H^2-Regularity of the solutions of a blood-flow model with stents

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Abstract

In this work, we study the fluid-dynamics of a blood in a stented artery connected to an aneurysmal sac. The presence of a vascular prosthesis of type stent can be considered as a local perturbation of a smooth boundary of flow, more precisely the walls artery can be seen as a strongly rough surface. We are mainly interested in controlling the H^2 regularity of a simplified model which takes into account the impact of these stents when the blood flow is controlled by a Poisson equation with a Dirichlet boundary condition, in a domain with a rough boundary (parametrized a small parameter ε). We analyse the existence and unicity of the solution of this model of blood-flow and we study the H^2 regularity using variational analysis methods. By a detailed study, we control the H^2 regularity of order $\mathcal{O}(\varepsilon^{-1})$. Moreover, we study of the regularity H^2 regularity using multi-scale analysis. We prove that the H^2 norm of the solution of this model is singular of order $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$.

Résumé

Dans-ce travail, nous nous intéressons à la régularité H^2 de la solution du problème de Poisson avec des conditions au bord de type Dirichlet dans un domaine rugueux (en fonction d'un petit paramètre ε). Les équations adjacentes modélisent la présence d'une endoprothèse vasculaire (stent) dans un écoulement. En effet, cette présence peut être considérée comme une perturbation locale d'un bord lisse de l'écoulement considéré. Plus précisément les parois de l'artère sont assimilées à une surface fortement rugueuse. Nous analysons l'existence et l'unicité de la solution de ce modèle d'écoulement sanguin et nous traitons la régularité H^2 par des techniques d'analyse variationnelle. Une étude minutieuse permet de contrôler la régularité H^2 en $\mathcal{O}(\varepsilon^{-1})$. Un deuxième axe est dédié à l'étude de la régularité H^2 par des analyses asymptotiques multi-échelles. Nous montrons que la norme H^2 de la solution de ce modèle d'écoulement sanguin est singulière en $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$.

Keywords: H^2-Regularity, rough domain, wall-laws, Poisson equation, asymptotic analysis, multi-scale modelling, stent, aneurysm, boundary layer approximation.

1. Introduction

Rupture of aneurysm are lethal pathologies of the cardiovascular system. A possible therapy consists in introducing a metallic multi-layered stent (see fig. 1), either as a supplementary protection of the arterial wall or in order to slow vortices in the aneurysm and to favor coagulation of the sac. In this paper we aim to investigate the fluid-dynamics of blood and the effects of the stent rugosity on the fluid flow. Starting from the Stokes system we simplify the problem by studying the axial velocity through the resolution of a specific Poisson problem. We choose to neglect the elasticity of the walls of the arteries and to consider the geometry of a 2-dimensional domain Ω_ε (see fig. 2). We point out that Fig. 2 represents a longitudinal cut

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through an 3D artery, where the rough base represents the shape of the wires of the stent. We are interested in the mathematical justification of the H^2-regularity of this simplified model, in terms of the rugosity.

The study of mathematical modeling and simulation of flows on rough domains developped at the beginning of the 19th century by the work of Nikuradze [18]. Subsequently, these results were extended by Prandtl and Schlichting [15] to flows on rough plates. Recently, several experimental studies and several methods of modeling the phenomenon have been presented in [12, 14], particularly in the context of turbulent flows. From an analytical point of view, the approach used to answer this problem is the wall law. Historically, wall laws have first been established in the context of turbulent flows on smooth plates [9]. Subsequently, the application of wall laws has been extended to rough domains, by modifying certain constants [12].

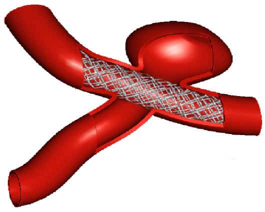

Figure 1: A sketch of stented arteries with an aneurysmal sac

Nevertheless, the approach lacks generality, since it cannot be used automatically for different shapes of roughness. The first attempt to implement a more general method to construct wall laws has been presented by Carrau [8] and Carrau-LeTallec [13] in the framework of compressible laminar flows on periodic rough walls. The strategy is based on a homogenization approach, starting from the decomposition of the domain into a local part, containing periodic roughnesses, and a global part where the wall law is imposed.

The first works proposing a rigorous analysis of wall laws concerns the Poisson problem with homogeneous Dirichlet boundary conditions over the edge of a rough domain, and are contained in the articles of Achdou, Pironneau, Valentin and Le Tallec [26, 23, 25]. Indeed, from the original idea of Carrau-LeTallec, a second approach was presented by Pironneau and Achdou [24] for the Laplace equation. From this approach, the wall laws are analytically established in a mathematical framework adapted to the error analysis. Thereafter, Achdou and al. [23] heard it to Stokes problem to first and second order.

On the other hand, we point out that W. Jäger and A. Mikelić [19, 20, 22] have been interested in the contact between a viscous fluid and a porous medium. The authors considered the same type of boundary conditions as in the aforementioned articles. We point out that techniques of extending solution to the zero-order on smooth domains are different from the ones used on rough domains. However, the last two strategies lead to the same average implied wall laws. Indeed, in [5, 6], Bresch and Milisic derived wall laws and established error estimates for a stent artery model with a periodic geometry. They examined the particular Poisson problem for the axial component $u_\varepsilon \in \mathbb{R}$ of the velocity of the fluid, with Dirichlet conditions on γ_ε and Γ^∞ as well as periodic incoming and outgoing conditions at the lateral edges Σ_e and Σ_s (see fig. 2). The case of the flow of a pressure-directed fluid is also treated by W. Jäger and A. Mikelić [19, 21] for a laminar flow of Poiseuille type. In the context of blood flow in stented arteries, the authors Bresch and al. applyied in [10] their previous results to the non-periodic case, with Neumann conditions at the lateral boundaries. The presence of Neumann conditions prevents a direct generalization of the results to Dirichlet conditions and requires more complicated estimates, called very low estimates [17]. The a priori estimates are then perfected by V. Milišić [16]. In this work, we aim to extend those results and study the H^2-regularity of this simplified model, in terms of the rugosity.

Next, we introduce in more details our main model and we state the main results of our work.

1.1. Geometry and setting of the problem

Geometry : In this paper we consider two space dimensions domains. We define :

- The macroscopic domain : Ω_ε denotes the rough domain in $\mathbb{R}^2$ depicted in figure 2. The spatial variable giving the position of a point in the above domain is a vector called $x = (x_1, x_2)$. In the interior of the domain one sets the square piecewise-smooth domain Ω_0, whose lower interface is denoted by $\gamma^0 = \{(x_1, x_2)\,;\, x_1 \in [0, L],\, x_2 = 0\}$. $\Omega_\varepsilon \backslash \Omega_0$ is the complementary rough subdomain. γ_ε is the rough boundary and $\Gamma^\infty = \{(x_1, x_2) \in \mathbb{R}^2\,;\, x_1 \in [0, L],\ x_2 = 1\}$ is the upper smooth boundary (see figure 2).

Hypotheses H 1. *The rough boundary γ_ε is described as a periodic repetition at the macroscopic scale. The latter can be parametrized as the graph of a bounded function $\tilde{\gamma} : \mathbb{R} \longrightarrow [-1, 0[$ of class C^2 such that*

$$\gamma_\varepsilon = \left\{(x_1, x_2)\,;\, x_1 \in [0, L]\,,\, x_2 = \varepsilon\tilde{\gamma}(\frac{x_1}{\varepsilon})\right\}. \tag{1.1}$$

The lateral boundaries are denoted by $\Sigma_e := \{0\} \times [\varepsilon\tilde{\gamma}(0), 1]$ and $\Sigma_s := \{L\} \times [\varepsilon\tilde{\gamma}(\frac{L}{\varepsilon}), 1]$. We assume that the ratio between L and the period ($T = 2\pi$) is always an integer denoted by N.

- The microscopic cell domain : The microscopic position variable is denoted by $y = \frac{x}{\varepsilon}$. The rough boundary γ_ε is described as a periodic repetition at the microscopic scale of a single boundary cell P^0 parametrized as
$$P^0 = \{(y_1, y_2) \in [0, 2\pi] \times [-1, 0[,\ y_2 = \tilde{\gamma}(y_1)\}.$$
$Z^+ \cup P$ denotes the microscopic cell (which is unbounded in the y_2-direction), where $Z^+ := [0, 2\pi] \times \mathbb{R}_+$ and $P = \{(y_1, y_2) \in \mathbb{R}^2 \text{ s.t. } y_1 \in [0, 2\pi]\ ,\ \tilde{\gamma}(y_1) < y_2 < 0\}$. We define Γ_f as the the fictitious interface such that $\Gamma_f = \{(y_1, y_2)\,;\, y_1 \in [0, 2\pi],\ y_2 = 0\}$ (see figure 2).

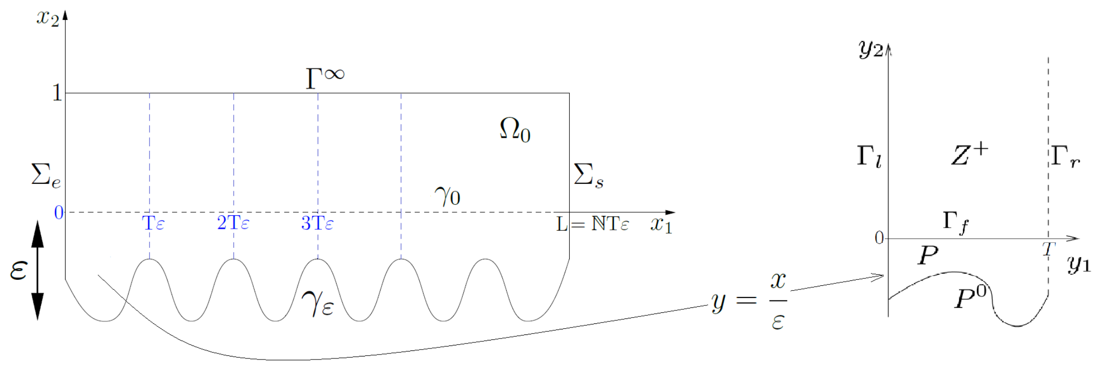

Figure 2: Rough domain Ω_ε (left) and cell domain (right)

Further notations : In the rest of the paper, we define the usual Sobolev spaces H^s ($s \geqslant 0$) by

$$H^s(\mathbb{R}^n) = \left\{u \in \mathrm{L}^2(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi < +\infty\right\}.$$

These Sobolev spaces $H^s(\mathbb{R}^n)$ are Hilbert spaces for the norm :

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi.$$

We refer to [1, 11] for a detailed study of these spaces.

1.2. The simplified problem

Instead of dealing directly with the full problem of Navier-Stokes flow, we consider a simplified setting that avoids theoretical difficulties and non-linear complications. We focus on problems related only to the roughness itself and not on the mixed character of the Stokes problem. Starting from the Stokes system, we consider a Poisson problem for the axial component of the velocity. The pressure gradient that forces the flow is represented in the right hand side by $C = -\frac{1}{\eta}\partial_{x_1} p$, where η denotes the dynamic viscosity of the fluid. For sake of conciseness, we consider only periodic inflow and outflow boundary conditions. The simplified formulation reads : find u_ε such that

$$\left\{ \begin{array}{lll} -\Delta u_\varepsilon = C & \text{in} & \Omega_\varepsilon \\ u_\varepsilon \quad = 0 & \text{on} & \Gamma^\infty \cup \gamma_\varepsilon \\ u_\varepsilon \text{ is } x_1 - \text{periodic} & \text{on} & \Sigma_e \cup \Sigma_s. \end{array} \right. \tag{1.2}$$

We note that the stent is modeled by the graph of a periodic function of size ε. This constitutes the rough boundary γ_ε of the domain. The existence and uniqueness of solutions of the system(1.2), are reviewed in the Appendix A1.
We underline that the stent could be seen as a local perturbation of a smooth boundary of the flow field, which leads to a certain singularity if we are interested in a higher order of regularity. It is mainly this biological phenomenon that led us to be interested in studying the H^2-regularity with respect to the roughness by different techniques.

1.3. Outline of our main results

Our first result mainly shows that the H^2-norm of the first order approximation of the exact solution u_ε in the whole rough domain is singular of order $\mathcal{O}(\varepsilon^{\frac{1}{2}})$.

Theorem 1.1. *Let $u^\infty_{\text{Ach},1}$ be the first order full boundary layer approximation. Then there exists two positive constants K_5 and K_6 independent of ε, such that for every $\varepsilon \in]0,1[$, we have*

$$\frac{K_5}{\sqrt{\varepsilon}} \leqslant \|u^\infty_{\text{Ach},1}\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{K_6}{\sqrt{\varepsilon}}. \tag{1.3}$$

The most important difficulty in the proof of Theorem 1.1 is to prove that the Hessian of the first boundary layer corrector on the microscopic scale (in unbounded cell domains) is controlled by a nonnegative constant independent of ε (see Proposition 3.3).

Theorem 1.2. *Let u_ε be the unique solution of the problem* (1.2)*. Then for all $0 < \varepsilon < 1$, the first order full boundary layer approximation $u^\infty_{\text{Ach},1}$ it satisfies*

$$\|u_\varepsilon - u^\infty_{\text{Ach},1}\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{K_7}{\sqrt{\varepsilon}}, \tag{1.4}$$

where the constant K_7 is independent of ε.

A key argument in the proof of the Theorem 1.2 is the use of the results of the study of H^2-regularity by the technique of the variational formulation in rough domains (see Appendix A2). The extension to the second order boundary layer approximation of the result of Theorem 1.1 is the following.

Theorem 1.3. *Let $u^\infty_{\text{Ach},2}$ be the second order full boundary layer approximation. Then, there exist $\varepsilon_0 > 0$ and two positive constants $\widetilde{\mathrm{C}}_5$ and $\widetilde{\mathrm{C}}_6$ independent of ε, such that for all $0 < \varepsilon < \varepsilon_0$, we have*

$$\frac{\widetilde{\mathrm{C}}_5}{\sqrt{\varepsilon}} \leqslant \|u^\infty_{\text{Ach},2}\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{\widetilde{\mathrm{C}}_6}{\sqrt{\varepsilon}}. \tag{1.5}$$

Now we give an error estimate :

Theorem 1.4. *Let u_ε be the solution of* (1.2). *Then there exists $\varepsilon_0 > 0$ and a positive constant $\widetilde{C}_7$ independent of ε, such that the second order full boundary layer approximation it satisfies*

$$\|u_\varepsilon - u^{\infty}_{\mathrm{Ach},2}\|_{H^2(\Omega_\varepsilon)} \leqslant \widetilde{C}_7 \, \mathrm{e}^{\frac{-1}{\varepsilon}}. \tag{1.6}$$

Note that the H^2-error estimate of Theorem 1.4 should be read as an improved regularity result. Indeed, the difference is exponentially small w.r.t. ε. Finally, our main theorem in this paper is the following.

Theorem 1.5. *The exact solution u_ε of the problem* (1.2) *is in $H^2(\Omega_\varepsilon)$. Moreover for every $\varepsilon \in [0, 1[$, we have*

$$\frac{\mathrm{C}_{\min}}{\sqrt{\varepsilon}} \leqslant \|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{\mathrm{C}_{\max}}{\sqrt{\varepsilon}}, \tag{1.7}$$

where the constants $\mathrm{C}_{\min}$, $\mathrm{C}_{\max}$ are independent of ε.

Then for every real $1 \leqslant s \leqslant 2$, we can deduce the result of the H^s-regularity of the exact solution of the problem (1.2).

Corollary 1.1. *Given a real number $s \in [1, 2]$, the solution u_ε of of the problem* (1.2) *satisfies the following estimate*

$$\|u_\varepsilon\|_{H^s(\Omega_\varepsilon)} \leqslant \mathrm{C}_{\mathrm{int}} \, \varepsilon^{\frac{(1-s)}{2}}, \tag{1.8}$$

where the constant $\mathrm{C}_{\mathrm{int}}$ does not depend on ε.

