No-arbitrage in discrete-time markets with proportional transaction costs and general information structure

B. Bouchard

LPMA University Paris 6 and CREST

Motivation

- Information delay, delay in execution of orders.

- Aim: characterisation of the no-arbitrage property when the agent's filtration $\mathbb{H} = (\mathcal{H}_t)_t$ does not contain the filtration $\mathbb{F} = (\mathcal{F}_t)_t$ induced by the price processes

- Useful to obtain dual formulation for the set of super-replicable claims.

The case of markets without friction

- Discrete-time model*: $t \in \{0, 1, \ldots, T\}$. The closure property of the set

$$A_T \ := \ \left\{ G \in L^0 \ : \ \exists\, \phi \text{ s.t. } \sum_{t=0}^{T-1} \phi_t'(S_{t+1} - S_t) \geq G \right\}$$

is done as in the "Teachers' note"†.

- Separation and exhaustion argument: $\mathbb{Q} \sim \mathbb{P}$ with $d\mathbb{Q}/d\mathbb{P} \in L^\infty$ such that

$$\mathbb{E}^{\mathbb{Q}}[\phi_t'(S_{t+1} - S_t)] \leq 0$$

for all $\mathcal{H}_t$-meas. $\phi_t \in L^\infty$. This implies $\mathbb{E}^{\mathbb{Q}}[S_{t+1} - S_t \mid \mathcal{H}_t] = 0$, i.e.

$$S^{\mathbb{Q}} := (\mathbb{E}^{\mathbb{Q}}[S_t \mid \mathcal{H}_t])_{t \leq T} \quad \text{is a } \mathbb{H}\text{-martingale under } \mathbb{Q}.$$

*Kabanov Y. and C. Stricker, The Dalang-Morton-Willinger theorem under delayed and restricted information, preprint 2003.

†Kabanov Y. and C. Stricker, A teachers' note on no-arbitrage criteria, *Séminaire de Probabilités XXXV*, Lect. Notes Math. 1755, Springer, 149-152, 2001.

Aim

Extend this result to markets with frictions.

Aim

Extend this result to markets with frictions.

Problem

The recent modelisation does not fit with the case where $\mathbb{H}$ does not contain $\mathbb{F}$.

The modelisation of Kabanov et al.

Wealth process and self-financing condition

- Wealth process described in quantities and not amounts: V_t^i = number of units of asset i in the portfolio at time t.

The modelisation of Kabanov et al.

Wealth process and self-financing condition

- Wealth process described in quantities and not amounts: V_t^i = number of units of asset i in the portfolio at time t.

- Strategy: $\xi = (\xi_t)_{t \leq T}$ where ξ_t^i = number of units of asset i bought (against an other asset) at time t.

The modelisation of Kabanov et al.

Wealth process and self-financing condition

- Wealth process described in quantities and not amounts: V_t^i = number of units of asset i in the portfolio at time t.

- Strategy: $\xi = (\xi_t)_{t \leq T}$ where ξ_t^i = number of units of asset i bought (against an other asset) at time t.

- Simple wealth dynamic: $V_t(\xi) = \sum_{s \leq t} \xi_s$.

The modelisation of Kabanov et al.

Wealth process and self-financing condition

• Wealth process described in quantities and not amounts: V_t^i = number of units of asset i in the portfolio at time t.

• Strategy: $\xi = (\xi_t)_{t \leq T}$ where ξ_t^i = number of units of asset i bought (against an other asset) at time t.

• Simple wealth dynamic: $V_t(\xi) = \sum_{s \leq t} \xi_s$.

• Abstract self-financing condition: $\xi_t \in -K_t$ for all t. Write $\xi \in -K$.

$\hookrightarrow$ $-K_t$ is the set of affordable exchanges at time t given the price of the assets, the transaction costs,...

The modelisation of Kabanov et al.

Wealth process and self-financing condition

• Wealth process described in quantities and not amounts: V_t^i = number of units of asset i in the portfolio at time t.

• Strategy: $\xi = (\xi_t)_{t \leq T}$ where ξ_t^i = number of units of asset i bought (against an other asset) at time t.

• Simple wealth dynamic: $V_t(\xi) = \sum_{s \leq t} \xi_s$.

• Abstract self-financing condition: $\xi_t \in -K_t$ for all t. Write $\xi \in -K$.

$\hookrightarrow$ $-K_t$ is the set of affordable exchanges at time t given the price of the assets, the transaction costs,...

• K_t is a.s. a closed convex polyhedral cone such that $\mathbb{R}_+^d \setminus \{0\} \subset \text{ri}(K_t)$ a.s.

Solvency region

- $V_t \in K_t \Rightarrow \exists\ \xi_t \in -K_t$ such that $V_t = -\xi_t$, i.e. $V_t + \xi_t = 0$.

Solvency region

- $V_t \in K_t \Rightarrow \exists\ \xi_t \in -K_t$ such that $V_t = -\xi_t$, i.e. $V_t + \xi_t = 0$.

$\hookrightarrow$ Up to an immediate transfer we can reduce to a portfolio with non-negative holdings.

Solvency region

- $V_t \in K_t \Rightarrow \exists\ \xi_t \in -K_t$ such that $V_t = -\xi_t$, i.e. $V_t + \xi_t = 0$.

$\hookrightarrow$ Up to an immediate transfer we can reduce to a portfolio with non-negative holdings.

- K_t is called the "solvency region".

Example 1:

• 1 cash account with zero interest rate ($S^1 = 1$). 1 risky asset with price S^2 in units of the asset 1.

• Proportional transaction costs of rate λ on the transacted amount.

$\hookrightarrow$ Buying 1 unit of S^2 costs $(1+\lambda)S^2_t$ units of S^1, one receives $(1-\lambda)S^2_t$ units of S^1 when selling one unit of S^2.

• ξ^i= number of units of S^i

$\hookrightarrow$ $-K_t(\omega) = \{(x^1, x^2) \ : \ x^1 + x^2 S^2_t(\omega) + \lambda |x^2| S^2_t(\omega) \leq 0\}$

Example 2:

- Modelisation of a d-dimensional market in terms of bid-ask spreads.

- π^{ij} = number of units of i from which one can obtain one unit of j.

