Diffusive limit approximation of pure jump optimal ergodic control problems

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April 28, 2024

Abstract

Motivated by the design of fast reinforcement learning algorithms, see [17], we study the diffusive limit of a class of pure jump ergodic stochastic control problems. We show that, whenever the intensity of jumps ε^{-1} is large enough, the approximation error is governed by the Hölder regularity of the Hessian matrix of the solution to the limit ergodic partial differential equation and is, indeed, of order $\varepsilon^{\frac{\gamma}{2}}$ for all $\gamma \in (0,1)$. This extends to this context the results of [1] obtained for finite horizon problems. Using the limit as an approximation, instead of directly solving the pre-limit problem, allows for a very significant reduction in the numerical resolution cost of the control problem. Additionally, we explain how error correction terms of this approximation can be constructed under appropriate smoothness assumptions. Finally, we quantify the error induced by the use of the Markov control policy constructed from the numerical finite difference scheme associated to the limit diffusive problem, which seems to be new in the literature and of independent interest.

1 Introduction

Let N be a random point process with predictable compensator $\eta\nu(\mathrm{d}e)\mathrm{d}t$, for some finite probability measure ν on $\mathbb{R}^{d'}$, $d' \in \mathbb{N}$, $\eta > 0$, and let $X^{x,\alpha}$ be the solution of

$$X^{x,\alpha} = x + \int_0^{\cdot} \int_{\mathbb{R}^{d'}} b(X^{x,\alpha}_{s-}, \alpha_s, e) N(\mathrm{d}e, \mathrm{d}s)\,,$$

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in which α belongs to the set $\mathcal{A}$ of predictable controls with values in some given compact set $\mathbb{A} \subset \mathbb{R}^m$ and the initial data $x \in \mathbb{R}^d$, $m \in \mathbb{N}$. Under some standard stability assumptions, see e.g. [3, 28], the value of the ergodic optimal control problem

$$\rho^* := \sup_{\alpha \in \mathcal{A}} \liminf_{T \to +\infty} \frac{1}{\eta T} \mathbb{E}\left[\int_0^T r(X_{s-}^{0,\alpha}, \alpha_s) \mathrm{d}N_s\right]$$

with $N_t := N(\mathbb{R}^{d'}, [0, t])$, $t \geq 0$, along with some continuous function w, solves the integro-differential equation

$$\rho^* = \sup_{a \in \mathbb{A}} \left\{ \eta \int [\mathrm{w}(\cdot + b(\cdot, a, e)) - \mathrm{w}] \nu(\mathrm{d}e) + r(\cdot, a) \right\} \text{ on } \mathbb{R}^d \tag{1.1}$$

possibly in the viscosity solution sense. This characterisation leads to numerical schemes for approximating the value of the problem and the Markovian optimal control.

However, (1.1) is non-local in nature which means that, unless ν is concentrated on a small number of points, the cost of numerical approximation is large, in particular when the intensity η is. This is a problem, e.g., for bidding problems (see e.g. [16]) in online display-ad auctions, where the system moves near-continuously in time, meaning that η is very large, and where unknown system parameters motivate the use of reinforcement learning to solve the control problem. Reinforcement learning compounds the cost by requiring computation of the optimal strategy for many plausible values of the parameters, namely at each time the value of the parameter is updated given the flow of new information, see [17] for more details.

On the other hand, when η is very large, asymptotic regimes exist which offer an alternative approximation path, notably the diffusive limit on which this paper focuses. Indeed, taking $\eta = \varepsilon^{-1}$ and $b(x, a, e) = \varepsilon b_1(x, a, e) + \varepsilon^{\frac{1}{2}} b_2(x, e)$, with $\int_{\mathbb{R}^{d'}} b_2(\cdot, e)\nu(\mathrm{d}e) = 0$, an immediate second order expansion suggests that (ρ^*, w) converges as $\varepsilon \to 0$ to the solution $(\bar{\rho}^*, \bar{\mathrm{w}})$ of

$$\bar{\rho}^* = \sup_{\bar{a} \in \mathbb{A}} \left\{ \int_{\mathbb{R}^{d'}} b_1^\top(\cdot, \bar{a}, e)\nu(\mathrm{d}e) \mathrm{D}\bar{\mathrm{w}} + \mathrm{Tr}\left[\int_{\mathbb{R}^{d'}} b_2 b_2^\top(\cdot, e)\nu(\mathrm{d}e) \mathrm{D}^2 \bar{\mathrm{w}}\right] + r(\cdot, a) \right\} \text{ on } \mathbb{R}^d. \tag{1.2}$$

Unlike (1.1), (1.2) is a local equation and much more easily solved numerically. Note that another possible limit regime, albeit less precise, is obtained via a first order expansion as in [18], which corresponds to considering a fluid limit.

For such a specification of the coefficients (η, b), the existence of a diffusive limit is expected, see e.g. [22] for general results on the convergence of stochastic processes. Stability of viscosity solutions, see e.g. [19, Section 3], can also be used to prove the convergence of the value function of stochastic control problems. This has been a subject of particular interest in insurance and queueing network literatures, see e.g. [8, 13, 14]. Nonetheless, these approaches do not permit the characterisation of the speed of convergence in the case of a (generic) ergodic optimal control problem as defined in Section 2 below, which is essential for regret analysis and design of efficient reinforcement learning algorithms. See [17] for a precise treatment of the Reinforcement Learning problem in this setting.

The aim of this paper is to characterize this convergence speed and explain how to numerically construct, in an efficient way, an approximation of the optimal control. A first step in this direction was done by [1] who considered finite time horizon problems. Such problems are easier to handle from a mathematical point of view, but are unfortunately not adapted to reinforcement learning algorithms that are based on a regret criterion that is intimately related to an ergodic control problem, see [17] again.

Still, a similar approach can be used, up to additional technicalities. As in [1], we study the regularity of $\bar{\mathrm{w}}$ in the solution couple $(\bar{\rho}^*, \bar{\mathrm{w}})$ to (1.2). We show that its second order derivative is (locally) γ-Hölder with a constant of at most linear growth in x, for some $\gamma \in (0,1]$, whenever the coefficients of (1.2) are uniformly Lipschitz in space, $\int b_1(\cdot, e)\nu(\mathrm{d}e)$ has linear growth, b_2 and r are continuous and bounded, and under a uniform ellipticity condition. By a second order Taylor expansion, this allows us to pass (rigorously) from (1.2) to (1.1) up to an error term of order $\varepsilon^{\frac{\gamma}{2}}$ (locally), and therefore provides the required convergence rate by verification. In general this rate can not be improved. As a by-product, the Markovian control taken from the Hamilton-Jacobi-Bellman equation of the diffusive limit problem provides an $\varepsilon^{\frac{\gamma}{2}}$-optimal control for the original pure-jump control problem. Under additional regularity assumptions, it can even be improved by constructing a first-order correction term.

In principle, this provides an efficient way of constructing an almost optimal Markovian control. However, it still remains to build up a pure numerical scheme. To complete the picture we therefore derive a convergence rate for a finite difference method for the numerical estimation of $\bar{\rho}^*$, depending again on γ. More importantly, we explain how to numerically construct an almost optimal Markovian control process based on a smoothed version of the numerical approximation of $\bar{\mathrm{w}}$ and we obtain a convergence rate towards $\bar{\rho}^*$, and therefore ρ^*, of the expected average gain associated to such a control. The latter seems to be (surprisingly) completely new and of own interest in the optimal control literature.

As an example of application, we consider in Section 5 a simplified repeated online auction bidding problem, where a buyer seeks to maximise its profit when facing both competition and a seller who adapts its price to incoming bids. Our numerical experiments show that our approximation permits a considerable gain in computation time relative to a direct resolution of the pure-jump problem (as expected).

Note that we restrict here to the case where b_2 does not depend on the value of the control, meaning that $\bar{\mathrm{w}}$ solves a semi-linear equation. In principle, the fully non-linear case could be studied along the same lines of arguments but the required regularity of the corresponding function $\bar{\mathrm{w}}$ would be much more complex to derive. We avoid considering this more general case for sake of simplicity.

Notations: We collect here some standard notations that will be used throughout this paper. We take $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_+^* := (0, +\infty)$ throughout. Any element x of $\mathbb{R}^d$ is viewed as a column vector. $\mathbb{M}^d$ (resp. $\mathbb{S}^d$) denotes the collection of (resp. symmetric) d-dimensional matrices. On $\mathbb{R}^d$ or $\mathbb{M}^d$, the superscript $^\top$ denotes transposition, we set $\langle x, y\rangle := x^\top y$ and $|x| := \sqrt{\langle x, x\rangle}$ for $x, y \in \mathbb{R}^d$. We let $\mathrm{Tr}[M]$ denote the trace of $M \in \mathbb{M}^d$ and $|M|$ be the Euclidean norm of M viewed as a vector of $\mathbb{R}^{d\times d}$. We denote by $B_\ell(x)$ the open ball centered at $x \in \mathbb{R}^d$ of radius $\ell > 0$. Given an open set $\mathcal{O} \subset \mathbb{R}^n$, $n \geq 1$, $p \in \{0, 1, 2\}$, we use the standard notation $\mathcal{C}^p(\mathcal{O})$ to denote the space of p-times continuously differentiable real-valued maps u on $\mathcal{O}$, and $\mathcal{C}^p_b(\mathcal{O})$ to denote the subspace of functions $u \in \mathcal{C}^p(\mathcal{O})$ such that

$$\|u\|_{\mathcal{C}^p_b(\mathcal{O})} := \sum_{j=0}^{p} \sup_{x\in\mathcal{O}} \left|\mathrm{D}^j u(x)\right| < \infty$$

in which $\mathrm{D}^0 u := u$, $\mathrm{D}^1 u$ is the gradient of u, as a line vector, $\mathrm{D}^2 u$ is the Hessian matrix of u. Given $\gamma \in [0, 1]$, we denote the γ-Hölder modulus of $u \in \mathcal{C}^0(\mathcal{O})$ on $\mathcal{O}$ as

$$[u]^\gamma_{\mathcal{C}^0(\mathcal{O})} := \sup_{x,x'\in\mathcal{O}} \frac{|u(x') - u(x)|}{|x' - x|^\gamma},$$

where we use the convention $0/0 = 0$. If $u = (u^1, \cdots, u^d)$ takes values in $\mathbb{R}^d$, $d \geq 1$, we use the same notation to denote the sum of the elements $\{[u^i]^\gamma_{\mathcal{C}^0(\mathcal{O})}, i \leq d\}$. We write $u \in \mathcal{C}^{p,\gamma}(\mathcal{O})$ if $\mathrm{D}^p u$ is γ-Hölder on each compact subset of $\mathcal{O}$, and $u \in \mathcal{C}^{p,\gamma}_b(\mathcal{O})$ if

$$\|u\|_{\mathcal{C}^{p,\gamma}_b(\mathcal{O})} := \|u\|_{\mathcal{C}^p_b(\mathcal{O})} + [\mathrm{D}^p u]^\gamma_{\mathcal{C}^0(\mathcal{O})} < \infty.$$

If u is restricted to take values in a subset $\mathcal{O}'$ of $\mathbb{R}$, we write $\mathcal{C}^p(\mathcal{O}; \mathcal{O}')$, $\mathcal{C}^p_b(\mathcal{O}; \mathcal{O}')$, $\mathcal{C}^{p,\gamma}(\mathcal{O}; \mathcal{O}')$ or $\mathcal{C}^{p,\gamma}_b(\mathcal{O}; \mathcal{O}')$ for the corresponding sets. We also use the notation $C^0_{\mathrm{lin}}(\mathcal{O})$ to denote the collection of continuous real-valued function u such that

$$[u]_{\mathcal{C}^0_{\mathrm{lin}}(\mathcal{O})} := \sup_{x\in\mathcal{O}} \frac{|u(x)|}{1 + |x|} < \infty.$$

In all the above notations, we omit $\mathcal{O}$ if it is equal to $\mathbb{R}^d$.

2 Pure jump Ergodic Optimal Control

In order to alleviate notations, we first consider the case where the intensity of the jump process is given, and recall rather standard results from the ergodic control litterature.

Let $\Omega = \mathbb{D}$ denote the space of d-dimensional càdlàg functions on $\mathbb{R}_+$ and $\mathcal{M}(\mathbb{R}^{d'} \times \mathbb{R}_+)$ denote the collection of positive finite measures on $\mathbb{R}^{d'} \times \mathbb{R}_+$, for some $d, d' \in \mathbb{N}^*$. Consider a measure-valued map $N : \mathbb{D} \mapsto \mathcal{M}(\mathbb{R}^{d'} \times \mathbb{R}_+)$ and a probability measure $\mathbb{P}$ on $\mathbb{D}$ such that N is a right-continuous real-valued $\mathbb{R}^{d'}$-marked point process with compensator $\eta\nu(\mathrm{d}e)\mathrm{d}t$, in which $\eta > 0$ and ν is a probability measure on $\mathbb{R}^{d'}$. See e.g. [12]. For ease of notations, we set $N_t := N(\mathbb{R}^{d'}, [0, t])$ for $t \geq 0$.

Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the $\mathbb{P}$-augmentation of the filtration generated by $(\int_0^t \int_{\mathbb{R}^{d'}} \exp(e) N(\mathrm{d}e, \mathrm{d}r))_{t \geq 0}$. Given a compact set $\mathbb{A} \subset \mathbb{R}^m$, $m \in \mathbb{N}$, let $\mathcal{A}$ be the collection of $\mathbb{F}$-predictable processes with values in $\mathbb{A}$. Throughout this paper, unless otherwise stated, we will work on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathcal{F} = \mathcal{F}_\infty$.

Given $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\alpha \in \mathcal{A}$, and a measurable map $(x, a, e) \in \mathbb{R}^d \times \mathbb{A} \times \mathbb{R}^{d'} \mapsto b(x, a, e) \in \mathbb{R}^d$, we define the càdlàg process $X^{x,\alpha}$ as the solution of

$$X^{x,\alpha}_\cdot = x + \int_0^\cdot \int_{\mathbb{R}^{d'}} b(X^{x,\alpha}_{s-}, \alpha_s, e) N(\mathrm{d}e, \mathrm{d}s)\,. \tag{2.1}$$

Note that, since the marked point process N has finite activity, the above can be defined path-wise by (a.s.) finite induction on the jump times.

We then consider the ergodic gain functional

$$\rho(x, \alpha) := \liminf_{T \to \infty} \frac{1}{\eta T} \mathbb{E}\left[\int_0^T r(X^{x,\alpha}_{t-}, \alpha_t) \mathrm{d}N_t\right],\ (x, \alpha) \in \mathbb{R}^d \times \mathcal{A}, \tag{2.2}$$

for some bounded measurable map $(x, a) \in \mathbb{R}^d \times \mathbb{A} \mapsto r(x, a) \in \mathbb{R}$. Note that this actually also pertains to the case where the reward function r depends on the mark e, by arguing as in Remark 2.2 below. By the same remark, the cost could have an extra component given in term of the Lebesgue measure.

In the above the scaling by $1/(\eta T)$ means that we consider the gain by average unit of time the controller acts on the system. Indeed, $\mathbb{E}[N_T] = \eta T$ and the control only affects the system at jump times of N.

This functional induces an infinite horizon control problem corresponding to finding the value function

$$\rho^* := \sup_{\alpha \in \mathcal{A}} \rho(\cdot, \alpha). \tag{2.3}$$

All throughout this paper, we make the following assumptions. First, we impose some control on the coefficients (b, r).

Assumption 1. *The map* (b, r) *is continuous. Moreover, there exists* $L_{b,r} > 0$ *such that*

$$[b(\cdot, a, e)]_{\mathcal{C}^0_{\rm lin}} + \|r(\cdot, a)\|_{\mathcal{C}^{0,1}_b} \leq L_{b,r}, \quad \textit{for all } (a, e) \in \mathbb{A} \times \mathbb{R}^{d'}.$$

The next assumption, known as asymptotic flatness, guarantees that each control process contracts all possible paths of (2.1) exponentially fast to a single trajectory. This is a sufficient condition to ensure that ρ^* is constant, i.e. that initial conditions are forgotten. See the proof of Lemma A.1 in the Appendix. It can be compared to standard assumptions used in the Brownian diffusion case as in e.g. [3, Proof of Lemma 7.3.4], up to a more abstract statement. We refer to Example 2.1 below for a case of application.

Assumption 2. *There is* $\zeta \in \mathcal{C}^0(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}_+)$ *such that*

(i) There exists $(\ell_\zeta, L_\zeta) \in (\mathbb{R}^*_+)^2$ *and* $p_\zeta \geq 1$ *for which*

$$\ell_\zeta |x - x'|^{p_\zeta} \leq \zeta(x, x') \leq L_\zeta |x - x'|^{p_\zeta}, \quad \textit{for all } x, x' \in \mathbb{R}^d.$$

(ii) There exists $C_\zeta > 0$ *such that for all* $x, x' \in \mathbb{R}^d$*,* $a \in \mathbb{A}$

$$\eta \int_{\mathbb{R}^{d'}} \{\zeta(x + b(x, a, e), x' + b(x', a, e)) - \zeta(x, x')\} \, \nu(\mathrm{d}e) \leq -C_\zeta \zeta(x, x') \,. \tag{2.4}$$

Our last assumption is typically required to control the long-time behavior of solutions of (2.1), see Lemma A.2 in the Appendix. It is a form of Lyapunov stability assumption, see e.g. [10, 21] for comparison.

Assumption 3. *There is* $\xi \in \mathcal{C}^0(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}_+)$ *such that*

(i) There exists $(\ell_\xi, L_\xi) \in (\mathbb{R}^*_+)^2$ *and* $p_\xi \geq 1$ *for which*

$$\ell_\xi |x|^{p_\xi} \leq \xi(x) \leq L_\xi |x|^{p_\xi}, \quad \textit{for all } x \in \mathbb{R}^d.$$

(ii) There exists $C^1_\xi > 0$ *and* $C^2_\xi \in \mathbb{R}$ *such that for all* $x \in \mathbb{R}^d$*,* $a \in \mathbb{A}$

$$\eta \int_{\mathbb{R}^{d'}} \{\xi(x + b(x, a, e)) - \xi(x)\} \, \nu(\mathrm{d}e) \leq -C^1_\xi \xi(x) + C^2_\xi. \tag{2.5}$$

Example 2.1. *Consider a bidding problem in a repeated auction with reserve (see e.g. [24] for an introduction to auctions), in which* X *stands for the current reserve price and* α *is the bid. We set* $e = (e_1, e_2, e_3, e_4) \in \mathbb{R}^4$ *and consider the dynamic induced by* $b(x, a, e) := e_1(ae_2 + e_3 - x)$ *for* $\mathbb{A} := [\underline{a}, \overline{a}] \subset \mathbb{R}_+$*. This means that the dynamic is* mean-reverting *around the level* $ae_2 + e_3$*. In this* *formula, e_2 corresponds to the retail value (the price at which the bidder will sell to the final client the product he bought) so that the value a of the control is the so-called shading factor. Then, $e_1>0$ is the realization of a random mean-reversion speed and e_3 is the realisation of an exogeneous noise. If the reserve price value x is smaller than the bid price ae_2 (up to the additional noise value e_3) then it moves up for the next auction, and the other way around if it is bigger. In a second price auction, with e_4 as the value of the competition bid, the natural reward function is*

$$r(x,a) = \int_{\mathbb{R}^4} (e_2 - x \vee e_4) 1_{\{ae_2 \geq x \vee e_4\}} \nu(\mathrm{d}e)\,.$$

We assume that $\nu([0,1]\times\mathbb{R}_+\times\mathbb{R}^2) = 1$, $1 - \int_{\mathbb{R}^4}(1-e_1)^{2p}\nu(de) =: m_1 \in (0,1]$ and that $\int_{\mathbb{R}^4} \sup_{a\in\mathbb{A}} |ae_1e_2 + e_1e_3|^{2p}\nu(de)<\infty$, for some integer $p \geq 1$. Then, Assumption 2 holds with $\zeta(x,x') := |x-x'|^{2p}$ and $C_\zeta = \eta m_1$, while Assumption 3 holds with $\xi(x) = |x|^{2p}$, $C^1_\xi = \frac{1}{2}\eta m_1$ and $C^2_\xi = \eta C_e$ for some $C_e > 0$ that does not depend on η.