Proof. Let u_ε be a weak solution to (1.2). We define the real number $s \in [s_1, s_2] := [1, 2]$ so that $s := \theta s_1 + (1-\theta)s_2 = 2 - \theta$, where $0 \leqslant \theta \leqslant 1$. Using the functional interpolation inequality, we have

$$\|u_\varepsilon\|_{H^s(\Omega_\varepsilon)} \leqslant \|u_\varepsilon\|^{\theta}_{H^1(\Omega_\varepsilon)} \|u_\varepsilon\|^{1-\theta}_{H^2(\Omega_\varepsilon)}.$$

So, thanks to inequalities (6.2) and (1.7), there exists a positive constant $\mathrm{C}_{\mathrm{int}}$ (independent of ε) such that

$$\|u_\varepsilon\|_{H^s(\Omega_\varepsilon)} \leqslant \sup(C^{\theta}, C^{1-\theta}_{\max}) \left(4(1+\varepsilon)^2 + 1\right)^{\theta} \left(\sqrt{\mathrm{mes}(\Omega_\varepsilon)}\right)^{\theta} \left(\frac{1}{\sqrt{\varepsilon}}\right)^{1-\theta} \leqslant \mathrm{C}_{\mathrm{int}} \, \varepsilon^{-\frac{(1-\theta)}{2}}.$$

We finish the proof by replacing the expression of $s = 2 - \theta$ in the last inequality. This ends the proof. ☐

1.4. Overview of the paper

In section 2 we give a brief summary related to full boundary layers approximations and H^1-error estimates [3]. These results are of constant use in our work. In section 3 we study the H^2-regularity of the exact solution of the problem (1.2) by using the first order approximation and prove Theorem 1.1 and Theorem 1.2 . In section 4 we continue the investigation of the H^2-regularity of the exact solution of the problem (1.2) by using the second order approximation. First, we prove Theorem 1.3 and Theorem 1.4 . Subsequently, we show that the H^2-norm of the solution is singular of order $\mathcal{O}(\varepsilon^{\frac{1}{2}})$. Concluding remarks and possible perspectives are given at the end of the paper. In Appendix A1, we give a proof of the existence and uniqueness of solution to the problem (1.2), while in Appendix A2, A3 and A4 we detail the results of the H^2-regularity by the variational technique in rough domains.

2. Preliminaries and useful estimates

2.1. Zero-order approximation

Passing to the limit formally w.r.t. ε in (1.2), the rough domain reduces to a smooth one. The one solution of system (1.2) in this limit is known and it is given by the Poiseuille profile :

$$u_0(x) = \frac{C}{2}\, x_2\, (1 - x_2) \qquad \text{in } \Omega_0. \tag{2.1}$$

We choose to extend u_0 by a linear function in $\Omega_\varepsilon \backslash \Omega_0$

$$u^0_{\mathrm{ext},1}(x) = u_0 \ \mathbb{1}_{[\Omega_0]} + \frac{\partial u_0}{\partial x_2}(x_1, 0) \ x_2 \ \mathbb{1}_{[\Omega_\varepsilon \backslash \Omega_0]} = \begin{cases} \frac{C}{2} \ x_2 \ (1 - x_2) & \text{in} \quad \Omega_0 \\ \frac{C}{2} \ x_2 & \text{in} \quad \Omega_\varepsilon \backslash \Omega_0, \end{cases} \tag{2.2}$$

where $\mathbb{1}_{[.]}$ represents the characteristic function of the set between brackets. Instead of extending only linearly the Poiseuille profile it is obvious that a quadratic term is missing to complete the approximation. In the following $u^0_{\mathrm{ext},2}$ denotes the second order extension of u_0 in $\Omega_\varepsilon \backslash \Omega_0$:

$$u^0_{\mathrm{ext},2}(x) = u_0 \ \mathbb{1}_{[\Omega_0]} + \left(\frac{\partial u_0}{\partial x_2}(x_1, 0) \ x_2 + \frac{\partial^2 u_0}{\partial x_2^2}(x_1, 0) \ \frac{x_2^2}{2} \right) \mathbb{1}_{[\Omega_\varepsilon \backslash \Omega_0]} = \frac{C}{2} \ x_2 \ (1 - x_2) \qquad \text{in } \Omega_\varepsilon. \tag{2.3}$$

Remark that the explicit solution of Poiseuille profile in Ω_0 is a polynomial function of order 2 which is equal to its Taylor expansion to the order 2.

2.2. The cell problems

2.2.1. The first order cell problem

The rough boundary is periodic at the microscopic scale and this leads to solve the microscopic cell problem: find β_1 such that

$$\begin{cases} -\Delta \beta_1 = \quad 0 & \text{in} \quad Z^+ \cup P \\ \quad \beta_1 = \quad -y_2 & \text{on} \quad P^0 \\ \beta_1 \quad \text{is } y_1 - \text{periodic} & \text{on} \quad \Gamma_l \cup \Gamma_r. \end{cases} \tag{2.4}$$

We define the microscopic average along the fictitious interface $\Gamma_f = \{y_1 \in [0, 2\pi]; \ y_2 = 0\}$ given by $\overline{\beta_1} = \frac{1}{2\pi} \int_0^{2\pi} \beta_1(y_1, 0) dy_1$. As $Z^+ \cup P$ is unbounded in the y_2 direction, we also define

$$D^{1,2}(Z^+ \cup P) = \left\{ v \in L^1_{loc}(Z^+ \cup P) \text{ s.t. } \nabla v \in L^2(Z^+ \cup P), v \text{ is } y_1\text{-periodic on } \Gamma_l \cup \Gamma_r \right\}. \tag{2.5}$$

See Fig 2 for Γ_l, Γ_r.

Proposition 2.1. *The problem* (2.4) *admits a unique solution β_1 belonging to $D^{1,2}(Z^+ \cup P)$.*

We underline that the solution β_1 restricted to Z^+ can be written explicitly as

$$(\beta_1)_{|Z^+} = \sum_{k=-\infty}^{+\infty} \left(\eta_k \ e^{-|k| y_2} \right) \ e^{iky_1}, \qquad \text{with} \quad \eta_k = \frac{1}{2\pi} \int_0^{2\pi} \beta_1(y_1, 0) e^{iky_1} \, dy_1.$$

Proposition 2.2. *Setting $\overline{\Gamma} = \frac{1}{2\pi} \int_0^{2\pi} (-\tilde{\gamma}(y_1)) \, dy_1$. Then, for all $\alpha \in]0, \frac{1}{2}[$ we have*

$$\left\| \beta_1 - \overline{\beta_1} + \overline{\Gamma} \right\|_{L^2(P, \ e^{\alpha y_2})} \leqslant \sqrt{\widetilde{C_2}} \ \left\| \nabla_y \beta_1 \right\|_{L^2(P, \ e^{\alpha y_2})}, \tag{2.6}$$

and

$$\left\| \beta_1 - \overline{\beta_1} \right\|_{L^2(Z^+, \ e^{\alpha y_2})} \leqslant \ \left\| \nabla_y \beta_1 \right\|_{L^2(Z^+, e^{\alpha y_2})}, \tag{2.7}$$

where the positive constant $\widetilde{C_2}$ depends on α.

2.2.2. The second order cell problem

The second order error on γ_ε should be corrected thanks to a new cell problem : find $\beta_2 \in D^{1,2}(Z^+ \cup P)$ solving

$$\begin{cases} -\Delta \beta_2 = \quad 0 & \text{in} \quad Z^+ \cup P \\ \quad \beta_2 = \quad -y_2^2 & \text{on} \quad P^0 \\ \beta_2 \quad \text{is } y_1 - \text{periodic} & \text{on} \quad \Gamma_l \cup \Gamma_r. \end{cases} \tag{2.8}$$

Again, the horizontal average is denoted by $\overline{\beta_2} = \frac{1}{2\pi} \int_0^{2\pi} \beta_2(y_1, 0) \, dy_1$. In the same way as for the first order cell problem, one can obtain a similar result:

Proposition 2.3. *The problem* (2.8) *admits a unique solution β_2 belonging to $D^{1,2}(Z^+ \cup P)$.*

2.3. The full boundary layers approximations

2.3.1. A first order approximation

Usually in the presentation of wall laws, we start by introducing the full boundary layer approximation. This approximation is an asymptotic expansion defined on the whole rough domain.

$$u^{\infty}_{\mathrm{Ach},1}(x) = u^0_{\mathrm{ext},1}(x) + \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right) - \overline{\beta_1}x_2\right). \tag{2.9}$$

This approximation satisfies a homogeneous Dirichlet boundary condition on γ_ε, and solves

$$\begin{cases} -\Delta u^{\infty}_{\mathrm{Ach},1} = C\ \mathbb{1}_{[\Omega_0]} & \text{in} \quad \Omega_\varepsilon \\ u^{\infty}_{\mathrm{Ach},1} = \frac{\varepsilon}{1+\varepsilon\langle\beta_1\rangle}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}) - \overline{\beta_1}\right) & \text{on} \quad \Gamma^\infty \\ u^{\infty}_{\mathrm{Ach},1} = 0 & \text{on} \quad \gamma_\varepsilon \\ u^{\infty}_{\mathrm{Ach},1} \quad \text{is } x_1-\text{periodic} & \text{on} \quad \Sigma_e\cup\Sigma_s. \end{cases} \tag{2.10}$$

For this first order full boundary layer approximation, we have the following error estimate [3] :

Proposition 2.4. *Setting* $r^{1,\infty}_\varepsilon = u_\varepsilon - u^{\infty}_{\mathrm{Ach},1}$. *The error obtained by the previous expansion reads :*

$$\|r^{1,\infty}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant K_1\ \varepsilon^{\frac{3}{2}}, \quad \textit{and} \quad \|r^{1,\infty}_\varepsilon\|_{L^2(\Omega_0)} \leqslant K_2\ \varepsilon^2, \tag{2.11}$$

where the positive constants K_1,K_2 *are independent of* ε.

2.3.2. A second order approximation

In order to cancel the non-homogenous boundary contributions at any order of ε, one constructs $u^\infty_{\mathrm{Ach},2}$ such that :

$$\begin{aligned} u^{\infty}_{\mathrm{Ach},2}(x) = {} & u^0_{\mathrm{ext},2}(x) + \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right) \\ & + \frac{\varepsilon^2}{2}\frac{\partial^2 u_0}{\partial x_2^2}(x_1,0)\left[\left(\beta_2\left(\frac{x}{\varepsilon}\right) - \overline{\beta_2}x_2\right) - \frac{\varepsilon\overline{\beta_2}}{1+\varepsilon\overline{\beta_1}}\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right)\right]. \end{aligned} \tag{2.12}$$

Our approximation satisfies the following boundary value problem

$$\begin{cases} -\Delta u^{\infty}_{\mathrm{Ach},2} = C & \text{in} \quad \Omega_\varepsilon \\ u^{\infty}_{\mathrm{Ach},2} = \omega_\varepsilon & \text{on} \quad \Gamma^\infty \\ u^{\infty}_{\mathrm{Ach},2} = 0 & \text{on} \quad \gamma_\varepsilon \\ u^{\infty}_{\mathrm{Ach},2} \quad \text{is } x_1-\text{periodic} & \text{on} \quad \Sigma_e\cup\Sigma_s, \end{cases} \tag{2.13}$$

$$\begin{aligned} \text{where} \quad \omega_\varepsilon = {} & \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right) \\ & + \frac{\varepsilon^2}{2}\frac{\partial^2 u_0}{\partial x_2^2}(x_1,0)\left[\left(\beta_2\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_2}\right) - \frac{\varepsilon\overline{\beta_2}}{1+\varepsilon\overline{\beta_1}}\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)\right]. \end{aligned}$$

We claim the following crucial estimate that implies exponential convergence of the full second order approximation.

Proposition 2.5. *Let* $r^{2,\infty}_\varepsilon := u_\varepsilon - u^{\infty}_{\mathrm{Ach},2}$. *Then*

$$\|r^{2,\infty}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C_1\ e^{-\frac{1}{\varepsilon}} \qquad \textit{and} \qquad \|r^{2,\infty}_\varepsilon\|_{L^2(\Omega_0)} \leqslant C_2\ \sqrt{\varepsilon}e^{-\frac{1}{\varepsilon}}, \tag{2.14}$$

where the positive constants C_1, C_2 *are independent of* ε.

3. H^2-Regularity by the first order approximation

3.1. H^2-Regularity of the zero-order extension $u^0_{\text{ext},1}$

The form of the solution u_0 is explicit in Ω_ε. Therefore, an explicit calculation gives :

Proposition 3.1. *Let $\varepsilon \in]0,1[$. There exists a positive constant K_0 independent of ε, such that*

$$\|u^0_{\text{ext},1}\|_{H^2(\Omega_\varepsilon)} \leqslant K_0. \tag{3.1}$$

3.2. H^2-Regularity of the first order boundary layer corrector

We focus here on the study of the H^2-regularity of the solution of the microscopic cell problem (2.4). Denote by D^2_x the Hessian in the rough domain Ω_ε, D^2_y the Hessian in the cell domain $Z^+ \cup P$. In the following proposition, we control the L^2 norm of the Hessian of the boundary layer corrector β_1 on the macroscopic scale its L^2 norm in the microscopic scale.

Proposition 3.2. *Let β_1 be the solution of the first order cell problem's* (2.4)*. Then, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, we have*

$$\frac{\mathrm{K}_2}{\sqrt{\varepsilon}} \leqslant \varepsilon \left\|D^2_x\beta_1\right\|_{L^2(\Omega_\varepsilon)} \leqslant \frac{\mathrm{K}_1}{\sqrt{\varepsilon}} \left\|D^2_y\beta_1\right\|_{L^2(Z^+\cup P)}, \tag{3.2}$$

where the positive constants K_1, K_2 are independent of ε.

Proof. We introduce the C^2-diffeomorphism $\phi_\varepsilon(x) = \left(\phi^1_\varepsilon(x), \phi^2_\varepsilon(x)\right) = (y_1, y_2) = \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$, whose Jacobian is given by $|J\phi_\varepsilon| = \frac{1}{\varepsilon^2}$. Denote by $E(\cdot)$ the integer part function. By changing the variable from the macroscopic scale to the microscopic scale, we obtain:

$$\left\|D^2_x\beta_1\right\|^2_{L^2(\Omega_\varepsilon)} = \sum_{k=1}^{E(\frac{L}{2\pi\varepsilon})} \left(\int_{(Z^+\cup P)_k} \sum_{i,j=1}^{2} \left|\frac{1}{\varepsilon^2}\partial^2_{y_iy_j}\ \beta_1(y)\right|^2 \varepsilon^2 dy\right) \leqslant \frac{L}{2\pi\varepsilon}\left(\int_{(Z^+\cup P)} \frac{1}{\varepsilon^2}\sum_{i,j=1}^{2}\left|\partial^2_{y_iy_j}\ \beta_1(y)\right|^2\ dy\right).$$

Consequently,

$$\left\|D^2_x\beta_1\right\|_{L^2(\Omega_\varepsilon)} \leqslant \frac{\sqrt{L}}{\sqrt{2\pi}\varepsilon^{\frac{3}{2}}} \left\|D^2_y\beta_1\right\|_{L^2(Z^+\cup P)}. \tag{3.3}$$

Multiplying the last inequality (3.3) by ε, we obtain the uper bound estimate with constant $\mathrm{K}_1 = \sqrt{\frac{L}{2\pi}}$. Now, we prove the lower bound estimate in (3.2). For this sake we use a property of the integer part, i.e. the following relation

$$\frac{L}{4\pi\varepsilon} < \frac{L}{2\pi\varepsilon} - 1 < \mathrm{E}\left(\frac{L}{2\pi\varepsilon}\right) \leqslant \frac{L}{2\pi\varepsilon} < \mathrm{E}\left(\frac{L}{2\pi\varepsilon}\right) + 1, \qquad \text{for all} \quad \varepsilon < \frac{L}{4\pi}.$$

So, one can easly see by a simple calculation that

$$\left\|D^2_x\beta_1\right\|^2_{L^2(\Omega_\varepsilon)} = \sum_{k=1}^{E(\frac{L}{2\pi\varepsilon})} \left(\int_{(Z^+\cup P)_k} \sum_{i,j=1}^{2} \left|\frac{1}{\varepsilon^2}\partial^2_{y_iy_j}\ \beta_1(y)\right|^2 \varepsilon^2 dy\right) \geqslant \frac{L}{4\pi\varepsilon}\left(\int_{(Z^+\cup P)} \frac{1}{\varepsilon^2}\sum_{i,j=1}^{2}\left|\partial^2_{y_iy_j}\ \beta_1(y)\right|^2\ dy\right).$$

Then, there exists a constant K_2 independent of ε, such that

$$\left\|D^2_x\beta_1\right\|_{L^2(\Omega_\varepsilon)} \geqslant \left(\frac{L}{4\pi\varepsilon^3}\right)^{\frac{1}{2}} \left\|D^2_y\beta_1\right\|_{L^2(Z^+\cup P)} \geqslant \frac{\mathrm{K}_2}{\varepsilon^{\frac{3}{2}}}. \tag{3.4}$$

We finish the proof by multiplying the inequality (3.4) by ε. This gives the desired estimate and the proof is complete. □

Next, we show that the Hessian of the boundary layer corrector β_1 on the microscopic scale is uniformly controlled in L^2.