- The set of affordable exchanges at time t is:

$$-K_t(\omega) = \{x \in \mathbb{R}^d \ : \ \exists\, a \in \mathbb{M}^d_+,\ x^i \le \sum_{j\le d} \left[a^{ji} - a^{ij}\pi^{ij}_t(\omega)\right]\ ,\ i \le d\}\ .$$

- Rem: if S^i is the price in term of a numeraire and λ^{ij} is the transaction cost paied in units of S^i when exchanging units of S^i to get some units of S^j, then $\pi^{ij} = (S^j/S^i)(1+\lambda^{ij})$

Efficient and non-efficient frictions

- Set $K_t^0 := K_t \cap (-K_t)$.

- $\xi_t \in -K_t$ belong to K_t implies that there is $\tilde{\xi} \in -K_t$ such that $\xi_t = -\tilde{\xi}_t$.

$\hookrightarrow$ K_t^0 is the set of "reversible" exchanges.

Efficient and non-efficient frictions

- Set $K_t^0 := K_t \cap (-K_t)$.

- $\xi_t \in -K_t$ belong to K_t implies that there is $\tilde{\xi} \in -K_t$ such that $\xi_t = -\tilde{\xi}_t$.

$\hookrightarrow$ K_t^0 is the set of "reversible" exchanges.

- No friction: $K_t^0 := -K_t \Rightarrow$ any exchange is perfectly reversible.

Efficient and non-efficient frictions

• Set $K_t^0 := K_t \cap (-K_t)$.

• $\xi_t \in -K_t$ belong to K_t implies that there is $\tilde{\xi} \in -K_t$ such that $\xi_t = -\tilde{\xi}_t$.

$\hookrightarrow$ K_t^0 is the set of "reversible" exchanges.

• No friction: $K_t^0 := -K_t \Rightarrow$ any exchange is perfectly reversible.

• Efficient friction: $K_t^0 := \{0\} \Rightarrow$ no reversible exchange, i.e. there is no couple of assets that can be exchanged freely. This is equivalent to K_t^* has non-empty interior where

$$K_t^*(\omega) \;=\; \{y \;:\; \langle y, x\rangle \geq 0\}\;.$$

Efficient and non-efficient frictions

• Set $K_t^0 := K_t \cap (-K_t)$.

• $\xi_t \in -K_t$ belong to K_t implies that there is $\tilde{\xi} \in -K_t$ such that $\xi_t = -\tilde{\xi}_t$.

$\hookrightarrow$ K_t^0 is the set of "reversible" exchanges.

• No friction: $K_t^0 := -K_t \Rightarrow$ any exchange is perfectly reversible.

• Efficient friction: $K_t^0 := \{0\} \Rightarrow$ no reversible exchange, i.e. there is no couple of assets that can be exchanged freely. This is equivalent to K_t^* has non-empty interior where

$$K_t^*(\omega) \;=\; \{y \;:\; \langle y, x\rangle \geq 0\} \;.$$

• Mixed case: $K_t^0 \notin \{\{0\}, -K_t\} \Rightarrow$ some couple of assets can be exchanged freely, some other can not.

Notions of No-Arbitrage in the full information case

- Set of wealth process at time t

$$A_t = \{V_t(\xi),\ \xi \in -K\}$$

- 1. *Weak no-arbitrage* property

$$NA^w : \quad A_T \cap L^0(\mathbb{R}^d_+; \mathcal{F}) = \{0\}$$

Notions of No-Arbitrage in the full information case

- Set of wealth process at time t

$$A_t = \{V_t(\xi), \ \xi \in -K\}$$

- 1. *Weak no-arbitrage* property

$$NA^w \ : \quad A_T \cap L^0(\mathbb{R}^d_+; \mathcal{F}) = \{0\}$$

- 2. *Strict no-arbitrage* property

$$NA^s : \ A_t \cap L^0(K_t; \mathcal{F}_t) \subset L^0(K^0_t; \mathcal{F}_t) \ \text{ for all } \ t \,.$$

(recall $K^0_t := K_t \cap (-K_t)$ is the set of reversible exchanges).

Notions of No-Arbitrage in the full information case

- Set of wealth process at time t

$$A_t = \{V_t(\xi),\ \xi \in -K\}$$

- 1. *Weak no-arbitrage* property

$$NA^w \ : \quad A_T \cap L^0(\mathbb{R}^d_+;\mathcal{F}) = \{0\}$$

- 2. *Strict no-arbitrage* property

$$NA^s : \ A_t \cap L^0(K_t;\mathcal{F}_t) \subset L^0(K^0_t;\mathcal{F}_t) \ \text{ for all } \ t\,.$$

(recall $K^0_t := K_t \cap (-K_t)$ is the set of reversible exchanges).

- 3. *Robust no-arbitrage* property

$$NA^r : \ NA^w \text{ holds for some } \tilde{K} \text{ which dominates } K,$$

here $\tilde{K}$ dominates K if $K_t \setminus K^0_t \subset \mathrm{ri}(\tilde{K}_t)$ for all t.

Interpretation of NA^r

- Take

$$-K_t(\omega) = \{x \in \mathbb{R}^d \ : \ \exists\, a \in \mathbb{M}^d_+,\ x^i \le \sum_{j \le d} \left[a^{ji} - a^{ij}\pi_t^{ij}(\omega)\right]\ ,\ i \le d\}\ .$$

- π^{ij} = # of units of i from which one can obtain one unit of j

- Bid-ask spread: $[1/\pi_t^{ji}\ ,\ \pi_t^{ij}]$

Interpretation of NA^r

- Take

$$-K_t(\omega) = \{x \in \mathbb{R}^d \ : \ \exists\, a \in \mathbb{M}^d_+,\ x^i \leq \sum_{j \leq d} \left[a^{ji} - a^{ij}\pi^{ij}_t(\omega)\right]\ ,\ i \leq d\}\ .$$

- $\pi^{ij} = \#$ of units of i from which one can obtain one unit of j

- Bid-ask spread: $[1/\pi^{ji}_t\ ,\ \pi^{ij}_t]$

- No friction between i and j if $1/\pi^{ji}_t = \pi^{ij}_t$.

Interpretation of NA^r

- Take

$$-K_t(\omega) = \{x \in \mathbb{R}^d \ : \ \exists\, a \in \mathbb{M}^d_+,\ x^i \le \sum_{j\le d} \left[a^{ji} - a^{ij}\pi^{ij}_t(\omega)\right]\ ,\ i \le d\}\ .$$

- $\pi^{ij} = \#$ of units of i from which one can obtain one unit of j

- Bid-ask spread: $[1/\pi^{ji}_t\ ,\ \pi^{ij}_t]$

- No friction between i and j if $1/\pi^{ji}_t = \pi^{ij}_t$.

