coupling around two K-roots succeeds.

4.1 Estimates regarding K-roots

In this section we show that for any starting vertex, $Y_{\tau_{\text{red}}^K}$ is a K-root with high probability.

Lemma 4.1. *For sufficiently large values of C_K the following holds. With high probability $G^{(n)}$ is such that for any vertex q we have $\mathbb{P}_q\Big(Y_{\tau_{\text{red}}^K} \in V_{K-\text{root}} \;\Big|\; G^{(n)}\Big) = 1 - o(1)$.*

Proof. Note that the number of vertices in a given R-ball is $\leq 2n^R$ and for any k the number of vertices in the first k levels of a quasi-tree is $\leq n^{2kR}$. Also, if there are s red edges revealed in $G^{(n)}$ so far and we are revealing the red edge of a given type, from a given vertex, the probability of the other endpoint taking a specific value is $\leq \frac{1}{2^{0.9n-1}-s}$. If $s \leq 2^{0.1n}$, then this is $\leq 4N^{-\frac{9}{10}}$.

Let us consider a given vertex q of $G^{(n)}$ and reveal its 'level $2K$ neighbourhood' in $G^{(n)}$ as follows. Firstly, let us reveal the green edges around it up to green distance R. Then for each vertex in this ball let us reveal the other endpoint of the red edges from it, except for the type $0.9n$ red edge from q. For each newly revealed endpoint, reveal the green edges around it up to green distance R. Then for each vertex in these newly revealed balls let us reveal the other endpoint of the red edges emenating form it. Let us continue similarly for K levels. We say that an overlap occurs if a newly revealed ball of green distance R intersects with any previously revealed ball.

From the above estimates we get that the probability that the level $2K$ neighbourhood of a given vertex q contains more than one an overlap is $\leq \mathbb{P}\Big(\text{Bin}\Big(n^{4RK}, \frac{4n^{3RK}}{N^{\frac{9}{10}}}\Big) > 1\Big) \lesssim \Big(\frac{n^{5RK}}{N^{\frac{9}{10}}}\Big)^2 \ll \frac{1}{N}$. If there is only 1 overlap, then whp the first vertex reached by the walk that is at long-range distance K from q is a K-root. Also, the probability of the walk backtracking any of the first K long-range edges crossed is $o(1)$, so whp crossing K red edges corresponds to reaching long-range distance K.

So whp $G^{(n)}$ is such that the level $2K$ neighbourhood of each q has at most 1 overlap, and on this event we have $\mathbb{P}_q\Big(Y_{\tau_{\text{red}}^K} \in V_{K-\text{root}} \;\Big|\; G^{(n)}\Big) = 1 - o(1)$. □

4.2 Estimates regarding the success of coupling

In this section we prove that with high probability $G^{(n)}$ is such that for any two K-roots in different type $(n-1)$ subgraphs, the coupling described in Section 3.2 has a positive probability of succeeding.

Note that in a given quasi-tree T the number of non-truncated long-range edges at a given level is $\leq \exp\left(\frac{1}{2}\log N + A\sqrt{(\log\log N)(\log N)}\right) = N^{\frac{1}{2}+A\sqrt{\frac{\log\log N}{\log N}}}$. In T_x and T_y the number of vertices up to level K is $\leq n^{2KR}$ and the number of explored vertices in each level of each $T(z_i)$ is $\leq N^{\frac{1}{2}+A\sqrt{\frac{\log\log N}{\log N}}}$. Hence the overall number of explored vertices up to level L is $\leq Ln^{2KR}N^{\frac{1}{2}+A\sqrt{\frac{\log\log N}{\log N}}} \ll N^{\frac{1}{2}+\alpha}$ for any $\alpha > 0$.

Lemma 4.2. *There exists a constant B with the following property. Let x and y be two vertices in different type $(n-1)$ subgraphs of $G^{(n)}$ and let $T_{x,0}$ and $T_{y,0}$ be possible realisations of the first K levels of quasi-trees. Let $\{T_{x,0}, T_{y,0}\}$ denote the event that some neighbourhoods of x and y in $G^{(n)}$ look like $T_{x,0}$ and $T_{y,0}$ respectively. Then the number* Bad *of bad vertices (defined in Section 3.2) satisfies*

$$\mathbb{P}(\text{Bad} > B \mid T_{x,0}, T_{y,0}) \quad \ll \quad \frac{1}{N^2}\,.$$

Proof. At each step of the exploration the probability of the newly revealed R-ball containing a z_i is $\lesssim \frac{n^{2RK}n^{2R}}{N^{\frac{9}{10}}}$. This is $\ll \frac{1}{N^{\frac{9}{10}-\alpha}}$ for any $\alpha > 0$. So

$$\mathbb{P}(\text{Bad} > B \mid T_{x,0}, T_{y,0}) \quad \leq \quad \binom{N^{\frac{1}{2}+\alpha}}{B}\left(\frac{1}{N^{\frac{9}{10}-\alpha}}\right)^B \quad \leq \quad N^{-B\left(\frac{2}{5}-2\alpha\right)},$$

which is $\ll \frac{1}{N^2}$ for α sufficiently small and B sufficiently large. $\square$

