Optimal reflection of diffusions and barrier options pricing under constraints

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Outline

I. Barrier options with constraints

II. Interpretation in terms of reflected process

III. Optimal control of reflection for SDEs

IV. Dual formulation for the hedging price

Part I: Barrier options pricing under constraints (sum up)

Financial market

- X d-risky assets

$$X_t = X_0 + \int_0^t \mathrm{diag}\,[X_s]\,\sigma(s, X_s)dW_s \quad , \quad t \le T$$

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Wealth process: $\phi \in K$ $\mathrm{d}t \times \mathrm{d}\mathbb{P} -$ a.s. and

$$Y_t \; = \; y + \int_0^t Y_s \phi_s' \mathrm{diag}\,[X_s]^{-1}\,dX_s \; = \; y + \int_0^t Y_s \phi_s' \sigma(s,X_s)dW_s \; ,$$

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Super-hedging problem: $\tau =$ exit time from $[0,T) \times \mathcal{O}$

$$v(0,X_0) \;:=\; \inf\left\{ y \in \mathbb{R} \;:\; Y_\tau^{y,\phi} \ge g(\tau, X_\tau) \;\; \text{for some } \phi \in \mathcal{K} \right\}$$

Vanilla options: Explosion of the hedge in BS

- Black and Scholes model: σ is constant

$$X_t \;=\; X_0 + \int_0^t \text{diag}\,[X_s]\,\sigma dW_s \;\;,\;\; t \le T$$

- If no contraints: $Y_t = v(t, X_t)$

$$dY_t \;=\; Y_t \phi_t' \sigma dW_t \;=\; dv(t, X_t) \;=\; v_x(t, X_t) X_t \sigma dW_t$$

$\Rightarrow \phi_t = X_t v_x(t, X_t)/v(t, X_t)$.

...Vanilla options : Explosion of the hedge in B

Example 1. Digital option in dimension 1

- $g(x) = 1_{x \geq \kappa}$
- $X_t = X_0 e^{-\sigma^2 T/2 + \sigma W_T}$
- $g(X_T) = \hat{g}(W_T) = \mathbf{1}_{W_T \geq \hat{\kappa}}$ with $\hat{\kappa} = [\ln(\kappa/X_0) + \sigma^2 T/2]/\sigma$.
- $\hat{v}(t,w) = v(t, X_0 e^{-\sigma^2 t/2 + \sigma w}) = \mathbb{P}\left[W_T - W_t \geq \hat{\kappa} - w\right]$

...Vanilla options : Explosion of the hedge in B

Example 1. Digital option in dimension 1

- Hedge: $\phi_t = X_t v_x(t, X_t)/v(t, X_t) = \widehat{v}_w(t, W_t)/\widehat{v}(t, W_t)$.
- $\widehat{v}_w(t, w) = f_{T-t}(\widehat{\kappa} - w) = (2\phi(T-t))^{\frac{1}{2}} \exp(-[\widehat{\kappa} - w]^2/[2(T-t)])$
- $\widehat{v}(t, w) \geq 1/2$ if $w \geq \widehat{\kappa}$
- For $\widehat{\kappa} \leq W_t \leq \widehat{\kappa} + C(T-t)^{\frac{1}{2}}$ but $T-t$ very small:

$$\phi_t = \widehat{v}_w(t, W_t)/\widehat{v}(t, W_t) \text{ very large !}$$

...Vanilla options : Explosion of the hedge in B

Example 2. Up-and-out call in dimension 1

• $\mathcal{O} = (0, U)$ and $g(t, x) = [x - \kappa]^+ \mathbf{1}_{t=T} \mathbf{1}_{x<U}$: similar problem when approaches $\{T, U\}$ if $U > \kappa$.

...Vanilla options : Explosion of the hedge in B

Usual practice

- For Vanilla option: smoothing of the payoff

- For Barrier options: move the barrier

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$\Rightarrow$ use portfolio constraint to rationalize these practices.

...Vanilla options : Explosion of the hedge in B

Usual practice

- For Vanilla option: smoothing of the payoff ok

- For Barrier options: move the barrier not so simple

$\Rightarrow$ use portfolio constraint to rationalize these practices.

Formal Derivation of the PDE

* $Y_t = v(t, X_t)$ and

$$\begin{aligned} dY_t &= Y_t \phi_t' \sigma(t, X_t) dW_t \geq dv(t, X_t) \\ &= \mathcal{L}v(t, X_t)dt + Dv(t, X_t)' \text{diag}\,[X_t] \sigma(t, X_t) dW_t \end{aligned}$$

where

$$\mathcal{L}v(t, x) = \frac{\partial}{\partial t} v(t, x) + \frac{1}{2} \text{Trace}[a(t, x) D^2 v(t, x)]$$

and $a(t, x) = \text{diag}\,[x]\, \sigma(t, x) \sigma(t, x)' \text{diag}\,[x]$

* $\phi_t \in K \Rightarrow \text{diag}\,[X_t]\, Dv(t, X_t)/v(t, X_t) \in K$.