- NA^r: there is $\tilde\pi$ such that $[1/\tilde\pi^{ji}_t\ ,\ \tilde\pi^{ij}_t] \subset$ ri$[1/\pi^{ji}_t\ ,\ \pi^{ij}_t]$ and NA^w holds for $\tilde\pi$

$\hookrightarrow$ there is no-arbitrage even in a model with slightly lower transaction costs in the directions where they are not equal to 0.

Dual variables $\sim$ "equivalent martingale measures"

- Assume $A_T \cap L^1$ is closed. By Hahn-Banach and NA^w, find some $Z \in L^\infty$ such that

$$\mathbb{E}\left[\langle Z, G \rangle\right] \leq 0 \quad \text{for all} \quad G \in A_T \cap L^1 \,.$$

Dual variables $\sim$ "equivalent martingale measures"

- Assume $A_T \cap L^1$ is closed. By Hahn-Banach and NA^w, find some $Z \in L^\infty$ such that

$$\mathbb{E}\left[\langle Z, G \rangle\right] \leq 0 \quad \text{for all } \; G \in A_T \cap L^1 \; .$$

- Since $-K_t \cap L^1(\mathcal{F}_t) \subset A_T \cap L^1$ for all t

$$\mathbb{E}\left[\langle Z_t, \xi_t \rangle\right] \leq 0 \quad \text{for all } \; \xi_t \in -K_t \cap L^1(\mathcal{F}_t) \; .$$

with $Z_t = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$.

Dual variables $\sim$ "equivalent martingale measures"

- Assume $A_T \cap L^1$ is closed. By Hahn-Banach and NA^w, find some $Z \in L^\infty$ such that

$$\mathbb{E}\left[\langle Z, G\rangle\right] \leq 0 \quad \text{for all} \quad G \in A_T \cap L^1 .$$

- Since $-K_t \cap L^1(\mathcal{F}_t) \subset A_T \cap L^1$ for all t

$$\mathbb{E}\left[\langle Z_t, \xi_t\rangle\right] \leq 0 \quad \text{for all} \quad \xi_t \in -K_t \cap L^1(\mathcal{F}_t) .$$

with $Z_t = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$.

$\hookrightarrow$ $Z_t \in K_t^*$ + exhaustion under additional conditions: $Z_t \in \mathrm{ri}(K_t^*)$.

- Dual variables: $\mathcal{Z}$ the set of bounded martingales Z such that $Z_t \in \mathrm{ri}(K_t^*)$.

Interpretation of $\mathcal{Z}$

- Take

$$-K_t(\omega) = \{x \in \mathbb{R}^d \ : \ \exists\, a \in \mathbb{M}^d_+,\ x^i \le \sum_{j \le d} \left[a^{ji} - a^{ij}\pi^{ij}_t(\omega)\right]\ ,\ i \le d\}\ .$$

- $Z_t \in \mathrm{ri}(K^*_t)$ means

$$\frac{\tilde{Z}^j_t}{\tilde{Z}^i_t} = \frac{Z^j_t}{Z^i_t} \in \mathrm{ri}[1/\pi^{ji}_t\ ,\ \pi^{ij}_t]$$

where $\tilde{Z} = Z/Z^1$.

$\hookrightarrow$ there is a fictitious price process in the numéraire corresponding to the first asset which is a martingale under $d\mathbb{Q} := Z^1_T d\mathbb{P}$ such that the corresponding exchange rates evolve in the ri of the bid-ask spreads.

Characterisation of No-Arbitrage

- Theorem:

1. $\mathcal{Z} \neq \emptyset \Leftrightarrow NA^r$
2. $\mathcal{Z} \neq \emptyset \Rightarrow NA^s$ and the converse is true if $K^0 = \{0\}$.
3. $\mathcal{Z} \neq \emptyset \Rightarrow (NA^w$ and A_T is closed in probability)

Characterisation of No-Arbitrage

- Theorem:

1. $\mathcal{Z} \neq \emptyset \Leftrightarrow NA^r$
2. $\mathcal{Z} \neq \emptyset \Rightarrow NA^s$ and the converse is true if $K^0 = \{0\}$.
3. $\mathcal{Z} \neq \emptyset \Rightarrow (NA^w$ and A_T is closed in probability)

Kabanov Y., C. Stricker and M. Rásonyi, No arbitrage criteria for financial markets with efficient friction, *Finance and Stochastics*, 6 (3), 2002.

Kabanov Y., C. Stricker and M. Rásonyi, On the closedness of sums of convex cones in L^0 and the robust no-arbitrage property, *Finance and Stochastics* 7 (3), 2003.

Schachermayer W., The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time, *Mathematical Finance*, 14 (1), 19-48, 2004.

Main problems: back to example 1

- The set of affordable exchanges at time t is:

$$-K_t(\omega) = \{(x^1, x^2) \ : \ x^1 + x^2 S_t^2(\omega) + \lambda |x^2| S_t^2(\omega) \leq 0\}$$

Main problems: back to example 1

- The set of affordable exchanges at time t is:

$$-K_t(\omega) = \{(x^1, x^2) \ : \ x^1 + x^2 S_t^2(\omega) + \lambda |x^2| S_t^2(\omega) \leq 0\}$$

- If π not $\mathbb{H}$-adapted neither is K ! What about the constraint $\xi \in -K$?

Main problems: back to example 1

- The set of affordable exchanges at time t is:

$$-K_t(\omega) = \{(x^1, x^2) \ : \ x^1 + x^2 S_t^2(\omega) + \lambda |x^2| S_t^2(\omega) \leq 0\}$$

- If π not $\mathbb{H}$-adapted neither is K ! What about the constraint $\xi \in -K$?

- To get one unit of 1: $\xi_t^2 = -[S_t^2(1-\lambda)]^{-1}$ is not $\mathcal{H}_t$-meas. if S_t^2 is not !

New modelisation: order process and conversion ma

- Orders: η is a *simple order process* if it belongs to $L^0(\mathbb{M}^d; \mathbb{H})$.

New modelisation: order process and conversion ma

- Orders: η is a *simple order process* if it belongs to $L^0(\mathbb{M}^d; \mathbb{H})$.

- Typically:

$\eta_t^{ij} > 0$ is the order: buy $|\eta_t^{ij}|$ units of j against units of i.

$\eta_t^{ij} < 0$ is the order: sell $|\eta_t^{ij}|$ units of j against some units of i.

New modelisation: order process and conversion ma

- Orders: η is a *simple order process* if it belongs to $L^0(\mathbb{M}^d;\mathbb{H})$.