Lemma 4.3. *For any $\theta \in (0,1)$ for sufficiently large value of C_R and sufficiently large values of n, for all $i \in \{1, 2, ..., L_x\}$, the following holds. Let us consider the coupling of $Y^{(1)}$ and $Z^{(1)}$ from z_i, between levels K and L, as in Section 3.2. Then we have*

$$\mathbb{P}_{z_i}\Big(\textit{coupling of } Y^{(1)} \textit{ and } Z^{(1)} \textit{ succeeds between levels } K \textit{ and } L \;\Big|\; \mathcal{F}_{i-1}\Big) \quad \geq \quad (1-\theta)\mathbf{1}_{\{z_i \ good\}}.$$

The analogous result holds for $i \in \{L_x+1, ..., L_x+L_y\}$ with $Y^{(2)}$ and $Z^{(2)}$ instead of $Y^{(1)}$ and $Z^{(1)}$.

Proof. In case z_i is good, we know that the R-ball of z_i does not intersect any of the already revealed R-balls. Now consider the exploration from z_i and the walks from z_i.

The probability that the walk $Z^{(j)}$ up to time $\tau_{\text{red}}^{(L,j)}$ crosses a long-range edge that is truncated due to an overlap is $\lesssim Ln^{2R}\frac{Ln^{2R}N^{\frac{1}{2}+\alpha}}{N^{\frac{9}{10}}} \ll 1$. Similarly, the probability of $Z^{(j)}$ crossing a long-range edge where the optimal coupling failed is also $\ll 1$.

The probability of $Z^{(j)}$ hitting the boundary of an R-ball is $\lesssim L\mathbb{P}\big(\text{Geom}_{\geq 0}\big(\frac{0.1n-1}{n}\big) > R\big) \lesssim \frac{n}{\log n}\left(\frac{9.1}{10}\right)^{C_5 \log n}$. This is $\ll 1$ for sufficiently large values of C_R.

The probability that up to the Lth time that $Y^{(j)}$ crosses a red edge, it ever backtracks the most recently crossed red edge is $\leq L\frac{1}{0.1n} \ll 1$. $\square$

Lemma 4.4. *For sufficiently large values of C_R the following holds. For any realisation of the first K levels of $T_{x,0}$ and $T_{y,0}$, the probability that the coupling of $Z^{(i)}$ and $Y^{(i)}$ is successful in the first K levels is $1 - o(1)$.*

Proof. The probability that the first red edge crossed by $Y^{(j)}$ is the type $0.9n$ one from $Y_0^{(j)}$ is $\leq \frac{1}{0.1n} \ll 1$.

The probabilities of the other ways of the coupling failing can be bounded analogously to the proof of Lemma 4.3. $\square$

Lemma 4.5. *For any $\theta \in (0,1)$, for sufficiently large values of the constants C_R and C_K, the following holds. For x, y and $\{T_{x,0}, T_{y,0}\}$ as in Lemma 4.2 we have*

$$\mathbb{P}\Big(\mathbb{P}_x\Big(\textit{the coupling of } Y^{(1)} \textit{ and } Z^{(1)} \textit{ succeeds} \;\Big|\; \mathcal{F}_{L_x+L_y}\Big) > 1-\theta \;\Big|\; T_{x,0}, T_{y,0}\Big) \quad \geq \quad 1 - o\left(\frac{1}{N^2}\right).$$

The analogous result holds for $Y^{(2)}$ and $Z^{(2)}$ starting from y.

Proof. Let

$$\widehat{V} := \quad \Big\{z_i : \mathbb{P}_x\Big(\xi_K^+ = z_i, \text{ coupling of } X^{(1)} \text{ and } Z^{(1)} \text{ fails} \;\Big|\; \mathcal{F}_{L_x+L_y}\Big) \geq \theta\Big\},$$

$$h(z_i) := \quad \mathbb{P}_x\big(\xi_K^+ = z_i \bigm| \mathcal{F}_{L_x+L_y}\big)\,,$$

where ξ_K^+ denotes the endpoint of ξ_K further from the root.

We will show that

$$\mathbb{P}\Big(h(\widehat{V}) > 3\theta \Bigm| T_{x,0}, T_{y,0}\Big) \quad \ll \quad \frac{1}{N^2}. \tag{6}$$

Once we have this, the proof is immediate (we have to replace θ by $\frac{1}{5}\theta$).

The proof of (6) is analogous to the proof of [9, Proposition 5.7]. We use that $h(z_i) \le \left(\frac{1}{0.1n-1}\right)^K \ll \frac{1}{n^2}$. □

5 Bounding the hit time

In this section we conclude the proof of Proposition 2.2, and then the proof of Proposition 1.6.

5.1 Proof of Proposition 2.2

For $u, v \in V_{K-\text{root}}$ in different type $(n-1)$ subgraphs of $G^{(n)}$, let $\mathcal{F}_{u,v} = \mathcal{F}_{L_u+L_v}$ be the σ-algebra generated by the explorations around them up to level L and let
$\Omega_{u,v} = \{$coupling of $(Y^{(1)}, Y^{(2)})$ and $(Z^{(1)}, Z^{(2)})$ from (u,v) succeeds$\}$.