* $\min_{\rho \in \text{dom}(\delta) \cap \partial B_1} \delta(\rho) v - \rho' \text{diag}\,[x]\, Dv \geq 0$ with

$$\delta(\rho) = \sup_{\xi \in K} \xi \cdot \rho$$

PDE characterization

(In BS model, under smoothness assumptions and $g(t,\cdot) = 0$ for $t < T$)

* **Inside the domain**

$$\min\{-\mathcal{L}v\ ,\ \inf_{\rho}\left(\delta(\rho)v - \rho'\mathrm{diag}\,[x]\,Dv\right)\} \;=\; 0\ .$$

* **On the time boundary** $\{T\} \times \bar{\mathcal{O}}$

$$\min\{v - g\ ,\ \inf_{\rho}\left(\delta(\rho)v - \rho'\mathrm{diag}\,[x]\,Dv\right)\} \;=\; 0\ .$$

* **On the spacial boundary** $[0, T) \times \partial\mathcal{O}$

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$$v(T,x) = \hat{g}(x)\ .$$

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$$\inf_{\rho} \Big(\delta(\rho)v - \rho' \mathrm{diag}\,[x]\, Dv \Big) \;=\; 0\,.$$

Neumann boundary condition with control on the direction of reflection !

Part II: Interpretation in terms of reflected process

Barrier options: Shortselling constraints in BS

- Schmock U., S. E. Shreve and U. Wystup (2002). Valuation of exotic options under shortselling constraints. *Finance and stochastics*, 6, 143-172.

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- Starting point: dual formulation of Cvitanič J. and I. Karatzas (1993)* and Föllmer H. and D. Kramkov (1997) †

*Hedging contingent claims with constrained portfolios. *Annals of Applied Probability*, 3, 652-681.

†Optional decomposition under constraints. *Probability Theory and Related Fields*, 109, 1-25.

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* **Set of dual variables:** $\tilde{\mathcal{K}}$: the set of bounded adapted processes ϑ taking values in $\tilde{K} = \mathrm{dom}(\delta)$.

◈ **"Super-martingale measures":** Associate the $\mathbb{P}$-equivalent probability measure $\mathbb{Q}^\vartheta$

$$\frac{\mathrm{d}\mathbb{Q}^\vartheta}{\mathrm{d}\mathbb{P}} = e^{-\frac{1}{2}\int_0^T |\sigma(t,X_t)^{-1}\vartheta_t|^2 dt + \int_0^T (\sigma(t,X_t)^{-1}\vartheta_t)' dW_t}$$

and denote by $\mathbb{E}^\vartheta$ the associated expectation operator.

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Discounting factor: $\mathcal{E}_r^\vartheta := e^{-\int_0^r \delta(\vartheta_t)dt}$.

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Dual formulation: $W^\vartheta = W - \int_0^\cdot \sigma(t,X_t)^{-1}\vartheta_t dt$

$$d(\mathcal{E}_t^\vartheta Y_t) = \mathcal{E}_t^\vartheta Y_t \left(\vartheta_t'\phi_t - \delta(\vartheta_t)\right) dt + \mathcal{E}_t^\vartheta Y_t \phi_t' \sigma(t,X_t) dW_t^\vartheta \qquad (\phi \in K \Rightarrow \vartheta'\phi \leq \delta(\vartheta))$$

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Dual formulation: $Y_\tau^{y,\phi} \geq g(\tau, X_\tau)$ implies $y \geq \mathbb{E}^\vartheta\left[\mathcal{E}_\tau^\vartheta g(\tau, X_\tau)\right]$

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* **Discounting factor:** $\mathcal{E}_r^\vartheta := e^{-\int_0^r \delta(\vartheta_t) dt}$.

Dual formulation: (assume $g > 0$ uniformly)

$$v(0, X_0) = \sup_{\vartheta \in \tilde{\mathcal{K}}} \mathbb{E}^\vartheta \left[\mathcal{E}_\tau^\vartheta g(\tau, X_\tau) \right] .$$

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- **A simple case:** 1-dim. BS model with $K = [-\alpha, \infty)$, $\mathcal{O} = (0, U)$, $g(t,x) = [x-\kappa]^+ \mathbf{1}_{t=T} \mathbf{1}_{x<U}$:

$$v(0, X_0) = \sup_{\vartheta \geq 0} \mathbb{E}\left[e^{-\alpha \int_0^T \vartheta_t dt} g(T, X_T e^{-\int_0^T \vartheta_t dt}) \mathbf{1}_{\{\sup_{s \leq T} X_s e^{-\int_0^s \vartheta_t dt} < U\}} \right]$$

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Extension and solution: Extend the class of dual variables to obtain existence

$$v(0, X_0) = \sup_{\phi \in \mathcal{R}} \mathbb{E}\left[e^{-\alpha \phi_T} g(T, X_T e^{-\phi_T}) \mathbf{1}_{\{\sup_{s \leq T} X_s e^{-\phi_s} < U\}} \right]$$

with $\mathcal{R}$ the set of non-decreasing continuous adapted processes with $\phi_0 = 0$.