Under a standard log-normal model for valuations (see e.g. [32]), and a uniform competition on $[0,\bar{c}]$ for some $\bar{c} > 0$, it is easily verified that Assumption 1 holds. This example is developped further in Section 5.

Under the above assumptions, we obtain the following classical result, Theorem 2.3 below whose proof is rather standard, but produced in the Appendix for completeness. To state it, we first need to introduce the following auxiliary optimal control problems, defined for all $x \in \mathbb{R}^d$, $\lambda, T > 0$ and $t \leq T$:

$$V_\lambda(x) := \sup_{\alpha\in\mathcal{A}} J_\lambda(x,\alpha) \quad \text{with} \quad J_\lambda(x,\alpha) := \frac{1}{\eta}\mathbb{E}\left[\int_0^\infty e^{-\lambda s} r(X^{x,\alpha}_{s-},\alpha_s)\mathrm{d}N_s\right] \tag{2.6}$$

and

$$V_T(t,x) := \sup_{\alpha\in\mathcal{A}} J_T(t,x,\alpha) \quad \text{with} \quad J_T(t,x,\alpha) := \frac{1}{\eta}\mathbb{E}\left[\int_t^T r(X^{t,x,\alpha}_{s-},\alpha_s)\mathrm{d}N_s\right] \tag{2.7}$$

where $X^{t,x,\alpha}$ is the solution of (2.1) on $[t,\infty)$ such that $X^{t,x,\alpha}_t = x$.

Remark 2.2. *Note that Assumption 1 implies that, for all $t \geq 0$, $(x,\alpha) \in \mathbb{R}^d \times \mathcal{A}$ $\sup_{[0,t]}|X^{x,\alpha}|$ has moments of any order. Also, it follows from the Assumption 1 again and the fact that ν is a probability measure that*

$$\rho(x,\alpha) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T r(X^{x,\alpha}_s,\alpha_s)ds\right],$$
$$J_\lambda(x,\alpha) = \mathbb{E}\left[\int_0^\infty e^{-\lambda s} r(X^{x,\alpha}_s,\alpha_s)ds\right], \quad \textit{and} \quad J_T(t,x,\alpha) = \mathbb{E}\left[\int_t^T r(X^{t,x,\alpha}_s,\alpha_s)ds\right].$$

For the same reason, we could consider expected gains of the more general form

$$\frac{1}{\eta T}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^{d'}} \tilde{r}(X^{x,\alpha}_{s-},\alpha_s,e)N(\mathrm{de},\mathrm{d}s)\right] = \frac{1}{T}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^{d'}} \tilde{r}(X^{x,\alpha}_{s},\alpha_s,e)\nu(\mathrm{de})\mathrm{d}s\right]$$

upon replacing r by $(x,a)\in\mathbb{R}^d\times\mathbb{A}\mapsto \int_{\mathbb{R}^{d'}}\tilde{r}(x,a,e)\nu(\mathrm{de})$.

Recall that ρ^* is defined in (2.3). In the following, we show that this function is actually a constant, i.e. $\rho^*=\rho^*(0)$ on $\mathbb{R}^d$.

Theorem 2.3. *Let Assumptions 1, 2 and 3 hold. Then, there exists sequences $(\lambda_n)_{n\ge1}$ going to 0 and $(T_n)_{n\ge1}$ going to $+\infty$ such that $(\lambda_n V_{\lambda_n})_{n\ge1}$ and $(T_n^{-1}V_{T_n}(0,\cdot))_{n\ge1}$ converge uniformly on compact sets to $\rho^*(0)$, and such that $(V_{\lambda_n}-V_{\lambda_n}(0))_{n\ge1}$ converges uniformly on compact sets to a function $\mathrm{w}\in\mathcal{C}^{0,1}$ that solves*

$$\rho^* = \sup_{a\in\mathbb{A}}\left\{\eta\int_{\mathbb{R}^{d'}}\left[\mathrm{w}(\cdot+b(\cdot,a,e))-\mathrm{w}\right]\nu(de)+r(\cdot,a)\right\},\quad \textit{on } \mathbb{R}^d. \tag{2.8}$$

Moreover, ρ^* is constant over $\mathbb{R}^d$ and, if $(\tilde{\mathrm{w}},\tilde\rho)\in C^0_{\mathrm{lin}}\times\mathbb{R}$, solves the ergodic equation

$$\tilde\rho = \sup_{a\in\mathbb{A}}\left\{\eta\int_{\mathbb{R}^{d'}}[\tilde{\mathrm{w}}(\cdot+b(\cdot,a,e))-\tilde{\mathrm{w}}]\nu(\mathrm{de})+r(\cdot,a)\right\},\quad \textit{on } \mathbb{R}^d,$$

then $\tilde\rho=\rho^*$.

Remark 2.4. *As a by-product of Theorem 2.3 and the first part of the proof of Lemma A.4, for all $x\in\mathbb{R}^d$, there exists an optimal Markovian control defined by $\hat\alpha:=\hat{\mathrm{a}}(X^{x,\hat\alpha}_{\cdot-})$ in which $\hat{\mathrm{a}}$ is a measurable map satisfying*

$$\eta\int_{\mathbb{R}^{d'}}\mathrm{w}(\cdot+b(\cdot,\hat{\mathrm{a}}(\cdot),e))\nu(\mathrm{de})+r(\cdot,\hat{\mathrm{a}}(\cdot)) = \max_{a\in\mathbb{A}}\left\{\eta\int_{\mathbb{R}^{d'}}\mathrm{w}(\cdot+b(\cdot,\mathrm{a},e))\nu(\mathrm{de})+r(\cdot,\mathrm{a})\right\},\quad \textit{on } \mathbb{R}^d.$$

Moreover,

$$\rho^* = \lim_{T\to\infty}\frac{1}{\eta T}\mathbb{E}\left[\int_0^T r(X^{x,\hat\alpha}_{t-},\hat\alpha_t)\mathrm{d}N_t\right].$$

3 Approximation for models with large activity

Given an $\varepsilon\in(0,1)$, we now replace η by

$$\eta_\varepsilon := \varepsilon^{-1}.$$

In the following, we omit the dependence of N and $X^{x,\alpha}$ on ε for ease of notations and set

$$\rho_\varepsilon^* := \sup_{\alpha\in\mathcal{A}} \liminf_{T\to\infty} \frac{1}{\eta_\varepsilon T}\mathbb{E}\left[\int_0^T r(X_{t-}^{0,\alpha},\alpha_t)\mathrm{d}N_t\right].$$

We shall see that ρ_ε^*, together with the associated optimal policy, can be approximated by considering its diffusive limit as $\varepsilon \to 0$, upon assuming that the jump coefficient $b := b_\varepsilon$ introduced in Section 2 is of the form

$$b_\varepsilon = \varepsilon b_1 + \sqrt{\varepsilon} b_2\,,$$

and making the following assumption.

Assumption 4. *We have $b = \varepsilon b_1 + \sqrt{\varepsilon} b_2$ for some continuous functions $b_1 : \mathbb{R}^d \times \mathbb{A} \times \mathbb{R}^{d'} \mapsto \mathbb{R}^d$ and $b_2 : \mathbb{R}^d \times \mathbb{R}^{d'} \mapsto \mathbb{R}^d$ such that:*

(i) *There exists $L_{b_1,b_2} > 0$ such that*

$$[b_1(\cdot,a,e)]_{\mathcal{C}^0_{\text{lin}}} + \|b_2(\cdot,e)\|_{\mathcal{C}^0_b} \le L_{b_1,b_2}$$

for all $(a,e) \in \mathbb{A} \times \mathbb{R}^{d'}$.

(ii) *There exists $\varsigma > 0$ such that*

$$\int_{\mathbb{R}^{d'}} b_2(\cdot,e)\nu(\mathrm{d}e) = 0 \text{ and } \int_{\mathbb{R}^{d'}} b_2(\cdot,e)b_2(\cdot,e)^\top\nu(\mathrm{d}e) \ge \varsigma \boldsymbol{I}_d$$

where $\boldsymbol{I}_d$ is the identity matrix.

(iii) *The map*

$$(x,a) \in \mathbb{R}^d \times \mathbb{A} \mapsto \mu(x,a) := \int_{\mathbb{R}^{d'}} b_1(x,a,e)\nu(\mathrm{d}e)$$

is Lipschitz in x uniformly in a, and there exists a Lipschitz $\mathbb{R}^{d\times d}$-valued function σ defined on $\mathbb{R}^d$ such that

$$\sigma\sigma^\top = \int_{\mathbb{R}^{d'}} b_2(\cdot,e)b_2^\top(\cdot,e)\nu(\mathrm{d}e).$$

(iv) *The estimates of Assumptions 1, 2 and 3 hold for each $(\eta_\varepsilon, b_\varepsilon, r)$ in place of (η, b, r), uniformly in $\varepsilon > 0$.*

Example 3.1. *Consider the context of Example 2.1 in which $\eta = \varepsilon^{-1}$ and*

$$b_\varepsilon(x,a,e) = e_1(\varepsilon(e_2 a - x) + \varepsilon^{\frac{1}{2}} e_3),\ (x,a,e) \in \mathbb{R}^d \times \mathbb{A} \times \mathbb{R}^4$$

with ν as in Example 2.1 such that in addition $\int_{\mathbb{R}^4} e_1 e_3 \nu(\mathrm{de}) = 0$. In this context, we obtain $\mu(x,a) = n_2 a - n_1 x$, with $n_1 := \int_{\mathbb{R}^4} e_1 \nu(de)$ and $n_2 := \int_{\mathbb{R}^4} e_1 e_2 \nu(\mathrm{de})$, and $\sigma(x)^2 = \int_{\mathbb{R}^4} |e_1 e_3|^2 \nu(\mathrm{de})$. Assume that $n_1 > 0$. Using a second order Taylor expansion around $\varepsilon = 0$, one easily checks that Assumption 3 holds with $\xi(x) = |x|^{2p}$, $p \geq 1$, for some C^1_ξ and C^2_ξ that do not depend on $\varepsilon > 0$. Similarly, Assumption 2 holds with $\zeta(x,x') = |x - x'|^{2p}$, $p \geq 1$, for some $C_\zeta > 0$, uniformly in $\varepsilon \in (0, \varepsilon_\circ)$, for some $\varepsilon_\circ > 0$ small enough.

3.1 Candidate diffusion limit

Let $\bar{\mathbb{P}}$ be a probability measure on $\mathbb{D}$ and let W be a stochastic process such that W is a $\bar{\mathbb{P}}$-Brownian motion, let $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_s)_{s\geq 0}$ be the $\bar{\mathbb{P}}$-augmentation of the filtration generated by W, and let $\bar{\mathcal{A}}$ be the collection of $\bar{\mathbb{F}}$-predictable processes taking values in $\mathbb{A}$. Given $\bar{\alpha} \in \bar{\mathcal{A}}$, we can then define $\bar{X}^{x,\bar{\alpha}}$ as the unique strong solution (see [34, Thm. 1]) of

$$\bar{X}^{x,\bar{\alpha}} = x + \int_0^\cdot \mu(\bar{X}^{x,\bar{\alpha}}_s, \bar{\alpha}_s)\mathrm{d}s + \int_0^\cdot \sigma(\bar{X}^{x,\bar{\alpha}}_s)\mathrm{d}W_s\,. \tag{3.1}$$

The corresponding ergodic control problem is defined by

$$\bar{\rho}^*(x) := \sup_{\bar{\alpha}\in\bar{\mathcal{A}}} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T r(\bar{X}^{x,\bar{\alpha}}_t, \bar{\alpha}_t)\mathrm{d}t\right],\ x \in \mathbb{R}^d.$$

As in Section 2, we define for $\lambda > 0$ and $x \in \mathbb{R}^d$

$$\bar{V}_\lambda(x) := \sup_{\bar{\alpha}\in\bar{\mathcal{A}}} \bar{J}_\lambda(x,\bar{\alpha}) \ \text{ with } \ \bar{J}_\lambda(x,\bar{\alpha}) := \mathbb{E}\left[\int_0^\infty e^{-\lambda s} r(\bar{X}^{x,\bar{\alpha}}_s, \bar{\alpha}_s)\mathrm{d}s\right],$$

and impose conditions corresponding to the estimates of Lemma A.1 and A.2.

Assumption 5. *There exists $L_{\bar{V}}, C_{\bar{X}} > 0$ and $p_{\bar{X}} \geq 1$ such that:*
(i) *For all $x, x' \in \mathbb{R}^d$ and $\lambda \in (0,1)$,*

$$|\bar{V}_\lambda(x) - \bar{V}_\lambda(x')| \leq L_{\bar{V}}|x - x'|.$$

(ii) *For all $x \in \mathbb{R}^d$ and $\bar{\alpha} \in \bar{\mathcal{A}}$,*

$$\mathbb{E}[|\bar{X}^{x,\bar{\alpha}}_t|^{p_{\bar{X}}}] \leq C_{\bar{X}}\left\{e^{-t/C_{\bar{X}}}|x|^{p_{\bar{X}}} + 1\right\},\ t \geq 0.$$

Remark 3.2. (i) *The condition* (i) *of Assumption 5 holds for instance under [3, Assumption 7.3.1]. Indeed, the latter implies a similar bound as* (A.3)*, see [3, Lemma 7.3.4], and the estimate of* (i) *then follows from the same arguments as in the proof of Lemma A.1. More generally, it suffices to find a family of $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$-functions $(\bar{\zeta}_\iota)_{\iota>0}$ that is locally bounded, satisfies*

$$\mathrm{D}\bar{\zeta}_\iota(x,x') \begin{pmatrix} \mu(x,a) \\ \mu(x',a) \end{pmatrix} + \frac{1}{2} \mathrm{Tr}\left[\Sigma(x,x')\mathrm{D}^2\bar{\zeta}_\iota(x,x')\right] \le -C_{\bar{\zeta}}\bar{\zeta}_\iota(x,x') + \varrho_\iota\,,\ x,x' \in \mathbb{R}^d,\ a \in \mathbb{A},\ \iota > 0, \tag{3.2}$$

in which $C_{\bar{\zeta}} > 0$, $\lim_{\iota\to 0} \varrho_\iota = 0$ and

$$\Sigma(x,x') := \begin{pmatrix} \sigma(x) \\ \sigma(x') \end{pmatrix} \begin{pmatrix} \sigma(x) \\ \sigma(x') \end{pmatrix}^\top,$$

and such that $(\bar{\zeta}_\iota)_{\iota>0}$ converges pointwise as $\iota \to 0$ to a map $\bar{\zeta} : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfing

$$\frac{1}{C_{\bar{\zeta}}}|x-x'|^{p_{\bar{\zeta}}} \le \bar{\zeta}(x,x') \le C_{\bar{\zeta}}|x-x'|^{p_{\bar{\zeta}}}\,,\ \textit{for all } x,x' \in \mathbb{R}^d,$$

for some $p_{\bar{\zeta}} \ge 1$. This follows from the arguments used in the proof of Lemma A.1 upon first applying Itô's lemma to $\bar{\zeta}_\iota$ and then sending $\iota \to 0$ to deduce the counterpart of (A.2) *before using the inequalities just above.*
(ii) *The condition* (ii) *of Assumption 5 holds for instance if we can find a smooth function $\bar{\xi}$ and constants $C^1_{\bar{\xi}} > 0$ and $C^2_{\bar{\xi}}$ such that*

$$\mathrm{D}\bar{\xi}(x)\mu(x,a) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top(x)\mathrm{D}^2\bar{\xi}(x)\right] \le -C^1_{\bar{\xi}}\bar{\xi}(x) + C^2_{\bar{\xi}}, \tag{3.3}$$

and

$$\frac{1}{C^2_{\bar{\xi}}}|x|^{p_{\bar{\xi}}} \le \bar{\xi}(x) \le C^2_{\bar{\xi}}|x|^{p_{\bar{\xi}}}, \tag{3.4}$$

for all $x \in \mathbb{R}^d$, for some $p_{\bar{\xi}} \ge 1$. This follows from the same arguments as in the proof of Lemma A.2. As in (i) *above, it suffices that* (3.3) *holds for a sequence of approximating smooth functions. In particular, condition* (ii) *of Assumption 5 holds under [3, Assumption 7.3.1], see [3, Lemma 7.6.3].*

Example 3.3. *Consider the context of Example 3.1 with σ constant, then it satisfies [3, Assumption 7.3.1], and therefore Assumption 5, by [3, Example 7.3.3].*

In order to state the counterpart of Theorem 2.3 for the diffusive limit ergodic control problem, we also define, for $T > 0$, $t \le T$ and $x \in \mathbb{R}^d$,

$$\bar{V}_T(t,x) := \sup_{\bar{\alpha}\in\bar{\mathcal{A}}} \bar{J}_T(t,x,\bar{\alpha}) \ \text{ with } \ \bar{J}_T(t,x,\bar{\alpha}) := \mathbb{E}\left[\int_t^T r(\bar{X}^{t,x,\bar{\alpha}}_s, \bar{\alpha}_s)\mathrm{d}s\right],$$

in which $\bar{X}^{t,x,\bar{\alpha}}$ is the solution of (3.1) on $[t,\infty)$ such that $\bar{X}_t^{t,x,\bar{\alpha}} = x$, and set

$$\bar{\mathcal{L}}^{\bar{a}}\varphi = \mathrm{D}\varphi\mu(\cdot,\bar{a}) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top \mathrm{D}^2\varphi],\ \bar{a}\in\mathbb{A},$$

for a smooth function $\varphi : \mathbb{R}^d \to \mathbb{R}$.

Theorem 3.4. *Let Assumptions 4 and 5 hold. Then, there exists sequences $(\lambda_n)_{n\geq 1}$ going to 0 and $(T_n)_{n\geq 1}$ going to $+\infty$ such that $(\lambda_n \bar{V}_{\lambda_n})_{n\geq 1}$ and $(T_n^{-1}\bar{V}_{T_n}(0,\cdot))_{n\geq 1}$ converge uniformly on compact sets to $\bar{\rho}^*(0)$, and such that $(\bar{V}_{\lambda_n} - \bar{V}_{\lambda_n}(0))_{n\geq 1}$ converges uniformly on compact sets to a function $\bar{\mathrm{w}} \in \mathcal{C}^2 \cap \mathcal{C}^0_{\mathrm{lin}}$ that satisfies*

$$\bar{\rho}^* = \sup_{\bar{a}\in\mathbb{A}} \left\{ \bar{\mathcal{L}}^{\bar{a}}\bar{\mathrm{w}} + r(\cdot,\bar{a}) \right\}\ ,\ \ \text{on } \mathbb{R}^d, \tag{3.5}$$

and

$$\|\bar{\mathrm{w}}\|_{\mathcal{C}_b^{0,1}} \leq L^\gamma_{\bar{\mathrm{w}}} \text{ and } \|\bar{\mathrm{w}}\|_{\mathcal{C}_b^{2,\gamma}(B_1(x))} \leq L^\gamma_{\bar{\mathrm{w}}}(1+|x|),\ \ \text{for all } x\in\mathbb{R}^d, \tag{3.6}$$

for some $L^\gamma_{\bar{\mathrm{w}}} > 0$, for all $\gamma \in (0,1)$. Moreover, $\bar{\rho}^*$ is constant over $\mathbb{R}^d$, and, if $(\tilde{\mathrm{w}},\tilde{\rho}) \in (\mathcal{C}^2 \cap \mathcal{C}^0_{\mathrm{lin}}) \times \mathbb{R}$ solves the ergodic equation

$$\tilde{\rho} = \sup_{\bar{a}\in\mathbb{A}} \left\{ \bar{\mathcal{L}}^{\bar{a}}\tilde{\mathrm{w}} + r(\cdot,\bar{a}) \right\},\ \ \text{on } \mathbb{R}^d, \tag{3.7}$$

then $\tilde{\rho} = \bar{\rho}^*$.