Proposition 3.3. *Let $\beta_1 \in D^{1,2}(Z^+ \cup P)$ a solution of cell problem's (2.4). Then, there exists is a constant* K_3 *independent of ε, such that*

$$\left\| D_y^2 \beta_1 \right\|_{L^2(Z^+ \cup P)} \leqslant \mathrm{K}_3. \tag{3.5}$$

Proof. **Step 1 :** We introduce the partition of the unit $\varphi(\cdot)$ such that

$$\begin{cases} \varphi(y_2) \in C_0^\infty\left(B_0\ ;\ [0,1]\right), & \text{where } B_0 =]\frac{1}{2}, \frac{5}{2}[\\ \varphi(y_2) \equiv 1 \quad \text{on } B_0^{'} =]1,2[. \end{cases}$$

Next, we define the function $\Phi_n(\cdot,\cdot)$ such that

$$\begin{cases} \Phi_n(y_1, y_2) = \varphi\left(\dfrac{y_2}{2^n}\right) \\ \Phi_n \in C_0^\infty\left(Q_n\ ;\ [0,1]\right), \qquad \text{with } Q_n =]0, 2\pi[\times]2^{n-1}, 5\left(2^{n-1}\right)[\\ \Phi_n \equiv 1 \quad \text{on } Q_n^{'} =]0, 2\pi[\times]2^n, 2^{n+1}[. \end{cases} \tag{3.6}$$

We notice that $\Phi_0(y_1, y_2) = \varphi\left(y_2\right)$ and $\Phi_n(y_1, y_2) = \Phi_0\left(y_1, \frac{y_2}{2^n}\right)$. The gradient of Φ_n satisfies

$$\nabla_y \Phi_n(y) = \begin{pmatrix} \dfrac{\partial \Phi_n}{\partial y_1} \\ \dfrac{\partial \Phi_n}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{2^n}\ \varphi^{'}\left(\frac{y_2}{2^n}\right) \end{pmatrix} = \frac{1}{2^n} \nabla \Phi_0.$$

We define the function $h_n(\cdot)$ by

$$h_n(y) = \varphi\left(\frac{y_2}{2^n}\right)(\beta_1(y) - \overline{\beta_1}). \tag{3.7}$$

We verify that $h_n(y) = \varphi\left(\dfrac{y_2}{2^n}\right)(\beta_1(y) - \overline{\beta_1}) \in H_0^1(Q_n)$ and h_n is the weak solution of the equation:

$$-\Delta_y h_n(y) = -\left(\frac{1}{2^n}\right)^2 \varphi^{''}\left(\frac{y_2}{2^n}\right)(\beta_1(y) - \overline{\beta_1}) - \left(\frac{2}{2^n}\right) \varphi^{'}\left(\frac{y_2}{2^n}\right) \frac{\partial \beta_1}{\partial y_2}(y) := F_n(y).$$

The variational formulation is given by

$$\int_{Q_n} \nabla h_n . \nabla v\ dy = \int_{Q_n} F_n\, v\ dy, \qquad \forall v \in H_0^1(Q_n).$$

Now, denote by Z the infinite half-band such that

$$Z = \ ]0, 2\pi[\ \times\ ]1, +\infty[\ \subset\ \left(\bigcup_n Q_n^{'}\right).$$

We have

$$\int_Z \left(\sum_{i,j=1}^2 \left|\partial^2_{y_i y_j} \beta_1\right|^2\right) dy = \int_Z \left(\sum_{i,j=1}^2 \left|\partial^2_{y_i y_j} (\beta_1 - \overline{\beta_1})\right|^2\right) dy \leqslant \sum_{n \in \mathbb{N}} \int_{Q_n^{'}} \left(\sum_{i,j=1}^2 \left|\partial^2_{y_i y_j} h_n\right|^2\right) dy. \tag{3.8}$$

On the other hand, using a change of variables, we have

$$\begin{aligned} \int_{Q_n^{'}} \left(\sum_{i,j=1}^2 \left|\partial^2_{y_i y_j} h_n(y)\right|^2\right) dy &= 2^n \int_{Q_n^{'}} \left(\sum_{i,j=1}^2 \left|\partial^2_{y_i y_j}\left(\varphi\left(\frac{y_2}{2^n}\right)(\beta_1(y) - \overline{\beta_1})\right)\right|^2\right) \frac{dy_2}{2^n}\, dy_1 \\ &= \left(2^n\right)^{-3} \int_{Q_0^{'}} \left(\sum_{i,j=1}^2 \left|\partial^2_{z_i z_j}\left(\varphi\left(z_2\right)\left(\beta_1(z_1, 2^n z_2) - \overline{\beta_1}\right)\right)\right|^2\right) dz_1\, dz_2, \end{aligned}$$

where $z_1 = y_1$ and $z_2 = \dfrac{y_2}{2^n}$. To simplify the presentation, we introduce $\widetilde{h}_n(z) = \varphi(z_2)\left(\beta_1(z_1, 2^n z_2) - \overline{\beta_1}\right)$. Consequently

$$\int_{Q_n'} \left(\sum_{i,j=1}^{2} \left| \partial^2_{y_i y_j} h_n(y) \right|^2 \right) dy = (2^n)^{-3} \int_{Q_0'} \left(\sum_{i,j=1}^{2} \left| \partial^2_{z_i z_j} \widetilde{h}_n(z) \right|^2 \right) dz_1\, dz_2. \tag{3.9}$$

We underline that we made a change of variables to turn the rectangle $Q_n^{'}$ into a domain independent of n. The Laplacian becomes a deformed Laplacian operator with variable coefficients depending on n.
Let us note by $\tilde{\Delta}$ the deformed Laplacian operator which is written as follows: $\tilde{\Delta} = 2^n \partial_{z_1^2} + \frac{1}{2^n}\partial_{z_2^2}$. Thus, $\widetilde{h}_n(\cdot)$ is a weak solution of the following elliptic problem

$$\begin{cases} -\tilde{\Delta}\widetilde{h}_n(z) = -2^n \varphi^{''}(z_2)\left(\beta_1(z_1, 2^n z_2) - \overline{\beta_1}\right) - 2\varphi^{'}(z_1)\dfrac{\partial \beta_1(z_1, 2^n z_2)}{\partial z_2} := \widetilde{F_n}(z) \quad \text{in } Q_0 \\ \\ \qquad \widetilde{h}_n = 0 \qquad \text{on } \partial Q_0. \end{cases} \tag{3.10}$$

Thanks to Proposition 6.2 and inequality (6.31), we show that there exists a positive constant K independent of n such that we have the following inequality

$$\sum_{i,j=1}^{2} \left\| \partial^2_{z_i z_j} \widetilde{h}_n \right\|_{L^2(Q_0')} \leqslant K (2^n)^2 \left(\left\| \widetilde{h}_n \right\|_{L^2(Q_0)} + \left\| \widetilde{F_n} \right\|_{L^2(Q_0)} \right). \tag{3.11}$$

Moreover, we have

$$\left\| \widetilde{h}_n \right\|_{L^2(Q_0)} = \left(\frac{1}{2^n} \int_{Q_n} \left| \varphi\left(\frac{y_2}{2^n}\right)\left(\beta_1(y) - \overline{\beta_1}\right) \right|^2 dy \right)^{\frac{1}{2}} = \left(\frac{1}{2^n}\right)^{\frac{1}{2}} \|h_n\|_{L^2(Q_n)},$$

and

$$\begin{aligned} \left\| \widetilde{F_n} \right\|_{L^2(Q_0)} &\leqslant 2 \left(\int_{Q_0} \left| 2^n \varphi^{''}(z_2)\left(\beta_1(z_1, 2^n z_2) - \overline{\beta_1}\right) \right|^2 dz + \int_{Q_0} \left| 2\varphi^{'}(z_1) \frac{\partial \beta_1}{\partial z_2}(z_1, 2^n z_2) \right|^2 dz \right)^{\frac{1}{2}} \\ &\leqslant K \left(\frac{1}{2^n} \int_{Q_n} \left| 2^n \left(\beta_1(y_1, y_2) - \overline{\beta_1}\right) \right|^2 dy + \frac{2}{2^n} \int_{Q_n} \left| \frac{\partial \beta_1}{\partial y_2}(y_1, y_2) \right|^2 dy \right)^{\frac{1}{2}} \\ &\leqslant K \left((2^n)^{\frac{1}{2}} \|h_n\|_{L^2(Q_n)} + \left(\frac{2}{2^n}\right)^{\frac{1}{2}} \|\nabla h_n\|_{L^2(Q_n)} \right). \end{aligned}$$

Combining these two estimates with (3.11), we deduce

$$\sum_{i,j=1}^{2} \left\| \partial^2_{z_i z_j} \widetilde{h}_n \right\|_{L^2(Q_0')} \leqslant K \left((2^n)^{\frac{1}{2}} \|h_n\|_{L^2(Q_n)} + \left(\frac{2}{2^n}\right)^{\frac{1}{2}} \|\nabla h_n\|_{L^2(Q_n)} \right). \tag{3.12}$$

Notice that K does not depend on n. Thanks to (3.9) and (3.12), we finaly obtain

$$\sum_{i,j=1}^{2} \left\| \partial^2_{y_i y_j} h_n \right\|_{L^2(Q_n')} \leqslant (2^n)^{-\frac{3}{2}} \sum_{i,j=1}^{2} \left\| \partial^2_{z_i z_j} \widetilde{h}_n \right\|_{L^2(Q_0')} \leqslant K\,(2^n) \left(\|h_n\|_{L^2(Q_n)} + \|\nabla h_n\|_{L^2(Q_n)} \right), \tag{3.13}$$

where K is a constant independent of n.
Step 2 : Combining (3.8) and (3.13), we get that

$$\begin{aligned}\sum_{i,j=1}^{2}\left\|\partial^2_{y_iy_j}\beta_1\right\|_{L^2(Z)} &\leqslant \sum_{n\in\mathbb{N}}\left(\sum_{i,j=1}^{2}\left\|\partial^2_{y_iy_j}h_n\right\|_{L^2(Q'_n)}\right)\\ &\leqslant K_1\left(2^n\right)\left(\sum_{n\in\mathbb{N}}\left\|\beta_1-\overline{\beta_1}\right\|_{L^2(Q_n)}+\sum_{n\in\mathbb{N}}\left\|\nabla\beta_1\right\|_{L^2(Q_n)}\right).\end{aligned} \tag{3.14}$$

To get rid of the weight $\mathcal{O}(2^n)$, we study the r.h.s. of the last inequality. Indeed, for all $\alpha\in\,]0,1[$, we have

$$\left\|\beta_1-\overline{\beta_1}\right\|_{L^2(Q_n)}\leqslant\left(e^{-2\alpha 2^{n-1}}\int_{Q_n}\left|e^{\alpha y_2}(\beta_1-\overline{\beta_1})\right|^2dy\right)^{\frac{1}{2}}:=\sqrt{e^{-\alpha 2^n}}\left\|e^{\alpha y_2}(\beta_1-\overline{\beta_1})\right\|_{L^2(Q_n)},$$

and given a real $\alpha\in\left]0,\frac{1}{2}\right[$, we also have

$$\left\|\nabla\beta_1\right\|_{L^2(Q_n)}\leqslant\left(e^{-2\alpha 2^{n-1}}\int_{Q_n}\left|e^{\alpha y_2}\nabla\beta_1\right|^2dy\right)^{\frac{1}{2}}:=\sqrt{e^{-\alpha 2^n}}\left\|e^{\alpha y_2}\nabla\beta_1\right\|_{L^2(Q_n)}.$$

Substituting these last calculations into (3.14), then for all $\alpha\in\left]0,\frac{1}{2}\right[$, we obtain

$$\sum_{i,j=1}^{2}\left\|\partial^2_{y_iy_j}\beta_1\right\|_{L^2(Z)}\leqslant K\sum_{n\in\mathbb{N}}\left(2^n\right)\sqrt{e^{-\alpha 2^n}}\left(\left\|e^{\alpha y_2}(\beta_1-\overline{\beta_1})\right\|_{L^2(Q_n)}+\left\|e^{\alpha y_2}\nabla\beta_1\right\|_{L^2(Q_n)}\right). \tag{3.15}$$

We notice that D'Alembert criterion for convergence of infinite series in expression (3.15) is verified. It follows then that there exists a positive constant K independent of n such that

$$\left\|D^2_y(\beta_1)\right\|_{L^2(Z)}\leqslant K,\qquad\text{with } Z=\,]0,2\pi[\,\times\,]1,+\infty[\,. \tag{3.16}$$

Furthermore, thanks to the result on the boundary H^2-regularity [11, theorem 4, p317], we conclude that the L^2 norm of the Hessian matrix of the boundary layer corrector β_1 at the macroscopic scale is bounded in $(Z^+\cup P)\setminus Z$. This ends the proof. □

In the following proposition, we compute the H^2-error estimate between the solution β_1 of problem (2.4) and the averaged first order corrector $\overline{\beta_1}$.