- Typically:

$\eta_t^{ij} > 0$ is the order: buy $|\eta_t^{ij}|$ units of j against units of i.

$\eta_t^{ij} < 0$ is the order: sell $|\eta_t^{ij}|$ units of j against some units of i.

- Conversion maps: $F = (F_t)$ a sequence of $\mathcal{F}$-meas. random continuous maps from $\mathbb{M}^d$ into $\mathbb{R}^d$.

$\hookrightarrow$ Converts order into net changes in the portfolio, i.e. to an order η_t associate $F_t(\eta_t)$ which is the impact on the portfolio of this order.

New modelisation: order process and conversion ma

• Orders: η is a *simple order process* if it belongs to $L^0(\mathbb{M}^d;\mathbb{H})$.

• Typically:

$\eta_t^{ij} > 0$ is the order: buy $|\eta_t^{ij}|$ units of j against units of i.

$\eta_t^{ij} < 0$ is the order: sell $|\eta_t^{ij}|$ units of j against some units of i.

• Conversion maps: $F = (F_t)$ a sequence of $\mathcal{F}$-meas. random continuous maps from $\mathbb{M}^d$ into $\mathbb{R}^d$.

$\hookrightarrow$ Converts order into net changes in the portfolio, i.e. to an order η_t associate $F_t(\eta_t)$ which is the impact on the portfolio of this order.

• F need not to be $\mathbb{H}$-adapted !

Assumptions on F

- HF_a : $\lambda F_t(a) = F_t(\lambda a)$ for all $\lambda \geq 0$ and $a \in \mathbb{M}^d$.

$\hookrightarrow$ the impact on the portfolio is proportional to the size of the order

Assumptions on F

- HF$_a$: $\lambda F_t(a) = F_t(\lambda a)$ for all $\lambda \geq 0$ and $a \in \mathbb{M}^d$.

$\hookrightarrow$ the impact on the portfolio is proportional to the size of the order

- HF$_b$: $F_t(a + a') \geq F_t(a) + F_t(a')$ for all $a,\ a' \in \mathbb{M}^d$.

$\hookrightarrow$ it is better to mutualise orders rather than to give them separately.

Assumptions on F

- HF$_a$: $\lambda F_t(a) = F_t(\lambda a)$ for all $\lambda \geq 0$ and $a \in \mathbb{M}^d$.

$\hookrightarrow$ the impact on the portfolio is proportional to the size of the order

- HF$_b$: $F_t(a + a') \geq F_t(a) + F_t(a')$ for all $a,\ a' \in \mathbb{M}^d$.

$\hookrightarrow$ it is better to mutualise orders rather than to give them separately.

- $F(\eta) = \sum_{i,j} \left[(\eta^{ij})^+ F(e_{ij}) + (\eta^{ij})^- F(-e_{ij}) \right]$ with $e_{ij}^{k,l} = \mathbf{1}_{(i,j)=(k,l)}$.

$\hookrightarrow$ the e_{ij} can be viewed as simple orders. Any order can be decomposed as the combination of simple ones. The impact is the corresponding combination of the impact of the simple orders.

Assumptions on F

• HF_a : $\lambda F_t(a) = F_t(\lambda a)$ for all $\lambda \geq 0$ and $a \in \mathbb{M}^d$.

$\hookrightarrow$ the impact on the portfolio is proportional to the size of the order

• HF_b : $F_t(a + a') \geq F_t(a) + F_t(a')$ for all $a,\ a' \in \mathbb{M}^d$.

$\hookrightarrow$ it is better to mutualise orders rather than to give them separately.

• $F(\eta) = \sum_{i,j} \left[(\eta^{ij})^+ F(e_{ij}) + (\eta^{ij})^- F(-e_{ij}) \right]$ with $e_{ij}^{k,l} = \mathbf{1}_{(i,j)=(k,l)}$.

$\hookrightarrow$ the e_{ij} can be viewed as simple orders. Any order can be decomposed as the combination of simple ones. The impact is the corresponding combination of the impact of the simple orders.

• HN^0: $F_t(\eta_t) \in N_t^0 \Rightarrow F_t(-\eta_t) = -F_t(\eta_t)$ where $N_t := \left\{ F_t(\eta),\ \eta \in L^0(\mathbb{M}^d; \mathcal{H}_t\right.$ and $N^0 = N \cap -N$.

Wealth process

- To $\xi \in N$, i.e. $\xi = F(\eta)$ for some order process η, we associate the wealth process

$$V_t(\xi) \ := \ \sum_{s \le t} \xi_s = \sum_{s \le t} F_s(\eta_s)$$

- Set of hedgeable claims with a strategy up to time t

$$A_t \ := \ \left\{ V_t(\xi) - r, \ \xi \in N, \ r \in L^0(\mathbb{R}^d_+) \right\}$$

Examples

Currency market #1

- F defined by

$$F_t^i(\eta_t) \; = \; \sum_{j=1}^{d} \left[\eta_t^{ji} - \eta_t^{ij} \left(\pi_t^{ij} \mathbf{1}_{\eta_t^{ij} \geq 0} + \frac{1}{\pi_t^{ji}} \mathbf{1}_{\eta_t^{ij} < 0} \right) \right] \quad ,$$

where

$$\pi^{ij} > 0 \; , \; \pi^{ii} \; = \; 1 \quad \text{and} \quad \pi^{ik}\pi^{kj} \; \geq \; \pi^{ij} \qquad \text{for all } i, j, k.$$

- $\pi^{ij} = \#$ of units of i from which one can obtain one unit of j

Currency market #2

- F defined by

$$F_t^i(\eta_t) \;=\; \sum_{j=1}^{d}\left[\eta_t^{ji} - \eta_t^{ij}\left(\pi_t^{ij}\mathbf{1}_{\eta_t^{ij}\geq 0} + \frac{1}{\pi_t^{ji}}\mathbf{1}_{\eta_t^{ij}<0}\right)\right] - \mathbf{1}_{i=1}\sum_{k\neq l}\lambda_t^{kl}|\eta_t^{kl}| \;\;,$$

where

$$\lambda^{ij} \geq 0\;,\; \pi^{ij} > 0\;,\; \pi^{ii} \;=\; 1 \quad \text{and} \quad \pi^{ik}\pi^{kj} \;\geq\; \pi^{ij} \qquad \text{for all } i,j,k.$$

- π^{ij} = # of units of i from which one can obtain one unit of j

- λ^{ij} = additional proportional cost paied in units of the first asset (e.g. execution cost paied in cash)

Stock market

- F defined by

$$F_t^1(\eta_t) = \sum_{1<i\leq d} \eta_t^{1i}\left(\pi_t^{i1}\mathbf{1}_{\eta_t^{1i}>0} + \pi_t^{1i}\mathbf{1}_{\eta_t^{1i}<0}\right) \quad \text{and} \quad F_t^i(\eta_t) = -\eta_t^{1i} \quad \text{for } i >$$

- Asset one is the numéraire (e.g. cash account).