Proof. The proof is exactly the same as the one of Theorem 2.3 upon replacing the estimates of Lemmas A.1 and A.2 by the ones of Assumption 5. See the Appendix. The only significant difference is that we have to show the estimate (3.6).

1. The fact that, for an appropriate sequence $(\lambda_n)_{n\geq 0}$ that converges to 0, $\lambda_n \bar{V}_{\lambda_n}(0) \to c \in \mathbb{R}$ and $\bar{V}_{\lambda_n} - \bar{V}_{\lambda_n}(0) \to \bar{\mathrm{w}}$ uniformly on compact sets for some $\bar{\mathrm{w}} \in \mathcal{C}^{0,1}$ follows from Assumption 5 and the same arguments as in the first part of the proof of Lemma A.3 below.

2. We now argue as in the proof of [3, Theorem 3.5.6]. Fix $n \geq 1$, let $\bar{\tau}_n^{x,\bar{\alpha}}$ be the first exit time of $\bar{X}^{x,\bar{\alpha}}$ from $B_n(0)$, for $(x,\bar{\alpha}) \in \mathbb{R}^d \times \bar{\mathcal{A}}$, and set

$$\bar{V}^n_\lambda(x) := \sup_{\bar{\alpha}\in\bar{\mathcal{A}}} \mathbb{E}\left[\int_0^{\bar{\tau}_n^{x,\bar{\alpha}}} e^{-\lambda s} r(\bar{X}_s^{x,\bar{\alpha}},\bar{\alpha}_s)\mathrm{d}s\right].$$

Then, $\bar{V}^n_\lambda \in \mathcal{C}^2(B_n(0))$ by the arguments in the proof of [3, Theorem 3.5.6]. Moreover, Assumption 5 and the linear growth of r (recall that it is assumed Lipschitz) imply that

$$\sup_{n\geq 1} [\bar{V}^n_\lambda]_{\mathcal{C}^0_{\mathrm{lin}}} \leq C_\lambda$$

for some $C_\lambda > 0$. Then, arguing as in the proof of [3, Theorem 3.5.6], we obtain that, for all $\lambda > 0$, $(\bar{V}^n_\lambda)_{n\geq 1}$ converges as $n \to \infty$ to a map $\psi_\lambda \in \mathcal{C}^2$ that solves

$$\lambda\psi_\lambda = \sup_{\bar{a}\in\mathbb{A}} \left\{\bar{\mathcal{L}}^{\bar{a}}\psi_\lambda + r(\cdot,\bar{a})\right\}, \quad \text{on } \mathbb{R}^d,$$

and has at most linear growth. Using this linear growth property, Assumption 5 and a verification argument, we deduce that $\psi_\lambda = \bar{V}_\lambda$.

Since $\bar{V}_\lambda \in \mathcal{C}^{0,1}_b$ by Assumption 5, it follows from Assumption 4 and Lemma B.2 that, given $\gamma \in (0,1)$, $\bar{V}_\lambda \in \mathcal{C}^{2,\gamma}$ and that there is $K > 0$ (depending on γ but not on $\lambda \in (0,1)$) such that

$$\|\Delta\bar{V}_\lambda\|_{\mathcal{C}^{2,\gamma}_b(B_1(x))} \leq K(1+|x|), \quad \text{for all } (x,\lambda) \in \mathbb{R}^d \times (0,1), \tag{3.8}$$

where $\Delta\bar{V}_\lambda := \bar{V}_\lambda - \bar{V}_\lambda(0)$ solves

$$\lambda\bar{V}_\lambda(0) + \lambda\Delta\bar{V}_\lambda = \sup_{\bar{a}\in\mathbb{A}} \left\{\bar{\mathcal{L}}^{\bar{a}}\Delta\bar{V}_\lambda + r(\cdot,\bar{a})\right\}, \quad \text{on } \mathbb{R}^d.$$

Let $(\lambda_n)_{n\geq 0}$ be as in step 1. Passing to the limit in the above leads to (3.5), with c defined in step 1. in place of $\bar{\rho}^*$, and to (3.6).

3. By the same arguments as in Lemma A.4, if $(\tilde{\mathrm{w}},\tilde{\rho}) \in (\mathcal{C}^2 \cap \mathcal{C}^0_{\mathrm{lin}}) \times \mathbb{R}$ solves (3.7) then $\tilde{\rho} = \bar{\rho}^*$. In particular, $\bar{\rho}^*$ is constant and $c = \bar{\rho}^*$ by step 2.

4. The fact that there exists $(T_n)_{n\geq 1}$ going to $+\infty$ such that $(T_n^{-1}\bar{V}_{T_n}(0,\cdot))_{n\geq 1}$ converge uniformly on compact sets to $\bar{\rho}^*(0)$ then follows from the same arguments as in Lemma A.5. □

3.2 First order approximation guarantees

We can now turn to the main part of this paper and quantify the approximation error due to passing to the diffusive limit in the original pure jump problem. We will show below that it controlled by the Hölder regularity of $\mathrm{D}^2\bar{\mathrm{w}}$, namely that the approximation error is of the order of $\varepsilon^{\frac{\gamma}{2}}$ for all $\gamma \in (0,1)$. In Section 3.3, we will see that it can be improved by considering appropriate correction terms.

The cornerstone of the analysis is the residual term of a second order Taylor expansion of $\bar{\mathrm{w}}$ performed on the Dynkin operator of the jump diffusion process (2.1), namely:

$$\delta r_\varepsilon(x,a) := \frac{1}{\varepsilon}\int_{\mathbb{R}^{d'}} [\bar{\mathrm{w}}(x+b_\varepsilon(x,a,e)) - \bar{\mathrm{w}}(x)]\,\nu(\mathrm{d}e) - \mathrm{D}\bar{\mathrm{w}}(x)\mu(x,a) - \frac{1}{2}\,\mathrm{Tr}[\sigma\sigma^\top(x)\mathrm{D}^2\bar{\mathrm{w}}(x)], \tag{3.9}$$

defined for $(x,a) \in \mathbb{R}^d \times \mathbb{A}$. The function δr^ε measures the error of the diffusion approximation explicitely in terms of the control problem, and thus will be shown to effectively control the error in all quantities of interest. Leveraging the regularity results in (3.6), the Hölder regularity of $\mathrm{D}^2\bar{\mathrm{w}}$ yields Proposition 3.5, which in turn yields Theorem 3.6.

Proposition 3.5. *Let Assumptions 4 and 5 hold with* $p_\xi \geq 3$*. Then, for every* $\gamma \in (0,1)$*, there exists* $L^{\gamma,1}_{\delta r}, L^{\gamma,2}_{\delta r} > 0$ *such that, for each* $0 < \varepsilon \leq \varepsilon_\circ := (L_{b_1,b_2})^{-2}$ *and* $(x,a) \in \mathbb{R}^d \times \mathbb{A}$*,*

$$|\delta r_\varepsilon(x,a)| \leq \varepsilon^{\frac{\gamma}{2}} L^{\gamma,1}_{\delta r}(1+|x|^3), \tag{3.10}$$

and

$$\sup_{t\geq 0}\sup_{\alpha\in\mathcal{A}} \mathbb{E}[|\delta r_\varepsilon(X^{x,\alpha}_t,\alpha_t)|] \leq \varepsilon^{\frac{\gamma}{2}} L^{\gamma,2}_{\delta r}(1+|x|^3)\ . \tag{3.11}$$

Proof. 1. We first prove the estimate (3.10) using (3.6). Namely,

$$\begin{aligned}\bar{\mathrm{w}}(x+b_\varepsilon(x,a,e)) - \bar{\mathrm{w}}(x) &= \bar{\mathrm{w}}(x+\varepsilon b_1(x,a,e)+\varepsilon^{\frac{1}{2}}b_2(x,e)) - \bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}}b_2(x,e))) \\ &\quad + \bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}}b_2(x,e)) - \bar{\mathrm{w}}(x)\end{aligned}$$

where

$$\begin{aligned}&\bar{\mathrm{w}}(x+\varepsilon b_1(x,a,e)+\varepsilon^{\frac{1}{2}}b_2(x,e)) - \bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}}b_2(x,e))) \\ &= \varepsilon \mathrm{D}\bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}}b_2(x,e))b_1(x,a,e) + \int_0^1 \frac{\varepsilon^2}{2} b_1(x,a,e)^\top \mathrm{D}^2\bar{\mathrm{w}}(\hat{x}^\varepsilon_1(u))b_1(x,a,e)\mathrm{d}u\end{aligned}$$

in which

$$\hat{x}^\varepsilon_1(u) := x + \varepsilon^{\frac{1}{2}}b_2(x,e) + u\varepsilon b_1(x,a,e)$$

is such that

$$\sup_{u\in[0,1]} |\hat{x}^\varepsilon_1(u)| \leq |x| + \varepsilon^{\frac{1}{2}}L_{b_1,b_2} + \varepsilon L_{b_1,b_2}(1+|x|),$$

by definition of L_{b_1,b_2} in Assumption 4. By (3.6) and Assumption 4, this implies that

$$\begin{aligned}&\left|\frac{\varepsilon^2}{2} b_1(x,a,e)^\top \mathrm{D}^2\bar{\mathrm{w}}(\hat{x}^\varepsilon_1(u))b_1(x,a,e)\right| \\ &\leq \frac{\varepsilon^2}{2}(L_{b_1,b_2})^2(1+|x|)^2 L^\gamma_{\bar{\mathrm{w}}}(1+|x|+\varepsilon^{\frac{1}{2}}L_{b_1,b_2}+\varepsilon L_{b_1,b_2}(1+|x|)).\end{aligned}$$

Moreover, since $\varepsilon^{\frac{1}{2}}L_{b_1,b_2} \leq 1$, we have

$$|\mathrm{D}\bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}}b_2(x,e)) - \mathrm{D}\bar{\mathrm{w}}(x)| \leq L^\gamma_{\bar{\mathrm{w}}}(1+|x|)\varepsilon^{\frac{1}{2}}L_{b_1,b_2}$$

by (i) of Assumption 4 and (3.6).

Using (ii) of Assumption 4, we next obtain that

$$\int_{\mathbb{R}^{d'}} \{\bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}}b_2(x,e)) - \bar{\mathrm{w}}(x)\}\nu(\mathrm{d}e) = \int_{\mathbb{R}^{d'}}\int_0^1 \frac{\varepsilon}{2} b_2(x,e)^\top \mathrm{D}^2\bar{\mathrm{w}}(\hat{x}^\varepsilon_2(u,e))b_2(x,e)\mathrm{d}u\,\nu(\mathrm{d}e)$$

in which

$$\hat{x}_2^{\varepsilon}(u,e) := x + u\varepsilon^{\frac{1}{2}} b_2(x,e) \in B_1(x)$$

since $\varepsilon^{\frac{1}{2}} L_{b_1,b_2} \le 1$ by assumption and (i) of Assumption 4. Then, by (3.6) again and (iii) of Assumption 4

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{d'}} \{\bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}} b_2(x,e)) - \bar{\mathrm{w}}(x)\}\nu(\mathrm{d}e) - \frac{\varepsilon}{2}\operatorname{Tr}[\sigma\sigma^\top(x)\mathrm{D}^2\bar{\mathrm{w}}(x)] \right| \\
&= \left| \int_{\mathbb{R}^{d'}} \{\bar{\mathrm{w}}(x+\varepsilon^{\frac{1}{2}} b_2(x,e)) - \bar{\mathrm{w}}(x)\}\nu(\mathrm{d}e) - \int_{\mathbb{R}^{d'}} \frac{\varepsilon}{2} b_2(x,e)^\top \mathrm{D}^2\bar{\mathrm{w}}(x) b_2(x,e)\nu(\mathrm{d}e) \right| \\
&\le \frac{\varepsilon}{2}(L_{b_1,b_2})^2 L_{\bar{\mathrm{w}}}^{\gamma}(1+|x|)(\varepsilon^{\frac{1}{2}} L_{b_1,b_2})^{\gamma}.
\end{aligned}$$

The estimate (3.10) is obtained by combining the above.

2. The estimate (3.11) follows from (3.10), Lemma A.2 and the fact that $p_\xi \ge 3$. □

We are now in position to state the main result of this section.

Theorem 3.6. *Let Assumptions 4 and 5 hold with $p_\xi \ge 3$. Then, for all $\gamma \in (0,1)$, there exists $L_{\delta\rho}^{\gamma} > 0$ such that*

$$|\bar{\rho}^* - \rho_\varepsilon^*| \le \varepsilon^{\frac{\gamma}{2}} L_{\delta\rho}^{\gamma} \ \textit{for all } \varepsilon \in (0,1).$$

Moreover, there exists a measurable map $\hat{\mathrm{a}} : \mathbb{R}^d \mapsto \mathbb{A}$ such that

$$\bar{\mathcal{L}}^{\hat{\mathrm{a}}}\bar{\mathrm{w}} + r(\cdot,\hat{\mathrm{a}}) = \sup_{\bar{a}\in\mathbb{A}} \left\{ \bar{\mathcal{L}}^{\bar{a}}\bar{\mathrm{w}} + r(\cdot,\bar{a}) \right\}, \quad \textit{on } \mathbb{R}^d$$

and

$$\rho_\varepsilon^* - \varepsilon^{\frac{\gamma}{2}} L_{\delta\rho}^{\gamma} \le \liminf_{T\to\infty} \frac{1}{\eta_\varepsilon T}\mathbb{E}\left[\int_0^T r(X_{t-}^{\hat{\mathrm{a}}}, \hat{\mathrm{a}}(X_{t-}^{\hat{\mathrm{a}}}))\mathrm{d}N_t\right], \quad \textit{for all } \varepsilon \in (0,1),$$

in which $X^{\hat{\mathrm{a}}}$ solves

$$X_\cdot^{\hat{\mathrm{a}}} = \int_0^\cdot \int_{\mathbb{R}^{d'}} b_\varepsilon(X_{s-}^{\hat{\mathrm{a}}}, \hat{\mathrm{a}}(X_{s-}^{\hat{\mathrm{a}}}), e) N(\mathrm{d}e,\mathrm{d}s)\,.$$

Proof. Fix $\gamma \in (0,1)$. Hereafter, we denote by w_ε the function w introduced in Theorem 2.3 for $\eta = \eta_\varepsilon = \varepsilon^{-1}$. By Theorems 2.3 and 3.4, $\Delta^\varepsilon := \bar{\mathrm{w}} - \mathrm{w}_\varepsilon$ solves

$$\bar{\rho}^* - \rho_\varepsilon^* \le \sup_{a\in\mathbb{A}} \left\{ \frac{1}{\varepsilon}\int_{\mathbb{R}^{d'}} \left[\Delta^\varepsilon(\cdot + b_\varepsilon(\cdot,a,e)) - \Delta^\varepsilon\right]\nu(\mathrm{d}e) - \delta r_\varepsilon(\cdot,a) \right\}, \quad \text{on } \mathbb{R}^d.$$

By the same arguments as in the proof of Lemma A.4, (3.11) applied with $x = 0$, (3.6), (A.4) and Lemma A.2, we deduce that

$$\bar{\rho}^* - \rho^*_\varepsilon \leq L^1_{\delta\rho} \varepsilon^{\frac{\gamma}{2}}$$

for some $L^1_{\delta\rho} > 0$ that does not depend on $\varepsilon \in (0,1)$. Replacing Δ^ε by $-\Delta^\varepsilon$ in this argument implies that

$$\rho^*_\varepsilon - \bar{\rho} \leq L^2_{\delta\rho} \varepsilon^{\frac{\gamma}{2}}$$

for some $L^2_{\delta\rho} > 0$ that does not depend on $\varepsilon \in (0,1)$.

The second assertion of the Theorem is then proved by following the arguments in the first part of the proof of Lemma A.4 and using the above. □

3.3 Higher order expansions

Under additional conditions, one can exhibit a first order correction term to improve the convergence speed in Theorem 3.6. It is in the spirit of the correction term introduced in [1, Section 3.5] but is formulated differently. In particular, the function $\delta\bar{\mathrm{w}}_\varepsilon$ introduced below depends on ε and the optimization in (3.12) is performed over the whole set $\mathbb{A}$. This approach can be iterated to higher order correction terms in an obvious manner, upon additional regularity conditions, without considering a coupled system of PDEs as in [1, Section 3.6].

From now on, we assume the following.

Assumption 6. *There exists $\gamma_\circ \in (0,1)$ and $(\delta\gamma, \delta C) \in (0,1) \times \mathbb{R}$ such that, for each $\varepsilon \in (0,1)$, we can find $\delta\bar{\rho}^*_\varepsilon \in \mathbb{R}$ and $\delta\bar{\mathrm{w}}_\varepsilon \in \mathcal{C}^0_{\mathrm{lin}}$ satisfying $\|\delta\bar{\mathrm{w}}_\varepsilon\|_{\mathcal{C}^{2,\delta\gamma}_b(B_1(x))} \leq \delta C(1+|x|)$ for all $x \in \mathbb{R}^d$ and*

$$\delta\bar{\rho}^*_\varepsilon = \sup_{\bar{a}\in\mathbb{A}} \left[\bar{\mathcal{L}}^{\bar{a}} \delta\bar{\mathrm{w}}_\varepsilon + \varepsilon^{-\frac{\gamma_\circ}{2}} [\delta r_\varepsilon + f](\cdot, \bar{a})\right] \text{ on } \mathbb{R}^d, \tag{3.12}$$

in which

$$f(\cdot, \bar{a}) := \bar{\mathcal{L}}^{\bar{a}} \bar{\mathrm{w}} + r(\cdot, \bar{a}) - \bar{\rho}^*.$$

Theorem 3.7. *Let the conditions of Theorem 3.6 and Assumption 6 hold. Assume further that $p_{\bar{X}} \geq 3$. Then,*

$$\limsup_{\varepsilon \downarrow 0} |\delta\bar{\rho}^*_\varepsilon| < \infty$$

and

$$\bar{\rho}^{*(1)}_\varepsilon := \bar{\rho}^* + \varepsilon^{\frac{\gamma_\circ}{2}} \delta\bar{\rho}^*_\varepsilon,\ \varepsilon \in (0,1),$$

satisfies

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{-\frac{\gamma_\circ+\delta\gamma}{2}} |\rho^*_\varepsilon - \bar\rho^{*(1)}_\varepsilon| < \infty.$$

Moreover, for each $\varepsilon \in (0,1)$*, there exists a measurable map* $\hat{\mathrm{a}}_\varepsilon : \mathbb{R}^d \mapsto \mathbb{A}$ *such that*

$$\bar{\mathcal{L}}^{\hat{\mathrm{a}}_\varepsilon}\delta\bar{\mathrm{w}}_\varepsilon + \varepsilon^{-\frac{\gamma_\circ}{2}}[\delta r_\varepsilon + f](\cdot,\hat{\mathrm{a}}_\varepsilon) = \sup_{\bar a\in\mathbb{A}} \left[\bar{\mathcal{L}}^{\bar a}\delta\bar{\mathrm{w}}_\varepsilon + \varepsilon^{-\frac{\gamma_\circ}{2}}[\delta r_\varepsilon + f](\cdot,\bar a)\right] \quad \textit{on } \mathbb{R}^d$$

and

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{-\frac{\gamma_\circ+\delta\gamma}{2}} |\rho^*_\varepsilon - \rho_\varepsilon(0,\hat{\mathrm{a}}_\varepsilon(X^{\hat{\mathrm{a}}_\varepsilon}_{\cdot-}))| < \infty,$$

in which $X^{\hat{\mathrm{a}}_\varepsilon}$ *solves*

$$X^{\hat{\mathrm{a}}_\varepsilon} = \int_0^\cdot \int_{\mathbb{R}^{d'}} b_\varepsilon(X^{\hat{\mathrm{a}}_\varepsilon}_{s-},\hat{\mathrm{a}}_\varepsilon(X^{\hat{\mathrm{a}}_\varepsilon}_{s-}),e)N(\mathrm{d}e,\mathrm{d}s)\,.$$

Proof. It follows from the same arguments as in Lemma A.4 and the fact that $f \le 0$ by (3.5) that

$$\delta\bar\rho^*_\varepsilon = \sup_{\bar\alpha\in\bar{\mathcal{A}}} \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \varepsilon^{-\frac{\gamma_\circ}{2}}[\delta r_\varepsilon + f](\bar X^{0,\bar\alpha}_s,\bar\alpha_s)\mathrm{d}s\right] \le \sup_{\bar\alpha\in\bar{\mathcal{A}}} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \varepsilon^{-\frac{\gamma_\circ}{2}}\delta r_\varepsilon(\bar X^{0,\bar\alpha}_s,\bar\alpha_s)\mathrm{d}s\right].$$

Let $\hat{\mathrm{a}}$ be as in Theorem 3.6. Then, $f(\cdot,\hat{\mathrm{a}}) = 0$ by (3.5). Hence,

$$\delta\bar\rho^*_\varepsilon \ge \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \varepsilon^{-\frac{\gamma_\circ}{2}}\delta r_\varepsilon(\bar X^{\hat{\mathrm{a}}}_s,\hat{\mathrm{a}}(\bar X^{\hat{\mathrm{a}}}_s))\mathrm{d}s\right]$$

in which $\bar X^{\hat{\mathrm{a}}}$ solves

$$\bar X^{\hat{\mathrm{a}}} = \int_0^\cdot \mu(\bar X^{\hat{\mathrm{a}}}_s,\hat{\mathrm{a}}(\bar X^{\hat{\mathrm{a}}}_s))\mathrm{d}s + \int_0^\cdot \sigma(\bar X^{\hat{\mathrm{a}}}_s)\mathrm{d}W_s.$$

Note that the existence of a solution of the above is guaranteed, upon considering another probability space and Brownian motion. Combining the above inequalities with (3.10), applied with $\gamma = \gamma_\circ$, and the second assertion of Assumption 5 with $p_{\bar X} \ge 3$ shows that $|\delta\bar\rho^*_\varepsilon| \le C'$ for some $C' > 0$ that does not depend on $\varepsilon \in (0,\varepsilon_\circ]$.