Proposition 3.4. *Let $\varepsilon\in]0,1[$. There exists a positive constant K_4 independent of ε, such that*

$$\varepsilon\left\|\beta_1-\overline{\beta_1}\right\|_{H^2(\Omega_\varepsilon)}\leqslant\frac{K_4}{\sqrt{\varepsilon}}. \tag{3.17}$$

Proof. We start from the following inequality

$$\varepsilon\left\|\beta_1-\overline{\beta_1}\right\|_{H^2(\Omega_\varepsilon)}\leqslant\varepsilon\left(\left\|\beta_1-\overline{\beta_1}\right\|^2_{L^2(\Omega_\varepsilon)}+\left\|\nabla_x(\beta_1-\overline{\beta_1})\right\|^2_{L^2(\Omega_\varepsilon)}+\left\|D^2_x(\beta_1-\overline{\beta_1})\right\|^2_{L^2(\Omega_\varepsilon)}\right)^{\frac{1}{2}}. \tag{3.18}$$

Using a change of variables, we verify that

$$\left\|\beta_1-\overline{\beta_1}\right\|^2_{L^2(\Omega_\varepsilon)}=\sum_{k=1}^{E(\frac{L}{2\pi\varepsilon})}\left(\int_{(Z^+\cup P)_k}\left|\beta_1(y)-\overline{\beta_1}\right|^2\ \varepsilon^2dy\right)\leqslant\frac{L}{2\pi\varepsilon}\left(\varepsilon^2\int_{(Z^+\cup P)}\left|\beta_1(y)-\overline{\beta_1}\right|^2dy\right), \tag{3.19}$$

where $E(\cdot)$ is the integer part. We define the average $\overline{\Gamma}$ of the function $(-\tilde{\gamma})$ along the period 2π, namely $\overline{\Gamma} = \frac{1}{2\pi}\int_0^{2\pi} (-\tilde{\gamma}(y_1))\ dy_1$. Using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we obtain, for all $\alpha \in]0, \frac{1}{2}[$

$$\begin{aligned}\int_{(Z^+\cup P)} \left|\beta_1(y) - \overline{\beta_1}\right|^2\ dy \leqslant 2\left(\int_P e^{-2\alpha y_2}\left|e^{\alpha y_2}\left(\beta_1(y) - \overline{\beta_1} + \overline{\Gamma}\right)\right|^2\ dy\right) + 2\left(\int_P \left|\overline{\Gamma}\right|^2\ dy\right)\\ + \int_{Z^+} e^{-2\alpha y_2}\left|e^{\alpha y_2}\left(\beta_1(y) - \overline{\beta_1}\right)\right|^2\ dy,\\ \leqslant 2\left(e^{-2\alpha\tilde{\gamma}(y_1)}\left\|\beta_1 - \overline{\beta_1} + \overline{\Gamma}\right\|^2_{L^2(P,\ e^{\alpha y_2})} + 2\pi|\tilde{\gamma}(y_1|\left(\overline{\Gamma}\right)^2\right) + \left\|\beta_1 - \overline{\beta_1}\right\|^2_{L^2(Z^+,\ e^{\alpha y_2})}.\end{aligned}$$

Thanks to the estimates (2.6) and (2.7), there exists two positive constants $\widetilde{C_2}$ and K independent of α such that

$$\left\|\beta_1 - \overline{\beta_1}\right\|^2_{L^2(Z^+\cup P)} \leqslant 2\left(e^{-2\alpha\tilde{\gamma}(y_1)}\widetilde{C_2}\left\|\nabla_y\beta_1\right\|^2_{L^2(P,\ e^{\alpha y_2})} + 2\pi\left(\overline{\Gamma}\right)^2\right) + \left\|\nabla_y\beta_1\right\|^2_{L^2(Z^+,\ e^{\alpha y_2})} \leqslant K.$$

Substituting the last estimate into (3.19), there exists a positive constant K independent of ε such that

$$\varepsilon\left\|\beta_1 - \overline{\beta_1}\right\|_{L^2(\Omega_\varepsilon)} \leqslant K\sqrt{\frac{L}{2\pi}}\ \varepsilon^{\frac{3}{2}}. \tag{3.20}$$

Now, we focus on the properties of the gradient. Thanks to the multi-scale structure of this corrector and the specific boundary layer properties of β_1, we have by a simple change of variables that

$$\varepsilon^2\left\|\nabla_x(\beta_1 - \overline{\beta_1})\right\|^2_{L^2(\Omega_\varepsilon)} = \varepsilon^2\sum_{k=1}^{E(\frac{L}{2\pi\varepsilon})}\left(\int_{(Z^+\cup P)_k}\sum_{i=1}^{2}\left|\frac{1}{\varepsilon}\partial_{y_i}\beta_1(y)\right|^2\varepsilon^2 dy_1 dy_2\right) \leqslant \frac{L\varepsilon}{2\pi}\left\|\nabla_y\beta_1\right\|^2_{L^2(Z^+\cup P)}, \tag{3.21}$$

where $E(\cdot)$ is the integer part. In a similar way, using again a simple change of variables that we get

$$\left\|D^2_x(\beta_1 - \overline{\beta_1})\right\|^2_{L^2(\Omega_\varepsilon)} = \sum_{k=1}^{E(\frac{L}{2\pi\varepsilon})}\left(\int_{(Z^+\cup P)_k}\sum_{i,j=1}^{2}\left|\frac{1}{\varepsilon^2}\partial^2_{y_iy_j}\ \beta_1(y)\right|^2\varepsilon^2 dy\right) \leqslant \frac{L}{2\pi\varepsilon^3}\left\|D^2_y\beta_1\right\|^2_{L^2(Z^+\cup P)}.$$

Thanks to Proposition 3.3 we get that

$$\varepsilon\left\|D^2_x(\beta_1 - \overline{\beta_1})\right\|_{L^2(\Omega_\varepsilon)} \leqslant \frac{\sqrt{L}}{\sqrt{2\pi\varepsilon}}\left\|D^2_y\beta_1\right\|_{L^2(Z^+\cup P)} \leqslant \frac{K_3\sqrt{L}}{\sqrt{2\pi\varepsilon}}. \tag{3.22}$$

We conclude by substituting the estimates (3.20), (3.21) and (3.22) into (3.18). Then there exists a positive constant K_4 independent of ε such that for every $0 < \varepsilon < 1$, we have

$$\varepsilon\left\|\beta_1 - \overline{\beta_1}\right\|_{H^2(\Omega_\varepsilon)} \leqslant K_4\left(\varepsilon^{\frac{3}{2}} + \sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\right) \leqslant \frac{3K_4}{\sqrt{\varepsilon}}.$$

This ends the proof. □

3.3. H^2-Regularity of the first order approximation and an error estimate

Here, we prove that the H^2-norm of the first order approximation $u^\infty_{\text{Ach},1}$ (see (2.9)) is singular of order $\mathcal{O}(\varepsilon^{\frac{1}{2}})$.

Proof of Theorem 1.1: We recall the following

$$\|u^\infty_{\text{Ach},1}\|^2_{H^2(\Omega_\varepsilon)} = \|u^\infty_{\text{Ach},1}\|^2_{H^1(\Omega_\varepsilon)} + \left\|D^2_x(u^\infty_{\text{Ach},1})\right\|^2_{L^2(\Omega_\varepsilon)}, \tag{3.23}$$

The first term of the r.h.s. is controlled thanks to Theorem 6.1 and Proposition 2.4. Using a triangular inequality, we obtain

$$\|u^{\infty}_{\mathrm{Ach},1}\|_{H^1(\Omega_\varepsilon)} \leqslant \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \|u^{\infty}_{\mathrm{Ach},1} - u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant K\left[\sqrt{\mathrm{mes}(\Omega_\varepsilon)}\left(4(1+\varepsilon)^2+1\right) + \varepsilon^{\frac{3}{2}}\right] \leqslant K. \quad (3.24)$$

Notice that the positive constant K does not depend on ε. In order to estimate the second term of the r.h.s in (3.23), let us consider the explicit expression of the first order approximation defined in (2.9). By a triangular inequality, we have

$$\begin{aligned}\left\|D_x^2(u^{\infty}_{\mathrm{Ach},1})\right\|_{L^2(\Omega_\varepsilon)} &\leqslant \left\|D_x^2(u^0_{\mathrm{ext},1})\right\|_{L^2(\Omega_\varepsilon)} + \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{C}{2}\left\|D_x^2\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right)\right\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant \left\|u^0_{\mathrm{ext},1}\right\|_{H^2(\Omega_\varepsilon)} + \frac{C}{2}\,\varepsilon\left\|D_x^2\beta_1\right\|_{L^2(\Omega_\varepsilon)} \leqslant \left\|u^0_{\mathrm{ext},1}\right\|_{H^2(\Omega_\varepsilon)} + \frac{K}{\sqrt{\varepsilon}}\left\|D_y^2\beta_1\right\|_{L^2(Z^+\cup P)},\end{aligned}$$

where the constant K is independent of ε for all $\varepsilon \in [0,1[$. Thanks to Theorem 3.3 and Proposition 3.1, we get the following estimate

$$\left\|D_x^2(u^{\infty}_{\mathrm{Ach},1})\right\|_{L^2(\Omega_\varepsilon)} \leqslant K\left(1+\frac{1}{\sqrt{\varepsilon}}\right) \leqslant \frac{2K}{\sqrt{\varepsilon}}. \quad (3.25)$$

We finally combine (3.23), (3.24) and (3.25), and get the desired upper bound estimate. For a lower bound for $\|u^{\infty}_{\mathrm{Ach},1}\|_{H^2(\Omega_\varepsilon)}$, we use a reverse triangle inequality :

$$\frac{C\varepsilon}{4}\left\|D_x^2\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right)\right\|_{L^2(\Omega_\varepsilon)} - \left\|D_x^2(u^0_{\mathrm{ext},1})\right\|_{L^2(\Omega_\varepsilon)} \leqslant \left\|D_x^2(u^{\infty}_{\mathrm{Ach},1})\right\|_{L^2(\Omega_\varepsilon)} \leqslant \|u^{\infty}_{\mathrm{Ach},1}\|_{H^2(\Omega_\varepsilon)}.$$

Thanks to inequality (3.4), we show that there exists a nonnegative constant K_2 independent of ε such that for every $0<\varepsilon<1$, we have

$$\left\|D_x^2\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right)\right\|_{L^2(\Omega_\varepsilon)} = \left\|D_x^2\beta_1\right\|_{L^2(\Omega_\varepsilon)} \geqslant \frac{\mathrm{K}_2}{\varepsilon^{\frac{3}{2}}}.$$

In light of the last two inequalities, we deduce that the norm $\|u^{\infty}_{\mathrm{Ach},1}\|_{H^2(\Omega_\varepsilon)}$ is of order $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$. This concludes the proof. □

Next we estimate the first order approximation $u^{\infty}_{\mathrm{Ach},1}$ with respect to the exact solution.

Proof of Theorem 1.2: We recall that the first order approximation $u^{\infty}_{\mathrm{Ach},1}$ satisfies:

$$\begin{cases} -\Delta u^{\infty}_{\mathrm{Ach},1} = C\ \mathbb{1}_{[\Omega_0]} & \text{in} \quad \Omega_\varepsilon \\ u^{\infty}_{\mathrm{Ach},1} = \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}) - \overline{\beta_1}\right) & \text{on} \quad \Gamma^\infty \\ u^{\infty}_{\mathrm{Ach},1} = 0 & \text{on} \quad \gamma_\varepsilon \\ u^{\infty}_{\mathrm{Ach},1} \text{ is } x_1-\text{periodic} & \text{on} \quad \Sigma_e\cup\Sigma_s. \end{cases}$$

Denote $r^{1,\infty}_\varepsilon = u_\varepsilon - u^{\infty}_{\mathrm{Ach},1}$ the error to estimate. This function is a solution of the problem:

$$\begin{cases} -\Delta r^{1,\infty}_\varepsilon = C\ \mathbb{1}_{[\Omega_\varepsilon\setminus\Omega_0]} & \text{in} \quad \Omega_\varepsilon \\ r^{1,\infty}_\varepsilon = -\frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}) - \overline{\beta_1}\right) & \text{on} \quad \Gamma^\infty \\ r^{1,\infty}_\varepsilon = 0 & \text{on} \quad \gamma_\varepsilon \\ r^{1,\infty}_\varepsilon \text{ is } x_1-\text{periodic} & \text{on} \quad \Sigma_e\cup\Sigma_s. \end{cases}$$

Notice that $\mathbb{1}_{[\cdot]}$ represents the characteristic function of the set between brackets and that the first order normal derivative of u_0 along the fictitious boundary γ^0 satisfies $\frac{\partial u_0}{\partial x_2}(x_1,0) = \frac{C}{2}$. Then, we define the function $\widetilde{R}_\varepsilon$ such that:

$$\widetilde{R}_\varepsilon(x_1,x_2) = r_\varepsilon^{1,\infty}(x_1,x_2) + \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\varphi(x_2)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right),$$

where, $\varphi(\cdot) \in C^\infty(]-1,1],[0,1])$ is a cut-off function satisfying

$$\begin{cases} 0 \leqslant \varphi \leqslant 1 & \text{in} \quad \left[\frac{1}{2},1\right] \\ \varphi \equiv 0 & \text{in} \quad \left]-1,\frac{1}{2}\right]. \end{cases}$$

This function $\widetilde{R}_\varepsilon$ satisfies a homogeneous Dirichlet boundary condition on $\Gamma^\infty \cup \gamma_\varepsilon$ and solves

$$\begin{cases} -\Delta\widetilde{R}_\varepsilon = C\mathbb{1}_{[\Omega_\varepsilon\setminus\Omega_0]} - \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\Delta\left(\varphi(x_2)\left(\beta_1(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}) - \overline{\beta_1}\right)\right) & \text{in} \quad \Omega_\varepsilon \\ \widetilde{R}_\varepsilon \quad = 0 & \text{on} \quad \Gamma^\infty\cup\gamma_\varepsilon \\ \widetilde{R}_\varepsilon \quad \text{is } x_1-\text{periodic} & \text{on} \quad \Sigma_e\cup\Sigma_s. \end{cases} \tag{3.26}$$

To simplify the presentation, we set

$$G_{\beta_1,\varepsilon}(x) = C\mathbb{1}_{[\Omega_\varepsilon\setminus\Omega_0]} - \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\ \Delta\left(\varphi(x_2)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)\right),$$

which represents the source term in the boundary value problem (3.26). By developing the Laplacian term in the above quantity, we get

$$G_{\beta_1,\varepsilon}(x) = C\mathbb{1}_{[\Omega_\varepsilon\setminus\Omega_0]} - \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{C}{2}\left[\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)\varphi''(x_2)\right]. \tag{3.27}$$

Remark that the difference $(\beta_1 - \overline{\beta_1})$ on Γ^∞ is exponentially small w.r.t. ε. Thus, we show that the function $G_{\beta_1,\varepsilon} \in L^2(\Omega_\varepsilon)$ and for every small ε, there exists a positive constant K independent of ε such that

$$\|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leqslant K\left(\sqrt{\operatorname{mes}(\Omega_\varepsilon\setminus\Omega_0)} + \varepsilon\left\|\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right\|_{L^2(\Gamma^\infty)}\right) \leqslant K\left(\sqrt{\varepsilon} + e^{-\frac{1}{\varepsilon}}\right) \leqslant 2K\sqrt{\varepsilon}. \tag{3.28}$$

According to the inequality (6.3) of Theorem 6.2, there exists a positive constant K independent of ε such that

$$\left\|\widetilde{R}_\varepsilon\right\|_{H^2(\Omega_\varepsilon)} \leqslant K\left(\|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon}\left\|\widetilde{R}_\varepsilon\right\|_{H^1(\Omega_\varepsilon)}\right). \tag{3.29}$$

Now we take the weak formulation of problem (3.26). Then, for every test function $v \in \mathcal{H}_{\text{per},0}(\Omega_\varepsilon)$, the Green formula yields

$$\left\langle\nabla\widetilde{R}_\varepsilon\ ,\ \nabla v\right\rangle_{\Omega_\varepsilon} = \left\langle-\Delta\widetilde{R}_\varepsilon\ ,\ v\right\rangle_{\Omega_\varepsilon} = \langle G_{\beta_1,\varepsilon}\ ,v\rangle_{\Omega_\varepsilon}.$$

It follows that

$$\left|\left\langle\nabla\widetilde{R}_\varepsilon\ ,\ \nabla v\right\rangle_{\Omega_\varepsilon}\right| \leqslant \left|\langle G_{\beta_1,\varepsilon}\ ,v\rangle_{\Omega_\varepsilon}\right|. \tag{3.30}$$

Then, using Cauchy-Schwarz and Poincaré inequalities in rough domains, we obtain the upper bound:

$$\left|\langle G_{\beta_1,\varepsilon}\ ,v\rangle_{\Omega_\varepsilon}\right| \leqslant \|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} \leqslant C_P(\varepsilon)\|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)}\|\nabla v\|_{L^2(\Omega_\varepsilon)}, \tag{3.31}$$

where $C_P(\varepsilon)$ denotes the Poincaré constant given here by $(1+\varepsilon)$. Taking $\widetilde{R}_\varepsilon = v$ in (3.30)-(3.31), we obtain

$$\alpha_\varepsilon \left\|\widetilde{R}_\varepsilon\right\|^2_{H^1(\Omega_\varepsilon)} \leqslant \left\|\nabla\widetilde{R}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon)} \leqslant |\langle G_{\beta_1,\varepsilon}\ , \widetilde{R}_\varepsilon\rangle_{\Omega_\varepsilon}| \quad \leqslant C_P(\varepsilon)\,\|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)}\,\|\nabla\widetilde{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{3.32}$$

where α_ε is the coercivity constant given by $\alpha_\varepsilon = \frac{1}{4(1+\varepsilon)^2+1}$. We underline that the solution $\widetilde{R}_\varepsilon \in L^2(\Omega_\varepsilon)$. Indeed, taking into account the estimate (2.11), we find that there exists a positive constant K independent of ε such that

$$\left\|\widetilde{R}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \leqslant K\left(\|r_\varepsilon^{1,\infty}\|_{L^2(\Omega_\varepsilon)} + \varepsilon\,\|\beta_1 - \overline{\beta_1}\|_{L^2(\Gamma^\infty)}\right) \leqslant K\varepsilon^{\frac{3}{2}}. \tag{3.33}$$

