# A stochastic target formulation for optimal switching problems in finite horizon

**B. Bouchard**

LPMA University Paris 6 and CREST

## OUTLINE

1. Optimal switching problem

2. Stochastic target problem

3. Equivalence property

# The optimal switching problem

## Problem formulation

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- **Controlled process:** $X^\xi$ defined by

$$X_t \;=\; X_0 + \int_0^t b(X_s, \xi_s)ds + \int_0^t a(X_s, \xi_s)dW_s + \sum_{\tau_i^\xi \leq t} \beta(X_{\tau_i^\xi -}, \xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi}) \,.$$

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- **Reward function:**

$$\Pi(\xi) \;:=\; g(X_T^\xi, \xi_T) + \int_0^T f(X_s^\xi, \xi_s) ds - \sum_{\tau_i^\xi \leq T} c\left( X^\xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi} \right)$$

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- **Assumptions:** $b, a, \beta$ Lipschitz, $g, f, c$ locally Lipschitz, with

$$\sup_{x \in \mathbb{R}^d} \left(|\beta(x,i,j)| \vee 1 + c(x,i,j)^+\right) \le \Psi(i,j)$$

$$\sup_{(x,i,j) \in \mathbb{R}^d \times E^2} \frac{|g(x,i)| + |f(x,i)| + |c(x,i,j)|}{1 + |x|^{\bar{p}}} < \infty$$

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- **Admissibility:**

$$\mathbb{E}[|\sum_i \Psi(\xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi})|^{2\bar p} \mathbf{1}_{\tau_i^\xi \le T}] < \infty ,$$

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- **Convention:** $\beta(\cdot, e, e) = 0$ and $c(\cdot, e, e) = 1$.

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- **Value function:**

$$v(0,X_0,e_0) := \sup_{\xi\in\mathcal{S}_0(e_0)} \mathbb{E}\left[\,\Pi(\xi)\,\right]\;.$$

## General references

• Bensoussan A. and J.-L. Lions (1984), *Impulse control and quasi-variational inequalities*, Gauthier-Villars.

• Tang S. and J. Yong (1993), Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach, *Stoch. and Stoch. Reports*, 45, 145-176.

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- Important points (compare with Tang and Yong):

- the switching process may have an impact on the terminal reward function and the size of the jumps of the diffusion,
- the jump coefficients may depend on the current value of the diffusion process $X$, (continuity of $v$ ?)
- the cost function $c$ is not assumed to be positive (nor non-negative) (as in Ly Vath and Pham 2006).

# PDE characterization

- **Dynamic programming:** $v$ is l.s.c. and for all stopping time $\theta \leq T$ $\mathbb{P}-$ a.s.

$$
\begin{aligned}
v(t,x,e) &\geq \mathbb{E}[v(\theta, X_\theta^{(t,x),\xi}, \xi_\theta) + \int_t^\theta f(X_s^{(t,x),\xi}, \xi_s)ds] \\
&- \mathbb{E}[\sum_{t<\tau_i^\xi \leq \theta} c\left(X_{\tau_i^\xi -}^{(t,x),\xi}, \xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi}\right)] \,.
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- **PDE:** Formal argument

1. Immediate switch: $v(t,x,e) \geq v(t, x+\beta(x,e,j), e, j) - c(x,e,j) \;\forall\; j$.

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• **PDE:** Formal argument

1. Immediate switch: $v(t,x,e) \geq v(t, x+\beta(x,e,j), e, j) - c(x,e,j) \ \forall \ j$.

2. No switch: $\mathcal{L}^e\varphi := \frac{\partial}{\partial t}\varphi + b(\cdot,e)'D\varphi + \frac{1}{2}\mathrm{Tr}\left[aa'(\cdot,e)D^2\varphi\right] + f(\cdot,e) = 0$.

## ...PDE characterization (definition and characterization)

- **Viscosity solutions:** If for all $e \in E$, $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and $(t,x) \in [0,T) \times \mathbb{R}^d$ which realizes a local minimum (resp. maximum) of $v(\cdot,e) - \varphi$, we have

$$\min \{-\mathcal{L}^e\varphi(t,x)\ ,\ \mathcal{G}^e v(t,x,e)\} \geq 0 \quad (\text{resp} \leq 0)\ . \quad (\text{PDE})$$

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- **Theorem** : If $v$ is locally bounded, then it is a discontinuous viscosity solution of (PDE) with terminal condition

$$\min\{v(T-,x,e) - g(x,e) \, , \, \mathcal{G}^e v(T-,x,e)\} = 0 \quad , \quad \text{for all } (x,e) \in \mathbb{R}^d \times E \, . \quad \text{(B)}$$

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- **Contruction of the smallest solution:** $G^0 = g$ and

$$G^{n+1}(x,e) \;=\; \max_{j\in E}\Big(G^n(x+\beta(x,e,i),j) - c(x,e,j)\mathbf{1}_{j\neq e}\Big) \longrightarrow G\ .$$

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• **Proposition** If there is a locally bounded supersolution $\psi$ of (B), then
(i) $G$ is locally bounded and is the smallest solution of (B).