Moreover, by Assumption 6 and the same arguments as in the proof of Proposition 3.5,

$$\delta r'_\varepsilon(x,a) := \frac{1}{\varepsilon}\int_{\mathbb{R}^{d'}} [\delta\bar{\mathrm{w}}_\varepsilon(x + b_\varepsilon(x,a,e)) - \delta\bar{\mathrm{w}}_\varepsilon(x)]\,\nu(\mathrm{d}e) - \bar{\mathcal{L}}^a\delta\bar{\mathrm{w}}_\varepsilon$$

satisfies

$$|\delta r'_\varepsilon(x,\cdot)| \le \varepsilon^{\frac{\delta\gamma}{2}} C''(1+|x|^3),\ x\in\mathbb{R}^d,$$

for some $C'' > 0$ that does not depend on $\varepsilon \in (0,\varepsilon_\circ]$. Since, by construction, $\bar{\mathrm{w}}^{(1)}_\varepsilon := \bar{\mathrm{w}} + \varepsilon^{\frac{\gamma_\circ}{2}}\delta\bar{\mathrm{w}}_\varepsilon$ solves

$$\bar\rho^{*(1)}_\varepsilon = \sup_{a\in\mathbb{A}} \left[\frac{1}{\varepsilon}\int_{\mathbb{R}^{d'}} \left[\bar{\mathrm{w}}^{(1)}_\varepsilon(\cdot + b_\varepsilon(\cdot,a)) - \bar{\mathrm{w}}^{(1)}_\varepsilon\right]\nu(\mathrm{d}e) - \varepsilon^{\frac{\gamma_\circ}{2}}\delta r'_\varepsilon(\cdot,a) + r(\cdot,a)\right] \text{ on } \mathbb{R}^d,$$

the same arguments as in the proof of Theorem 3.6 then imply that $|\bar\rho_\varepsilon^{*(1)} - \rho_\varepsilon^*| \le L\varepsilon^{\frac{\gamma_\circ+\delta\gamma}{2}}$, for some $L > 0$ that does not depend on $\varepsilon \in (0, \varepsilon_\circ]$, and also lead to the last assertion of the Theorem.

□

4 Numerical resolution of the ergodic diffusive problem

The numerical resolution of (3.5) can be done by using standard finite difference schemes as explained in [26, Chapter 7]. We focus on the one-dimensional case $d = 1$ for simplicity, and also because similar schemes in higher dimension often have to be constructed on a case by case basis, see e.g. [26, Chapter 5].

Given $\kappa \in \mathbb{N}$, $\kappa \ge 3$, and $h > 0$, we consider the space grid $\mathcal{M}_h^\kappa := \{z_i := -\kappa h + (i-1)h,\ 1 \le i \le 2\kappa+1\}$. We use the notation $\mathring{\mathcal{M}}_h^\kappa := \mathcal{M}_h^\kappa \setminus \{z_1, z_{2\kappa+1}\}$ and denote by L_h^κ the collection of real-valued maps φ defined on $\mathcal{M}_h^\kappa$. For $\varphi \in \mathrm{L}_h^\kappa$, we define the usual finite (central) differences operators:

$$\Delta_h\varphi(x) := \frac{\varphi(x+h) - \varphi(x-h)}{2h},\ \Delta_h^2\varphi(x) := \frac{\varphi(x+h) + \varphi(x-h) - 2\varphi(x)}{h^2},\ x \in \mathring{\mathcal{M}}_h^\kappa,$$

and set

$$\bar{\mathcal{L}}_h^{\bar a}\varphi := \mu(\cdot, \bar a)\Delta_h\varphi + \frac{1}{2}\sigma^2\Delta_h^2\varphi,\ \bar a \in \mathbb{A}. \tag{4.1}$$

Then, we approximate the solution $(\bar\rho^*, \bar{\mathrm{w}})$ of (3.5) by a solution $(\bar\rho_h^{\kappa,*}, \bar{\mathrm{w}}_h^\kappa) \in \mathbb{R} \times \mathrm{L}_h^\kappa$ of

$$\bar\rho_h^{\kappa,*} = \sup_{\bar a \in \mathbb{A}} \left\{ \bar{\mathcal{L}}_h^{\bar a}\bar{\mathrm{w}}_h^\kappa + r(\cdot, \bar a) \right\},\ \text{on } \mathring{\mathcal{M}}_h^\kappa, \tag{4.2}$$

with a suitable reflecting boundary at z_1 and $z_{2\kappa+1}$, see below. Note that $\bar{\mathrm{w}}_h^\kappa$ is defined only up to a constant and that we can, and will, set $\bar{\mathrm{w}}_h^\kappa(0) = \bar\rho_h^{\kappa,*}\Delta t_h$ in the following, with

$$\Delta t_h := \frac{h^2}{(L_{b_1,b_2})^2}.$$

Let us now denote by A the collection of measurable maps from $\mathbb{R}$ to $\mathbb{A}$ and identify, given $\bar{\mathrm{a}} \in \mathrm{A}$, $\bar{\mathrm{w}}_h^\kappa$ and $r(\cdot, \bar{\mathrm{a}}(\cdot))$ on $\mathcal{M}_h^\kappa$ to column vectors $\bar{\mathcal{W}}_h^\kappa := (\bar{\mathrm{w}}_h^\kappa(z_i))_{1 \le i \le 2\kappa+1}$ and $\mathcal{R}(\bar{\mathrm{a}}) := (r(z_i, \bar{\mathrm{a}}(z_i)))_{1 \le i \le 2\kappa+1}$ of $\mathbb{R}^{2\kappa+1}$. Then, to solve (4.2) on $\mathcal{M}_h^\kappa$ with $\bar{\mathrm{w}}_h^\kappa(0) = \bar\rho_h^{\kappa,*}\Delta t_h$, including a suitable reflection term on the boundary $\{z_1, z_{2\kappa+1}\}$, we search for $(\bar\rho_h^{\kappa,*}, \bar{\mathcal{W}}_h^\kappa) \in \mathbb{R} \times \mathbb{R}^{2\kappa+1}$ that satisfies

$$\bar{\mathcal{W}}_h^\kappa = \sup_{\bar{\mathrm{a}} \in \mathrm{A}} \bar Q_h^{\bar{\mathrm{a}}} \left\{ \bar{\mathcal{W}}_h^\kappa - e\bar\rho_h^{\kappa,*}\Delta t_h + \mathcal{R}(\bar{\mathrm{a}})\Delta t_h \right\},\ \text{on } \mathcal{M}_h^\kappa \tag{4.3}$$

$$\bar{\mathrm{w}}_h^\kappa(0) = \bar\rho_h^{\kappa,*}\Delta t_h \tag{4.4}$$

where e is the column vector of $\mathbb{R}^{2\kappa+1}$ with all entries equal to 1, and $\bar{Q}_h^{\bar{a}} = ((\bar{Q}_h^{\bar{a}})^{i,j})_{1\le i,j\le 2\kappa+1}$ is the matrix with all entries null except for

$$(\bar{Q}_h^{\bar{a}})^{i,i-1} := q_h^-(z_i, \bar{a}(z_i))\,,\ (\bar{Q}_h^{\bar{a}})^{i,i} := q_h(z_i, \bar{a}(z_i))\,,\ \text{ and }\ (\bar{Q}_h^{\bar{a}})^{i,i+1} := q_h^+(z_i, \bar{a}(z_i))\,,$$

for $1 < i < 2\kappa + 1$, with

$$q_h := 1 - \frac{\sigma^2}{(L_{b_1,b_2})^2},\ q_h^+ := \frac{\mu h + \sigma^2}{2(L_{b_1,b_2})^2}\,,\ \text{ and }\ q_h^- := \frac{-\mu h + \sigma^2}{2(L_{b_1,b_2})^2},$$

and except for

$$\begin{aligned}(\bar{Q}_h^{\bar{a}})^{1,j} &:= (\bar{Q}_h^{\bar{a}})^{3,j} \text{ for } j = 2,3,4\\ (\bar{Q}_h^{\bar{a}})^{2\kappa+1,j} &:= (\bar{Q}_h^{\bar{a}})^{2\kappa-1,j} \text{ for } j = 2\kappa-2, 2\kappa-1, 2\kappa.\end{aligned}$$

The above scheme is of the form of [26, Chapter 7 (2.3)], and (4.2) and (4.3) are the same on $\mathring{\mathcal{M}}_h^\kappa$.

Without loss of generality, one can assume from now on that

$$L_{b_1,b_2} > \|\sigma\|_{\mathcal{C}_b^0}.$$

Consequently, recalling (i)-(ii) of Assumption 4, $\bar{Q}_h^{\bar{a}}$ defines a transition probability matrix satisfying

$$\min_{1\le i\le 2\kappa+1}\ \min_{1\vee(i-1)\le j\ne i\le(2\kappa+1)\wedge(i+1)} (\bar{Q}_h^{\bar{a}})^{i,j} =: \underline{p}_h > 0$$

whenever

$$L_{b_1,b_2}(1 + \kappa h)h < \varsigma. \tag{4.5}$$

Given $\bar{a} \in \mathrm{A}$, let $(Z_t^{x,\bar{a}})_{t\in\mathbb{N}}$ be the Markov chain starting from $x \in \mathcal{M}_h^\kappa$ and such that

$$\mathbb{P}[Z_{t+1}^{x,\bar{a}} = z_j | Z_t^{x,\bar{a}} = z_i] = (\bar{Q}_h^{\bar{a}})^{i,j},\ 1 \le i,j \le 2\kappa+1,\ t \in \mathbb{N}.$$

Then,

$$\mathbb{P}[Z_\kappa^{x,\bar{a}} = 0] \ge (\underline{p}_h)^\kappa > 0, \tag{4.6}$$

under (4.5). Further assuming that

$$(b,r)(x,\cdot) : \mathbb{A} \mapsto \mathbb{R}^2 \text{ is continuous for all } x \in \mathbb{R}, \tag{4.7}$$

it follows that the conditions of [26, Chapter 7 Theorem 2.1] hold so that $(\bar{\rho}_h^{\kappa,*}, \bar{\mathcal{W}}_h^\kappa)$ is well-defined and can be computed by using the iterative scheme of [26, Chapter 7 (2.3)].

Under the following conditions, one can exhibit an upper-bound on the convergence rate of the above numerical scheme.

Assumption 7. *There exists a function* $\bar{\xi} \in \mathcal{C}^3(\mathbb{R})$, $p_{\bar{\xi}} \geq 2$, *and constants* $C^1_{\bar{\xi}} > 0$ *and* $C^2_{\bar{\xi}} \in \mathbb{R}$ *such that* (3.3) *and* (3.4) *hold for all* $x \in \mathbb{R}^d$. *Moreover, there are constants* $L > 0$, $\Upsilon > 0$, *and* $C_\Upsilon > 0$, *such that* $|\mathrm{D}^2\bar{\xi}(x)| + |\mathrm{D}^3\bar{\xi}(x)| \leq L(1 + |x|^{p_{\bar{\xi}}-1})$ *for all* $x \in \mathbb{R}$, *and* $\mathrm{sgn}(x)\mathrm{D}\bar{\xi}(x) \geq C_\Upsilon\,|x|^{p_{\bar{\xi}}-1}$ *for all* $|x| \geq \Upsilon$, *where* $\mathrm{sgn}(\cdot)$ *is the sign function.*

Proposition 4.1. *Let Assumptions 4, 5 and 7 hold with* $p_{\bar{\xi}} \geq 3$. *Assume further that* (4.7) *is satisfied. Then, there exists* $L_{\mathrm{num}} > 0$ *and* $h_{\mathrm{num}} > 0$ *such that, for all* $(h, \kappa) \in (0, h_{\mathrm{num}}) \times \mathbb{N}$, *satisfying* (4.5), $\kappa h^2 \leq 1$ *and* $(\kappa - 3)h \geq \Upsilon$, *we have*

$$|\bar{\rho}^{\kappa,*}_h - \bar{\rho}^*| \leq L_{\mathrm{num}}(h^\gamma + h^{-1}|\kappa h|^{-|p_{\bar{\xi}}-1|}).$$

In particular,

$$|\bar{\rho}^{\kappa,*}_h - \rho^*_\varepsilon| \leq L_{\mathrm{num}}(h^\gamma + h^{-1}|\kappa h|^{-|p_{\bar{\xi}}-1|}) + \varepsilon^{\frac{\gamma}{2}} L^\gamma_{\delta\rho} \text{ for all } \varepsilon \in (0,1).$$

Proof. Given $\bar{\mathrm{a}} \in \mathrm{A}$ and $x \in \mathcal{M}^\kappa_h$, let $\tilde{X}^{x,\bar{\mathrm{a}}}$ be the pure jump continuous time Markov chain defined by a sequence of jump times $(\tau_n)_{n\geq 1}$ such that the increments $(\tau_{n+1} - \tau_n)_{n\geq 0}$ (with the convention $\tau_0 = 0$) are independent and identically distributed according to the exponential law of mean Δt_h and such that, for $n \geq 1$,

$$\mathbb{P}[\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_n} = z_i | (\tilde{X}^{x,\bar{\mathrm{a}}}_0, \tau_0), \ldots, (\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_{n-1}}, \tau_{n-1}), \tau_n] = (\bar{Q}^{\bar{\mathrm{a}}}_h)^{i, j(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_{n-1}})},$$

with

$$j(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_{n-1}}) \in \mathbb{N} \text{ s.t. } z_{j(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_{n-1}})} = \tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_{n-1}},$$

and $\tilde{X}^{x,\bar{\mathrm{a}}} = \tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_{n-1}}$ on $[\tau_{n-1}, \tau_n)$.

1. First note that, by construction, $\bar{\mathrm{w}}^\kappa_h$ is bounded on the finite set $\mathcal{M}^\kappa_h$. Then, by the arguments in the proof of Lemma A.4 and (4.3), we have

$$\bar{\rho}^{\kappa,*}_h = \sup_{\bar{\mathrm{a}} \in \mathrm{A}} \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T r(\tilde{X}^{x,\bar{\mathrm{a}}}_s, \bar{\mathrm{a}}(\tilde{X}^{x,\bar{\mathrm{a}}}_s)) ds\right]. \tag{4.8}$$

2. We now prove that there exists $C^{1'}_{\bar{\xi}}, C^{2'}_{\bar{\xi}}, h_{\mathrm{num}} > 0$ such that, for all $x \in \mathbb{R}$, $\bar{\mathrm{a}} \in \mathrm{A}$, $0 < h \leq h_{\mathrm{num}}$ and κ such that (4.5) holds, $\kappa h^2 \leq 1$ and $(\kappa - 3)h \geq \Upsilon$, we have

$$\mathbb{E}[|\tilde{X}^{x,\bar{\mathrm{a}}}_t|^{p_{\bar{\xi}}}] \leq C^2_{\bar{\xi}} \left\{ e^{-C^{1'}_{\bar{\xi}} t} C^2_{\bar{\xi}} |x|^{p_{\bar{\xi}}} + \frac{C^{2'}_{\bar{\xi}}}{C^{1'}_{\bar{\xi}}} (1 - e^{-C^{1'}_{\bar{\xi}} t}) \right\}, \; t \geq 0. \tag{4.9}$$

Using Assumption 7 and Assumption 4, and Taylor expansions of first and second orders, we first deduce that, for $x \in \mathring{\mathcal{M}}^\kappa_h$,

$$\mathrm{D}\bar{\xi}(x)\mu(x, \bar{\mathrm{a}}(x)) + \frac{1}{2}\sigma^2(x)\mathrm{D}^2\bar{\xi}(x) = \frac{1}{\Delta t_h}\mathbb{E}[\bar{\xi}(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_1}) - \bar{\xi}(x)] - c(x)h,$$

in which $|c(x)| \leq C(1+|x|^{p_{\bar{\xi}}}) \leq C(1+C^2_{\bar{\xi}}\bar{\xi}(x))$ for some $C > 0$ independent on $x \in \mathbb{R}$, $\bar{\mathrm{a}} \in \mathrm{A}$, h and κ. Using (3.3), this implies that, for $x \in \mathring{\mathcal{M}}^{\kappa}_h$,

$$\frac{1}{\Delta t_h}\mathbb{E}[\bar{\xi}(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_1}) - \bar{\xi}(x)] \leq -\left(C^1_{\bar{\xi}} - hCC^2_{\bar{\xi}}\right)\bar{\xi}(x) + C^2_{\bar{\xi}} + Ch. \tag{4.10}$$

Consider now the case $x = z_1$, the other boundary case, $x = z_{2\kappa+1}$, being symmetric. Let Ξ be a discrete random variable taking value $k \in \{1,2,3\}$ with probability $(\bar{Q}^{\bar{\mathrm{a}}}_h)^{1,k}$. Using Assumption 7 and (3.4), we obtain that, for some random variable $\hat{z}_\Xi$ such that $\hat{z}_\Xi \in [z_1, z_1 + \Xi h]$ a.s.,

$$\begin{aligned}
\frac{1}{\Delta t_h}\mathbb{E}[\bar{\xi}(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_1}) - \bar{\xi}(x)] &= \frac{1}{\Delta t_h}\mathbb{E}[\xi(z_1 + \Xi h) - \xi(z_1)] \\
&= \frac{1}{\Delta t_h}\mathbb{E}[\Xi h \mathrm{D}\xi(\hat{z}_\Xi)] \\
&\leq -\frac{L^2_{b_1,b_2}}{h}C_\Upsilon \mathbb{E}[\Xi\,|\hat{z}_\Xi|^{p_{\bar{\xi}}-1}] \\
&\leq -L^2_{b_1,b_2}C_\Upsilon \mathbb{E}[\kappa h\,|\hat{z}_\Xi|^{p_{\bar{\xi}}-1}] \\
&\leq -C'\bar{\xi}(x)
\end{aligned} \tag{4.11}$$

when $|(\kappa-3)h| \geq \Upsilon$ and $\kappa h^2 \leq 1$, in which $C' > 0$ does not depend on κ nor h. The above also holds with $z_{2\kappa+1}$ in place of z_1. Combining (4.10)-(4.11), we obtain

$$\frac{1}{\Delta t_h}\mathbb{E}[\bar{\xi}(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_1}) - \bar{\xi}(x)] \leq -\left((C^1_{\bar{\xi}} - hCC^2_{\bar{\xi}}) \wedge C'\right)\bar{\xi}(x) + C^2_{\bar{\xi}} + Ch \leq -C^{1'}_{\bar{\xi}}\bar{\xi}(x) + C^{2'}_{\bar{\xi}},$$

for all $x \in \mathcal{M}^\kappa_h$, whenever $h \leq h_{\mathrm{num}}$, in which $C^{1'}_{\bar{\xi}}, C^{2'}_{\bar{\xi}}, h_{\mathrm{num}} > 0$ do not depend on κ nor h. One can then argue as in the proof of Lemma A.2 to obtain (4.9).