From (3.32) and (3.33), we deduce

$$\alpha_\varepsilon \left\|\widetilde{R}_\varepsilon\right\|^2_{H^1(\Omega_\varepsilon)} \leqslant C_P(\varepsilon)\,\|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)}\left(\|\widetilde{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla\widetilde{R}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right). \tag{3.34}$$

Thus, using (3.28) and (3.34), we deduce:

$$\left\|\widetilde{R}_\varepsilon\right\|_{H^1(\Omega_\varepsilon)} \leqslant \frac{C_P(\varepsilon)\,\|G_{\beta_1,\varepsilon}\|_{L^2(\Omega_\varepsilon)}}{\alpha_\varepsilon} \leqslant K\left(\frac{C_P(\varepsilon)}{\alpha_\varepsilon}\right)\sqrt{\varepsilon}. \tag{3.35}$$

Substituting the last estimate into (3.29), there exists a positive constant K independent of ε such that for every small ε, we have

$$\left\|\widetilde{R}_\varepsilon\right\|_{H^2(\Omega_\varepsilon)} \leqslant K\left(\sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\right) \leqslant \frac{2K}{\sqrt{\varepsilon}}. \tag{3.36}$$

Moreover, by computing along Γ^∞ the H^2-error estimate between the corrector β_1 and the microscopic average $\overline{\beta_1}$, we get

$$\left\|\frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\varphi(x_2)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)\right\|_{H^2(\Gamma^\infty)} \leqslant K\varepsilon e^{-\frac{1}{\varepsilon}}. \tag{3.37}$$

Notice that the positive constant K does not depend on ε. Finally, combining (3.36)-(3.37) and a triangular inequality, we show the existence of a positive constant K independent of ε such that for every small ε, we have

$$\begin{aligned}\|r_\varepsilon^{1,\infty}\|_{H^2(\Omega_\varepsilon)} &\leqslant \left\|\widetilde{R}_\varepsilon\right\|_{H^2(\Omega_\varepsilon)} + \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left\|\varphi(x_2)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)\right\|_{H^2(\Gamma^\infty)} \\ &\leqslant K\left(\left\|\widetilde{R}_\varepsilon\right\|_{H^2(\Omega_\varepsilon)} + \varepsilon\, e^{-\frac{1}{\varepsilon}}\right) \leqslant \frac{K}{\sqrt{\varepsilon}}.\end{aligned}$$

□

Remark 3.1. *Using the triangular inequality, we show that* $\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)}$ *is of order* $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$. *In other words, we prove the existence of a nonnegative constant* $\mathrm{C}_{\max}$ *independent of* ε, *such that*

$$\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant \|u_\varepsilon - u^\infty_{\mathrm{Ach},1}\|_{H^2(\Omega_\varepsilon)} + \|u^\infty_{\mathrm{Ach},1}\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{K_6+K_7}{\sqrt{\varepsilon}} \leqslant \frac{\mathrm{C}_{\max}}{\sqrt{\varepsilon}}.$$

It is interesting to ask the following question : can we obtain a more optimal estimate for the H^2*-norm of the exact solution* u_ε *when we use the approximation* $u^\infty_{\mathrm{Ach},2}$ *? This observation motivates the next section.*

4. H^2-Regularity by the second order approximation

In this section we continue the investigation in the sense introduced above (see Remark 3.1). We start by recalling the expression of the second order approximation defined in (2.12) :

$$u^{\infty}_{\mathrm{Ach},2}(x) = u^0_{\mathrm{ext},2}(x) + \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right)$$
$$+\frac{\varepsilon^2}{2}\frac{\partial^2 u_0}{\partial x_2^2}(x_1,0)\left[\left(\beta_2\left(\frac{x}{\varepsilon}\right) - \overline{\beta_2}x_2\right) - \frac{\varepsilon\overline{\beta_2}}{1+\varepsilon\overline{\beta_1}}\left(\beta_1\left(\frac{x}{\varepsilon}\right) - \overline{\beta_1}x_2\right)\right].$$

4.1. H^2-Regularity of the zero-order approximation and the second order corrector

We give here a first result of the H^2-norm for the second order extension of u_0 and the second order boundary layer corrector β_2.

Proposition 4.1. *There exists $\varepsilon_0 > 0$ and two positive constants $\widetilde{\mathrm{C}}_0$, $\widetilde{\mathrm{C}}_4$ independent of ε, such that for every $0 < \varepsilon < \varepsilon_0$, we have*

$$\|u^0_{\mathrm{ext},2}\|_{\mathrm{H}^2(\Omega_\varepsilon)} \leqslant \widetilde{\mathrm{C}}_0, \qquad \varepsilon\left\|\beta_2 - \overline{\beta_2}\right\|_{\mathrm{H}^2(\Omega_\varepsilon)} \leqslant \frac{\widetilde{\mathrm{C}}_4}{\sqrt{\varepsilon}}. \tag{4.1}$$

The proof follows similar arguments as in the proof of Proposition 3.1 and Proposition 3.4 . We refer to [4] for a rigorous proof of this proposition.

4.2. H^2-Regularity of the second order approximation

Proof of Theorem 1.3 : The proof follows the same lines as in Theorem 1.1 . In the same way, we show that the norm $\|u^{\infty}_{\mathrm{Ach},2}\|_{H^2(\Omega_\varepsilon)}$ is of order $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$. □

4.3. The error estimate

The gain obtained when using the second order approximation is of order $\exp(\frac{-1}{\varepsilon})$. Indeed, the following error estimate holds.

Proof of Theorem 1.4 : The proof is identical to the one of Theorem 1.2 . First, we consider the second order approximation $u^{\infty}_{\mathrm{Ach},2}$ satisfying the folowing boundary value problem

$$\begin{cases} -\Delta u^{\infty}_{\mathrm{Ach},2} = C & \text{in} \quad \Omega_\varepsilon \\ u^{\infty}_{\mathrm{Ach},2} = \omega_\varepsilon & \text{on} \quad \Gamma^\infty \\ u^{\infty}_{\mathrm{Ach},2} = 0 & \text{on} \quad \gamma_\varepsilon \\ u^{\infty}_{\mathrm{Ach},2} \text{ is } x_1-\text{periodic} & \text{on} \quad \Sigma_e \cup \Sigma_s, \end{cases}$$

where ω_ε is the contribution of the microscopic correctors on Γ^∞ and reads :

$$\omega_\varepsilon(x_1) = \frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{\partial u_0}{\partial x_2}(x_1,0)\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)$$
$$+\frac{\varepsilon^2}{2}\frac{\partial^2 u_0}{\partial x_2^2}(x_1,0)\left[\left(\beta_2\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_2}\right) - \frac{\varepsilon\overline{\beta_2}}{1+\varepsilon\overline{\beta_1}}\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right) - \overline{\beta_1}\right)\right].$$

We define the error function $r^{2,\infty}_\varepsilon = u_\varepsilon - u^{\infty}_{\mathrm{Ach},2}$. Then, $r^{2,\infty}_\varepsilon$ is a solution of the following boundary value problem:

$$\begin{cases} -\Delta r^{2,\infty}_\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \\ r^{2,\infty}_\varepsilon = -\omega_\varepsilon & \text{on} \quad \Gamma^\infty \\ r^{2,\infty}_\varepsilon = 0 & \text{on} \quad \gamma_\varepsilon \\ r^{2,\infty}_\varepsilon \text{ is } x_1-\text{periodic} & \text{on} \quad \Sigma_e \cup \Sigma_s. \end{cases} \tag{4.2}$$

We define the function $\mathcal{E}_\varepsilon(\cdot)$ as

$$\mathcal{E}_\varepsilon(x) = r_\varepsilon^{2,\infty}(x) + \eta(x_2)\,\omega_\varepsilon(x_1),$$

where the cut-off function $\eta(\cdot)$ is $C^\infty(]-1,1],[0,1])$, and satisfies

$$\begin{cases} 0 \leqslant \eta \leqslant 1 & \text{on} \quad \left[\dfrac{1}{2}, 1\right] \\ \quad\eta \equiv 0 & \text{on} \quad \left]-1, \dfrac{1}{2}\right]. \end{cases}$$

This function $\mathcal{E}_\varepsilon(\cdot)$ satisfies a homogeneous Dirichlet boundary condition on $\Gamma^\infty \cup \gamma_\varepsilon$ and solves

$$\begin{cases} -\Delta\mathcal{E}_\varepsilon = -\Delta\left(\eta\,\omega_\varepsilon\right) & \text{in} \quad \Omega_\varepsilon \\ \mathcal{E}_\varepsilon = 0 & \text{on} \quad \Gamma^\infty \cup \gamma_\varepsilon \\ \mathcal{E}_\varepsilon \text{ is } x_1-\text{periodic} & \text{on} \quad \Sigma_e \cup \Sigma_s. \end{cases} \tag{4.3}$$

In the sequel we denote by $\mathrm{F}^\varepsilon_{\beta_1,\beta_2}(\cdot)$ the source function from the boundary value problem (4.3), that is

$$\begin{aligned} \mathrm{F}^\varepsilon_{\beta_1,\beta_2}(x) := -\Delta\left(\eta\,\omega_\varepsilon\right) = &-\frac{\varepsilon}{1+\varepsilon\overline{\beta_1}}\frac{C}{2}\left[\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right)-\overline{\beta_1}\right)\eta^{''}(x_2)\right] + \frac{C\varepsilon^2}{2}\left[\left(\beta_2\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right)-\overline{\beta_2}\right)\eta^{''}(x_2)\right] \\ &-\frac{C\varepsilon^2}{2}\frac{\varepsilon\overline{\beta_2}}{1+\varepsilon\overline{\beta_1}}\left[\left(\beta_1\left(\frac{x_1}{\varepsilon},\frac{1}{\varepsilon}\right)-\overline{\beta_1}\right)\eta^{''}(x_2)\right]. \end{aligned}$$

Taking into account that the difference $(\beta_1 - \overline{\beta_1})$ on Γ^∞ is exponentially small w.r.t. ε, we get that the function $\mathrm{F}^\varepsilon_{\beta_1,\beta_2} \in L^2(\Omega_\varepsilon)$. Thus, we can show the existence of a positive constant K independent of ε, such that

$$\left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)} \leqslant K\,\mathrm{e}^{-\frac{1}{\varepsilon}}. \tag{4.4}$$

Using similar arguments as in the last proof, we apply the result of Theorem 6.2 to the boundary value problem (4.3) to get

$$\|\mathcal{E}_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant K\left(\left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon}\|\mathcal{E}_\varepsilon\|_{H^1(\Omega_\varepsilon)}\right). \tag{4.5}$$

Notice that the positive constant K does not depend on ε. The weak formulation of (4.3) reads :

$$\langle\nabla\mathcal{E}_\varepsilon\ ,\ \nabla v\rangle_{\Omega_\varepsilon} = \langle-\Delta\mathcal{E}_\varepsilon\ ,\ v\rangle_{\Omega_\varepsilon} = \left\langle\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\ , v\right\rangle_{\Omega_\varepsilon}, \qquad \forall\, v \in \mathcal{H}_{\mathrm{per},0}(\Omega_\varepsilon).$$

By Cauchy-Schwartz and Poincaré inequalities in rough domains, we have the following estimates :

$$\left|\left\langle\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\ , v\right\rangle_{\Omega_\varepsilon}\right| \leqslant \left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} \leqslant C_P(\varepsilon)\left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)}\|\nabla v\|_{L^2(\Omega_\varepsilon)}. \tag{4.6}$$

Note that $C_P(\varepsilon)$ is the Poincaré constant given by $(1+\varepsilon)$. Taking $v = \mathcal{E}_\varepsilon$ in (4.6), we obtain

$$\alpha_\varepsilon\,\|\mathcal{E}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leqslant \|\nabla\mathcal{E}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leqslant |\langle\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\ , \mathcal{E}_\varepsilon\rangle_{\Omega_\varepsilon}| \quad \leqslant C_P(\varepsilon)\left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathcal{E}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{4.7}$$

where α_ε is the coercivity constant given by $\alpha_\varepsilon = \frac{1}{4(1+\varepsilon)^2+1}$. By a triangular inequality, we check that $\mathcal{E}_\varepsilon(\cdot)$ is exponentially small w.r.t. ε for $L^2(\Omega_\varepsilon)$ norm. Indeed,

$$\|\mathcal{E}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant K\left(\left\|r_\varepsilon^{2,\infty}\right\|_{L^2(\Omega_\varepsilon)} + \|\omega_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right) \leqslant K\mathrm{e}^{-\frac{1}{\varepsilon}}. \tag{4.8}$$

Thus (4.7) and (4.8) imply

$$\alpha_\varepsilon\,\|\mathcal{E}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leqslant C_P(\varepsilon)\left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)}\left(\|\mathcal{E}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla\mathcal{E}_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right). \tag{4.9}$$

Using (4.4) and (4.9), we deduce

$$\|\mathcal{E}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant \frac{C_P(\varepsilon)\left\|\mathrm{F}^\varepsilon_{\beta_1,\beta_2}\right\|_{L^2(\Omega_\varepsilon)}}{\alpha_\varepsilon} \leqslant K\left(\frac{C_P(\varepsilon)}{\alpha_\varepsilon}\right)\mathrm{e}^{-\frac{1}{\varepsilon}}. \tag{4.10}$$

This last estimate (4.10) and (4.5) yield $\|\mathcal{E}_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant K\mathrm{e}^{-\frac{1}{\varepsilon}}$, where K is a positive constant independent of ε. Moreover, by similar computations, we can show that the convergence of the quantity $\|\eta\,\omega_\varepsilon\|_{H^2(\Gamma^\infty)}$ is exponentially small w.r.t. ε. Finally, by a triangular inequality we show the existence of a positive constant K independent of ε such that

$$\left\|r^{2,\infty}_\varepsilon\right\|_{H^2(\Omega_\varepsilon)} \leqslant \|\mathcal{E}_\varepsilon\|_{H^2(\Omega_\varepsilon)} + \|\eta\,\omega_\varepsilon\|_{H^2(\Gamma^\infty)} \leqslant K\mathrm{e}^{\frac{-1}{\varepsilon}}.$$

This concludes the proof. □

Proof of Theorem 1.5 : Theorem 1.4 compares $u^\infty_{\mathrm{Ach},2}$ and u_ε for the H^2-norm. This estimate is crucial: the difference is exponentially small. By the triangular inequality, it is not difficult to show that in fact there exist two positive constants $\mathrm{C_{min}}$ and $\mathrm{C_{max}}$ independent of ε such that

$$\begin{aligned}\frac{\mathrm{C_{min}}}{\sqrt{\varepsilon}} \leqslant \frac{\widetilde{C}_5}{\sqrt{\varepsilon}} - \widetilde{C}_7\ \mathrm{e}^{\frac{-1}{\varepsilon}} &\leqslant \|u^\infty_{\mathrm{Ach},2}\|_{\mathrm{H}^2(\Omega_\varepsilon)} - \|u_\varepsilon - u^\infty_{\mathrm{Ach},2}\|_{\mathrm{H}^2(\Omega_\varepsilon)} \\ &\leqslant \|u_\varepsilon\|_{\mathrm{H}^2(\Omega_\varepsilon)} \leqslant \|u^\infty_{\mathrm{Ach},2}\|_{\mathrm{H}^2(\Omega_\varepsilon)} + \|u_\varepsilon - u^\infty_{\mathrm{Ach},2}\|_{\mathrm{H}^2(\Omega_\varepsilon)} \leqslant \frac{\widetilde{C}_6}{\sqrt{\varepsilon}} + \widetilde{C}_7\ \mathrm{e}^{\frac{-1}{\varepsilon}} \leqslant \frac{\mathrm{C_{max}}}{\sqrt{\varepsilon}},\end{aligned}$$

which ends the proof. □

5. Conclusion

In this work we studied the H^2-regularity for the exact solution of a blood flow problem in arough domain with periodic lateral boundary conditions. We established the H^2-norm for the exact solution by using boundary-layer approximations. The estimate of the H^2 norm using the first boundary layer approximation is of order $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$ while it is singular in $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$ by using the second boundary layer approximation.