Lemma 5.1. *For any $\theta \in (0,1)$, there exists $\theta' \in (0,1)$ with the following property. For any u and v in different type $(n-1)$ subgraphs of $G^{(n)}$, on the event that $u, v \in V_{K-\text{root}}$ and $\mathbb{P}(\Omega_{u,v} \mid \mathcal{F}_{u,v}) > \theta$, we have*

$$\mathbb{P}\left(\sum_{w,z} \mathbb{P}_{(u,v)}\left(Y^{(1)}_{\tau_{\text{red}}^{(L,1)}-1} = w, Y^{(2)}_{\tau_{\text{red}}^{(L,2)}-1} = z, \Omega_{u,v} \,\middle|\, G^{(n)}\right) \mathbf{1}_{\eta(w)=z} < \frac{\theta'}{N} \,\middle|\, \mathcal{F}_{u,v}\right) \quad \ll \quad \frac{1}{N^2}\,.$$

Proof. By the definition of $\mathrm{Tr}'(\cdot)$ we have

$$\mathbb{P}_v\left(Y^{(1)}_{\tau_{\text{red}}^{(L,1)}-1} = w, \Omega_{u,v} \,\middle|\, \mathcal{F}_{u,v}\right) \le \exp\left(-\frac{1}{2}\log N - \sqrt{(\log N)(\log\log N)}\right) < \frac{1}{\sqrt{N}\log N}$$

for all w, and similarly $\mathbb{P}_v\left(Y^{(2)}_{\tau_{\text{red}}^{(L,2)}-1} = z, \Omega_{u,v} \,\middle|\, \mathcal{F}_{u,v}\right) < \frac{1}{\sqrt{N}\log N}$ for all z. Also, conditional on $\mathcal{F}_{u,v}$, the type n blue edges form a uniform random matching between the two type $(n-1)$ subgraphs.

For each i and j in these type $(n-1)$ subgraphs let $w_{i,j} = \mathbb{P}_{(u,v)}\left(Y^{(1)}_{\tau_{\text{red}}^{(L,1)}-1} = i, Y^{(2)}_{\tau_{\text{red}}^{(L,2)}-1} = j, \Omega_{u,v} \,\middle|\, \mathcal{F}_{u,v}\right)$. Also, let $m = 2a = \frac{1}{2^{n-1}}\sum_{i,j} w_{i,j} \asymp \frac{1}{N}$ and $b = \max_{i,j} w_{i,j} \le \frac{1}{N(\log N)^2}$. Then with a proof analogous to the proof of [4, Lemma 5.1] we get that

$$\mathbb{P}\left(\sum_i w_{i,\eta_n(i)} \le m - a \,\middle|\, \mathcal{F}_{u,v}\right) \quad \le \quad \exp\left(-\frac{a^2}{4bm}\right) \quad \ll \quad \frac{1}{N^2}.$$

Note that the probability $\mathbb{P}_{(u,v)}\left(Y^{(1)}_{\tau_{\text{red}}^{(L,1)}-1} = w, Y^{(2)}_{\tau_{\text{red}}^{(L,2)}-1} = z, \Omega_{u,v} \,\middle|\, G^{(n)}\right)$ is $\mathcal{F}_{u,v}$-measurable, so we can change the conditioning on $G^{(n)}$ to a conditioning on $\mathcal{F}_{u,v}$. This finishes the proof. □

Let us choose $C_{L,2}$, C_R and C_K such that Lemmas 4.1 and 4.5 are satisfied.

Let θ_1 be the constant from Lemma 4.1, let θ_2 be an arbitrary constant in $(0,1)$ and let θ_3 be the corresponding constant θ' from Lemma 5.1.

Let Ω_0 be the high probability event that $G^{(n)}$ satisfies all of the following.

- For all q we have $\mathbb{P}_q\Big(Y_{\tau^K_{\text{red}}} \in V_{K-\text{root}} \,\Big|\, G^{(n)}\Big) \geq \theta_1$. (See Lemma 4.1.)
- For all $u,v \in V_{K-\text{root}}$ in different type $(n-1)$ subgraphs, we have $\mathbb{P}(\Omega_{u,v} \mid \mathcal{F}_{u,v}) > \theta_2$. (See Lemma 4.5.)
- For all $u,v \in V_{K-\text{root}}$ in different type $(n-1)$ subgraphs with $\mathbb{P}(\Omega_{u,v} \mid \mathcal{F}_{u,v}) > \theta_2$, we have $\sum_{w,z} \mathbb{P}_{(u,v)}\Big(Y^{(1)}_{\tau^{(L,1)}_{\text{red}}-1} = w, Y^{(2)}_{\tau^{(L,2)}_{\text{red}}-1} = z, \Omega_{u,v} \,\Big|\, G^{(n)}\Big) \mathbf{1}_{\eta(w)=z} \geq \frac{\theta_3}{N}$. (See Lemma 5.1.)

Proof of Proposition 2.2. Let us work on the high probability event Ω_0.