3. From now on, we denote by $C > 0$ a generic constant, which may change from line to line, but does not depend on κ or h. We now appeal to (3.6) and the Lipschitz continuity of (μ, σ^2), and use the fact that $h \leq 1$ to deduce by consistency arguments that, for $x \in \mathcal{M}^\kappa_h$,

$$\bar{\mathcal{L}}^{\bar{\mathrm{a}}(x)}\bar{\mathrm{w}}(x) = \frac{1}{\Delta t_h}\mathbb{E}[\bar{\mathrm{w}}(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_1}) - \bar{\mathrm{w}}(x)] + \delta r_h(x, \mathrm{a}(x))$$

in which

$$|\delta r_h(x, \bar{\mathrm{a}}(x))| \leq C((1+|x|)h + h^\gamma)(1+|x|) + Ch^{-1}(1+|x|)1_{\{|x|=\kappa h\}}.$$

The above combined with (3.5) implies that

$$\bar{\rho}^* = \frac{1}{\Delta t_h}\sup_{\bar{a}\in \mathrm{A}}\mathbb{E}\left[\bar{\mathrm{w}}(\tilde{X}^{x,\bar{a}}_{\tau_1}) - \bar{\mathrm{w}}(x) + \{r(x,\bar{a}) + \delta r_h(x,\bar{a})\}\Delta t_h\right] \quad \text{for } x \in \mathcal{M}^\kappa_h.$$

Arguing again as in the proof of Lemma A.4, recalling 4.8, and combining (4.9) with Hölder's and Markov's inequality, we deduce that we can find $C, C', C'' > 0$ such that

$$
\begin{aligned}
|\bar\rho_h^{\kappa,*} - \bar\rho^*| &\le \sup_{\bar{\mathrm{a}}\in \mathrm{A}} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T |\delta r_h|(\tilde X_s^{0,\bar{\mathrm{a}}}, \bar{\mathrm{a}}(\tilde X_s^{0,\bar{\mathrm{a}}}))\left(1_{\{|\tilde X_s^{0,\bar{\mathrm{a}}}|<\kappa h\}} + 1_{\{|\tilde X_s^{0,\bar{\mathrm{a}}}|=\kappa h\}}\right) ds\right] \\
&\le \sup_{\bar{\mathrm{a}}\in \mathrm{A}} \limsup_{T\to\infty} \frac{C}{T}\int_0^T \mathbb{E}\left[(h + h|\tilde X_s^{0,\bar{\mathrm{a}}}|^2 + h^\gamma |\tilde X_s^{0,\bar{\mathrm{a}}}|) + h^{-1}(1+|\tilde X_s^{0,\bar{\mathrm{a}}}|)1_{\{|\tilde X_s^{0,\bar{\mathrm{a}}}|=\kappa h\}}\right]\mathrm{d}s \\
&\le C'(h+h^\gamma) + \sup_{\bar{\mathrm{a}}\in \mathrm{A}} \limsup_{T\to\infty} h^{-1}\frac{C}{T}\int_0^T \mathbb{E}[(1+|\tilde X_s^{0,\bar{\mathrm{a}}}|)^{p_{\bar\xi}}]^{\frac{1}{p_{\bar\xi}}}\left(\frac{\mathbb{E}[|\tilde X_s^{0,\bar{\mathrm{a}}}|^{p_{\bar\xi}}]}{(\kappa h)^{p_{\bar\xi}}}\right)^{\frac{p_{\bar\xi}-1}{p_{\bar\xi}}} \mathrm{d}s \\
&\le C''(h^\gamma + h^{-1}|\kappa h|^{-|p_{\bar\xi}-1|}) .
\end{aligned}
$$

It remains to appeal to Theorem 3.6 to complete the proof. □

One can also construct from the above scheme an almost optimal control for the original pure jump problem. We first extend the definition of $\bar{\mathrm{w}}_h^\kappa$ by setting

$$
\bar{\mathrm{w}}_h^\kappa(x) := \bar{\mathrm{w}}_h^\kappa(\Pi_h^\kappa(x)),\ x\in\mathbb{R},
$$

in which $\Pi_h^\kappa(x) := \inf\{z \in \mathcal{M}_h^\kappa : z \ge x\} \wedge z_{2\kappa+1}$. Then, we let ϕ be a smooth density function with support $(-1,1)$ such that $\|\phi\|_{C_b^2} \le 1$. Given $n\ge 1$, let

$$
\bar{\mathrm{w}}_h^{\kappa,n}(x) := \int (\bar{\mathrm{w}}_h^\kappa(y) - \bar\rho_h^{\kappa,*}\Delta t_h)\phi(n(y-x))\mathrm{d}y,\ x\in\mathbb{R},
$$

with the convention that $\bar{\mathrm{w}}_h^\kappa = \bar{\mathrm{w}}_h^\kappa(z_1) - \bar\rho_h^{\kappa,*}\Delta t_h$ on $(-\infty, z_1)$ and $\bar{\mathrm{w}}_h^\kappa = \bar{\mathrm{w}}_h^\kappa(z_{2\kappa+1}) - \bar\rho_h^{\kappa,*}\Delta t_h$ on $(z_{2\kappa+1},\infty)$.

Let $\bar{\mathrm{a}}_h^{\kappa,n} \in \mathrm{A}$ be such that

$$
\bar{\mathrm{a}}_h^{\kappa,n} \in \arg\max_{a\in\mathbb{A}}[\bar{\mathcal{L}}^a \bar{\mathrm{w}}_h^{\kappa,n} + r(\cdot,a)], \quad \text{on } \mathbb{R}, \tag{4.12}
$$

and set $\hat\alpha_h^{\kappa,n} = \bar{\mathrm{a}}_h^{\kappa,n}(\hat X^{\kappa,n,h})$ with

$$
\hat X_\cdot^{\kappa,n,h} = \int_0^\cdot \int_{\mathbb{R}^{d'}} b_\varepsilon(\hat X_{s-}^{\kappa,n,h}, \bar{\mathrm{a}}_h^{\kappa,n}(\hat X_{s-}^{\kappa,n,h}), e)N(\mathrm{d}e, \mathrm{d}s) .
$$

The control $\bar{\mathrm{a}}_h^{\kappa,n}$ can be computed numerically at low cost, e.g. via first order conditions; Proposition 4.2 gives the associated error bounds. This approach seems novel in the literature, and is of independent methodological interest.

Proposition 4.2. *Let the conditions of Proposition 4.1 hold. Then, there exists $C > 0$ such that, for all $K > 0$, $n \geq 1$ and $\varepsilon \in (0,1)$,*

$$|\rho^*_\varepsilon(0, \hat{\alpha}^{\kappa,n}_h) - \rho^*_\varepsilon| \leq C \left(n^{-\gamma} + \varepsilon^{\frac{\gamma}{2}} + n \sup_{x \in B_K(0)} |\bar{\mathrm{w}}^\kappa_h - \bar{\rho}^{\kappa,*}_h \Delta t_h - \bar{\mathrm{w}}|(x) + nK^{-1} \right). \tag{4.13}$$

If, moreover,
(i) *σ is constant,*
(ii) *there exists $c > 0$ such that $\mu(x) - \mu(x') \leq -c(x - x')$ if $x \geq x' \in \mathbb{R}$,*
(iii) *there exists $R > 0$ such that*

$$\sup_{|x|>R} \sup_{\bar{a} \in \mathbb{A}} \mu(x, \bar{a})x < -\frac{1}{2}\sigma^2, \tag{4.14}$$

then

$$\limsup_{h \to 0} \sup_{x \in B_K(0)} |\bar{\mathrm{w}}^{\kappa_h}_h - \bar{\rho}^{\kappa_h,*}_h \Delta t_h - \bar{\mathrm{w}}|(x) = 0$$

for any family $(\kappa_h)_{h>0} \subset \mathbb{N}$ such that $\lim_{h \downarrow 0} \kappa_h h^2 = 0$ and $\lim_{h \downarrow 0} \kappa_h h^{\frac{p_{\bar{\xi}}}{p_{\bar{\xi}}-1}} = \infty$.

Proof. 1. We first note that

$$\mathrm{D}\bar{\mathrm{w}}^{\kappa,n}_h(x) = \int \mathrm{D}\bar{\mathrm{w}}(y)\phi(n(y-x))dy - \int (\bar{\mathrm{w}}^\kappa_h - \bar{\rho}^{\kappa,*}_h \Delta t_h - \bar{\mathrm{w}})(y) n\phi'(n(y-x))dy$$
$$\mathrm{D}^2\bar{\mathrm{w}}^{\kappa,n}_h(x) = \int \mathrm{D}^2\bar{\mathrm{w}}(y)\phi(n(y-x))dy + \int (\bar{\mathrm{w}}^\kappa_h - \bar{\rho}^{\kappa,*}_h \Delta t_h - \bar{\mathrm{w}})(y) n^2\phi''(n(y-x))dy, \ x \in \mathbb{R},$$

in which ϕ' and ϕ'' stand for the first and second order derivatives of ϕ. Hence, it follows from (3.6), (i) of Assumption 4 and (3.5) that

$$\begin{aligned} \bar{\mathcal{L}}^{\bar{\mathrm{a}}^{\kappa,n}_h(\cdot)} \bar{\mathrm{w}}^{\kappa,n}_h + r(\cdot, \bar{\mathrm{a}}^{\kappa,n}_h(\cdot)) &= \max_{a \in \mathbb{A}} [\bar{\mathcal{L}}^a \bar{\mathrm{w}}^{\kappa,n}_h + r(\cdot, a)] \\ &\geq \max_{a \in \mathbb{A}} [\bar{\mathcal{L}}^a \bar{\mathrm{w}} + r(\cdot, a)] - \frac{1}{2}\delta r^{\kappa,n}_h \\ &= \bar{\rho}^* - \frac{1}{2}\delta r^{\kappa,n}_h \end{aligned}$$

in which $\delta r^{\kappa,n}_h$ satisfies, for some $C > 0$ independent on n, κ and h,

$$0 \leq \delta r^{\kappa,n}_h(x) \leq C(1 + |x|) \left[n^{-\gamma} + 2n^2 \int_{B_{\frac{1}{n}}(x)} |\bar{\mathrm{w}}^\kappa_h - \bar{\rho}^{\kappa,*}_h \Delta t_h - \bar{\mathrm{w}}|(y) dy \right].$$

Similarly,

$$\begin{aligned} \bar{\rho}^* - \frac{1}{2}\delta r^{\kappa,n}_h &\leq \bar{\mathcal{L}}^{\bar{\mathrm{a}}^{\kappa,n}_h(\cdot)} \bar{\mathrm{w}}^{\kappa,n}_h + r(\cdot, \bar{\mathrm{a}}^{\kappa,n}_h(\cdot)) \\ &\leq \bar{\mathcal{L}}^{\bar{\mathrm{a}}^{\kappa,n}_h(\cdot)} \bar{\mathrm{w}} + r(\cdot, \bar{\mathrm{a}}^{\kappa,n}_h(\cdot)) + \frac{1}{2}\delta r^{\kappa,n}_h. \end{aligned}$$

Recalling (3.9) and Theorem 3.6, we deduce that

$$\rho^*_\varepsilon - \varepsilon^{\frac{\gamma}{2}} L^\gamma_{\delta\rho} \le \frac{1}{\varepsilon}\int_{\mathbb{R}^{d'}} [\bar{\mathrm{w}}(x + b_\varepsilon(x, \bar{\mathrm{a}}^{\kappa,n}_h(x), e)) - \bar{\mathrm{w}}(x)]\,\nu(\mathrm{d}e) + r(x, \bar{\mathrm{a}}^{\kappa,n}_h(x)) + \delta r^{\kappa,n}_h(x) - \delta r_\varepsilon(x, \bar{\mathrm{a}}^{\kappa,n}_h(x))$$

for all $x \in \mathbb{R}$. We then deduce (4.13) by the same arguments as in the proof of Theorem 3.6.

2. It remains to prove the second assertion of the proposition. For ease of notations, we do not write the dependence of κ with respect to h, but we keep in mind that we can consider h small and that κ can be adjusted as soon as the following results can apply to sequences such that $\lim_{h\downarrow 0} \kappa_h h^2 = 0$ and $\lim_{h\downarrow 0} \kappa_h h^{\frac{p_{\bar{\xi}}}{p_{\bar{\xi}}-1}} = \infty$.

2.a. We first prove that $[\bar{\mathrm{w}}^\kappa_h]_{\mathcal{C}^0_{\mathrm{lin}}(\mathcal{M}^\kappa_h)}$ does not depend on κ nor h. To this end, we adapt the arguments of Lemma A.1 and Theorem 2.3, and actually prove that it is Lipschitz, uniformly in κ and h. Let $(\xi_j)_{j\ge 1}$ be a sequence of i.i.d. random variables following the uniform distribution on $[0,1]$ and let $(\tau_n)_{n\ge 1}$ be a random sequence, independent of $(\xi_j)_{j\ge1}$, such that the increments $(\tau_{n+1} - \tau_n)_{n\ge 0}$ (with the convention $\tau_0 = 0$) are independent and identically distributed according to the exponential law of mean Δt_h. Given $(x, \bar{a}, y) \in \mathbb{R} \times \mathbb{A} \times \mathbb{R}$, set

$$\Delta\mathrm{x}(x, \bar{a}, y) := h 1_{\{y \le q^+_h(x,\bar{a})\}} - h 1_{\{q^+_h(x,a) < y \le (q^+_h + q^-_h)(x,\bar{a})\}}, \text{ if } x \in \mathring{\mathcal{M}}^\kappa_h,$$

and

$$\Delta\mathrm{x}(z_1, \bar{a}, y) := 2h + \Delta\mathrm{x}(z_3, \bar{a}, y)\ ,\ \Delta\mathrm{x}(z_{2\kappa+1}, \bar{a}, y) := -2h + \Delta\mathrm{x}(z_{2\kappa-1}, \bar{a}, y).$$

Let $\check{\mathcal{A}}$ denote the collection of $\mathbb{A}$-valued processes that are predictable with respect to the filtration generated by $t \mapsto \sum_{i\ge 1} \xi_i 1_{\{\tau_i \le t\}}$ Given $\check{\alpha} \in \check{\mathcal{A}}$ and $x \in \mathcal{M}^\kappa_h$, let $\tilde{X}^{x,\check{\alpha}}$ be the pure jump continuous time Markov chain defined by

$$\tilde{X}^{x,\check{\alpha}}_{\tau_{i+1}} = \tilde{X}^{x,\check{\alpha}}_{\tau_i} + \Delta\mathrm{x}(\tilde{X}^{x,\bar{\mathrm{a}}}_{\tau_i}, \check{\alpha}_{\tau_i}, \xi_{i+1})$$

and $\tilde{X}^{x,\check{\alpha}} = \tilde{X}^{x,\check{\alpha}}_{\tau_i}$ on $[\tau_i, \tau_{i+1})$, $i \ge 0$. It has the same law as the process introduced at the beginning of the proof of Proposition 4.1, and in particular

$$\bar{\rho}^{\kappa,*}_h = \sup_{\check{\alpha}\in\check{\mathcal{A}}} \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T r(\tilde{X}^{x,\bar{\mathrm{a}}}_s, \check{\alpha}_s) ds\right].$$

We set

$$\check{V}_\lambda(x) := \sup_{\check{\alpha}\in\check{\mathcal{A}}} \mathbb{E}\left[\int_0^\infty e^{-\lambda s} r(\check{X}^{x,\check{\alpha}}_s, \check{\alpha}_s) ds\right].$$

2.a.(i) We first need to obtain contraction estimates similar to the ones obtained in the proof of Lemma A.1. We restrict for the moment to the case where the distance between the initial data are

in $2h\mathbb{Z}$.
Let us first observe that, for h small enough for condition (4.5) to hold, we have $q_h^+(x) < (q_h^+ + q_h^-)(x') = \sigma^2/(L_{b_1,b_2})^2 =: m$ and conversely. Although recall that, by Assumption, $\mu(x) - \mu(x') \leq -c(x-x') \leq 0$ and therefore $q_h^+(x) \leq q_h^+(x')$ if $x \geq x' \in \mathbb{R}$. Keeping this in mind, direct computations show that, if $x - x' \in 2h\mathbb{Z}$ and $x, x' \in \overset{\circ}{\mathcal{M}}{}_h^\kappa$, and $\bar{a} \in \mathbb{A}$, then

$$
\begin{aligned}
&\frac{1}{\Delta t_h}\mathbb{E}\left[|x + \Delta \mathrm{x}(x,\bar{a},\xi_1) - x' - \Delta \mathrm{x}(x',\bar{a},\xi_1)| - |x - x'|\right] \\
&= (|x - x' + 2h| - |x - x'|)\,\frac{q_h^+(x) \wedge m - q_h^+(x) \wedge q_h^+(x')}{\Delta t_h} \\
&\quad + (|x - x' - 2h| - |x - x'|)\,\frac{q_h^+(x') \wedge m - q_h^+(x') \wedge q_h^+(x)}{\Delta t_h} \\
&= (|x - x' + 2h| - |x - x'|)\,\frac{\mu(x) - \mu(x) \wedge \mu(x')}{2h} + (|x - x' - 2h| - |x - x'|)\,\frac{\mu(x') - \mu(x) \wedge \mu(x')}{2h} \\
&= \left[1_{\{x \geq x'\}}(\mu(x) - \mu(x')) + 1_{\{x' > x\}}(\mu(x') - \mu(x))\right] \\
&\leq -c|x - x'|\,.
\end{aligned}
$$

On the other hand, if $x = z_1$, $x' \in \overset{\circ}{\mathcal{M}}{}_h^\kappa$ and $z_1 - x' \in 2h\mathbb{Z}$, then

$$
\begin{aligned}
&\frac{1}{\Delta t_h}\mathbb{E}\left[|x + \Delta \mathrm{x}(x,\bar{a},\xi_1) - x' - \Delta \mathrm{x}(x',\bar{a},\xi_1)| - |x - x'|\right] \\
&= \frac{1}{\Delta t_h}\left(-2hq_h^+(x',\bar{a}) - 4h(q_h^+(z_3,\bar{a})) - q_h^+(x',\bar{a})) - 2h(1 - q_h^+(z_3,\bar{a}))\right) 1_{\{x' \geq z_1 + 4h\}} \\
&\quad - \frac{1}{\Delta t_h}\,|x - x'|\, 1_{\{x' < z_1 + 4h\}} \\
&\leq -c|z_3 - x'| \\
&\leq -\frac{c}{2}|x - x'|\,.
\end{aligned}
$$