Studying the H^2-regularity in the spirit of weak solution but in the weighted context improves the L^2 and H^1 estimates [10] but requires an extra amount of work not presented here. This is done in [2], we present the H^2-regularity for the exact solution of a blood flow problem in a rough domain with non-periodic lateral boundary conditions.

6. Appendix A

A1 Existence and uniqueness results for (1.2)

We define the Sobolev space $\mathcal{H}_{\mathrm{per},0}$ by :

$$\mathcal{H}_{\mathrm{per},0} = \{v \in H^1(\Omega_\varepsilon) : \; v = 0 \text{ on } \Gamma^\infty \cup \gamma_\varepsilon \text{ and } v \text{ is } x_1 - \text{periodic on } \Sigma_e \cup \Sigma_s\}. \tag{6.1}$$

These spaces $\mathcal{H}_{\mathrm{per},0}$ are Hilbert spaces for the norm $\| \, . \, \|_{\mathcal{H}_{\mathrm{per},0}} = \sqrt{\langle \cdot, \cdot \rangle}$ where

$$\langle u, v \rangle = \int_{\Omega_\varepsilon} (\nabla u \, \nabla v + u \, v) \, dy,$$

In this section, we study the existence and uniqueness of solutions for the Laplace equation with Dirichlet boundary conditions in the rough domain Ω_ε. More precisely, we are interested in the following problem

$$(\mathrm{E}) \quad \begin{cases} -\Delta u_\varepsilon = g & \text{in} \quad \Omega_\varepsilon \\ u_\varepsilon \quad = 0 & \text{on} \quad \Gamma^\infty \cup \gamma_\varepsilon \\ u_\varepsilon \quad \text{is } x_1 - \text{periodic} & \text{on} \quad \Sigma_e \cup \Sigma_s, \end{cases}$$

where the function $g \in L^2(\Omega_\varepsilon)$. We recall that, Ω_ε denotes the rough domain in $\mathbb{R}^2$ depicted in figure 2. γ_ε is the rough boundary and Γ^∞ is the upper smooth one (see figure 2). Σ_e is the vertical entry of the domain and Σ_s its output.

Theorem 6.1. *Under hypotheses* **H1***, the problem* (E) *admits a unique solution u_ε belonging to $\mathcal{H}_{\mathrm{per},0}$. Moreover, we have for every $\varepsilon > 0$,*

$$\|u_\varepsilon\|_{\mathcal{H}_{\mathrm{per},0}} \leqslant \left(4(1+\varepsilon)^2 + 1\right) \|g\|_{L^2(\Omega_\varepsilon)}. \tag{6.2}$$

Proof. The equivalent variational form of the boundary value problem (E) reads

$$a_\varepsilon(u_\varepsilon, v) = l(v), \qquad \forall v \in \mathcal{H}_{\mathrm{per},0},$$

where $a_\varepsilon(u_\varepsilon, v) = \left(\int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \; \nabla v(x) \; dx\right)$ and $l(v) = \left(\int_{\Omega_\varepsilon} g(x) \; v(x) \; dx\right)$. These forms are obviously continuous and bilinear (resp. linear) on $\mathcal{H}_{\mathrm{per},0} \times \mathcal{H}_{\mathrm{per},0}$ (resp. $\mathcal{H}_{\mathrm{per},0}$). Due to the homogeneous boundary condition, the semi-norm of the gradient is a norm. By Lax-Milgram theorem [7], we conclude that the problem (E) admits a unique solution in $\mathcal{H}_{\mathrm{per},0}$. Furthermore, we have

$$\alpha_\varepsilon \|u_\varepsilon\|^2_{\mathcal{H}_{\mathrm{per},0}} \leqslant |a_\varepsilon(u_\varepsilon, u_\varepsilon)| = |l(u_\varepsilon)| \leqslant \|g\|_{L^2} \; \|u_\varepsilon\|_{\mathcal{H}_{\mathrm{per},0}},$$

where $\alpha_\varepsilon = \left(4(1+\varepsilon)^2 + 1\right)^{-1}$. Then the desired result follows. □

A2 H^2 Regularity by variational formulation in rough domains

Theorem 6.2. *Under hypotheses* **H1***, the solution u_ε of the boundary value problem* (E) *is in $H^2(\Omega_\varepsilon)$. Moreover, there exists $K > 0$ such that, for every $\varepsilon > 0$,*

$$\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant K \left(\|g\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon} \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \right). \tag{6.3}$$

Proof :

1. Partition of unity :

The domain Ω_ε is a bounded open set whose boundary $\partial\Omega_\varepsilon$ is compact, then we can as usual cover $\partial\Omega_\varepsilon$ with finitely many sets $(V_p)_{0 \le p \le N_0}$ (with N_0 a fixed number independent of ε). We introduce the following sets

1. Let V_{int} be an open set of Ω_ε. We assume that it is strictly included in Ω_ε. It allows us to study the H^2-regularity inside Ω_ε away from the boundary.

2. $V_\infty := D(x_\infty, \frac{L^2+1}{4})$ is the disk centered at $x_\infty = (\frac{L}{2}, \frac{L^2+3}{4})$ and of radius $\frac{L^2+1}{4}$. It allows us to treat the H^2-regularity near the smooth boundary Γ^∞.

3. $V_e := D(x_e, r_e)$ is the disk centered at $x_e \in \Sigma_e$ and of radius $r_e > 0$. It allows us to study the H^2-regularity near the smooth boundary Σ_e.

4. $V_s := D(x_s, r_s)$ is the disk centered at $x_s \in \Sigma_s$ and of radius $r_s > 0$. It allows us to study the H^2-regularity near the smooth boundary Σ_s.

5. $V_{\text{rug}} := D(x_{\text{rug}}, R_{\text{rug}})$ is the disk centered at $x_{\text{rug}} = (\frac{L}{2}\ ,\ \varepsilon\tilde{\gamma}(\frac{L}{2\varepsilon}) + \frac{1-L^2}{4})$ and of radius $R_{\text{rug}} = \frac{L^2+1}{4}$. It allows us to study the H^2-regularity near the rough boundary $\gamma_\varepsilon \subset V_{\text{rug}}$.

Therefore, we can write the solution u_ε of the problem (E) in the following form:

$$u_\varepsilon := u_{\text{int}} + u_{\text{s}} + u_{\text{e}} + u_\infty + u^\varepsilon_{\text{rug}} = \varphi_{\text{int}}\, u_\varepsilon + \varphi_{\text{s}}\, u_\varepsilon + \varphi_{\text{e}}\, u_\varepsilon + \varphi_\infty\, u_\varepsilon + \varphi_{\text{rug}}\, u_\varepsilon, \tag{6.4}$$

where $\varphi_{\text{int}} \in C^\infty_c(V_{\text{int}}, [0,1]), \varphi_\infty \in C^\infty_c(V_\infty, [0,1]), \varphi_{\text{e}} \in C^\infty_c(V_e, [0,1]), \varphi_{\text{s}} \in C^\infty_c(V_s, [0,1]), \varphi_{\text{rug}} \in C^\infty_c(V_{\text{rug}}, [0,1])$ and such that $\varphi_{\text{int}} + \varphi_{\text{e}} + \varphi_{\text{s}} + \varphi_\infty + \varphi_{\text{rug}} = 1 \quad \text{in} \quad \overline{\Omega_\varepsilon}$.

2. Interior H^2-regularity in Ω_ε :

We verify that $u_{\text{int}} = \varphi_{\text{int}} u_\varepsilon$ is a weak solution in Ω_ε of the following equation:

$$-\Delta u_{\text{int}} = \varphi_{\text{int}}\, g - 2\nabla\varphi_{\text{int}}.\nabla u_\varepsilon - (\Delta\varphi_{\text{int}})u_\varepsilon := G_{\text{int}}.$$

It is straightforward to check that $G_{\text{int}} \in L^2(\Omega_\varepsilon)$. Applying Theorem 1 [[11, p.309]] to the last equation, we show that there exists $K > 0$ such that $u_{\text{int}} \in H^2_{loc}(\Omega_\varepsilon)$ and we have the following estimate:

$$\|u_{\text{int}}\|_{H^2(V_{\text{int}})} \leqslant K \left(\|G_{\text{int}}\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right) \leqslant K \left(\|g\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\right). \tag{6.5}$$

3. H^2-Regularity near the smooth boundaries $\Gamma^\infty \cup \Sigma_e \cup \Sigma_s$:

We denote by $D_k = \Omega_\varepsilon \cap D(x_k, r_k)$ with $(k = s, e, \infty)$ the intersection between the rough domain Ω_ε and the disk centered at x_k and of radius r_k. We can verify repeatedly that $u_k = \varphi_k u_\varepsilon \in H^1_0(D_k)$ with $(k = s, e, \infty)$ and that u_k is a weak solution on D_k of the equation:

$$-\Delta u_k = \varphi_k\ g - 2\nabla\varphi_k.\nabla u_\varepsilon - (\Delta\varphi_k)u_\varepsilon := G_k, \qquad \text{for} \quad (k = s, e, \infty), \tag{6.6}$$

where $G_k \in L^2(D_k)$. In light of Theorem 4 [[11, p.317]], we see that $u_k \in H^2(D_k)_{(k=s,e,\infty)}$ and that

$$\|u_k\|_{H^2(D_k)} \leqslant K\left[\|G_k\|_{L^2(D_k)} + \|u_\varepsilon\|_{L^2(D_k)}\right] \leqslant K\left[\|g\|_{L^2(D_k)} + \|u_\varepsilon\|_{H^1(D_k)}\right]. \tag{6.7}$$

4. H^2-Regularity near the rough boundary γ_ε : We present our approach to study the H^2-regularity near the rough boundary γ_ε. We start by change a of variable a to straighten the rough boundary γ_ε into a domain no longer depending on ε. Then, the Laplacian becomes an operator with variable coefficients dependent on ε. Next, we establish the variational formulation in the adjusted domain in order to control the bilinear form associated with the variational formulation by choosing particular test functions. After establishing the H^2-estimate of the solution u_ε in the adjusted domain, we can show the result of the H^2-regularity by returning to the rough domain. The following calculations are technical.

Step I : Straightening of the rough boundary

We consider the point x_{rug} of coordinates $(\frac{L}{2}\ ,\ \varepsilon\tilde{\gamma}(\frac{L}{2\varepsilon}) + \frac{1-L^2}{4})$ and we denote by $D_{\text{rug}} = \Omega_\varepsilon \cap D(x_{\text{rug}}, R_{\text{rug}})$ the intersection between the rough domain Ω_ε and the disk centered at x_{rug} and of radius $R_{\text{rug}} = \frac{L^2+1}{4} > 0$.

Since $\varphi_{\rm rug}$ has a compact support in $D_{\rm rug}(x_{\rm rug}, R_{\rm rug})$, we verify that $u^\varepsilon_{\rm rug} = \varphi_{\rm rug}\, u_\varepsilon \in H^1_0\,(D_{\rm rug})$ and that $u^\varepsilon_{\rm rug}$ is a weak solution in $D_{\rm rug}$ of equation :

$$-\Delta u^\varepsilon_{\rm rug} = \varphi_{\rm rug}\; g - 2\nabla\varphi_{\rm rug}.\nabla u_\varepsilon - (\Delta\varphi_{\rm rug})u_\varepsilon := F_{\rm rug}, \tag{6.8}$$

where $F_{\rm rug} \in L^2\,(D_{\rm rug})$. Thanks to the variational inequalities we have $\|F_{\rm rug}\|_{L^2(D_{\rm rug})} \leqslant C\|g\|_{L^2(D_{\rm rug})}$. Then, we multiply (6.8) by a regular test function $v \in H^1_0(D_{\rm rug})$ and we integrate by parts. We get by the Green formula

$$\int_{D_{\rm rug}} \nabla u^\varepsilon_{\rm rug} \nabla v\, dx = \int_{D_{\rm rug}} F_{\rm rug}\; v\, dx, \tag{6.9}$$

We recall that $D_{\rm rug} = \Omega_\varepsilon \cap D\,(x_{\rm rug}, R_{\rm rug}) = \{x \in D\,(x_{\rm rug}, R_{\rm rug})\ ;\ x_2 > \tilde{\gamma}_\varepsilon(x_1)\}$, where $\tilde{\gamma}_\varepsilon$ is a C^∞-function given by $\gamma_\varepsilon(x_1) = \varepsilon\tilde{\gamma}(\frac{x_1}{\varepsilon})$. We now define the C^2-diffeomorphism $\phi(x) = y = \big(\phi^1(x), \phi^2(x)\big)$ by

$$\left\{\begin{array}{l} \phi^1(x) = y_1 = x_1 \\ \phi^2(x) = y_2 = x_2 - \tilde{\gamma}_\varepsilon(x_1) = x_2 - \varepsilon\tilde{\gamma}(\dfrac{x_1}{\varepsilon}), \end{array}\right. \tag{6.10}$$

and $\phi^{-1} = \psi : \widetilde{D}_{\rm rug} \longrightarrow D_{\rm rug}$ by:

$$\left\{\begin{array}{l} \psi^1(y) = x_1 = y_1 \\ \psi^2(y) = x_2 = y_2 + \tilde{\gamma}_\varepsilon(y_1) = y_2 + \varepsilon\tilde{\gamma}(\dfrac{y_1}{\varepsilon}). \end{array}\right. \tag{6.11}$$

In the sequel we choose the real $s > 0$ so that the half-disk $\widetilde{U} = D(0,s) \cap \{(y_1, y_2)\,; y_2 > 0\}$ covers $\widetilde{D}_{\rm rug}$, this yields $\phi(D_{\rm rug}) \subset \widetilde{U}$. We define the half-disk $\widetilde{V} \subset \widetilde{U}$ such that $\widetilde{V} = D(0, \frac{s}{2}) \cap \{y = (y_1, y_2)\,;\, y_2 > 0\}$.

Step II : Variational formulation after straightening
We denote ∇ the gradient such that $\nabla\tilde{u}^\varepsilon_{\rm rug} = \left(\frac{\partial\tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}, \frac{\partial\tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\right)$. We carry out the following change of notations

$$\left\{\begin{array}{ll} \tilde{u}^\varepsilon_{\rm rug}(y) = u^\varepsilon_{\rm rug}(\psi(y)) & \text{for} \quad y \in \widetilde{U} \\ \tilde{v}(y) = v(\psi(y)) & \text{for} \quad y \in \widetilde{U}. \end{array}\right. \tag{6.12}$$

Similarly, we set

$$\left\{\begin{array}{ll} \tilde{u}^\varepsilon_{\rm rug}(\phi(x)) = u^\varepsilon_{\rm rug}(x) & \text{for} \quad x \in D_{\rm rug} \\ \tilde{v}(\phi(x)) = v(x) & \text{for} \quad x \in D_{\rm rug}. \end{array}\right. \tag{6.13}$$

Taking into account the above notations (6.12) and (6.13), we have $\tilde{u}^\varepsilon_{\rm rug} \in H^1_0(\widetilde{U})$ and

$$\big\langle \mathrm{G}_\varepsilon(y_1)\; \nabla\tilde{u}^\varepsilon_{\rm rug}\ ,\ \nabla\tilde{v}\big\rangle = \langle \widetilde{F}_{\rm rug}\ ,\ \tilde{v}\rangle \qquad \forall\ \tilde{v} \in H^1_0(\widetilde{U}), \tag{6.14}$$

where $\widetilde{F}_{\rm rug} = (F_{\rm rug} \circ \psi)|J\psi| \in L^2(\widetilde{U})$ and $\mathrm{G}_\varepsilon(y_1)$ is the metric defined in (6.33).