Consider any x and y that are in different type $(n-1)$ subgraphs. Let ℓ be such that x is in $G^{(n-1,\ell)}$. Then by (4) we have

$$\begin{aligned}
&\mathbb{P}_x\Big(\widetilde{X}_\tau = y \,\Big|\, G^{(n)}\Big) \\
\geq\ & \left(\inf_{q \in G^{(n-1,\ell)}} \mathbb{P}_q\Big(Y_{\tau^K_{\text{red}}} \in V_{K-\text{root}} \,\Big|\, G^{(n)}\Big)\right) \mathbb{P}_y\Big(Y_{\tau^K_{\text{red}}} \in V_{K-\text{root}} \,\Big|\, G^{(n)}\Big) \\
&\cdot \left(\inf_{\substack{u \in V_{K-\text{root}} \cap G^{(n-1,\ell)} \\ v \in V_{K-\text{root}} \cap G^{(n-1,3-\ell)}}} \sum_{w,z} \mathbb{P}_u\Big(Y_{\tau^L_{\text{red}}-1} = w \,\Big|\, G^{(n)}\Big) \mathbb{P}_v\Big(Y_{\tau^L_{\text{red}}-1} = z \,\Big|\, G^{(n)}\Big) \mathbf{1}_{\eta(w)=z}\right).
\end{aligned}$$

By the definition of Ω_0 this is $\geq \theta_1^2 \theta_3 \frac{1}{N}$. This finishes the proof. □

5.2 Proof of Proposition 1.6

Let θ be as in Proposition 2.2 and let us work on the corresponding high probability event. Then we have

$$1 - d_{\text{TV}}\left(\mathbb{P}_x\Big(\widetilde{X}_\tau = \cdot \,\Big|\, G^{(n)}\Big), \mathcal{U}(\cdot)\right) = \sum_y \frac{1}{N} \wedge \mathbb{P}_x\Big(\widetilde{X}_\tau = y \,\Big|\, G^{(n)}\Big) \geq N \cdot \frac{\theta}{N} = \theta,$$

hence, also using Lemma 2.4 (with $\frac{1}{2}\theta$), we get

$$\begin{aligned}
d_{\text{TV}}\left(\mathbb{P}_x\Big(X_\tau = \cdot \,\Big|\, G^{(n)}\Big), \mathcal{U}(\cdot)\right) &\leq \mathbb{P}_x\Big(X_\tau \neq \widetilde{X}_\tau \,\Big|\, G^{(n)}\Big) + d_{\text{TV}}\left(\mathbb{P}_x\Big(\widetilde{X}_\tau = \cdot \,\Big|\, G^{(n)}\Big), \mathcal{U}(\cdot)\right) \\
&\leq \frac{1}{2}\theta + (1-\theta) = 1 - \frac{1}{2}\theta.
\end{aligned}$$

From Lemma 2.3 we know that there exists C such that $\mathbb{P}_x\big(\tau > Cn \,\big|\, G^{(n)}\big) \leq \frac{1}{4}\theta$ for all x.

Let $\alpha = 1 - \frac{1}{8}\theta$ and let us consider any $x \in V^{(n)}$ and $A \subseteq V^{(n)}$ with $\mathcal{U}(A) \geq \alpha$.

Then $\mathbb{P}_x\big(\tau_A > Cn \,\big|\, G^{(n)}\big) \leq \mathbb{P}_x\big(\tau > Cn \,\big|\, G^{(n)}\big) + \mathbb{P}_x\big(\tau_A > \tau \,\big|\, G^{(n)}\big)$. We have $\mathbb{P}_x\big(\tau > Cn \,\big|\, G^{(n)}\big) \leq \frac{1}{4}\theta$ and $\mathbb{P}_x\big(\tau_A > \tau \,\big|\, G^{(n)}\big) \leq \mathbb{P}_x\big(X_\tau \notin A \,\big|\, G^{(n)}\big) \leq \mathcal{U}(A^c) + d_{\text{TV}}\left(\mathbb{P}_x\big(X_\tau = \cdot \,\big|\, G^{(n)}\big), \mathcal{U}(\cdot)\right) \leq (1-\alpha) + \big(1 - \frac{1}{8}\theta\big)$. So overall we get that $\mathbb{P}_x\big(\tau_A > Cn \,\big|\, G^{(n)}\big) \leq 1 - \frac{1}{8}\theta$

This shows that $\text{hit}_{1-\frac{1}{8}\theta}\left(1 - \frac{1}{8}\theta\right) \leq Cn$, and so finishes the proof.

6 Bounding the absolute relaxation time

In this section we prove Proposition 1.7.

We will prove that whp the graph $G^{(n)}$ is such that for any partition (A, B) of the vertices, at least a constant proportion of the edges of $G^{(n)}$ run within A or within B. Then we show that this property implies $t_{\text{rel}}^{\text{abs}} \asymp t_{\text{rel}}$.

First we prove the following lemma.

Lemma 6.1. *There exists a positive constant c with the following property.*

Let $m \in \mathbb{Z}_{\geq 1}$ and let V_1, V_2, ... V_8 be disjoint sets of size m each. Let us consider a random graph H with vertex set $V = \bigcup_{i=1}^{8} V_i$ and edge set E consisting of a uniformly chosen perfect matching between V_{2i-1} and V_{2i} for $i = 1, 2, 3, 4$, a uniform perfect matching between $V_{4i-3} \cup V_{4i-2}$ and $V_{4i-1} \cup V_{4i}$ for $i = 1, 2$ and a uniform perfect matching between $\bigcup_{i=1}^{4} V_i$ and $\bigcup_{i=5}^{8} V_i$. (These matchings are independent of each other.)