In the case, $x' = z_{2\kappa+1}$ (with $\kappa \geq 4$ which we can assume here w.l.o.g.), then

$$
\begin{aligned}
&\frac{1}{\Delta t_h}\mathbb{E}\left[|x + \Delta \mathrm{x}(x,\bar{a},\xi_1) - x' - \Delta \mathrm{x}(x',\bar{a},\xi_1)| - |x - x'|\right] \\
&= \frac{1}{\Delta t_h}\left[-4hq_h^+(x',\bar{a}) - 6h(q_h^+(z_3,\bar{a})) - q_h^+(z_{2\kappa-1},\bar{a})) - 4h(1 - q_h^+(z_3,\bar{a})))\right] \\
&\leq -c|z_3 - z_{2\kappa-1}| \\
&\leq -\frac{c}{2}|x - x'|,
\end{aligned}
$$

in which the last inequalities follows from the fact that $\kappa \geq 4$. A similar analysis can be done when $x' = z_{2\kappa+1}$ and $x \in \mathcal{M}_h^\kappa$. The above implies that, for h small enough,

$$
\frac{1}{\Delta t_h}\mathbb{E}\left[|x + \Delta \mathrm{x}(x,\bar{a},\xi_1) - x' - \Delta \mathrm{x}(x',\bar{a},\xi_1)| - |x - x'|\right] \leq -\frac{c}{2}|x - x'| \,\forall\, x, x' \in \mathcal{M}_h^\kappa \text{ s.t. } x - x' \in 2h\mathbb{Z},
$$

which is the required contraction property, whenever $x - x' \in 2h\mathbb{Z}$. The key property is that $\check{X}^{x,\check{\alpha}} - \check{X}^{x',\check{\alpha}}$ remains in $2h\mathbb{Z}$ whenever $x - x' \in 2h\mathbb{Z}$ (by the above calculations jumps of $\check{X}^{x,\check{\alpha}} - \check{X}^{x',\check{\alpha}}$ lie in $\{-6h, -4h, -2h, 0, 2h, 4h, 6h\}$). Then, the same arguments as in the proof of Lemma A.1 imply that one can find $\check{L} > 0$, that only depends on c, such that

$$|\check{V}_\lambda(x) - \check{V}_\lambda(x')| \le \check{L}|x - x'|, \quad \text{for } x, x' \in \mathcal{M}_h^\kappa \text{ s.t. } x - x' \in (2h\mathbb{Z}). \tag{4.15}$$

In particular,

$$|\check{V}_\lambda(x) - \check{V}_\lambda(0)| \le \check{L}|x|, \quad \text{for } x \in \mathcal{M}_h^\kappa \cap (2h\mathbb{Z}). \tag{4.16}$$

2.a.(ii) We now turn to the general case in which the distance between the initial data does not belong to $2h\mathbb{Z}$. Take $x \in \{x_\circ - h, x_\circ + h\} \cap \mathring{\mathcal{M}}_h^\kappa$, for some $x_\circ \in \mathcal{M}_h^\kappa \cap (2h\mathbb{Z})$. Let θ_1 be the first time at which $|\check{X}_{\theta_1}^{x,\check{\alpha}} - x| = h$. By the dynamic programming principle,

$$\begin{aligned}|\check{V}_\lambda(x) - \check{V}_\lambda(x_\circ)| \le& \sup_{\check{\alpha} \in \check{\mathcal{A}}} \mathbb{E}\left[\frac{1}{\lambda}(1 - e^{-\lambda\theta_1})\|r\|_{C_b^0} + e^{-\lambda\theta_1}|\check{V}_\lambda(\check{X}_{\theta_1}^{x,\check{\alpha}}) - \check{V}_\lambda(x_\circ)|\right] \\ &+ \mathbb{E}\left[(1 - e^{-\lambda\theta_1})|\check{V}_\lambda(x_\circ)|\right]\end{aligned}$$

in which $\check{X}_{\theta_1}^{x,\check{\alpha}} - x_\circ \in \{-2h, 0, 2h\}$ and therefore $|\check{V}_\lambda(\check{X}_{\theta_1}^{x,\check{\alpha}}) - \check{V}_\lambda(x_\circ)| \le 2\check{L}|h|$ by (4.15). By exhaustive enumeration, one can compute

$$\begin{aligned}\mathbb{E}[e^{-\lambda\theta_1}] &= \sum_{k\ge1} q_h(x)^{k-1}(1 - q_h(x)\left(\int_0^\infty e^{-\lambda y}\frac{1}{\Delta t_h}e^{-\Delta t_h^{-1}y}dy\right)^k \\ &= \sum_{k\ge1} q_h(x)^{k-1}(1 - q_h(x))\,(\lambda\Delta t_h + 1)^{-k} \\ &= (\lambda\Delta t_h + 1)^{-1}\,(1 - q_h(x))\frac{\lambda\Delta t_h + 1}{\lambda\Delta t_h + 1 - q_h(x_\circ)} \\ &= \frac{1 - q_h(x)}{\lambda\Delta t_h + 1 - q_h(x)} \le 1.\end{aligned}$$

Since $1 - q_h(x) \ge (\sigma/L_{b_1,b_2})^2 \ge (\varsigma/L_{b_1,b_2})^2 > 0$ for all h, by Assumption 4, the above implies that, for some $C > 0$, independent on λ, κ and h,

$$\begin{aligned}|\check{V}_\lambda(x) - \check{V}_\lambda(x_\circ)| \le& \sup_{a\in\mathbb{A}} \mathbb{E}\left[\frac{\Delta t_h}{\lambda\Delta t_h + 1 - q_h(x)}\|r\|_{C_b^0} + 2\frac{1 - q_h(x)}{\lambda\Delta t_h + 1 - q_h(x)}\check{L}|h|\right] \\ &+ \mathbb{E}\left[\frac{\lambda\Delta t_h}{\lambda\Delta t_h + 1 - q_h(x)}|\check{V}_\lambda(x_\circ)|\right] \\ =& C(\Delta t_h + h + \lambda\Delta t_h|\check{V}_\lambda(x_\circ)|).\end{aligned}$$

Note that $\lambda \check{V}_\lambda$ is bounded by $\|r\|_{\mathcal{C}^0_b} < \infty$, while $\Delta t_h \le h \le |x|$, for $x \neq 0$ and h small enough. Since $x_\circ \in \mathcal{M}^\kappa_h \cap (2h\mathbb{Z})$, the above, combined with (4.16), thus shows that

$$|\check{V}_\lambda(x) - \check{V}_\lambda(0)| \le \check{L}'|x|,\ \forall\, x \in \mathring{\mathcal{M}}^\kappa_h, \tag{4.17}$$

for some $\check{L}' > 0$ that does not depend on λ, h nor κ. In the case where $x \in \{z_1, z_{2\kappa+1}\}$, we can conduct a similar analysis by considering the first time θ_1 at which $\check{X}^{x,\check{\alpha}}$ jumps. In this case, $\check{X}^{x,\check{\alpha}}_{\theta_1} \in \mathring{\mathcal{M}}^\kappa_h$ by construction and $|\check{X}^{x,\check{\alpha}}_{\theta_1} - x_\circ| \le 2h$. Given (4.15), we retrieve a similar estimate as (4.17). Hence,

$$|\check{V}_\lambda(x) - \check{V}_\lambda(0)| \le \check{L}'|x|,\ \forall\, x \in \mathcal{M}^\kappa_h, \tag{4.18}$$

for some $\check{L}' > 0$ that does not depend on λ, h nor κ.

2.a.(iii) We are now in position to show that $[\bar{\mathrm{w}}^\kappa_h]_{\mathcal{C}^0_{\rm lin}(\mathcal{M}^\kappa_h)}$ does not depend on κ nor h. Using (4.18) and the arguments of Lemma A.3, we obtain that, after possibly passing to a subsequence, $(\check{V}_\lambda - \check{V}_\lambda(0))_{\lambda>0}$ converges pointwise, as $\lambda \to 0$, to $\bar{\mathrm{w}}^\kappa_h - \bar{\rho}^{\kappa,*}_h \Delta t_h$ and that the latter satisfies

$$|\bar{\mathrm{w}}^\kappa_h(x) - \bar{\rho}^{\kappa,*}_h \Delta t_h| \le \check{L}'|x|,\ x \in \mathcal{M}^\kappa_h. \tag{4.19}$$

2.b. To complete the proof, it remains to appeal to the stability of viscosity solutions, and use comparison results in the class of semi-continuous super/sub-solutions with linear growth. Let $(\kappa_h)_{h>0}$ be as in the statement of the Proposition. By (4.19), $(\bar{\mathrm{w}}^{\kappa_h}_h - \bar{\rho}^{\kappa_h,*}_h \Delta t_h)_{h>0}$ admits locally bounded relaxed semi-limits

$$\bar{\mathrm{w}}^{\infty*}_0(x) := \limsup_{x'\to x,\, h\downarrow 0} \bar{\mathrm{w}}^{\kappa_h}_h(x') - \bar{\rho}^{\kappa_h,*}_h \Delta t_h\ ,\ \bar{\mathrm{w}}^{\infty}_{0*}(x) := \liminf_{x'\to x,\, h\downarrow 0} \bar{\mathrm{w}}^{\kappa_h}_h(x') - \bar{\rho}^{\kappa_h,*}_h \Delta t_h.$$

which take the value 0 at 0, recall (4.4), and have linear growth. We can then use that (4.2) is satisfied on any bounded set B for h small enough with respect to B, Proposition 4.1 and standard stability arguments for viscosity solutions, see e.g. [7, Theorem 2.1] for the case of numerical schemes or [5, Theorem 6.2, p.77] for general stability results, to deduce that $\bar{\mathrm{w}}^\infty_{0*}$ and $\bar{\mathrm{w}}^{\infty*}_0$ are respectively viscosity super- and subsolutions of (3.5) in the classical sense of viscosity solutions for elliptic equations (the constant $\bar{\rho}^*$ being already given). We claim that $\bar{\mathrm{w}}^{\infty*}_0 = \bar{\mathrm{w}} + g$ for some $g \in \mathbb{R}$. Then, we will deduce that $g = \bar{\mathrm{w}}^{\infty*}_0(0) - \bar{\mathrm{w}}(0) = 0$ by construction. The same argument can be used to prove that $\bar{\mathrm{w}}^\infty_{0*} = \bar{\mathrm{w}}$. To prove the above, we follow the arguments of [6, Proof of Theorem 3.1]. We first fix $R > 0$ and let $B_R := B_R(0)$ be the open ball of radius R centered at 0. Set $g := \max_{\partial B_R}(\bar{\mathrm{w}}^{\infty*}_0 - \bar{\mathrm{w}})$. Since $\Phi := \bar{\mathrm{w}}^{\infty*}_0 - \bar{\mathrm{w}} - g$ has linear growth, see (3.6) and above, we can fix $\iota > 0$, independently of R, such that $x \mapsto \Phi(x) - \iota|x|^2$ has a maximum point $\hat{x}_R$ on $(B_R)^c$. If $\sup_{(B_R)^c} \Phi > 0$, then, for $\iota > 0$ small enough, we have $\Phi(\hat{x}_R) - \iota|\hat{x}_R|^2 > 0$ and therefore $\hat{x}_R$ lies in the interior of $(B_R)^c$. We now use the subsolution property of $\bar{\mathrm{w}}_0^{\infty *}$ and the fact that $\bar{\mathrm{w}}$ is a smooth solution of (3.5) to obtain

$$\begin{aligned}0 \leq& \sup_{\bar{a}\in\mathbb{A}} \left\{\bar{\mathcal{L}}^{\bar{a}}\bar{\mathrm{w}}(\hat{x}_R) + r(\hat{x}_R,\bar{a}) - \bar{\rho}^* + \iota\left(2\mu(\hat{x}_R,\bar{a})\hat{x}_R + \sigma^2\right)\right\}\\ \leq& \iota \sup_{\bar{a}\in\mathbb{A}} \left\{2\mu(\hat{x}_R,\bar{a})\hat{x}_R + \sigma^2\right\}.\end{aligned}$$

Using (4.14), we get a contradiction for R large enough. This shows that $\sup_{(B_R)^c}\Phi \leq 0$. Now the fact that $\max_{B_R\cup\partial B_R}\Phi = 0$ follows by the maximum principle applied to (3.5) on B_R with Dirichlet boundary conditions on ∂B_R. Moreover, Φ is a viscosity subsolution of

$$0 \leq \sup_{\bar{a}\in\mathbb{A}} \bar{\mathcal{L}}^{\bar{a}}\Phi.$$

We can thus now appeal to the strong maximum principle, see e.g. [23, Theorem 1], to deduce that $\bar{\mathrm{w}}_0^{\infty *} - \bar{\mathrm{w}} - g = \Phi \equiv 0$, with $g = \bar{\mathrm{w}}_0^{\infty *}(0) - \bar{\mathrm{w}}(0) = 0$.

As mentioned above, similar reasoning can be applied to $\bar{\mathrm{w}}_{0*}^{\infty}$, showing that

$$\limsup_{x'\to x,\, h\downarrow 0} |\bar{\mathrm{w}}_h^{\kappa_h} - \bar{\rho}_h^{\kappa_h,*}\Delta t_h - \bar{\mathrm{w}}|(x') = 0,\ x\in\mathbb{R},$$

so that the convergence is uniform on compact sets. □

5 Application to high-frequency auctions

5.1 Motivation and setting

Web display advertising is a typical example of real-world high-frequency pure jump control problems [18]. The ad spaces are sold by algorithmic platforms in automated auctions which occur at the dozen microsecond scale [32]. The frequency imposes computational issues on optimisation problems in this industry, while at the same time, the volume creates a significant monetary incentive for all parties to engage in revenue maximisation.

Consequently, the question of the strategic behaviour of bidders in repeated auctions in the face of learning sellers has been a popular topic in contemporary auction theory, see e.g. [30, § 4] for a survey. A rich line of work has focused on asymmetric problems where one player is significantly more patient than the other [2, 31]. This asymmetry reduces game theoretic considerations in the analysis to optimisation or control problems. In this example, we take interest in the case where the buyer is infinitely patient (it optimises an ergodic objective), while the seller's algorithm has effectively finite memory.

Given these horizons, the format of the auction will strongly influence the behaviour of bidders and sellers when they seek to maximise their profit, see e.g. [24] for some generic examples. While it is a sub-optimal auction format for the seller [29], we choose to focus on the second price auction format here. Indeed, there are unsurmountable difficulties in learning the optimal auction format [27], and second-price is in practice a common compromise between tractability and optimality [33].

Recalling the notations introduced in Example 2.1, in a second price auction with reserve, the bidder wins if it outbids the maximum of the competing bids denoted by $e_4 \in \mathbb{R}_+$ and the reserve price x, and pays the smallest bid which still wins the auction, i.e. $x \vee e_4$. As a result of the time scale, there is little time in practice to perform computations to determine the bid, and one typically relies on using a precomputed function of the value to bid when an ad-slot arrives and its value $e_2 \in \mathbb{R}_+$ is revealed. More formally: the bid should be predictable. For simplicity, in this example, we consider a linear shading of the value: ae_2, where the control input value a is the shading factor. Consequently, we have the (expected) reward function for the bidder in a single auction

$$r(x,a) := \int (e_2 - x \vee e_4) 1_{\{ae_2 \geq x \vee e_4\}} \nu(\mathrm{d}e) \,. \tag{5.1}$$

Such auctions are well-defined only for positive bids. Thus, we impose $a \in \mathbb{R}_+$.

Within the constraints of a second price auction, maximising profits corresponds to tuning the reserve price x. Dynamically optimising the reserve price is a difficult problem even for a stationary bidder, see e.g. [2, 16]. To simplify, we consider the mean-reverting dynamic introduced in Example 2.1. For some $\eta = \varepsilon^{-1}$ fixed, this dynamic is given by (2.1) with

$$b := b_\epsilon = \varepsilon b_1 + \sqrt{\varepsilon} b_2 \text{ where } b_1(x,a,e) := e_1(ae_2 - x) \text{ and } b_2(x,e) := e_1 e_3. \tag{5.2}$$

In the above framework, the noise realisation of e_1 encodes seller aggressivity as an exogenous randomness, while the noise realisation of e_3 encodes the realisation of the seller's internal randomisation aimed at increasing robustness to strategic play.

Under the conditions outlined in Example 2.1, we can choose for simplicity

$$\nu(\mathrm{d}e) = \prod_{i=1}^{4} f_i(e_i)\mathrm{d}e_i$$

in which

$$f_1 \sim \texttt{Unif}(0,1) \quad \text{and} \quad f_3 \sim \mathcal{N}(0, \sigma_0^2)$$

with $\sigma_0 = \frac{1}{2}$.

Second price auctions without reserve leave the most revenue on the table when the buyers are highly asymmetrical, we therefore study

$$f_2 \sim \texttt{LogNorm}(\mu_1, \sigma_1) \quad \text{and} \quad f_4 \sim \texttt{Unif}(0,1)$$

with $\mu_1 = 0$ and $\sigma_1 = \frac{1}{2}$. Note that empirical observations [32] suggest log-normals are a realistic model for values.

Assumption 1, and the remaining conditions in Example 2.1 for Assumptions 2 and 3 are easily seen to hold under the above choices. Therefore, this pure jump process admits, and converges to, a diffusion limit by Theorem 3.6, in particular, it is easily checked that the coefficients of the limit diffusion are given by

$$\mu(x,a) := \frac{1}{2}\,(aC - x) \;\text{ and }\; \sigma(x) := \frac{\sigma_0}{\sqrt{3}}\,, \tag{5.3}$$

where $C := \exp\left(\mu_1 + \frac{\sigma_1^2}{2}\right)$. It is clear from (5.1) and (5.3), that values of a larger than 1 cannot be optimal, therefore we fix $\mathbb{A} = [0,1]$.

5.2 Numerical Resolution of the HJB Equations

Using this example motivated by high-frequency auctions we illustrate in this section the benefits of the diffusion limit problem in regards to numerical computation. We use the method detailed in Section 4 to solve numerically (3.5), with parameters μ and σ given by (5.3). Throughout, we will take $\kappa_h := h^{-1/4}$, for which $h \leq \left(\frac{\sigma}{2}\right)^{\frac{8}{3}}$ suffices to uphold condition (4.5) since we have $[\mu]_{C^0_{\text{lin}}} \leq (1 + e^{1/8})/2$. Note that, with f_1, f_2, f_3 as above, p in Example 2.1 and Example 3.3 can be taken to be any positive real number.

In comparison to (3.5), solving (2.8) with coefficients given by (5.2) is complicated by the computation of the integral term. In many situations, when ν is a non-atomic measure with a known closed form, quadrature would be the preferred method for resolution, see e.g. [15]. In this example, this quadrature would be 4-dimensional, which is somewhat expensive.

In contrast, the relatively simple form of the combination of independent noise sources makes Monte Carlo simulation competitive in this specific example. Fixing a grid $\mathcal{M}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}$ analogous to the one in Section 4, we compute the empirical transition distribution $p^{\varepsilon,h_\varepsilon}_{N_\varepsilon} : (x,a) \in \mathcal{M}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon} \times \mathbb{A} \to p^{\varepsilon,h_\varepsilon}_{N_\varepsilon}(\cdot; x, a) \in \Delta_{2\kappa_\varepsilon+1}$, where $\Delta_{2\kappa_\varepsilon+1}$ is the $2\kappa_\varepsilon + 1$-dimensional probability simplex, based on N_ε independent samples from each law, by projecting sample transitions onto $\mathcal{M}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}$. We then approximate (2.8) by solving the analogue of (4.3), i.e. finding $(\rho^{\kappa_\varepsilon,*}_{\varepsilon,h_\varepsilon}, \mathcal{W}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon})$, $\mathcal{W}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon} := (\mathrm{w}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}(z_i))_{1\leq i\leq 2\kappa_\varepsilon+1}$, solving

$$0 = \max_{\mathrm{a}\in\mathrm{A}} \left\{ \frac{1}{\varepsilon}(P^{\mathrm{a}}_{N_\varepsilon,\varepsilon,h_\varepsilon} - \boldsymbol{I}_{2\kappa_\varepsilon+1})\mathcal{W}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon} - e\mathrm{w}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}(0) + \mathcal{R}(\mathrm{a}) \right\} \tag{5.4}$$

$$\rho^{\kappa_\varepsilon,*}_{\varepsilon,h_\varepsilon} = \mathrm{w}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}(0) \tag{5.5}$$

by policy iteration, where $P^{\mathrm{a}}_{N_\varepsilon,\varepsilon,h_\varepsilon} = (p^{\varepsilon,h_\varepsilon}_{N_\varepsilon}(z_j; z_i, \mathrm{a}(z_i)))_{1\le i,j\le 2\kappa_\varepsilon+1}$, $\boldsymbol{I}_{2\kappa_\varepsilon+1}$ is the $2\kappa_\varepsilon + 1$-dimensional identity matrix, and $\mathcal{R}(\mathrm{a})$ is as in Section 4.