Step III : Choice of the test function
We define the differential quotient D^h_1 by

$$D^h_1\tilde{u}^\varepsilon_{\rm rug} = \frac{\tilde{u}^\varepsilon_{\rm rug}(y + he_1) - \tilde{u}^\varepsilon_{\rm rug}(y)}{h} \qquad \forall h \in \mathbb{R}^*_+.$$

For the rest of the proof, we define the cut-off function $\xi \in C^\infty_c(D(0,s), [0,1])$ as follows :

$$\left\{\begin{array}{lll} \xi \equiv 1 & \text{in} & D\left(0, \dfrac{s}{2}\right) \\ 0 \leqslant \xi \leqslant 1 & \text{in} & D(0,s) \setminus D\left(0, \dfrac{s}{2}\right) \\ \xi \equiv 0 & \text{in} & \mathbb{R}^2 \setminus D(0,s). \end{array}\right. \tag{6.15}$$

Note in particular that $\xi \equiv 1$ in $\widetilde{V} = D(0, \frac{s}{2}) \cap \{y = (y_1, y_2)\,;\, y_2 > 0\}$. Now, let $|h|$ be small and choose the following test function

$$\boxed{\tilde{v} = -D_1^{-h}(\xi^2 D_1^h \tilde{u}^\varepsilon_{\rm rug})}\,. \tag{6.16}$$

We have

$$\begin{aligned}\tilde{v}(y) &= \frac{-1}{h} D_1^{-h}\left(\xi^2(y)[\tilde{u}^\varepsilon_{\rm rug}(y+he_1) - \tilde{u}^\varepsilon_{\rm rug}(y)]\right)\\ &= \frac{-1}{h^2}\left(\xi^2(y-he_1)[\tilde{u}^\varepsilon_{\rm rug}(y) - \tilde{u}^\varepsilon_{\rm rug}(y-he_1)] - \xi^2(y)[\tilde{u}^\varepsilon_{\rm rug}(y+he_1) - \tilde{u}^\varepsilon_{\rm rug}(y)]\right).\end{aligned}$$

We observe that if $\tilde{u}^\varepsilon_{\rm rug} \in H^1_0(\widetilde{U})$ then $D_1^h \tilde{u}^\varepsilon_{\rm rug} \in H^1_0(\widetilde{U})$.

We may therefore substitute $\tilde{v}$ into the identity (6.14) and write the resulting expression as

$$\mathrm{A}\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{v}\right) = \mathrm{B}(\tilde{v}), \tag{6.17}$$

where

$$\mathrm{A}\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{v}\right) = \int_{\widetilde{U}} \left[\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_1} + \frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_2}\left(1 + \left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)^2\right) - \frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_2}\tilde{\gamma}'(\frac{y_1}{\varepsilon}) - \frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_1}\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right] dy, \tag{6.18}$$

and

$$\mathrm{B}(\tilde{v}) = \int_{\widetilde{U}} \widetilde{F}_{\rm rug}\, \tilde{v}\, dy. \tag{6.19}$$

We can now estimate the terms $\mathrm{A}(\cdot,\cdot)$ and $\mathrm{B}(\cdot)$.

Step IV : Minimization of the bilinear form in the variational formulation $\mathrm{A}(\cdot,\cdot)$
We have

$$\begin{aligned}\mathrm{A}\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right) = &- \int \frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\frac{\partial D_1^{-h}(\xi^2 D_1^h \tilde{u}^\varepsilon_{\rm rug})}{\partial y_1}\, dy + \int \tilde{\gamma}'(\frac{y_1}{\varepsilon})\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\frac{\partial D_1^{-h}(\xi^2 D_1^h \tilde{u}^\varepsilon_{\rm rug})}{\partial y_2}\, dy\\ &+ \int \tilde{\gamma}'(\frac{y_1}{\varepsilon})\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\frac{\partial D_1^{-h}(\xi^2 D_1^h \tilde{u}^\varepsilon_{\rm rug})}{\partial y_1}\, dy - \int \left(1 + \tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)^2 \frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\frac{\partial D_1^{-h}(\xi^2 D_1^h \tilde{u}^\varepsilon_{\rm rug})}{\partial y_2}\, dy.\end{aligned} \tag{6.20}$$

Here we used that

$$\left\{\begin{array}{l}\displaystyle\int \tilde{v}\, D_1^{-h}\tilde{u}^\varepsilon_{\rm rug}\, dy = -\int \tilde{u}^\varepsilon_{\rm rug}\, D_1^h \tilde{v}\, dy\\ D_1^h(\tilde{v}\tilde{u}^\varepsilon_{\rm rug}) = \tilde{v}^h\, D_1^h(\tilde{u}^\varepsilon_{\rm rug}) + \tilde{u}^\varepsilon_{\rm rug}\, D_1^h(\tilde{v}), \qquad \text{where } \tilde{v}^h = \tilde{v}(y+he_1).\end{array}\right.$$

We can write $\mathrm{A}\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right) = \mathrm{A}_1\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right) + \mathrm{A}_2\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right) + \mathrm{A}_3\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right)$, where

$$\mathrm{A}_1\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right) = \left\langle \xi^2 \begin{pmatrix} 1 & -\tilde{\gamma}'(\frac{y_1}{\varepsilon})\\ -\tilde{\gamma}'(\frac{y_1}{\varepsilon}) & 1 + (\tilde{\gamma}'(\frac{y_1}{\varepsilon}))^2\end{pmatrix}\begin{pmatrix} D_1^h\left(\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\right)\\ D_1^h\left(\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\right)\end{pmatrix}, \begin{pmatrix} D_1^h\left(\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\right)\\ D_1^h\left(\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\right)\end{pmatrix}\right\rangle, \tag{6.21}$$

$$\mathrm{A}_2\left(\tilde{u}^\varepsilon_{\rm rug}, \tilde{u}^\varepsilon_{\rm rug}\right) = \left\langle 2\,\xi \begin{pmatrix} 1 & -\tilde{\gamma}'(\frac{y_1}{\varepsilon})\\ -\tilde{\gamma}'(\frac{y_1}{\varepsilon}) & 1 + (\tilde{\gamma}'(\frac{y_1}{\varepsilon}))^2\end{pmatrix}\begin{pmatrix} D_1^h\left(\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_1}\right)\\ D_1^h\left(\frac{\partial \tilde{u}^\varepsilon_{\rm rug}}{\partial y_2}\right)\end{pmatrix}, \begin{pmatrix} \frac{\partial \xi}{\partial y_1} D_1^h \tilde{u}^\varepsilon_{\rm rug}\\ \frac{\partial \xi}{\partial y_2} D_1^h \tilde{u}^\varepsilon_{\rm rug}\end{pmatrix}\right\rangle, \tag{6.22}$$

and

$$\mathrm{A}_3\left(\tilde{u}^\varepsilon_{\mathrm{rug}},\tilde{u}^\varepsilon_{\mathrm{rug}}\right)=\left\langle \xi^2\begin{pmatrix}0 & D^h_1\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)\\ D^h_1\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right) & D^h_1\left(1+\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)^2\right)\end{pmatrix}\begin{pmatrix}\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\\ \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\end{pmatrix},\begin{pmatrix}\frac{\partial D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\\ \frac{\partial D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\end{pmatrix}\right\rangle$$
$$+\left\langle 2\xi\begin{pmatrix}0 & D^h_1\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)\\ D^h_1\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right) & D^h_1\left(1+\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)^2\right)\end{pmatrix}\begin{pmatrix}\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\\ \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\end{pmatrix},\begin{pmatrix}\frac{\partial \xi}{\partial y_1}D^h_1(\tilde{u}^\varepsilon_{\mathrm{rug}})\\ \frac{\partial \xi}{\partial y_2}D^h_1(\tilde{u}^\varepsilon_{\mathrm{rug}})\end{pmatrix}\right\rangle. \tag{6.23}$$

Applying Proposition 6.2 (uniform ellipticity condition) to the vector $w=\xi\begin{pmatrix}D^h_1\left(\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\right)\\ D^h_1\left(\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\right)\end{pmatrix}$, one gets:

$$\mathrm{A}_1\left(\tilde{u}^\varepsilon_{\mathrm{rug}},\tilde{u}^\varepsilon_{\mathrm{rug}}\right)\geqslant\alpha_0\int\xi(y)^2\,|D^h_1\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}|^2\,dy\;. \tag{6.24}$$

Note that α_0 is an ellipticity constant given by $\alpha_0:=\frac{1}{2+\|\tilde{\gamma}'\|^2_{L^\infty}}$. Now, we define the vectors w_1 and w_2 such that :

$$w_1=\begin{pmatrix}D^h_1\left(\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\right)\\ D^h_1\left(\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\right)\end{pmatrix}\quad\text{and}\quad w_2=2\begin{pmatrix}\frac{\partial\xi}{\partial y_1}D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}\\ \frac{\partial\xi}{\partial y_2}D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}\end{pmatrix},$$

Then, thanks to Proposition 6.3, we show that there exists a constant $\mathrm{C}_{\mathrm{A}_2}>0$ such that, for all real $\alpha>0$, we have

$$\mathrm{A}_2\left(\tilde{u}^\varepsilon_{\mathrm{rug}},\tilde{u}^\varepsilon_{\mathrm{rug}}\right)\geqslant-\frac{\alpha}{4}\int\xi(y)^2\left|D^h_1\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy-\frac{\mathrm{C}_{\mathrm{A}_2}}{\alpha}\int\left|D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy.$$

More precisely, the constant $\mathrm{C}_{\mathrm{A}_2}$ is given by $\mathrm{C}_{\mathrm{A}_2}=2\left\|\frac{\partial\xi}{\partial y}\right\|_{L^\infty}\left(2+\|\tilde{\gamma}'\|^2_{L^\infty}\right)$. Moreover, according to [7, page 184] we have the estimate

$$\int|D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}|^2\,dy\leqslant\int|\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}|^2\,dy. \tag{6.25}$$

By applying (6.25), we deduce that

$$\mathrm{A}_2\left(\tilde{u}^\varepsilon_{\mathrm{rug}},\tilde{u}^\varepsilon_{\mathrm{rug}}\right)\geqslant-\frac{\alpha}{4}\int\xi(y)^2\left|D^h_1\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy-\frac{\mathrm{C}_{\mathrm{A}_2}}{\alpha}\int\left|\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy,\quad\forall\alpha>0. \tag{6.26}$$

Let us finally estimate the quantity (6.23). We consider the vectors w_3, w_4 and w_5 such that

$$w_3=\xi\begin{pmatrix}\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\\ \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\end{pmatrix},\qquad w_4=2\begin{pmatrix}\frac{\partial\xi}{\partial y_1}D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}\\ \frac{\partial\xi}{\partial y_2}D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}\end{pmatrix}\qquad\text{and}\quad w_5=\xi\begin{pmatrix}\frac{\partial D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\\ \frac{\partial D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\end{pmatrix}.$$

Applying Proposition 6.5 to the vectors w_3, w_4 and w_5, we show that there exists a constant $C_1(\alpha)>0$ such that, for all real $\alpha>0$, we have

$$\mathrm{A}_3\left(\tilde{u}^\varepsilon_{\mathrm{rug}},\tilde{u}^\varepsilon_{\mathrm{rug}}\right)\geqslant-\frac{\alpha}{4}\int\xi(y)^2\left|D^h_1\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy-\frac{1}{2}\int\left|2\frac{\partial\xi}{\partial y}D^h_1\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy-\frac{C_1}{\varepsilon^2}\int\left|\xi\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}\right|^2dy.$$

Let us specify that the constant C_1 is independent of ε. Moreover, we can establish that $C_1(\alpha)=\left(\frac{1}{\alpha}+\frac{1}{2}\right)\left(|\mathrm{I}_1|+|\mathrm{I}_2|\right)^2$ where I_1 and I_1 are the following integrals: $\mathrm{I}_1=\displaystyle\int_0^1\tilde{\gamma}''\left(\frac{y_1}{\varepsilon}+\frac{sh}{\varepsilon}\right)ds$ and

$\mathrm{I}_2 = \displaystyle\int_0^1 2\, \tilde{\gamma}^{'} \left(\frac{y_1}{\varepsilon} + \frac{sh}{\varepsilon} \right) \tilde{\gamma}^{''} \left(\frac{y_1}{\varepsilon} + \frac{sh}{\varepsilon} \right) ds$. Thanks to the inequality (6.25) and after some calculations, we show that there exists a constant $\mathrm{C}_{\mathrm{A}_3} > 0$ such that for all real $\alpha > 0$, we have

$$\mathrm{A}_3\left(\tilde{u}^{\varepsilon}_{\mathrm{rug}}, \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right) \geqslant -\frac{\alpha}{4} \int \xi(y)^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy - \frac{\mathrm{C}_{\mathrm{A}_3}}{\varepsilon^2} \int \left|\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy. \tag{6.27}$$

Note that the constant $\mathrm{C}_{\mathrm{A}_3}$ is given by $\mathrm{C}_{\mathrm{A}_3} = \max\Big(C_1(\alpha), \left\|\frac{\partial \xi}{\partial y}\right\|_{L^\infty}\Big)$. We finally combine (6.24), (6.26) and (6.27) to get that there exists a constant $\mathrm{C}_{\mathrm{A}} > 0$ such that for all real $\alpha > 0$ we have

$$\mathrm{A}\left(\tilde{u}^{\varepsilon}_{\mathrm{rug}}, \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right) \geqslant \left(\alpha_0 - \frac{\alpha}{2}\right) \int \xi(y)^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy - \frac{\mathrm{C}_{\mathrm{A}}}{\varepsilon^2} \int |\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}|^2 \, dy. \tag{6.28}$$

Let us specify that the constant C_{A} is given by $\mathrm{C}_{\mathrm{A}} = \left(\frac{1}{\alpha}\mathrm{C}_{\mathrm{A}_2} + \mathrm{C}_{\mathrm{A}_3}\right)$.

Step V : Upper bounds

Thanks to inequality (6.25) and elementary calculations, one gets

$$\begin{aligned}
\int |\tilde{v}|^2 \, dy = \int \left|D_1^{-h}(\xi^2 D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}})\right|^2 dy \leqslant \int \left|D(\xi^2 D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}})\right|^2 dy &\leqslant \int \left|\nabla(\xi^2)\, D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}} + \xi^2\, \nabla \left(D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right)\right|^2 dy \\
&\leqslant 2 \int \left(\left| 2\xi \frac{\partial \xi}{\partial y}\, D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}} \right|^2 + \left|\xi^2 D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 \right) dy \\
&\leqslant \mathrm{C}_{\mathrm{B}} \int \left(\left|D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 + \xi^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 \right) dy.
\end{aligned}$$

We underline that the cut-off $\xi \in [0,1]$ and the constant C_{B} is given by $\mathrm{C}_{\mathrm{B}} = 2\max\left(1, 4\left\|\left(\frac{\partial \xi}{\partial y}\right)^2\right\|_{L^\infty}\right)$. Thus, the last inequality and the Cauchy's inequality $\left(a\, b \leqslant \dfrac{\alpha}{4\, \mathrm{C}_{\mathrm{B}}} a^2 + \dfrac{\mathrm{C}_{\mathrm{B}}}{\alpha} b^2\right)$ imply

$$\begin{aligned}
\left|B(\tilde{u}^{\varepsilon}_{\mathrm{rug}})\right| \leqslant \int \left|\widetilde{F}_{\mathrm{rug}}\right| |\tilde{v}| \, dy &\leqslant \frac{\alpha}{4\mathrm{C}_{\mathrm{B}}} \int |\tilde{v}|^2 \, dy + \frac{\mathrm{C}_{\mathrm{B}}}{\alpha} \int \left|\widetilde{F}_{\mathrm{rug}}\right|^2 dy \\
&\leqslant \frac{\alpha}{4} \int \left(\left|D_1^h \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 + \xi^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 \right) dy + \frac{\mathrm{C}_{\mathrm{B}}}{\alpha} \int \left|\widetilde{F}_{\mathrm{rug}}\right|^2 dy \qquad (6.29) \\
&\leqslant \frac{\alpha}{4} \int \xi^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy + \frac{\alpha}{4} \int \left|\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy + \frac{\mathrm{C}_{\mathrm{B}}}{\alpha} \int \left|\widetilde{F}_{\mathrm{rug}}\right|^2 dy.
\end{aligned}$$