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- **Proposition** If there is a locally bounded supersolution $\psi$ of (B), then

(i) $G$ is locally bounded and is the smallest solution of (B).

(ii) Any subsolution $\varphi$ of (B) satisfies $\varphi \le G$ if

**H3** : If $(e_i, x_i)_{0\le i\le k}$, $k\ge 1$, is a sequence in $E\times\mathbb{R}^d$ such that $x_i = x_{i-1}+\beta(x_{i-1},e_{i-1},e_i)$ for $1\le i\le k$ and $e_k=e_0$, then $x_k = x_0$ and $\sum_{i=1}^k c(x_{i-1},e_{i-1},e_i)>0$.

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- **Proposition** Under **H3**, the only solution of (B) is $G$.

## ...PDE characterization (uniqueness)

**H1** : $v^+$ satisfies the growth condition

$$\sup_{(t,x,e)\in[0,T]\times\mathbb{R}^d\times E} |\psi(t,x,e)|/(1+|x|^\gamma) \; < \; \infty \, .$$

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**H2** : There is a function $\Lambda$ on $\mathbb{R}^d \times E$ satisfying

(i) $\Lambda(\cdot,e) \in C^2(\mathbb{R}^d)$ for all $e \in E$,

(ii) $b'D\Lambda + \frac{1}{2}\mathrm{Tr}\left[aa'D^2\Lambda\right] \leq \varrho\Lambda$ on $\mathbb{R}^d \times E$, for some $\varrho > 0$,

(iii) $\mathcal{G}^e\Lambda(x,e) \geq q(x)$ on $\mathbb{R}^d \times E$ for some continuous function $q > 0$,

(iv) $\Lambda \geq g^+$ ,

(v) $\Lambda(x,e)/|x|^\gamma \to \infty$ as $|x| \to \infty$ for all $e \in E$.

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(v) $\Lambda(x,e)/|x|^\gamma \to \infty$ as $|x| \to \infty$ for all $e \in E$.

• **Theorem** : Under **H1-H2**, $v$ is continuous and is the unique (discontinuous) viscosity solution of (PDE)-(B) satisfying the growth property of **H1**.

## Sufficient conditions for H1

- **Linear growth if:** $g(x,e) \leq C_1 + \eta' x$, $[\eta' b + f]^+ \leq C_2$ and $\eta'\beta - c \leq 0$.

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- **Growth with coefficient $\gamma \geq 1$ if:** $\exists$ a supersolution $w$ to (B) satisfying **H1** such that $w(\cdot, e) \in C^2(\mathbb{R}^d)$ for each $e$ and $(\mathcal{L}w)^+ + |Dw' a|$ is uniformly bounded.

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- **Growth with coefficient $\gamma = \bar{p}$ if:** $c \ge 0$ and $\beta = 0$ on $\{x \in \mathbb{R}^d \;:\; |x| \ge K\} \times E^2$.

# Sufficient condition for H2

- **General condition:** For some $\gamma \geq \bar{p}$, $\exists\ (d_i)_{i \in E}$ and $\alpha > 0$ such that

$$-\alpha \quad < \quad |x + \beta(x,i,j)|^{2\gamma} - |x|^{2\gamma} \qquad \text{for all} \ \ (x,i,j) \in \mathbb{R}^d \times E^2$$

$$\eta \quad := \quad \min_{i,j \in E} \inf_{x \in \mathbb{R}^d} \frac{d_i - d_j + c(x,i,j)}{|x + \beta(x,i,j)|^{2\gamma} - |x|^{2\gamma} + \alpha} > 0 \,.$$

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• **In this case, take:** $\Lambda(t,x,e) := (d + \eta|x|^{2\gamma} + d_e)$ for some $d > 0$ large enough so that $\Lambda \geq g^+$.

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- **Examples:**

1. $c \geq \varepsilon$ for some $\varepsilon > 0$ and $\beta$ has a compact support.

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- **Examples:**

1. $c \geq \varepsilon$ for some $\varepsilon > 0$ and $\beta$ has a compact support.

2. $c$ independent of $x$, satisfies a strict triangular condition

$$c(i,j) + c(j,k) \quad > \quad c(i,k) \quad , \; i,j,k \in E \; ,$$

and $\beta$ has a compact support.

# The stochastic target problem (super-hedging)

## References

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- **Reachability set:** $\Gamma := \{$ initial conditions $z$ such that $Z_T^{z,\nu} \in G$ $\mathbb{P}-$a.s. for some $\nu$ $\}$.

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- **Simplification:** $Z^{z,\nu} = (Y^{(y,x),\nu}, X^{x,\nu}) \in \mathbb{R} \times \mathbb{R}^d$ and $G = \mathcal{E}pi(g)$ with $g : \mathbb{R}^d \mapsto \mathbb{R}$. Generalization of super-hedging problems for fixed $x$.

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## References

- $Z^{z,\nu}$: controlled (by $\nu$) process with initial condition $z$.

- $G \subset \mathbb{R}^d$: target.

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- **Brownian diffusions + jumps:** Bouchard B. (2002), Stochastic Targets with Mixed diffusion processes, *Stochastic Processes and their Applications*, 101, 273-302.