As $\varepsilon \to 0$, all the transitions concentrate into a ball of size $\varepsilon^{\frac{1}{2}}$ with a drift of size ε, meaning that the mesh must refine faster than ε, in order to avoid degeneracy. Therefore, we consider the sequence of grids $\{\mathcal{M}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}\}_{\varepsilon\ge 0}$, with $\mathcal{M}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon} = \{y_i = -10 + (i-1)h_\varepsilon, 1 \le i \le 2\kappa_\varepsilon + 1\}$ with $h_\varepsilon = \varepsilon^2$ and $\kappa_\varepsilon = 20/h_\varepsilon$. We take $N_\varepsilon := \kappa_\varepsilon^2 \wedge 10^6$ to reduce noise while capping unmanageable computation cost once $\varepsilon < \sqrt{2}/10$. Note that the refinement of the grid $\mathcal{M}^{\kappa_\varepsilon}_{\varepsilon,h_\varepsilon}$ as $\varepsilon \to 0$ does not imply that the accuracy of the scheme increases as $\varepsilon \to 0$, the increasingly fine resolution is a *cost* incurred due to η_ε. The increase in this cost becomes impossible to maintain as ε becomes small, this is illustrated by Figure 1: it rises at a rate ε^{-4}.

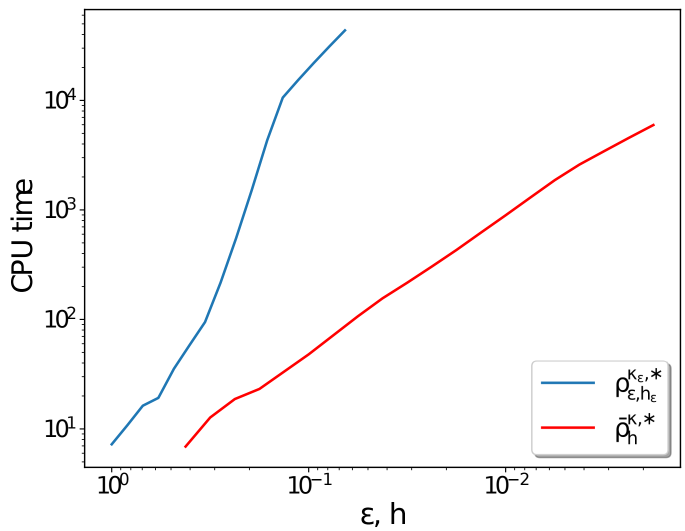

Figure 1: Comparison of computation costs for (5.4) ($\rho^{\kappa_\varepsilon,*}_{\varepsilon,h_\varepsilon}$) and (4.3) ($\bar{\rho}^{\kappa,*}_h$).

In contrast, using the diffusion limit by combining Sections 3 and 4, allows us to solve the problem to a high precision for relatively cheap. Figure 2 demonstrates the convergence in value of Theorem 3.6, with a rate of $\varepsilon^{\frac{1}{2}}$.

Explicit computation for approximately optimal controls using (4.12) is impractical for the r given in (5.1), due to its lack of a closed-form derivative to apply first-order conditions. Nevertheless, in order to illustrate the bounds in Propositions 4.2, we resort to numerical approximation. We fix a grid $\mathbb{A}_\Gamma := \{i\Gamma^{-1}, 0 \le i \le \Gamma\}$ on $\mathbb{A} = [0,1]$, fixing $\Gamma = 1000$, and then solve the maximum in (4.12) on $\mathbb{A}_\Gamma$ instead of $\mathbb{A}$. Contrary to Section 4, we only compute it on $\mathcal{M}^{\kappa_h}_h$. This yields a map $\breve{a}^\Gamma_h : \mathcal{M}^{\kappa_h}_h \to \mathbb{A}_\Gamma$, which can be viewed as a vector of controls associated to $\mathcal{M}^{\kappa_h}_h$.

From here, we extend the definition of $\breve{\mathrm{a}}^\Gamma_h$ by setting $\breve{\mathrm{a}}^\Gamma_h(x) := \breve{a}^\Gamma_h(\Pi^{\kappa_h}_h(x))$, for $x \in \mathbb{R}$, where

$\Pi_h^{\kappa_h}(x) := \inf\{z \in \mathcal{M}_h^{\kappa_h} : z \geq x\} \wedge z_{2\kappa_h+1}$, so that $\breve{\mathrm{a}}_h^\Gamma \in \mathrm{A}$. We now consider the solution $\breve{X}^{\kappa_h,h,\Gamma}$ of

$$\breve{X}^{\kappa_h,h,\Gamma}_\cdot = \int_0^\cdot \int_{\mathbb{R}^{d'}} b_\varepsilon(\breve{X}^{\kappa_h,h,\Gamma}_{s-}, \breve{\mathrm{a}}_h^\Gamma(\breve{X}^{\kappa_h,h,\Gamma}_{s-}), e) N(\mathrm{d}e, \mathrm{d}s)\,,$$

and evaluate $\rho_\varepsilon(0, \breve{\alpha}_h^\Gamma)$ for each ε, where $\breve{\alpha}_h^\Gamma := \breve{\mathrm{a}}_h^\Gamma(\breve{X}^{\kappa_h,h,\Gamma}_{\cdot-})$. In practice, we fix $T = 1000$ and compute

$$\frac{\varepsilon}{T}\mathbb{E}\left[\int_0^T r(\breve{X}^{\kappa_h,h,\Gamma}_{t-}, \breve{\mathrm{a}}_h^\Gamma(\breve{X}^{\kappa_h,h,\Gamma}_{t-}))\mathrm{d}N_t\right]$$

by Monte Carlo with $1000\varepsilon^{-1}$ trajectories[1] of $\breve{X}^{\kappa_h,h,\Gamma}$ for each ε.

In spite of the noise and the simple approximate control scheme, we recover the bounds of Propositions 4.2, in terms of ε in Figure 3, with $h = 0.0035570$, the smallest h on Figure 1. Note that this convergence rate matches the one of Figure 2.

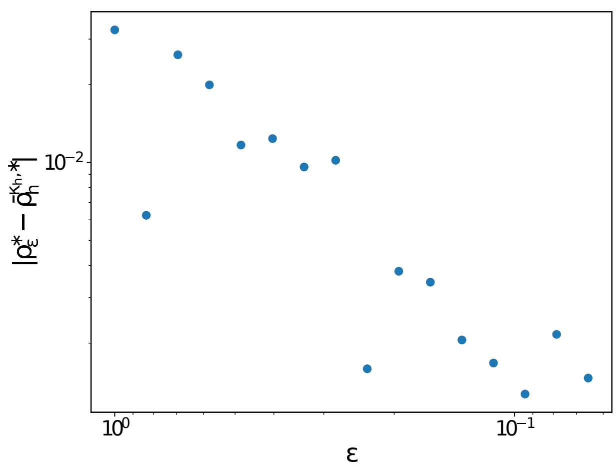

Figure 2: Approximation error of ρ^*_ε by $\bar{\rho}_h^{\kappa_h,*}$.

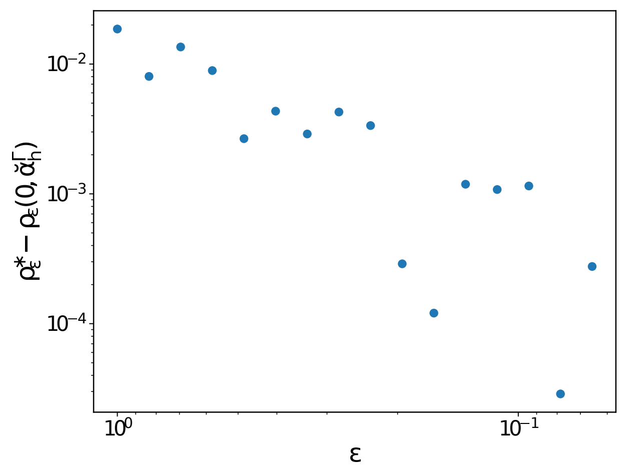

Figure 3: Suboptimality of the diffusive control relative to ρ^*_ε.

Appendix

A Proof of Theorem 2.3

In this Appendix, we first provide the proof of Theorem 2.3. It follows a standard route. We adapt arguments of [4] and [3] to our context.

[1]Computing an ergodic average over each trajectory is very numerically expensive for small values of ε, reducing the feasible amount of samples. In spite of the noise, a slope faster than $\frac{1}{2}$ is still visible in Figure 3.

We first show that $(V_\lambda)_{\lambda\in(0,1)}$ is equi-Lipschitz continuous, under the contraction condition of Assumption 2.

Lemma A.1. *Let Assumptions 1 and 2 hold, then*

$$|V_\lambda(x) - V_\lambda(x')| \le L_V|x - x'|, \text{ for } x, x' \in \mathbb{R}^d, \lambda \in (0,1),$$

in which

$$L_V := \frac{L_{b,r}p_\zeta}{C_\zeta}\left(\frac{L_\zeta}{\ell_\zeta}\right)^{\frac{1}{p_\zeta}}.$$

Proof. Fix $x, x' \in \mathbb{R}^d$, together with $\alpha \in \mathcal{A}$. By applying (2.4) of Assumption 2, we have

$$\eta\int_{\mathbb{R}^{d'}} \left\{\zeta(X_{\cdot-} + b(X_{\cdot-},\alpha,e), X'_{-} + b(X'_{\cdot-},\alpha,e)) - \zeta(X_{\cdot-},X'_{\cdot-})\right\}\nu(\mathrm{d}e) \le -C_\zeta\zeta(X_{\cdot-},X'_{\cdot-}) \tag{A.1}$$

in which $(X, X') := (X^{x,\alpha}, X^{x',\alpha})$ and $(X_{\cdot-}, X'_{\cdot-})$ is its left-limit. Applying Itô's Lemma[2] then implies

$$\zeta(X_t, X'_t) = \zeta(x,x') + \int_0^t \int_{\mathbb{R}^{d'}} (\zeta(X_{s-} + b(X_{s-},\alpha_s,e), X'_s + b(X'_{s-},\alpha_s,e)) - \zeta(X_{s-},X'_{s-}))N(\mathrm{d}e,\mathrm{d}s)$$

for $t \ge 0$. Combining the above with (A.1) yields

$$\zeta(X_t, X'_t) \le \zeta(x,x') - C_\zeta \int_0^t \zeta(X_s, X'_s)\mathrm{d}s + M_t$$

in which

$$M := \int_0^\cdot \int_{\mathbb{R}^{d'}} (\zeta(X_{s-} + b(X_{s-},\alpha_s,e), X'_s + b(X'_{s-},\alpha_s,e)) - \zeta(X_{s-},X'_{s-}))(N(\mathrm{d}e,\mathrm{d}s) - \eta\nu(\mathrm{d}e)\mathrm{d}s),$$

is a local martingale. Upon using a localisation argument, recall (i) of Assumption 2, taking the expectation and using an immediate comparison result for ODEs leads to

$$\mathbb{E}[\zeta(X_t, X'_t)] \le \zeta(x,x')e^{-C_\zeta t}, \ t \ge 0. \tag{A.2}$$

It remains to use (i) of Assumption 2 to deduce that

$$\mathbb{E}[|X_t - X'_t|^{p_\zeta}] \le \frac{L_\zeta}{\ell_\zeta}|x - x'|^{p_\zeta}e^{-C_\zeta t}, t \ge 0. \tag{A.3}$$

[2]Which, in this setting, is equivalent to writing $\zeta(X_t, X'_t)$ as the sum of its jumps.

Combining the above with Remark 2.2, the Lipschitz continuity assumption on r, Assumption 1, and using Jensen's inequality then leads to

$$\begin{aligned}
|J_\lambda(x,\alpha) - J_\lambda(x',\alpha)| &\le L_{b,r}\int_0^\infty e^{-\lambda t}\mathbb{E}\left[|X_t - X'_t|\right]\mathrm{d}t \\
&\le L_{b,r}\int_0^\infty e^{-\lambda t}\mathbb{E}\left[|X_t - X'_t|^{p_\zeta}\right]^{\frac{1}{p_\zeta}}\mathrm{d}t \\
&\le L_{b,r}\left(\frac{L_\zeta}{\ell_\zeta}\right)^{\frac{1}{p_\zeta}}\int_0^\infty |x-x'|\, e^{-\lambda t - \frac{C_\zeta}{p_\zeta}t}\mathrm{d}t \\
&\le \frac{L_{b,r}p_\zeta}{C_\zeta + \lambda p_\zeta}\left(\frac{L_\zeta}{\ell_\zeta}\right)^{\frac{1}{p_\zeta}}|x-x'|\,.
\end{aligned}$$

Since $|V_\lambda(x) - V_\lambda(x')| \le \sup_{\alpha\in\mathcal{A}}|J_\lambda(x,\alpha) - J_\lambda(x',\alpha)|$ and $\lambda p_\zeta \ge 0$, this completes the proof. □

We now use Assumption 3 to provide a uniform (in time and the control) estimate on the diffusion (2.1).

Lemma A.2. *Let Assumptions 1 and 3 hold. Then, for all $(x,\alpha)\in\mathbb{R}^d\times\mathcal{A}$,*

$$\mathbb{E}[|X_t^{x,\alpha}|^{p_\xi}] \le \frac{1}{\ell_\xi}\left\{e^{-C_\xi^1 t}L_\xi|x|^{p_\xi} + \frac{C_\xi^2}{C_\xi^1}(1-e^{-C_\xi^1 t})\right\},\ t\ge 0.$$

Proof. Fix $(x,\alpha)\in\mathbb{R}^d\times\mathcal{A}$ and let us write X for $X^{x,\alpha}$. By (2.5) and the same arguments as in the proof of Lemma A.1,

$$\mathbb{E}[\xi(X_t)] \le \xi(x) + \int_0^t \mathbb{E}[-C_\xi^1\xi(X_s) + C_\xi^2]ds,\ t\ge 0,$$

which implies that

$$\mathbb{E}[\xi(X_t)] \le e^{-C_\xi^1 t}\xi(x) + \frac{C_\xi^2}{C_\xi^1}(1-e^{-C_\xi^1 t}),\ t\ge 0.$$

We conclude with (i) of Assumption 3. □

We can now prove a first convergence result.

Lemma A.3. *Let Assumptions 1 and 2 hold. Then there is $c\in\mathbb{R}$ and a sequence $(\lambda_n)_{n\ge1}$ going to 0 such that $(\lambda_n V_{\lambda_n})_{n\ge1}$ converges uniformly on compact sets to c, and such that $(V_{\lambda_n} - V_{\lambda_n}(0))_{n\ge1}$ converges uniformly on compact sets to a function $\mathrm{w}\in\mathcal{C}^{0,1}$ that solves*

$$c = \sup_{a\in\mathbb{A}}\left\{\eta\int_{\mathbb{R}^{d'}}[\mathrm{w}(\cdot + b(\cdot,a,e)) - \mathrm{w}]\,\nu(de) + r(\cdot,a)\right\},\quad \text{on } \mathbb{R}^d,$$

and satisfies

$$|\mathrm{w}(x)| \le L_V |x|,\ x \in \mathbb{R}^d. \tag{A.4}$$

Proof. The proof applies classical arguments from [4] to the pure jump setting. By Lemma A.1, $(V_\lambda - V_\lambda(0))_{\lambda>0}$ is equicontinuous in the Lipschitz sense and, in particular, $|V_\lambda(x) - V_\lambda(0)| \le L_V\,|x|$ for all $x \in \mathbb{R}^d$ and $\lambda > 0$. Hence, $(\lambda(V_\lambda - V_\lambda(0)))_{\lambda\ge 0}$ converges uniformly on compact sets to 0 as $\lambda \to 0$. Since $(\lambda V_\lambda(0))_{\lambda\ge 0}$ is bounded, recall Lemma A.2 and Assumption 1, there is a sequence $(\lambda_n)_{n\ge 1}$ converging to 0 such that $\lambda_n V_{\lambda_n}(0) \to c \in \mathbb{R}$ as $n \to \infty$. Thus, $\lambda_n V_{\lambda_n} \to c$ uniformly on compact sets.

By Lemma A.1, $(V_\lambda - V_\lambda(0))_{\lambda>0}$ is locally bounded. Then, a diagonalisation argument allows one to extract a further subsequence (also denoted $(\lambda_n)_{n\ge 0}$) such that $V_{\lambda_n} - V_{\lambda_n}(0) \to \mathrm{w}$ on $\mathbb{Q}^d$ for some $\mathrm{w} : \mathbb{Q}^d \to \mathbb{R}$. By the uniform equicontinuity of $(V_\lambda)_{\lambda\in(0,1)}$, w can be extended to $\mathbb{R}^d$ and $V_{\lambda_n} - V_{\lambda_n}(0) \to \mathrm{w}$ uniformly on compact sets. Moreover, w is L_V-Lipschitz and $\mathrm{w}(0) = 0$, which implies (A.4).

Next, it follows from standard arguments, see e.g. [11], that V_{λ_n} solves for each $n \ge 1$

$$0 = \sup_{a\in\mathbb{A}} \left\{ \eta \int_{\mathbb{R}^{d'}} [V_{\lambda_n}(\cdot + b(\cdot, a, e)) - V_{\lambda_n}]\nu(\mathrm{d}e) + r(\cdot, a) \right\} - \lambda_n V_{\lambda_n},\ \text{ on } \mathbb{R}^d. \tag{A.5}$$

Hence,

$$\begin{aligned}\lambda_n V_{\lambda_n}(0) =& -\lambda_n(V_{\lambda_n} - V_{\lambda_n}(0)) \\ &+ \sup_{a\in\mathbb{A}} \left\{ \eta \int_{\mathbb{R}^{d'}} [V_{\lambda_n}(\cdot + b(\cdot, a, e)) - V_{\lambda_n}(0) - (V_{\lambda_n} - V_{\lambda_n}(0))]\nu(\mathrm{d}e) + r(\cdot, a) \right\},\ \text{ on } \mathbb{R}^d,\end{aligned}$$

and passing to the limit (recall Assumption 1 and that ν is a probability measure) implies that

$$c = \sup_{a\in\mathbb{A}} \left\{ \eta \int_{\mathbb{R}^{d'}} [\mathrm{w}(\cdot + b(\cdot, a, e)) - \mathrm{w}]\nu(\mathrm{d}e) + r(\cdot, a) \right\},\ \text{ on } \mathbb{R}^d.$$

□

We now have to prove that the constant c defined above equals $\rho^*(0)$ and that only $(\mathrm{w}, \rho^*(0))$ solves (2.8), up to restricting to functions with linear growth taking the value 0 at 0.

Lemma A.4. *Let Assumptions 1, 2 and 3 hold. Let $(\tilde{\mathrm{w}}, \tilde\rho) \in \mathcal{C}^0_{\mathrm{lin}} \times \mathbb{R}$ be a solution of the ergodic equation*

$$\tilde\rho = \sup_{a\in\mathbb{A}} \left\{ \eta \int_{\mathbb{R}^{d'}} [\tilde{\mathrm{w}}(\cdot + b(\cdot, a, e)) - \tilde{\mathrm{w}}]\nu(\mathrm{d}e) + r(\cdot, a) \right\},\ \text{ on } \mathbb{R}^d.$$

Then, ρ^* is constant and equal to $\tilde\rho$. In particular, the constant c of Lemma A.3 is equal to ρ^*.