Then with probability $\geq 1 - 2^{-\frac{1}{4}m}$ the graph H is such that for any partition (A, B) of V we have $|E(A)| + |E(B)| \geq c|E|$. Here $E(A)$ denotes the set of edges with both endpoints in A, and $E(B)$ is defined analogously.

Proof. Consider a partition (A, B) of V. Let $A_0 := A \cap (V_1 \cup V_2)$, $B_0 := B \cap (V_1 \cup V_2)$. Let $\widetilde{A}$ consist of A_0 and the neighbours of B_0 in $V_3 \cup V_4$. Let $\widetilde{B} = \left(\bigcup_{i=1}^{4} V_i\right) \setminus \widetilde{A}$. Let $\widehat{A}$ consist of the neighbours of $\widetilde{B}$ in $\bigcup_{i=5}^{8} V_i$, and let $\widehat{B} = \left(\bigcup_{i=5}^{8} V_i\right) \setminus \widehat{A}$. Fix some constant $\delta \in \left(0, \frac{1}{4}\right)$ such that $C(\delta)$ in [9, Lemma 6.5] satisfies $C(\delta) \leq \frac{1}{8}$.

We will consider the following events.

$$\begin{aligned}
\Omega_1 :=& \quad \left\{|A_0| \notin \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right) |V_1 \cup V_2|\right\}, \\
\Omega_2 :=& \quad \left\{\geq 2\delta m \text{ edges run between } \widetilde{B} \cap V_3 \text{ and } \widetilde{B} \cap V_4\right\}, \\
\Omega_3 :=& \quad \left\{\geq 4\delta m \text{ edges run between } \widehat{B} \cap (V_5 \cup V_6) \text{ and } \widehat{B} \cap (V_7 \cup V_8)\right\}.
\end{aligned}$$

Note that these events only depend on A_0 and the graph H. We will show that if any of them holds, then for any choice of (A, B) with the given A_0, there are $\geq \delta m$ edges with both endpoints in A or both endpoints in B. Then we will use [9, Lemma 6.5] to bound the probability of these events failing for a given A_0, and take a union bound over the possible choices of A_0.

If Ω_1 holds, then by pigeonhole principle at least δm edges of the matching between V_1 and V_2 have both endpoints in A_0 or both endpoints in B_0.

If Ω_2 holds and $|\widetilde{B} \cap A \cap (V_3 \cup V_4)| \leq \delta m$ then at least δm of the edges between $\widetilde{B} \cap V_3$ and $\widetilde{B} \cap V_4$ have both endpoints in B. If Ω_2 holds and $|\widetilde{B} \cap A \cap (V_3 \cup V_4)| > \delta m$ then there are at least δm edges between A_0 and $\widetilde{B} \cap A \cap (V_3 \cup V_4)$. So in either case there are $\geq \delta m$ edges with both endpoints in A or both endpoints in B.

If Ω_3 holds and $|\widehat{B} \cap A| \leq 2\delta m$ then at least $2\delta m$ of the edges between $\widehat{B} \cap (V_5 \cup V_6)$ and $\widehat{B} \cap (V_7 \cup V_8)$ have both endpoints in B. If Ω_3 holds and $|\widehat{B} \cap A| > 2\delta m$ then there are at least $2\delta m$ edges between $\widetilde{A}$ and $\widehat{B} \cap A$. If $|\widetilde{A} \cap B| \leq \delta m$ then at least δm of these edges have both endpoints in A. If $|\widetilde{A} \cap B| > \delta m$ then there are at least δm edges between B_0 and $\widetilde{A} \cap B$. So in all cases there are $\geq \delta m$ edges with both endpoints in A or both endpoints in B.

In what follows assume that $|A_0| \in \left(\frac{1}{2}-\delta, \frac{1}{2}+\delta\right)|V_1 \cup V_2|$. Also note that $|\widetilde{A}| = \frac{1}{2}\left|\bigcup_{i=1}^4 V_i\right|$.

Let σ be a matching on $V_1 \cup V_2$ that matches vertices x and y if and only if their neighbours in $V_3 \cup V_4$ are connected by an edge. Note that σ is a uniformly chosen perfect matching on $V_1 \cup V_2$. (The uniform randomness of σ comes from the randomness of edges between $V_1 \cup V_2$ and $V_3 \cup V_4$, for any realisation of the other edges.) By [9, Lemma 6.5] we know that with probability $\geq 1 - 2^{-2m\left(\frac{1}{2}-C(\delta)\right)}$ there are at least $2\delta m$ pairs in σ with both ends in A_0, i.e. Ω_2 holds.

Now consider a matching η of $\bigcup_{i=1}^4 V_i$ that matches vertices x and y if and only if their neighbours in $\bigcup_{i=5}^8 V_i$ are connected by an edge between $V_5 \cup V_6$ and $V_7 \cup V_8$. This is a uniform perfect matching on $\bigcup_{i=1}^4 V_i$, independently of σ. By [9, Lemma 6.5] we know that with probability $\geq 1 - 2^{-4m\left(\frac{1}{2}-C(\delta)\right)}$ there are at least $4\delta m$ pairs in η with both ends in $\widetilde{A}$, i.e. Ω_3 holds.