Step VI : H^2-Regularity in the adjusted domain

We recall the expression (6.17) of the variational formulation : $\mathrm{A}\left(\tilde{u}^{\varepsilon}_{\mathrm{rug}}, \tilde{v}\right) = \mathrm{B}(\tilde{v})$ for every $\tilde{v} \in H^1_0(\widetilde{U})$. We select $\alpha = \alpha_0$ in inequalities (6.28) and (6.29). Then we obtain

$$\begin{aligned}
&\frac{\alpha_0}{2} \int_{\widetilde{U}} \xi^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy - \frac{\mathrm{C}_{\mathrm{A}}}{\varepsilon^2} \int_{\widetilde{U}} |\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}(y)|^2 dy \\
&\qquad \leqslant \frac{\alpha_0}{4} \left(\int_{\widetilde{U}} \xi^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy + \int_{\widetilde{U}} \left|\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy \right) + \frac{\mathrm{C}_{\mathrm{B}}}{\alpha_0} \int_{\widetilde{U}} \left|\widetilde{F}_{\mathrm{rug}}\right|^2 dy.
\end{aligned}$$

Since the cut-off $\xi \equiv 1$ in $\widetilde{V}$ and after some calculations, one gets

$$\int_{\widetilde{V}} \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy \leqslant \int_{\widetilde{U}} \xi^2 \left|D_1^h \nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy \leqslant \frac{4}{\alpha_0} \left(\frac{\alpha_0}{4} + \frac{\mathrm{C}_{\mathrm{A}}}{\varepsilon^2} \right) \int_{\widetilde{U}} \left|\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy + \frac{4\mathrm{C}_{\mathrm{B}}}{\alpha_0^2} \int_{\widetilde{U}} \left|\widetilde{F}_{\mathrm{rug}}\right|^2 dy.$$

Consequently,

$$\sum_{\substack{l,k=1 \\ k+l<4}}^{2} \left\| \frac{\partial^2 \tilde{u}^{\varepsilon}_{\mathrm{rug}}}{\partial y_k \partial y_l} \right\|_{L^2(\widetilde{V})} \leqslant \left(\frac{\mathrm{C}_{\mathrm{R}_1}}{\varepsilon} + 1 \right) \left(\int_{\widetilde{U}} \left|\nabla \tilde{u}^{\varepsilon}_{\mathrm{rug}}\right|^2 dy \right)^{\frac{1}{2}} + \mathrm{C}_{\mathrm{R}_2} \left(\int_{\widetilde{U}} \left|\widetilde{F}_{\mathrm{rug}}\right|^2 dy \right)^{\frac{1}{2}}, \tag{6.30}$$

where the constants $\mathrm{C}_{\mathrm{R}_1}$, $\mathrm{C}_{\mathrm{R}_2}$ are given by $\mathrm{C}_{\mathrm{R}_1} = \frac{2\sqrt{\mathrm{C}_{\mathrm{A}}}}{\sqrt{\alpha_0}}$ and $\mathrm{C}_{\mathrm{R}_2} = \frac{2\sqrt{\mathrm{C}_{\mathrm{B}}}}{\alpha_0}$. We must now complete the estimate (6.30) with an estimate of the L^2-norm of $\left(\tilde{u}^\varepsilon_{\mathrm{rug}}\right)_{y_2y_2}$ over $\widetilde{V}$. For this, we consider a test function $\tilde{v} \in C^1_c(\widetilde{U})$ such that:

$$\int_{\widetilde{U}} \left[\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_1} + \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_2}\ \left(1+\left(\tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right)^2\right) - \ \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_2}\ \tilde{\gamma}'\,(\frac{y_1}{\varepsilon}) - \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_1}\ \tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right] dy = \int_{\widetilde{U}} \widetilde{F}_{\mathrm{rug}}\,\tilde{v}\,dy,$$

where $\widetilde{F}_{\mathrm{rug}} = (F_{\mathrm{rug}}\circ\psi)|J\psi| \in L^2(\widetilde{U})$. It follows that

$$\int_{\widetilde{U}} \widetilde{F}_{\mathrm{rug}}\tilde{v}dy - \int_{\widetilde{U}} \left[\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_1} - \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_2}\tilde{\gamma}'\,(\frac{y_1}{\varepsilon}) - \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_1}\tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right] dy = \int_{\widetilde{U}} \left(1+\left(\tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right)^2\right)\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_2}dy.$$

At this point, we remark that $1+\left(\tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right)^2 > 1$. Using a triangular inequality, we obtain

$$\begin{aligned}\left|\int_{\widetilde{U}} \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_2}dy\right| &\leqslant \left|\int_{\widetilde{U}} \widetilde{F}_{\mathrm{rug}}\tilde{v}dy\right| + \left|\int_{\widetilde{U}} \left[\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_1} - \ \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_2}\tilde{\gamma}'\,(\frac{y_1}{\varepsilon}) - \frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_1}\tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right] dy\right| \\ &\leqslant \max\left(1, \left\|\tilde{\gamma}'\,(\frac{y_1}{\varepsilon})\right\|_{L^\infty}\right)\int_{\widetilde{U}}\left(\left|\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_1}\right| + \left|\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_1}\frac{\partial \tilde{v}}{\partial y_2}\right| + \left|\frac{\partial \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_2}\frac{\partial \tilde{v}}{\partial y_1}\right|\right) dy + \int_{\widetilde{U}}\left|\widetilde{F}_{\mathrm{rug}}\tilde{v}\right| dy.\end{aligned}$$

Using this last estimate in inequality (6.30), we conclude that $\tilde{u}^\varepsilon_{\mathrm{rug}} \in H^2(\widetilde{V})$ and that there exists a $\mathrm{C}_{\mathrm{R}} > 0$ independent of ε, such that

$$\sum_{l,k=1}^{2}\left\|\frac{\partial^2 \tilde{u}^\varepsilon_{\mathrm{rug}}}{\partial y_k \partial y_l}\right\|_{L^2(\widetilde{V})} \leqslant \mathrm{C}_{\mathrm{R}}\left(\frac{1}{\varepsilon}\left\|\nabla\tilde{u}^\varepsilon_{\mathrm{rug}}\right\|_{L^2(\widetilde{U})} + \left\|\widetilde{F}_{\mathrm{rug}}\right\|_{L^2(\widetilde{U})}\right). \tag{6.31}$$

Step VII : Local estimate of the H^2-norm near the rough boundary γ_ε

We now recall the geometric frame. The domain V is defined as $V = \psi(\widetilde{V})$, where $\widetilde{V}$ is the positive half-disk such that $\widetilde{V} = D(0, \frac{s}{2}) \cap \{(y_1, y_2)\,;y_2 > 0\}$ and ψ is the the reciprocal function of the C^2-diffeomorphism defined in (6.11). The domain $D_{\mathrm{rug}} = \Omega_\varepsilon \cap D(x_{\mathrm{rug}}, R_{\mathrm{rug}})$ is the intersection between the rough domain Ω_ε and the disk centered at $x_{\mathrm{rug}} = \left(\frac{L}{2}\ ,\ \varepsilon\tilde{\gamma}(\frac{L}{2\varepsilon}) + \frac{1-L^2}{4}\right)$ and of radius $R_{\mathrm{rug}} = \frac{L^2+1}{4}$. Applying the technical Lemma 6.1 to the solution $u^\varepsilon_{\mathrm{rug}}$ and using the estimate (6.31), we show after some calculations the existence of a constant $\mathrm{C}_{\mathrm{F}} > 0$, independent of ε, such that

$$\sum_{i,j=1}^{2}\left\|\frac{\partial^2 u^\varepsilon_{\mathrm{rug}}(x)}{\partial x_i \partial x_j}\right\|_{L^2(V)} \leqslant \mathrm{C}_{\mathrm{F}}\left[\left(\frac{1}{\varepsilon}\right)\left\|\nabla u^\varepsilon_{\mathrm{rug}}\right\|_{L^2(D_{\mathrm{rug}})} + \|g\|_{L^2(D_{\mathrm{rug}})}\right]. \tag{6.32}$$

Step VIII : Conclusion

Since $\partial\Omega_\varepsilon$ is compact, we sum the resulting estimates in the vicinity of the rough boundary (6.32), the smooth edges (6.7) and the interior estimate (6.5), to find $u_\varepsilon \in H^2(\Omega_\varepsilon)$, with the inequality

$$\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant K\left(\|g\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon}\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\right),$$

where $K > 0$ is a constant independent on ε. This ends the proof. □

A3 : The properties of the modified metric

Proposition 6.1. *We define the matrix* $\mathrm{G}_\varepsilon(y_1)$ *by*

$$\mathrm{G}_\varepsilon(y_1) = \begin{pmatrix} 1 & -\tilde{\gamma}'\,(\frac{y_1}{\varepsilon}) \\ -\tilde{\gamma}'\,(\frac{y_1}{\varepsilon}) & 1+(\tilde{\gamma}'\,(\frac{y_1}{\varepsilon}))^2 \end{pmatrix}. \tag{6.33}$$

Then, the set of the eigenvalues of $\mathrm{G}_\varepsilon(y_1)$ *is :* $\mathrm{Sp}(\mathrm{G}_\varepsilon(y_1)) = \{\lambda_{\min}(\varepsilon, y_1), \lambda_{\max}(\varepsilon, y_1)\}$ *where*

$$\lambda_{\min}(\varepsilon, y_1) = \frac{[2 + (\tilde{\gamma}'(\frac{y_1}{\varepsilon}))^2] - \sqrt{[2 + (\tilde{\gamma}'(\frac{y_1}{\varepsilon}))^2)]^2 - 4}}{2} \quad \text{is the smallest eigenvalue of } \mathrm{G}_\varepsilon(y_1),$$
$$\lambda_{\max}(\varepsilon, y_1) = \frac{[2 + (\tilde{\gamma}'(\frac{y_1}{\varepsilon}))^2] + \sqrt{[2 + (\tilde{\gamma}'(\frac{y_1}{\varepsilon}))^2)]^2 - 4}}{2} \quad \text{is the largest eigenvalue of } \mathrm{G}_\varepsilon(y_1).$$

Proposition 6.2. *There exists* α_0, $\alpha_1 > 0$ *independent of* ε*, such that*

$$\alpha_1 \, \|w\|^2_{L^2} \geqslant \langle \mathrm{G}_\varepsilon(y_1) \, w, \, w \rangle \geqslant \alpha_0 \, \|w\|^2_{L^2} \, , \qquad \forall w \in C^0_c(\widetilde{U}).$$

Remark 6.1. α_0 *is an ellipticity constant given by* $\alpha_0 := \frac{1}{2+\|\tilde{\gamma}'\|^2_{L^\infty}}$*. It is a lower bound of the smallest eigenvalue of* $\mathrm{G}_\varepsilon(y_1)$*. Moreover, we can take* $\alpha_1 := 2 + \|\tilde{\gamma}'\|^2_{L^\infty}$*. This is an upper bound of the largest eigenvalue of* $\mathrm{G}_\varepsilon(y_1)$*.*

Proposition 6.3. *For any real* $\alpha > 0$*, we have*

$$|\langle \mathrm{G}_\varepsilon(y_1) \, w_1, \, w_2 \rangle| \leqslant \frac{\alpha}{4} \int_{\widetilde{U}} |w_1|^2 \, dy + \frac{\alpha_1^2}{\alpha} \int_{\widetilde{U}} |w_2|^2 \, dy, \quad \text{for } w_1, w_2 \in C^0_c(\widetilde{U}), \tag{6.34}$$

where $\widetilde{U} = D(0, s) \cap \{(y_1, y_2) \, ; \, y_2 > 0\}$*.*

Proposition 6.4. *We define the matrix* $\widehat{\mathrm{G}_\varepsilon(y_1)}$ *by*

$$\widehat{\mathrm{G}_\varepsilon(y_1)} = \begin{pmatrix} 0 & D_1^h\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right) \\ D_1^h\left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right) & D_1^h\left(1 + \left(\tilde{\gamma}'(\frac{y_1}{\varepsilon})\right)^2\right) \end{pmatrix}. \tag{6.35}$$

Then, the set of the eigenvalues of $\widehat{\mathrm{G}_\varepsilon(y_1)}$ *is :* $\mathrm{Sp}\;(\widehat{\mathrm{G}_\varepsilon(y_1)}) = \{\widehat{\lambda}_{\min}(\varepsilon, y_1), \widehat{\lambda}_{\max}(\varepsilon, y_1)\}$*, where*

$$\begin{cases} \widehat{\lambda}_{\min}(\varepsilon, y_1) = \dfrac{\mathrm{I}_2 - \sqrt{\mathrm{I}_2^2 + 4\mathrm{I}_1^2}}{2\,\varepsilon} & \text{is the smallest eigenvalue of } \widehat{\mathrm{G}_\varepsilon(y_1)}, \\ \widehat{\lambda}_{\max}(\varepsilon, y_1) = \dfrac{\mathrm{I}_2 + \sqrt{\mathrm{I}_2^2 + 4\mathrm{I}_1^2}}{2\,\varepsilon} & \text{is the largest eigenvalue of } \widehat{\mathrm{G}_\varepsilon(y_1)}, \end{cases} \tag{6.36}$$

where $\mathrm{I}_1 = \displaystyle\int_0^1 \tilde{\gamma}''\left(\frac{y_1}{\varepsilon} + \frac{sh}{\varepsilon}\right) ds$ *and* $\mathrm{I}_2 = \displaystyle\int_0^1 2\,\tilde{\gamma}'\left(\frac{y_1}{\varepsilon} + \frac{sh}{\varepsilon}\right) \tilde{\gamma}''\left(\frac{y_1}{\varepsilon} + \frac{sh}{\varepsilon}\right) ds.$

Proposition 6.5. *Let* w_3, w_4, $w_5 \in C^0_c(\widetilde{U})$*. Then, there exists a constant* $C_1(\alpha) > 0$ *independent of* ε *such that for every real* $\alpha > 0$*, we have:*

$$\left|\langle \widehat{\mathrm{G}_\varepsilon(y_1)} \, w_3, \, w_4 + w_5 \rangle\right| \leqslant \frac{\alpha}{4} \int_{\widetilde{U}} |w_5|^2 \, dy + \frac{1}{2} \int_{\widetilde{U}} |w_4|^2 \, dy + \frac{C_1}{\varepsilon^2} \int_{\widetilde{U}} |w_3|^2 \, dy. \tag{6.37}$$

A4 : Calculus

Lemma 6.1. *Let* f *be a function belonging to* $C^\infty(D_{\mathrm{rug}})$*. We set* $\tilde{f}(\phi(x)) = f(x)$ *for every* $x \in D_{\mathrm{rug}}$*, where* ϕ *is the* C^2*-diffomorphism defined in (6.10). Then we have*

$$
\begin{cases}
\dfrac{\partial^2 f}{\partial x_1 \partial x_1} = \dfrac{\partial^2 \tilde{f}}{\partial y_1 \partial y_1} - 2\ \tilde{\gamma}'\left(\dfrac{y_1}{\varepsilon}\right)\dfrac{\partial^2 \tilde{f}}{\partial y_1 \partial y_2} + \left(\tilde{\gamma}'\left(\dfrac{y_1}{\varepsilon}\right)\right)^2 \dfrac{\partial^2 \tilde{f}}{\partial y_2 \partial y_2} - \boxed{\dfrac{1}{\varepsilon}\tilde{\gamma}''\left(\dfrac{y_1}{\varepsilon}\right)\dfrac{\partial \tilde{f}}{\partial y_2}}, \\
\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = \dfrac{\partial^2 \tilde{f}}{\partial y_1 \partial y_2} - \tilde{\gamma}'\left(\dfrac{y_1}{\varepsilon}\right)\dfrac{\partial^2 \tilde{f}}{\partial y_2 \partial y_2}, \\
\dfrac{\partial^2 f}{\partial x_2 \partial x_1} = \dfrac{\partial^2 \tilde{f}}{\partial y_1 \partial y_2} - \tilde{\gamma}'\left(\dfrac{y_1}{\varepsilon}\right)\dfrac{\partial^2 \tilde{f}}{\partial y_2 \partial y_2}, \\
\dfrac{\partial^2 f}{\partial x_2 \partial x_2} = \dfrac{\partial^2 \tilde{f}}{\partial y_2 \partial y_2}.
\end{cases}
\tag{6.38}
$$

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