- π^{i1} = number of physical units of asset 1 one receives when selling one unit of i

- π^{1i} = number of units of asset 1 one pays to buy one unit of i.

- Assume $\pi_t^{1i} \geq \pi_t^{i1}$.

Natural set of dual variables

- Assume that $A_T \cap L^1$ is closed and that

$$\mathrm{NA}^w \; : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \; .$$

Natural set of dual variables

- Assume that $A_T \cap L^1$ is closed and that

$$\mathsf{NA}^w \ : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \ .$$

- Then, by Hahn-Banach, there is $Z \in L^\infty$ such that

$$\mathbb{E}\left[\langle Z, G\rangle\right] \ \leq \ 0 \quad \text{for all} \ \ G \in A_T \cap L^1 \ .$$

Natural set of dual variables

- Assume that $A_T \cap L^1$ is closed and that

$$\text{NA}^w \; : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \; .$$

- Then, by Hahn-Banach, there is $Z \in L^\infty$ such that

$$\mathbb{E}\left[\langle Z, G\rangle\right] \;\leq\; 0 \quad \text{for all} \;\; G \in A_T \cap L^1 \; .$$

- For all bounded order process η: $F_t(\eta_t) \in A_T \cap L^1$. Thus,

$$\mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle\right] \;\leq\; 0 \;\; \forall \, \eta_t \quad \text{and thus} \quad \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right] \;\leq\; 0 \;\; \forall \, \eta_t \; .$$

Natural set of dual variables

- Assume that $A_T \cap L^1$ is closed and that

$$\mathsf{NA}^w \; : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \,.$$

- Then, by Hahn-Banach, there is $Z \in L^\infty$ such that

$$\mathbb{E}\left[\langle Z, G\rangle\right] \;\le\; 0 \quad \text{for all} \;\; G \in A_T \cap L^1 \,.$$

- For all bounded order process η: $F_t(\eta_t) \in A_T \cap L^1$. Thus,

$$\mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle\right] \;\le\; 0 \;\; \forall\, \eta_t \quad \text{and thus} \quad \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right] \;\le\; 0 \;\; \forall\, \eta_t \,.$$

- Under additional assumptions, one also get that $Z^i > 0$ for all i and

$$F_t(\eta_t)\mathbf{1}_{\{\mathbb{E}[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t]\}=0} \in N^0_t \,.$$

Definition and Interpretation

- We define $\bar{F}_t(\eta_t; Z) := \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right]$.

Definition and Interpretation

- We define $\bar{F}_t(\eta_t; Z) := \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right]$.

- Set of dual variables: $\mathcal{D}$ is the collection of elements Z of $L^\infty((0,\infty)^d)$ satisfying

$$\bar{F}_t(\eta_t; Z) \leq 0 \quad \text{and} \quad F_t(\eta_t)\mathbf{1}_{\bar{F}_t(\eta_t;Z)=0} \in N_t^0\,, \quad \text{for all } \eta \in L^0(\mathbb{M}^d; \mathbb{H}) \quad \text{and}$$

Definition and Interpretation

- We define $\bar{F}_t(\eta_t; Z) := \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right]$.

- Set of dual variables: $\mathcal{D}$ is the collection of elements Z of $L^\infty((0,\infty)^d)$ satisfying

$$\bar{F}_t(\eta_t; Z) \leq 0 \quad \text{and} \quad F_t(\eta_t)\mathbf{1}_{\bar{F}_t(\eta_t;Z)=0} \in N_t^0\,, \quad \text{for all } \eta \in L^0(\mathbb{M}^d; \mathbb{H}) \quad \text{and}$$

- Define $\widehat{F}$ by $\widehat{F}_t^i(\eta_t; Z) := \mathbb{E}\left[Z^i F_t^i(\eta_t) \mid \mathcal{H}_t\right] / \mathbb{E}\left[Z^i \mid \mathcal{H}_t\right]$

and $\bar{Z}_t = \mathbb{E}\left[Z \mid \mathcal{H}_t\right]$, $\tilde{Z} = \bar{Z}/\bar{Z}^1$ and $d\mathbb{Q} = \bar{Z}_T^1 d\mathbb{P}$.

Definition and Interpretation

- We define $\bar{F}_t(\eta_t; Z) := \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right]$.

- Set of dual variables: $\mathcal{D}$ is the collection of elements Z of $L^\infty((0,\infty)^d)$ satisfying

$$\bar{F}_t(\eta_t; Z) \le 0 \quad \text{and} \quad F_t(\eta_t)\mathbf{1}_{\bar{F}_t(\eta_t;Z)=0} \in N_t^0\,, \quad \text{for all } \eta \in L^0(\mathbb{M}^d; \mathbb{H}) \quad \text{and}$$

- Define $\hat{F}$ by $\hat{F}_t^i(\eta_t; Z) := \mathbb{E}\left[Z^i F_t^i(\eta_t) \mid \mathcal{H}_t\right] / \mathbb{E}\left[Z^i \mid \mathcal{H}_t\right]$

and $\bar{Z}_t = \mathbb{E}\left[Z \mid \mathcal{H}_t\right]$, $\tilde{Z} = \bar{Z}/\bar{Z}^1$ and $d\mathbb{Q} = \bar{Z}_T^1 d\mathbb{P}$.

- $\hat{F}(\cdot; Z)$ is the $(Z, \mathbb{H})$-expected impact of the order on the portfolio.

Definition and Interpretation

- We define $\bar{F}_t(\eta_t; Z) := \mathbb{E}\left[\langle Z, F_t(\eta_t)\rangle \mid \mathcal{H}_t\right]$.