Taking a union bound over the $\leq 2^{2m}$ choices of $A_0 \subseteq (V_1 \cup V_2)$ with $|A_0| \in \left(\frac{1}{2}-\delta, \frac{1}{2}+\delta\right)|V_1 \cup V_2|$ finishes the proof. $\square$

We use the above lemma for different subgraphs of $G^{(n)}$ to show the following.

Lemma 6.2. *There exists a positive constant c' with the following property. With high probability $G^{(n)}$ is such that for any partition (A, B) of its vertices we have $|E(A)| + |E(B)| \geq c'|E|$.*

Proof. Let $\frac{n}{2} + 5 \leq m \leq n$ and consider the set of edges $E_{m,i}$ at levels $m-2$, $m-1$ and m in graph $G^{(m,i)}$ for each $i \in \{1, 2, ..., 2^{n-m}\}$. Using Lemma 6.1 for each $E_{m,i}$ and taking a union bound we get that with probability $\geq 1 - 2^{n-m} \cdot 2^{-\frac{1}{4} \cdot 2^{m-3}} \geq 1 - 2^{\frac{n}{2} - 2^{\frac{n}{2}}}$ each $E_{m,i}$ is such that for any partition (A, B) of the vertices at least c proportion of the edges have both endpoints in A or both endpoints in B.

Taking a union bound over m and using that a constant proportion of the edges of $G^{(n)}$ is at levels $\geq \frac{n}{2} + 3$ we get that with probability $\geq 1 - \frac{n}{2} \cdot 2^{\frac{n}{2} - 2^{\frac{n}{2}}} = 1 - o(1)$ the graph $G^{(n)}$ has the required property. $\square$

Finally, we use a result stating that the above property implies that $t_{\text{rel}} \asymp t_{\text{rel}}^{\text{abs}}$.

Proposition 6.3. *Let $G = (V, E)$ be a graph satisfying that there is a constant $c > 0$ such that for any partition of V into two sets A and B, we have $|E(A)| + |E(B)| > c|E|$. Then a simple random walk on G satisfies that $t_{\text{rel}} \asymp t_{\text{rel}}^{\text{abs}}$.*

The proof of Proposition 6.3 relies on a result from [8] and it is presented in [1].

Proof of Proposition 1.7. The proof follows directly from Lemma 6.2, Proposition 6.3 and as from [5, Proposition 3] we know that $t_{\text{rel}} \asymp n$. $\square$

7 The cover time

In this section we present the proof of Theorem 2.

In what follows we consider a simple random walk. Once we prove the bounds for this, the bounds for a lazy random walk follow by noting that the cover time of a lazy walk is twice the cover time of a simple random walk.

The lower bound follows from a general lower bound from [7] which states that the cover time of a simple random walk on a graph with N vertices satisfies $t_{\text{cov}} \geq N \log N$.

For the upper bound, we use Matthew's theorem (see [10, Theorem 11.2]) and a standard bound on the maximal hitting time in terms of the maximal hitting time from the invariant distribution (see [10, Lemma 10.2], which give that $t_{\text{cov}} \leq \max_{x,y \in V^{(n)}} \mathbb{E}_y[\tau_x] \log N \leq 2 \max_{x \in V^{(n)}} \mathbb{E}_\pi[\tau_x] \log N$, where τ_x denotes the hitting time of vertex x.

Therefore, it only remains to prove that

$$\max_{x \in V^{(n)}} \mathbb{E}_\pi[\tau_x] \quad \lesssim \quad N \quad = \quad 2^n. \tag{7}$$

We know that $\mathbb{E}_x[\tau_x^+] = \frac{1}{\pi(x)} = 2^n$ for all x, where τ_x^+ denotes the first return time to x, and we will use this to prove an upper bound of the same order on any hitting time.

In what follows we work on the high probability event that the mixing times of $G^{(n)}$, $G^{(n-1,1)}$ and $G^{(n-1,2)}$ all satisfy the bounds in Theorem 1. Let us fix a vertex x of $G^{(n)}$ and for notational convenience assume that x is in $G^{(n-1,1)}$.

Note that in each step a random walk has probability $\leq \frac{1}{n}$ of stepping to x and probability $\frac{1}{n}$ of stepping to $G^{(n-1,2)}$, hence for any positive constant A there exists a positive constant ε_A such that $\mathbb{P}_x(\tau_x^+ \geq An, \tau_{G^{(n-1,2)}} \leq An) \geq \varepsilon_A$.

Let Y be a simple random walk on $G^{(n-1,2)}$ and let $t := 2t_{\text{mix}}^{G^{(n-1,2)}}\left(\frac{1}{16}\right)$ be two times its $\frac{1}{16}$-mixing time. From the assumption that $G^{(n-1,2)}$ satisfies the bounds in Theorem 1 we know that $t \leq Cn$ where C is a positive constant. From [10, Lemma 6.17] we also know that at time t the separation distance $s(t) := \max_{y,z \in V^{(n-1,2)}} \left(1 - \frac{\mathbb{P}_y(Y_t = z)}{\pi_{G^{(n-1,2)}}(z)}\right)$ of Y satisfies $s(t) \leq \frac{1}{4}$. Hence we get that $\mathbb{P}_y(Y_t = z) \geq \frac{3}{4}\pi_{G^{(n-1,2)}}(z) = \frac{3}{2N}$ for all y and z in $G^{(n-1,2)}$.