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• $\Rightarrow$ **Hedges the reward associated to a random policy.**

# Dynamic programming

## Dynamic programming

- **Supersolution property:** $y > u(0, X_0, e_0) \Rightarrow \exists\, \phi$ s.t. for all stopping times $\theta \leq T$ $\mathbb{P}-$a.s.

$$Y_\theta^{y,\phi} \;\geq\; u(\theta, X_\theta^N, N_\theta) + \int_0^\theta f(X_s^N, N_s)ds - \sum_{\tau_i^N \leq \theta} c(X_{\tau_i^N-}^N, N_{\tau_i^N-}, N_{\tau_i^N}) \;\; \mathbb{P}-\text{a.s.}$$

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- **Subsolution property:** $y < u(0, X_0, e_0) \Rightarrow \forall\, \phi$ and stopping times $\theta \le T$ $\mathbb{P}-$a.s.

$$\mathbb{P}\left[ Y_\theta^{y,\phi} > u(\theta, X_\theta^N, N_\theta) + \int_0^\theta f(X_s^N, N_s)ds - \sum_{\tau_i^N \le \theta} c(X^N_{\tau_i^N -}, N_{\tau_i^N -}, N_{\tau_i^N}) \right] < 1\ .$$

## PDE characterization

- **Formal argument:** Set $y = u(0,x,e)$ and "$X_{0-} = x$", "$N_{0-} = e$".

$$
\begin{aligned}
0 &\leq d\left(Y_0^{y,\phi} - u(0,X_0,N_0) - \int_0^{0+} f(X_s^N,N_s)ds + \sum_{\tau_i^N \leq 0+} c(X_{\tau_i^N -},N_{\tau_i^N -},N_{\tau_i^N})\right) \\
&= (\phi_0 - a\nabla u)\cdot dW_0 \\
&- \mathcal{L}^e u dt \\
&- \mathbf{1}_{\tau_1^N = 0}\left\{u\left(0,x+\beta(x,e,N_0),\right) - u\left(0,x,e\right) + c(x,e,N_0)\right\} .
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## ...PDE characterization

• **Theorem** : If $u$ is locally bounded, then it is a discontinuous viscosity solution of

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**H1'** : $u^+$ satisfies the growth condition

$$\sup_{(t,x,e)\in[0,T]\times\mathbb{R}^d\times E} |\psi(t,x,e)|/(1+|x|^\gamma) \; < \; \infty \; .$$

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**H2** : There is a function $\Lambda$ on $\mathbb{R}^d \times E$ satisfying

(i) $\Lambda(\cdot, e) \in C^2(\mathbb{R}^d)$ for all $e \in E$,

(ii) $b' D\Lambda + \frac{1}{2}\mathrm{Tr}\left[aa' D^2\Lambda\right] \leq \varrho\Lambda$ on $\mathbb{R}^d \times E$, for some $\varrho > 0$,

(iii) $\mathcal{G}^e\Lambda(x,e) \geq q(x)$ on $\mathbb{R}^d \times E$ for some continuous function $q > 0$,

(iv) $\Lambda \geq g^+$ ,

(v) $\Lambda(x,e)/|x|^\gamma \to \infty$ as $|x| \to \infty$ for all $e \in E$.

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• **Theorem** : Under **H1'-H2**, $u$ is continuous and is the unique (discontinuous) viscosity solution of (PDE)-(B) satisfying the growth property of **H1'**.

# Equivalence property

## Equivalence through PDEs

• **Theorem** If $u$ and $v$ are locally bounded, then both are discontinuous viscosity solutions of

$$\min\left\{-\mathcal{L}^e\varphi(t,x)\ ,\ \mathcal{G}^e V(t,x,e)\right\} = 0\ .$$

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$$\min\left\{V(T-,x,e) - g(x,e)\ ,\ \mathcal{G}^e V(T-,x,e)\right\} = 0\ ,\quad \text{for all}\ \ (x,e) \in \mathbb{R}^d \times E\ .$$

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• **Theorem** If $u^+$ and $v^+$ have polynomial growth and **H2** holds, then $u = v$, i.e.

$$\sup_{\xi\in\mathcal{S}_0(e_0)} \mathbb{E}[\,\Pi(\xi)\,] \;=\; \text{Super-hedging price of } \Pi(N)$$

Economic value of the firm = Hedging value of the total reward for a policy viewed as random

Recall

$$\Pi(\xi) = g(X_T^\xi, \xi_T) + \int_0^T f(X_s^\xi, \xi_s)ds - \sum_{\tau_i^\xi \le T} c\left(X^\xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi}\right)\ .$$

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# An other dual formulation

## An other dual formulation

- **Dual variables:** Let $\mathcal{U}$ = set of predictable essentially bounded processes $\nu = (\nu^0, \ldots, \nu^\kappa)$ with values in $(0, \infty)^{\kappa+1}$. Set

$$H^\nu \ := \ \mathcal{E}\left(\int_0^T \int_E (\nu_t^\sigma - 1)\tilde{\mu}(d\sigma, dt)\right) \mapsto \mathbb{Q}^\nu \sim \mathbb{P} \ .$$

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