- Set of dual variables: $\mathcal{D}$ is the collection of elements Z of $L^\infty((0,\infty)^d)$ satisfying

$$\bar{F}_t(\eta_t; Z) \le 0 \quad \text{and} \quad F_t(\eta_t)\mathbf{1}_{\bar{F}_t(\eta_t;Z)=0} \in N_t^0\,, \quad \text{for all } \eta \in L^0(\mathbb{M}^d; \mathbb{H}) \quad \text{and}$$

- Define $\hat{F}$ by $\hat{F}_t^i(\eta_t; Z) := \mathbb{E}\left[Z^i F_t^i(\eta_t) \mid \mathcal{H}_t\right] / \mathbb{E}\left[Z^i \mid \mathcal{H}_t\right]$

and $\bar{Z}_t = \mathbb{E}\left[Z \mid \mathcal{H}_t\right]$, $\tilde{Z} = \bar{Z}/\bar{Z}^1$ and $d\mathbb{Q} = \bar{Z}_T^1 d\mathbb{P}$.

- $\hat{F}(\cdot; Z)$ is the $(Z, \mathbb{H})$-expected impact of the order on the portfolio.

- The $(\mathbb{Q}, \mathbb{H})$-martingale $\tilde{Z}$ can be viewed as the expected price process in a model without transaction costs where the first asset is taken as a numéraire.

- In this “expected model”:

1. $\langle \tilde{Z}_t, \widehat{F}_t(\eta; Z)\rangle \leq 0$. The expected changes in unit in the portfolio multiplied by the expected price is non-positive

$\hookrightarrow$ Self-financing condition.

- In this "expected model":

1. $\langle \tilde{Z}_t, \widehat{F}_t(\eta; Z) \rangle \leq 0$. The expected changes in unit in the portfolio multiplied by the expected price is non-positive

$\hookrightarrow$ Self-financing condition.

2. $\langle \tilde{Z}_t, \widehat{F}_t(\eta; Z) \rangle = 0 \Leftrightarrow F_t(\eta_t)$ is reversible

$\hookrightarrow$ The self-financing condition is bind if the order is reversible, i.e. order between freely exchangeable assets.

No-arbitrage conditions

- 1. *Weak no-arbitrage* property:

$$NA^w \ : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \, .$$

No-arbitrage conditions

- 1. *Weak no-arbitrage* property:

$$NA^w \ : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \ .$$

- 2. *Strict no-arbitrage* property:

$$NA^s \ : \quad A_t \cap (-N_t + L^0(\mathbb{R}^d_+)) \subset N^0_t \quad \text{for all} \ \ t \ .$$

No-arbitrage conditions

- 1. *Weak no-arbitrage* property:

$$NA^w \ : \quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\} \ .$$

- 2. *Strict no-arbitrage* property:

$$NA^s \ : \quad A_t \cap (-N_t + L^0(\mathbb{R}^d_+)) \subset N^0_t \quad \text{for all} \ \ t \ .$$

- 3. *Robust no-arbitrage* property NA^r: there is a conversion map G such that for all $\eta \in L^0(\mathbb{M}^d; \mathbb{H})$ and t:

1. $G_t(\eta_t) \geq F_t(\eta_t)$
2. $F_t(\eta_t) \notin N^0_t \ \Rightarrow \ \mathbb{P}\left[\exists \, k \text{ such that } G^k_t(\eta_t) > F^k_t(\eta_t)\right] > 0$
3. NA^w holds for G.

No-arbitrage conditions

- 1. *Weak no-arbitrage* property:

$$NA^w \;:\quad A_T \cap L^0(\mathbb{R}^d_+) = \{0\}\ .$$

- 2. *Strict no-arbitrage* property:

$$NA^s \;:\quad A_t \cap (-N_t + L^0(\mathbb{R}^d_+)) \subset N_t^0 \quad \text{for all} \ \ t\ .$$

- 3. *Robust no-arbitrage* property NA^r: there is a conversion map G such that for all $\eta \in L^0(\mathbb{M}^d;\mathbb{H})$ and t:

  1. $G_t(\eta_t) \geq F_t(\eta_t)$
  2. $F_t(\eta_t) \notin N_t^0 \ \Rightarrow\ \mathbb{P}\left[\exists\, k \text{ such that } G_t^k(\eta_t) > F_t^k(\eta_t)\right] > 0$
  3. NA^w holds for G.

- The *efficient friction* assumption is $EF\ :\ N_t^0 = \{0\} \quad \text{for all} \ \ t\ .$

Remark on the robust no-arbitrage

- Take F defined by

$$F_t^i(\eta_t) \;=\; \sum_{j=1}^{d}\left[\eta_t^{ji} - \eta_t^{ij}\left(\pi_t^{ij}\mathbf{1}_{\eta_t^{ij}\geq 0} + \frac{1}{\pi_t^{ji}}\mathbf{1}_{\eta_t^{ij}<0}\right)\right] \;,$$

where

$$\pi^{ij} > 0\;,\;\pi^{ii} \;=\; 1 \quad \text{and} \quad \pi^{ik}\pi^{kj} \;\geq\; \pi^{ij} \qquad \text{for all } i,j,k.$$

- $\pi^{ij} = \#$ of units of i from which one can obtain one unit of j.

- The *robust no-arbitrage* property holds in the sense of Scachermayer if there is $\tilde{\pi}$ such that $[1/\tilde{\pi}_t^{ji}\,,\,\tilde{\pi}_t^{ij}] \subset \text{ri}[1/\pi_t^{ji}\,,\,\pi_t^{ij}]$ and NA^w holds for $\tilde{\pi}$.

$\hookrightarrow$ In the case where π is $\mathbb{H}$-adapted, the two definitions are equivalent.

Characterisation of the no-arbitrage properties

- Theorem:

1. If either NA^r or (NA^s and EF) hold, then $\mathcal{D} \neq \emptyset$ and A_T is closed in probability.

2. If $\mathcal{D} \neq \emptyset$ then NA^w, NA^s and NA^r hold.

Alternative characterization of $\mathcal{D}$

- $F_t(e_{ij})$: impact on the portfolio of the order: buy one unit of j against units of i.

- $F_t(-e_{ij})$: impact on the portfolio of the order: sell one unit of j against units of i.

Alternative characterization of $\mathcal{D}$

- $F_t(e_{ij})$: impact on the portfolio of the order: buy one unit of j against units of i.

- $F_t(-e_{ij})$: impact on the portfolio of the order: sell one unit of j against units of i.

- Set $-\widehat{K}_t(Z)(\omega) = \text{cone}\{\widehat{F}_t(e_{ij}; Z)(\omega)\ ,\ \widehat{F}_t(-e_{ij}; Z)(\omega)\ ,\ i,j\} - \mathbb{R}^d_+$,
$\hookrightarrow$ Affordable exchanges in expectation.