Let X be a simple random walk on $G^{(n)}$. Note that in each step X crosses a type n edge with probability $\frac{1}{n}$, regardless of its location, hence starting from y, we have $\tau_{G^{(n-1,1)}} \sim \text{Geom}_{\geq 1}\left(\frac{1}{n}\right)$ and conditional on the value of $\tau_{G^{(n-1,1)}}$, the walk $(X_k)_{k<\tau_{G^{(n-1,1)}}}$ is distributed like a simple random walk on $G^{(n-1,2)}$. Let θ be a sufficiently small constant such that $\mathbb{P}\big(\text{Geom}_{\geq 1}\left(\frac{1}{n}\right) > t\big) \geq \theta$. Then we have

$$\begin{aligned}
N \quad = \quad \mathbb{E}_x\left[\tau_x^+\right] \quad &\geq \sum_{y,z \in V^{(n-1,2)}} \mathbb{P}_x\Big(\tau_x^+ \geq An, \tau_{G^{(n-1,2)}} \leq An, X_{\tau_{G^{(n-1,2)}}} = y\Big) \\
&\qquad\qquad \cdot \mathbb{P}_y(X_t = z, t < \tau_{G^{(n-1,1)}}) \cdot \mathbb{E}_z[\tau_x] \\
&\geq \sum_{y,z \in V^{(n-1,2)}} \mathbb{P}_x\Big(\tau_x^+ \geq An, \tau_{G^{(n-1,2)}} \leq An, X_{\tau_{G^{(n-1,2)}}} = y\Big) \\
&\qquad\qquad \cdot \mathbb{P}_y(Y_t = z)\, \mathbb{P}_y(t < \tau_{G^{(n-1,\ell)}}) \cdot \mathbb{E}_z[\tau_x] \\
&\geq \sum_{y,z \in V^{(n-1,2)}} \mathbb{P}_x\Big(\tau_x^+ \geq An, \tau_{G^{(n-1,2)}} \leq An, X_{\tau_{G^{(n-1,2)}}} = y\Big) \\
&\qquad\qquad \cdot \frac{3}{2N}\mathbb{P}\left(\text{Geom}_{\geq 1}\left(\frac{1}{n}\right) > t\right) \mathbb{E}_z[\tau_x] \\
&\geq \sum_{z \in V^{(n-1,2)}} \varepsilon_A \frac{3}{2N} \theta \mathbb{E}_z[\tau_x] \quad = \quad \frac{3}{4}\varepsilon_A \theta \mathbb{E}_{\pi_{G^{(n-1,2)}}}[\tau_x]\,.
\end{aligned}$$

This shows that $\mathbb{E}_{\pi_{G^{(n-1,2)}}}[\tau_x] \leq \frac{4}{3\varepsilon_A \theta} N$.

To obtain a bound on $\mathbb{E}_\pi[\tau_x]$, note that $\mathbb{P}_{\pi_{G(n-1,1)}}\left(X_{\tau_{G(n-1,2)}} = y\right) = \pi_{G(n-1,2)}(y)$, hence

$$\begin{aligned}\mathbb{E}_\pi[\tau_x] \quad &= \quad \frac{1}{2}\mathbb{E}_{\pi_{G(n-1,1)}}[\tau_x] + \frac{1}{2}\mathbb{E}_{\pi_{G(n-1,2)}}[\tau_x] \quad \leq \quad \mathbb{E}_{\pi_{G(n-1,2)}}[\tau_x] + \frac{1}{2}\mathbb{E}_{\pi_{G(n-1,1)}}[\tau_{G(n-1,1)}] \\ &\leq \quad \frac{4}{3\varepsilon_A\theta}N + \frac{1}{2}n \quad \leq \quad \left(\frac{4}{3\varepsilon_A\theta} + 1\right)N.\end{aligned}$$

This finishes the proof of (7), hence completing the proof of Theorem 2.

8 The chromatic number

In this section we prove the following auxiliary statement, which immediately implies Theorem 3.

Lemma 8.1. *For any positive integer c there exists a graph H_c such that $\chi(H_c) > c$ and with high probability the graph $G^{(n)}$ contains a copy of H_c.*

We prove Lemma 8.1 by induction on c.

For $c = 1$ the statement holds with H_1 consisting of two vertices connected via an edge.

Assume that the statement holds for $c = k$ and let us construct H_{k+1} as follows.

Let M_k be the number of vertices of H_k. Let A be a set of kM_k vertices and let $H_k^{(1)}$, ..., $H_k^{\left(\binom{kM_k}{M_k}\right)}$ be $\binom{kM_k}{M_k}$ disjoint copies of H_k. Let $A^{(1)}$, ..., $A^{\left(\binom{kM_k}{M_k}\right)}$ be an enumeration of all size M_k subsets of A, and let H_{k+1} be a graph on $kM_k + \binom{kM_k}{M_k}$ vertices obtained by considering A and all $H^{(i)}$, and for each i adding M_k edges according to a perfect matching between the M_k vertices in $A^{(i)}$ and the M_k vertices in $H^{(i)}$.