Proof. Let us fix $x \in \mathbb{R}^d$.

a. By Lemma A.3 and [9, Proposition 7.33, p.153], we can find a measurable map $x' \in \mathbb{R}^d \to \hat{\mathrm{a}}(x') \in \mathbb{A}$ such that

$$\tilde{\rho} = \eta \int_{\mathbb{R}^{d'}} [\tilde{\mathrm{w}}(\cdot + b(\cdot, \hat{\mathrm{a}}(\cdot), e)) - \tilde{\mathrm{w}}]\nu(\mathrm{d}e) + r(\cdot, \hat{\mathrm{a}}(\cdot)), \quad \text{on } \mathbb{R}^d.$$

Let $\hat{X}$ denote the solution of (2.1) associated to $\hat{\alpha} := \hat{\mathrm{a}}(\hat{X}_{\cdot-})$ and the initial condition x. Then, Itô's Lemma implies that

$$\mathbb{E}\left[\tilde{\mathrm{w}}(\hat{X}_t) - \tilde{\mathrm{w}}(x) + \frac{1}{\eta}\int_0^t r(\hat{X}_{s-}, \hat{\alpha}_s)\mathrm{d}N_s\right] = \tilde{\rho}t,\ t \geq 0.$$

Moreover, since $\tilde{\mathrm{w}}$ has linear growth, there exists $C > 0$ such that

$$\mathbb{E}[|\tilde{\mathrm{w}}(\hat{X}_t) - \tilde{\mathrm{w}}(x)|] \leq C\mathbb{E}[1+|\hat{X}_t| + |x|].$$

By Lemma A.2, $\mathbb{E}[|\hat{X}_t|]/t \to 0$ as $t \to \infty$ since $p_\xi \geq 1$. Then, the above implies that

$$\lim_{t\to\infty} \frac{1}{\eta t}\mathbb{E}\left[\int_0^t r(\hat{X}_{s-}, \hat{\alpha}_s)\mathrm{d}N_s\right] = \tilde{\rho}.$$

b. Conversely, for any $\alpha \in \mathcal{A}$,

$$\mathbb{E}\left[\tilde{\mathrm{w}}(X_t^{x,\alpha}) - \tilde{\mathrm{w}}(x) + \frac{1}{\eta}\int_0^t r(X_{s-}^{x,\alpha}, \alpha_s)\mathrm{d}N_s\right] \leq \tilde{\rho}t,\ t \geq 0.$$

By Lemma A.2 and the linear growth of $\tilde{\mathrm{w}}$ again, we deduce that

$$\limsup_{t\to\infty} \frac{1}{\eta t}\mathbb{E}\left[\int_0^t r(X_{s-}^{x,\alpha}, \alpha_s)\mathrm{d}N_s\right] \leq \tilde{\rho}.$$

c. Combining a. and b. implies that $\tilde{\rho} = \rho^*(x)$. By arbitrariness of $x \in \mathbb{R}^d$, ρ^* is constant. $\square$

We are now in position to prove our second convergence result, and therefore to complete the proof of Theorem 2.3.

Lemma A.5. *Let Assumptions 1, 2 and 3 hold. Then, there exists a sequence $(T_n)_{n\geq 1}$ going to $+\infty$ such that $(T_n^{-1}V_{T_n}(0,\cdot))_{n\geq 1}$ converges uniformly on compact sets to $\rho^*(0)$.*

Proof. The proof follows from the same arguments as in [4, Prop. VI.1] except that in their case the convergence holds uniformly on $\mathbb{R}^d$. Let $(\lambda_n)_{n\geq 1}$ be as in Lemma A.3 and set $T_n := \delta/\lambda_n$ for some

$\delta \in (0,1)$, so that $\lambda_n \to 0$ and $T_n \to \infty$ as $n \to \infty$. Fix $x \in \mathbb{R}^d$. By Lemma A.1 and Lemma A.2, we can find $C > 0$ such that $\mathbb{E}[|V_{\lambda_n}(X_t^{x,\alpha}) - V_{\lambda_n}(x)|] \le C(1+|x|)$ uniformly in $\alpha \in \mathcal{A}$ and for all $x \in \mathbb{R}^d$, and $t \ge 0$. Arguing as in the proof of [4, Prop. VI.1], we then deduce from the dynamic programming principle applied to V_{λ_n}, see e.g. [11], Lemma A.1, Lemma A.2 and Assumption 1 that, for some $C' > 0$ that does not depend on n,

$$|\rho^*(1-e^{-\delta}) - \frac{\delta}{T_n} V_{T_n}(0,x)| \le 2|\lambda_n V_{\lambda_n}(x) - \rho^*| + \lambda_n C'(1+|x|).$$

It remains to divide the above by δ, send $n \to \infty$ and use Lemmas A.3 and A.4 to obtain that

$$\rho^* \frac{(1-e^{-\delta})}{\delta} \le \liminf_{n\to\infty} \frac{1}{T_n} V_{T_n}(0,x) \le \limsup_{n\to\infty} \frac{1}{T_n} V_{T_n}(0,x) \le \rho^* \frac{(1-e^{-\delta})}{\delta},$$

and we conclude by arbitrariness of $\delta \in (0,1)$. The fact that the convergence is uniform on compact sets follows from the above and Lemma A.3. □

B Estimates for elliptic Hamilton-Jacobi-Bellman equations without control on the volatility part

In this section, we collect standard estimates on elliptic Hamilton-Jacobi-Bellman equations associated to infinite horizon optimal control problems of a diffusion, in which there is no control on the volatility part. This is a specific class of quasi-linear equations whose analysis is standard. Our focus here is on the growth rate of local $\mathcal{C}_b^{2,1}$-estimates in the case where the solution is already known to be Lipschitz. We follow closely the arguments of [20] that considers compact domains and insist only on the points where the Lipschitz continuity property is used.

As usual, we first consider linear equations of the form

$$0 = \langle \mathfrak{b}, \mathrm{D}u^\top \rangle + \frac{1}{2}\mathrm{Tr}\left[\mathfrak{a}\mathrm{D}^2 u\right] - \lambda u - \mathfrak{f} \text{ on } \mathbb{R}^d. \tag{B.1}$$

We fix $M > 0$ and a modulus of continuity ϱ (i.e. a real valued map on $\mathbb{R}^d$ that is continuous at 0 and such that $\varrho(0) = 0$). We let $\mathfrak{S}(M,\varrho)$ denote the collections of real-valued maps $u \in \mathcal{C}^2$ such that $u(0) = 0$, $|\mathrm{D}u| \le M$ and that are strong solutions of (B.1) with coefficients satisfying:

(i) $\lambda \in [0,1]$,

(ii) $(\mathfrak{b},\mathfrak{f}) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$ is measurable and $[\mathfrak{b}]_{\mathcal{C}^0_{\text{lin}}} + \|\mathfrak{f}\|_{\mathcal{C}^0_b} \le M$,

(iii) $\mathfrak{a} : \mathbb{R}^d \to \mathbb{S}^d$ is bounded by M and admits ϱ as a modulus of continuity,

(iv) $\inf\{\xi^\top \mathfrak{a}(x)\, \xi : (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, |\xi| = 1\} \geq 1/M$.

Hereafter, we use the convention $0/0 = 0$.

Lemma B.1. *For each $\gamma \in (0,1)$, there exists $K^\gamma_{M,\varrho} > 0$ such that any $u \in \mathfrak{S}(M,\varrho)$ satisfies*

$$\|u\|_{\mathcal{C}^{1,\gamma}_b(B_2(x))} \leq K^\gamma_{M,\varrho}(1 + |x|), \quad \text{for all } x \in \mathbb{R}^d.$$

Proof. 1. Given $p > 1$, we first estimate $\|u\|_{W^{2,p}(B_2(x))}$ in which $\|\cdot\|_{W^{2,p}(B_2(x))}$ denotes the norm associated to the Sobolev space $W^{2,p}(B_2(x))$. We follow the proof of [20, Theorem 9.11]. Fix $x_0 \in B_2(x)$. By [20, (9.37)], for any $v \in W^{2,p}(B_3(x_0))$ supported in some $B_R(x_0) \subset B_3(x)$, $R > 0$, there is $C_1 > 0$, that depends only on p, such that

$$\left\|\mathrm{D}^2 v\right\|_{L^p(B_R(x_0))} \leq C_1 M \left(\sup_{B_R(x_0)} |\mathfrak{a} - \mathfrak{a}(x_0)| \left\|\mathrm{D}^2 v\right\|_{L^p(B_R(x_0))} + \left\|\mathrm{Tr}[\mathfrak{a}\mathrm{D}^2 v]\right\|_{L^p(B_R(x_0))} \right),$$

in which $\|\cdot\|_{L^p(B_R(x_0))}$ denotes the usual norm of the L^p-space associated to the Lebesgues measure on $B_R(x_0)$.

The uniform continuity of $\mathfrak{a}$ implies that there exists $R > 0$ small enough, that only depends on p, M and ϱ, such that $|\mathfrak{a} - \mathfrak{a}(x_0)| \leq (2C_1 M)^{-1}$ on $B_R(x_0)$, so that the above implies that

$$\left\|\mathrm{D}^2 v\right\|_{L^p(B_R(x_0))} \leq 2C_1 M \left\|\mathrm{Tr}[\mathfrak{a}\mathrm{D}^2 v]\right\|_{L^p(B_R(x_0))}. \tag{B.2}$$

Take $u \in \mathfrak{S}(M,\varrho)$ a solution to (B.1) in $B_3(x)$, applying (B.2) yields

$$\left\|\mathrm{D}^2 u\right\|_{L^p(B_R(x_0))} \leq C_2(\|\mathfrak{f}\|_{\mathcal{C}^0_b(B_3(x))} + \lambda \|u\|_{\mathcal{C}^0_b(B_3(x))} + \|\mathfrak{b}\|_{\mathcal{C}^0_b(B_3(x))} \left\|\mathrm{D}u^\top\right\|_{\mathcal{C}^0_b(B_3(x))})$$

for some $C_2 > 0$ that only depends on M, p and ϱ. From the definition of $\mathfrak{S}(M,\varrho)$, it follows that there is $C_3 > 0$, independent of x_0, such that

$$\|u\|_{W^{2,p}(B_R(x_0))} \leq C_3(1 + |x|),$$

and, by covering $B_2(x)$ with finitely many balls of radius less that R, one obtains

$$\|u\|_{W^{2,p}(B_2(x))} \leq C_4(1 + |x|)$$

for some C_4 that depends only on p, M and ϱ.

2. Using an imbedding theorem, see *e.g.* [20, Theorem 7.26], we can find $\bar{K}^{\gamma,p} > 0$ such that

$$\|u\|_{\mathcal{C}^{1,\gamma}_b(B_2(x))} \leq \bar{K}^{\gamma,p} \|u\|_{W^{2,p}(B_2(x))}, \; \forall\, u \in \mathfrak{S}(M,\varrho), \; x \in \mathbb{R}^d,$$

for all $p \in \mathbb{N}$ such that $0 < d/p < 1$ and $\gamma \in (0, 1 - d/p)$. Given $\gamma \in (0,1)$, the required result follows by combining the above for some p large enough. □

We now turn to the quasilinear case

$$0 = \hat{\mathfrak{b}}(\cdot, \mathrm{D}u^\top) + \frac{1}{2}\mathrm{Tr}\left[\mathfrak{a}\mathrm{D}^2 u\right] - \lambda u \text{ on } \mathbb{R}^d, \tag{B.3}$$

in which

$$\hat{\mathfrak{b}}(x,y) := \langle \mathfrak{b}(x,y), y\rangle - \mathfrak{f}(x,y),\ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

We again fix $M > 0$, and $\rho = (\rho_1, \rho_2) \in (0,1]^2$, and let $\tilde{\mathfrak{S}}(M,\rho)$ denote the collection of real-valued maps $u \in \mathcal{C}^2$ such that $u(0) = 0$, $|\mathrm{D}u| \le M$, and that are solutions of (B.3) for some coefficients satisfying:

(a.) $\lambda \in [0,1]$,

(b.) $(\mathfrak{b}, \mathfrak{f}) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$ is measurable and $[\mathfrak{b}]_{\mathcal{C}^0_{\mathrm{lin}}(\mathbb{R}^{2d})} + \|\mathfrak{f}\|_{\mathcal{C}^0_b(\mathbb{R}^{2d})} \le M$,

(c.) $\mathfrak{a} : \mathbb{R}^d \to \mathbb{S}^d$ is measurable and bounded by M.

(d.) $\inf\{\xi^\top \mathfrak{a}(x)\, \xi : (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, |\xi| = 1\} \ge 1/M$,

(e.) for all $x, x' \in \mathbb{R}^d$ such that $|x - x'| \le 1$ and all $y, y' \in \mathbb{R}^d$:

$$|\mathfrak{a}(x) - \mathfrak{a}(x')| + |\hat{\mathfrak{b}}(x,y) - \hat{\mathfrak{b}}(x',y')| \le M\left(|x-x'|^{\rho_1} + |y-y'|^{\rho_2}\right).$$

Lemma B.2. *Fix $\gamma \in (0, \rho_1 \wedge \rho_2)$. Then, there exists $\tilde{K}^\gamma_{M,\rho} > 0$ such that any $u \in \tilde{\mathfrak{S}}(M,\rho)$ satisfies*

$$\|u\|_{\mathcal{C}^{2,\gamma}_b(B_1(x))} \le \tilde{K}^\gamma_{M,\rho}(1 + |x|), \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. Fix $x \in \mathbb{R}^d$. Since $|\mathrm{D}u| \le M$, by Lemma B.1 applied to the coefficient $x' \in \mathbb{R}^d \mapsto (\mathfrak{b}(x', \mathrm{D}u(x')), \mathfrak{a}(x'), \mathfrak{f}(x', \mathrm{D}u(x')))$ in place of $(\mathfrak{b}, \mathfrak{a}, \mathfrak{f})$, for each $\gamma \in (0,1)$, we can find $C_\gamma > 0$ such that

$$\|u\|_{\mathcal{C}^{1,\gamma}_b(B_2(x))} \le C_\gamma(1 + |x|) \text{ for all } x \in \mathbb{R}^d. \tag{B.4}$$

It then follows from [20, Theorem 9.19] that $u \in \mathcal{C}^{2,\gamma}_b(B_2(x))$ for any $\gamma \in (0, \rho_1 \wedge \rho_2)$.
To obtain an associated estimate, we turn to the proof of [20, Theorem 6.2] which we apply to the solution $w = u$ of the linear equation $Lw := \frac{1}{2}\mathrm{Tr}\left[\mathfrak{a}\mathrm{D}^2 w\right] = -\hat{\mathfrak{b}}(\cdot, \mathrm{D}w^\top) + \lambda w$, in our particular setting. Fix $x_0 \in B_2(x)$, and consider the constant coefficient equation $L_0 w := \frac{1}{2}\mathrm{Tr}\left[\mathfrak{a}(x_0)\mathrm{D}^2 w\right] = F$ where $F(z) := \frac{1}{2}\mathrm{Tr}\left[(\mathfrak{a}(x_0) - \mathfrak{a}(z))\mathrm{D}^2 u(z)\right] - \hat{\mathfrak{b}}(z, \mathrm{D}u^\top(z)) + \lambda u(z)$, $z \in \mathbb{R}^d$.

We first introduce some notations. For $\Omega \subset \mathbb{R}^d$, $\gamma \in (0,1)$, and $f \in \mathcal{C}^{2,\gamma}(\Omega)$ define the following norm and Schauder semi-norm respectively as follows:

$$|f|^{(2)}_{0,\gamma,\Omega} := \sup_{z\in\Omega} d_z^2\,|f(z)| + \sup_{(z,z')\in\Omega^2} d_{z,z'}^{2+\gamma}\frac{|f(z)-f(z')|}{|z-z'|^\gamma}$$

$$[f]^*_{2,\gamma,\Omega} := \sup_{(z,z')\in\Omega^2} d_{z,z'}^{2+\gamma}\frac{|\mathrm{D}^2 f(z)-\mathrm{D}^2 f(z')|}{|z-z'|^\gamma} \tag{B.5}$$

$$[f]^*_{2,\Omega} := \sup_{z\in\Omega} d_z^2\left|\mathrm{D}^2 f(z)\right|\ , \tag{B.6}$$

where d_z is the distance of z to the boundary of Ω and $d_{z,z'} := d_z \wedge d_{z'}$ for any $(z,z')\in\Omega^2$.

We now fix $\gamma \in (0,\rho_1\wedge\rho_2)$. Let $\mu \in (0,\frac{1}{2}]$ and set $\Omega := B_2(x)$. Fix $y_0 \in B_2(x)$ such that $d_{x_0} \le d_{y_0}$ (without loss of generality) and set $B := B_{\mu d_{x_0}}(x_0)$. Then, [20, Lemma 6.1 (a.)] (see [20, (6.16)] for details) applied to $L_0 w = F$ implies that

$$d_{x_0,y_0}^{2+\gamma}\frac{|\mathrm{D}^2u(x_0)-\mathrm{D}^2u(y_0)|}{|x_0-y_0|^\gamma} = d_{x_0}^{2+\gamma}\frac{|\mathrm{D}^2u(x_0)-\mathrm{D}^2u(y_0)|}{|x_0-y_0|^\gamma} \le \frac{C_1^\gamma}{\mu^{2+\gamma}}(\|u\|_{\mathcal{C}^0_b(B_2(x))} + |F|^{(2)}_{0,\gamma,B}) + \frac{4}{\mu^\gamma}[u]^*_{2,B_2(x)}$$

for some $C_1^\gamma > 0$, which only depends on $\gamma \in (0,\rho_1\wedge\rho_2)$. Then, using [20, (6.8)] yields

$$d_{x_0,y_0}^{2+\gamma}\frac{|\mathrm{D}^2u(x_0)-\mathrm{D}^2u(y_0)|}{|x_0-y_0|^\gamma} \le \frac{C_1^\gamma}{\mu^{2+\gamma}}\left(\|u\|_{\mathcal{C}^0_b(B_2(x))} + |F|^{(2)}_{0,\gamma,B}\right) + 4\left(C_1(\mu)\|u\|_{\mathcal{C}^0_b(B_2(x))} + \mu^\gamma[u]^*_{2,\gamma,B_2(x)}\right)$$

for some $C_1(\mu) > 0$ that only depends on μ. The Schauder estimate then comes from bounding term by term $|F|^{(2)}_{0,\gamma,B}$. First, we argue as for [20, (6.19)], using (c.) and (e.) in the definition of $\tilde{\mathfrak{S}}(M,\rho)$, to obtain

$$\left|\mathrm{Tr}\left[(\mathfrak{a}(x_0)-\mathfrak{a})\mathrm{D}^2u\right]\right|^{(2)}_{0,\gamma,B} \le C_2^\gamma\mu^{2+\gamma}\left[C_2(\mu)\|u\|_{\mathcal{C}^0_b(B_2(x))} + \mu^\gamma[u]^*_{2,\gamma,B_2(x)}\right]$$

for some $C_2^\gamma, C_2(\mu) > 0$ which only depend on γ and μ. Second, we combine (B.4) with items (a.) and (e.) in the definition of $\tilde{\mathfrak{S}}(M,\rho)$ to obtain that

$$\left|\hat{\mathfrak{b}}(\cdot,\mathrm{D}u^\top) - \lambda u\right|^{(2)}_{0,\gamma,B_2(x)} \le C_3^\gamma(1+|x|)$$

for some $C_3^\gamma > 0$, that only depends on γ.

Combining the above with (B.4) and using the arbitrariness of $x_0, y_0 \in B_2(x)$ leads to

$$[u]^*_{2,\gamma,B_2(x)} \le \frac{C_1^\gamma}{\mu^{2+\gamma}}\left(\|u\|_{\mathcal{C}^0_b(B_2(x))} + C_3^\gamma(1+|x|)\right) + \frac{C_1^\gamma C_2^\gamma}{2}\left[C_2(\mu)\|u\|_{\mathcal{C}^0_b(B_2(x))} + \mu^\gamma[u]^*_{2,\gamma,B_2(x)}\right] + 4\left(C_1(\mu)\|u\|_{\mathcal{C}^0_b(B_2(x))} + \mu^\gamma[u]^*_{2,\gamma,B_2(x)}\right)\ .$$

We now take $\mu > 0$ small enough and recall (B.4) to obtain, for each $0 < \gamma < \rho_1 \wedge \rho_2$, a constant $C_4^\gamma > 0$, independent on x, such that

$$[u]^*_{2,\gamma,B_2(x)} \leq C_4^\gamma \,(1 + |x|)$$

and we conclude by using [20, (6.9)] and the fact that the distance between a point of $B_1(x)$ and the boundary of $B_2(x)$ is a least 1. □

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