Alternative characterization of $\mathcal{D}$

- $F_t(e_{ij})$: impact on the portfolio of the order: buy one unit of j against units of i.

- $F_t(-e_{ij})$: impact on the portfolio of the order: sell one unit of j against units of i.

- Set $-\widehat{K}_t(Z)(\omega) = \text{cone}\{\widehat{F}_t(e_{ij}; Z)(\omega)\ ,\ \widehat{F}_t(-e_{ij}; Z)(\omega)\ ,\ i, j\} - \mathbb{R}^d_+$,
$\hookrightarrow$ Affordable exchanges in expectation.

- Consider its polar: $\widehat{K}^*_t(Z)(\omega) = \{y \in \mathbb{R}^d \ : \ \langle x, y\rangle \geq 0 \ \text{ for all } x \in \widehat{K}_t(Z)(\omega)\}$.

Alternative characterization of $\mathcal{D}$

• $F_t(e_{ij})$: impact on the portfolio of the order: buy one unit of j against units of i.

• $F_t(-e_{ij})$: impact on the portfolio of the order: sell one unit of j against units of i.

• Set $-\widehat{K}_t(Z)(\omega) = \text{cone}\{\widehat{F}_t(e_{ij}; Z)(\omega)\ ,\ \widehat{F}_t(-e_{ij}; Z)(\omega)\ ,\ i,j\} - \mathbb{R}^d_+$,
$\hookrightarrow$ Affordable exchanges in expectation.

• Consider its polar: $\widehat{K}_t^*(Z)(\omega) = \{y \in \mathbb{R}^d \ : \ \langle x,y\rangle \geq 0$ for all $x \in \widehat{K}_t(Z)(\omega)\}$.

• Characterization of $\mathcal{D}$:
1. $Z \in \mathcal{D}$ implies then $\bar{Z} \in \text{ri}(\widehat{K}^*(Z))$.
2. If $\mathbb{P}\left[F_t^k(\pm e_{ij}) > 0\right]\mathbb{P}\left[F_t^k(\pm e_{ij}) < 0\right] = 0$ then $\bar{Z} \in \text{ri}(\widehat{K}^*(Z))$ implies $Z \in \mathcal{D}$.

- Typically:

$F_t^k(e_{ij}) = 0$ if $k \notin \{i, j\}$, $-$ price of buying one unit j with units of i if $k = i$ and 1 if $k = j$

$F_t^k(-e_{ij}) = 0$ if $k \notin \{i, j\}$, gain for selling one unit j to get units of i if $k = i$ and -1 if $k = j$.

- Typically:

$F_t^k(e_{ij}) = 0$ if $k \notin \{i,j\}$, $-$ price of buying one unit j with units of i if $k = i$ and 1 if $k = j$

$F_t^k(-e_{ij}) = 0$ if $k \notin \{i,j\}$, gain for selling one unit j to get units of i if $k = i$ and -1 if $k = j$.

- In the perfect information case $\mathbb{H} = \mathbb{F}$.

1. $\widehat{F} = F$, $\bar{Z} = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$,

- Typically:

$F^k_t(e_{ij}) = 0$ if $k \notin \{i, j\}$, $-$ price of buying one unit j with units of i if $k = i$ and 1 if $k = j$

$F^k_t(-e_{ij}) = 0$ if $k \notin \{i, j\}$, gain for selling one unit j to get units of i if $k = i$ and -1 if $k = j$.

- In the perfect information case $\mathbb{H} = \mathbb{F}$.

1. $\widehat{F} = F$, $\bar{Z} = \mathbb{E}[Z \mid \mathcal{F}_t]$,

2. $-\widehat{K}_t(Z)(\omega) = -K_t = \text{cone}\{F_t(e_{ij})(\omega)\ ,\ F_t(-e_{ij})(\omega)\ ,\ i, j\} - \mathbb{R}^d_+$.

3. $\bar{Z} \in \text{ri}(K^*_t)$.

$\hookrightarrow$ We retrieve the characterization of the full information case.

Examples

Stock market

- F defined by

$$F_t^1(\eta_t) = \sum_{1<i\leq d} \eta_t^{1i} \left(\pi_t^{i1} \mathbf{1}_{\eta_t^{1i}>0} + \pi_t^{1i} \mathbf{1}_{\eta_t^{1i}<0} \right) \quad \text{and} \quad F_t^i(\eta_t) = -\eta_t^{1i} \quad \text{for } i >$$

Examples

Stock market

- F defined by

$$F_t^1(\eta_t) \;=\; \sum_{1<i\leq d} \eta_t^{1i}\left(\pi_t^{i1}\mathbf{1}_{\eta_t^{1i}>0} + \pi_t^{1i}\mathbf{1}_{\eta_t^{1i}<0}\right) \quad \text{and} \quad F_t^i(\eta_t) \;=\; -\eta_t^{1i} \quad \text{for } i >$$

- Set $\widehat{\pi}_t^{ij} := \mathbb{E}\left[Z^1\pi_t^{ij} \mid \mathcal{H}_t\right] / \bar{Z}_t^1$

Examples

Stock market

- F defined by

$$F_t^1(\eta_t) \;=\; \sum_{1<i\le d} \eta_t^{1i}\left(\pi_t^{i1}\mathbf{1}_{\eta_t^{1i}>0} + \pi_t^{1i}\mathbf{1}_{\eta_t^{1i}<0}\right) \quad \text{and} \quad F_t^i(\eta_t) \;=\; -\eta_t^{1i} \quad \text{for } i >$$

- Set $\hat{\pi}_t^{ij} := \mathbb{E}\left[Z^1\pi_t^{ij} \mid \mathcal{H}_t\right]/\bar{Z}_t^1$

$\hookrightarrow$ $\bar{Z}_t \in \mathrm{ri}(\widehat{K}_t^*(Z))$ if and only if $\bar{Z}_t^1\hat{\pi}_t^{i1} \le \bar{Z}_t^i \le \bar{Z}_t^1\hat{\pi}_t^{1i}$ with strict inequalities on $\{\hat{\pi}_t^{1i} > \hat{\pi}_t^{i1}\} = \{\mathbb{P}\left[\pi_t^{1i} > \pi_t^{i1} \mid \mathcal{H}_t\right] > 0\}$.