Firstly, we show that $\chi(H_{k+1}) > k+1$. Assume for contradiction that there is a proper colouring of H_{k+1} with $(k+1)$ colours. Then by the pigeonhole principle there is an i such that all M_k vertices in $A^{(i)}$ are of the same colour. [3] Since each vertex in $H_k^{(i)}$ is connected to a vertex in $A^{(i)}$, this colour cannot appear in the vertices of $H_k^{(i)}$. This means that we have a proper colouring of $H_k^{(i)}$ with k colours, which contradicts $\chi(H_k) > k$. So we must have $\chi(H_{k+1}) > k+1$.

Now we turn to showing that whp $G^{(n)}$ contains a copy of H_{k+1}.

We know that whp $G^{(n)}$ contains a copy of H_k, hence there is a positive integer n_k such that $\mathbb{P}\big(G^{(n_k)}$ contains a copy of $H_k\big) \geq \frac{1}{2}$. Let $m = n_k + \binom{kM_k}{M_k}$. We will show that $\mathbb{P}\big(G^{(m)}$ contains a copy of $H_{k+1}\big) \geq 2^{-(mkM_k+1)\binom{kM_k}{M_k}} > 0$. Once we have this, using that the number of independent copies of $G^{(m)}$ in $G^{(n)}$ diverges as $n \to \infty$, we can conclude that $G^{(n)}$ contains H_{k+1} with high probability.

We can prove the lower bound on $\mathbb{P}\big(G^{(m)}$ contains a copy of $H_{k+1}\big)$ as follows.

Let A be a fixed set of size kM_k in $G^{(n_k,1)}$ and let $A^{(0)}$, ..., $A^{\left(\binom{kM_k}{M_k}-1\right)}$ be an enumeration of its size M_k subsets. Note that for each i we have $\mathbb{P}\big(G^{(n_k+i,2)}$ contains a copy of $H_k\big) \geq \frac{1}{2}$, since $G^{(n_k+i,2)}$ contains copies of $G^{(n_k)}$.

[3] We use that $M_k \geq k+1$, hence $kM_k \geq (k+1)(M_k - 1) + 1$.

On the event that $G^{(n_k+i,2)}$ contains a copy $H_k^{(i)}$ of H_k for each $i \in \left\{0, 1\ldots, \binom{kM_k}{M_k}\right\}$, the probability that the vertices of each $H_k^{(i)}$ are matched to the vertices of the corresponding $A^{(i)}$ is $\prod_{i=0}^{\binom{kM_k}{M_k}-1} \binom{2^{n_k+i}}{kM_k}^{-1} \geq 2^{-mkM_k\binom{kM_k}{M_k}}$. We used here that for each i there is a perfect matching between the vertices of $G^{(n_c+i,2)}$ and the vertices of $G^{(n_c+i,1)}$, the latter graph contains $G^{(n_k,1)}$ containing A, and these matchings are independent for different i.

This finishes the proof that H_{k+1} has the desired properties, hence finishing the proof of Lemma 8.1.

Remark 8.2. *The above proof, if carried out with a bit more care, can provide a quantitative lower bound on the growth of the chromatic number as $n \to \infty$, however since this bound would be extremely weak, we decided not to work out its details. It remains an open question to find a sensible lower bound on $\chi\left(G^{(n)}\right)$ as a function of n that holds with high probability.*

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References

[1] Zsuzsanna Baran, Jonathan Hermon, Anđela Šarković, Allan Sly, and Perla Sousi. Random walk on the small-world network model in 3 or more dimensions. *In preparation.*

[2] Zsuzsanna Baran, Jonathan Hermon, Anđela Šarković, and Perla Sousi. Phase transition for random walks on graphs with added weighted random matching. *Probability Theory and Related Fields*, 2024.

[3] Riddhipratim Basu, Jonathan Hermon, and Yuval Peres. Characterization of cutoff for reversible Markov chains. *The Annals of Probability*, 45(3):1448 – 1487, 2017.

[4] Anna Ben-Hamou and Justin Salez. Cutoff for nonbacktracking random walks on sparse random graphs. *The Annals of Probability*, 45(3):1752–1770, 2017.

[5] Itai Benjamini, Yotam Dikstein, Renan Gross, and Maksim Zhukovskii. Randomly twisted hypercubes – between structure and randomness. *Random Structures & Algorithms*, 66, 2025.

[6] Andrzej Dudek, Xavier Pérez-Giménez, Paweł Prałat, Hao Qi, Douglas West, and Xuding Zhu. Randomly twisted hypercubes. *European Journal of Combinatorics*, 70:364–373, 2018.

[7] Uriel Feige. A tight lower bound on the cover time for random walks on graphs. *Random Structures & Algorithms*, 1995.

[8] Jonathan Hermon, Ben Langer, and Jens Malmquist. On quantitative near-bipartiteness: mixing, parity breaking, max-cut and eigenvalue gap. *In preparation.*

[9] Jonathan Hermon, Allan Sly, and Perla Sousi. Universality of cutoff for graphs with an added random matching. *The Annals of Probability*, 50(1):203 – 240, 2022.

[10] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times.* American Mathematical Society, 2006.