Examples

Stock market

- F defined by

$$F_t^1(\eta_t) = \sum_{1<i\le d} \eta_t^{1i}\left(\pi_t^{i1}\mathbf{1}_{\eta_t^{1i}>0} + \pi_t^{1i}\mathbf{1}_{\eta_t^{1i}<0}\right) \quad \text{and} \quad F_t^i(\eta_t) = -\eta_t^{1i} \quad \text{for } i >$$

- Set $\hat{\pi}_t^{ij} := \mathbb{E}\left[Z^1\pi_t^{ij} \mid \mathcal{H}_t\right]/\bar{Z}_t^1$

$\hookrightarrow$ $\bar{Z}_t \in \mathrm{ri}(\widehat{K}_t^*(Z))$ if and only if $\bar{Z}_t^1\hat{\pi}_t^{i1} \le \bar{Z}_t^i \le \bar{Z}_t^1\hat{\pi}_t^{1i}$ with strict inequalities on $\{\hat{\pi}_t^{1i} > \hat{\pi}_t^{i1}\} = \{\mathbb{P}\left[\pi_t^{1i} > \pi_t^{i1} \mid \mathcal{H}_t\right] > 0\}$.

- Set $d\mathbb{Q}/d\mathbb{P} = Z^1/\mathbb{E}\left[Z^1\right]$. Then, $\hat{\pi}_t = \mathbb{E}^{\mathbb{Q}}[\pi_t \mid \mathcal{H}_t]$.

Examples

Stock market

- F defined by

$$F_t^1(\eta_t) \;=\; \sum_{1<i\leq d} \eta_t^{1i}\left(\pi_t^{i1}\mathbf{1}_{\eta_t^{1i}>0} + \pi_t^{1i}\mathbf{1}_{\eta_t^{1i}<0}\right) \quad \text{and} \quad F_t^i(\eta_t) \;=\; -\eta_t^{1i} \quad \text{for } i >$$

- Set $\widehat{\pi}_t^{ij} := \mathbb{E}\left[Z^1\pi_t^{ij} \mid \mathcal{H}_t\right]/\bar{Z}_t^1$

$\hookrightarrow$ $\bar{Z}_t \in \mathrm{ri}(\widehat{K}_t^*(Z))$ if and only if $\bar{Z}_t^1\widehat{\pi}_t^{i1} \leq \bar{Z}_t^i \leq \bar{Z}_t^1\widehat{\pi}_t^{1i}$ with strict inequalities on $\{\widehat{\pi}_t^{1i} > \widehat{\pi}_t^{i1}\} = \{\mathbb{P}\left[\pi_t^{1i} > \pi_t^{i1} \mid \mathcal{H}_t\right] > 0\}$.

- Set $d\mathbb{Q}/d\mathbb{P} = Z^1/\mathbb{E}\left[Z^1\right]$. Then, $\widehat{\pi}_t = \mathbb{E}^{\mathbb{Q}}[\pi_t \mid \mathcal{H}_t]$.

$\hookrightarrow$ There is a $(\mathbb{Q},\mathbb{H})$-martingale $\bar{Z}/\bar{Z}^1$ such that each component i evolves in the ri of the "estimated" bid-ask spread $[\widehat{\pi}_t^{i1}, \widehat{\pi}_t^{1i}]$.

- In the “no frictions” case, i.e. $\pi^{i1} = \pi^{1i}$, then $\bar{Z}_t^1 \widehat{\pi}_t^{i1} = \bar{Z}_t^i = \bar{Z}_t^1 \widehat{\pi}_t^{1i}$. There is $\mathbb{Q} \sim \mathbb{P}$ under which the optional projection $\widehat{\pi}$ of the discounted price processes π on $\mathbb{H}$ are $(\mathbb{Q}, \mathbb{H})$-martingales.

Currency market

- F defined by

$$F_t^i(\eta_t) \;=\; \sum_{j=1}^{d} \left[\eta_t^{ji} - \eta_t^{ij} \left(\pi_t^{ij} \mathbf{1}_{\eta_t^{ij} \geq 0} + \frac{1}{\pi_t^{ji}} \mathbf{1}_{\eta_t^{ij} < 0} \right) \right] \;.$$

Currency market

- F defined by

$$F_t^i(\eta_t) \;=\; \sum_{j=1}^{d}\left[\eta_t^{ji} - \eta_t^{ij}\left(\pi_t^{ij}\mathbf{1}_{\eta_t^{ij}\geq 0} + \frac{1}{\pi_t^{ji}}\mathbf{1}_{\eta_t^{ij}<0}\right)\right] \quad .$$

- Expected prices

$$\widehat{\pi}_t^{ij,b} := \mathbb{E}\left[Z^i(\pi_t^{ji})^{-1} \mid \mathcal{H}_t\right]/\bar{Z}_t^i \quad \text{and} \quad \widehat{\pi}_t^{ij,a} := \mathbb{E}\left[Z^i\pi_t^{ij} \mid \mathcal{H}_t\right]/\bar{Z}_t^i \ .$$

Currency market

- F defined by

$$F_t^i(\eta_t) \;=\; \sum_{j=1}^{d}\left[\eta_t^{ji} - \eta_t^{ij}\left(\pi_t^{ij}\mathbf{1}_{\eta_t^{ij}\geq 0} + \frac{1}{\pi_t^{ji}}\mathbf{1}_{\eta_t^{ij}<0}\right)\right] \;.$$

- Expected prices

$$\hat{\pi}_t^{ij,b} := \mathbb{E}\left[Z^i(\pi_t^{ji})^{-1} \mid \mathcal{H}_t\right]/\bar{Z}_t^i \quad \text{and} \quad \hat{\pi}_t^{ij,a} := \mathbb{E}\left[Z^i\pi_t^{ij} \mid \mathcal{H}_t\right]/\bar{Z}_t^i \,.$$

- Then $\bar{Z}_t \in \mathrm{ri}(\hat{K}_t^{ij*}(Z))$ if and only if

$$\bar{Z}_t^i\hat{\pi}_t^{ij,b} \leq \bar{Z}_t^j \leq \bar{Z}_t^i\hat{\pi}_t^{ij,a} \;,$$

with strict inequalities on $\{\hat{\pi}_t^{ij,a} > \hat{\pi}_t^{ij,b}\} = \{\mathbb{P}\left[\pi_t^{ij}\pi_t^{ji} > 1 \mid \mathcal{H}_t\right] > 0\}$.

Super-hedging with partial information

- Let $G \in L^0$ be such that $G - 1c \in K_T$ for some $c \in \mathbb{R}$. Then,

$$G \in A_T \quad \text{if and only if} \quad \mathbb{E}\left[\langle Z, G \rangle\right] \leq 0 \quad \text{for all} \quad Z \in \mathcal{D}\,.$$