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$\Rightarrow$ $\phi^* =$ local time which causes reflection of X on U.

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Dual formulation in terms of reflected process:

$$v(0, X_0) = \mathbb{E}\left[e^{-\alpha \phi_T^*} g(T, X_T e^{-\phi_T^*}) \mathbf{1}_{\{\sup_{s \leq T} X_s e^{-\phi_s^*} < U\}} \right]$$

$\Rightarrow$ $\phi^* =$ local time which causes reflection of X on U.

Part III: Optimal control of reflection for SDEs

Problem formulation

◈ **Controlled SDE:** Given a "control" process $\beta = (\alpha, \epsilon)$, let $(X^{\alpha,\epsilon}, L^{\alpha,\epsilon})$ be a continuous adapted process with $L^{\alpha,\epsilon} \in \mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ non-decreasing satisfying

$$
\begin{aligned}
X(s) &= x + \int_t^s \mu(X(r), \beta_r) dr + \int_t^s \sigma(X(r), \beta_r) dW(r) + \int_t^s \gamma(X(r), \epsilon_r) dL(r) \\
L(s) &= \int_t^s \mathbf{1}_{\{X(r) \in \partial\mathcal{O}\}} \, d|L|(r) \ , \quad t \le s \le T \ .D
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\end{aligned}
$$

Control problem:

$$
u(t,x) \ := \ \sup_{(\alpha,\epsilon)} J(t,x;\alpha,\epsilon)
$$

where

$$
\begin{aligned}
J(t,x;\alpha,\epsilon) &:= \mathbb{E}\left[\beta^{\alpha,\epsilon}_{t,x}(T) g\left(X^{\alpha,\epsilon}_{t,x}(T)\right) + \int_t^T \beta^{\alpha,\epsilon}_{t,x}(s) f\left(X^{\alpha,\epsilon}_{t,x}(s), \alpha(s), \epsilon(s)\right) ds\right] \\
\beta^{\alpha,\epsilon}_{t,x}(s) &:= e^{-\int_t^s \rho(X^{\alpha,\epsilon}_{t,x}(r), \epsilon(r)) dL^{\alpha,\epsilon}_{t,x}(r)}\ .
\end{aligned}
$$

References

- Lions P.-L. and A-S. Sznitman (1984). Stochastic Differential Equations with Reflecting Boundary Conditions, *Comm. Pure Appl. Math.*

- Dupuis P. and H. Ishii (1993). SDEs with oblique reflection on nonsmooth domains. *The Annals of Probability*.

- Ding D. (2000). A note on stochastic optimal control of reflected diffusions with jumps. *Applied Mathematics and Mechanics*.

- Kushner H.-J. (1996). A numerical method for reflected diffusions: Control of the reflection directions and applications. *Applied Mathematics and Optimization*.

...Problem formulation

- **Existence of $(X^{\alpha,\epsilon}, L^{\alpha,\epsilon})$?**

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◈ **Dynamic programming for u ?**

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- **PDE characterization ?**

Existence of the controlled reflected SDE

Theorem (Dupuis and Ishii Fix $\gamma \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ with $|\gamma| = 1$. Assume that $\mathcal{O}$ is open and bounded and that there exists some $r \in (0, 1)$ for which

$$\bigcup_{0 \leq \lambda \leq r} B\left(x - \lambda\gamma(x), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } x \in \partial\mathcal{O} \, .$$

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◈ Then, for all $\psi \in C([0,T], \mathbb{R}^d)$ satisfying $\psi(0) \in \bar{\mathcal{O}}$, there exists $(\phi, \eta) \in C([0,T], \bar{\mathcal{O}}) \times \mathrm{BV}([0,T], \mathbb{R}_+)$ such that

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s))d\eta(s) \;,\; \eta(t) = \int_0^t \mathbf{1}_{\{\phi(s) \in \partial\mathcal{O}\}} d|\eta|(s) \quad,\; t \le T \,.$$

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Then, for all $\psi \in C([0,T], \mathbb{R}^d)$ satisfying $\psi(0) \in \bar{\mathcal{O}}$, there exists $(\phi, \eta) \in C([0,T], \bar{\mathcal{O}}) \times \mathrm{BV}([0,T], \mathbb{R}_+)$ such that

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Moreover, $(\phi(t), \eta(t)) \in \sigma(\psi(s), s \le t)$ for all $t \le T$, and uniqueness holds if $\psi \in \mathrm{BV}([0,T], \mathbb{R}^d)$.

...Existence of the controlled reflected SDE

- Lemma (Dupuis and Ishii) Let X be a continuous semimartingale with values in $\bar{\mathcal{O}}$. Assume that Y is a continuous semimartingale with values in $\bar{\mathcal{O}}$ satisfying for $t \leq T$

$$Y(t) \;=\; X(0) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(Y(s))dL(s) \; ,$$

where L is an element of $\mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ such that

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Let X' be an other continuous semimartingales with values in $\bar{\mathcal{O}}$ and assume that (Y', L') satisfies the same properties as (Y, L) with X' in place of X.

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Let X' be an other continuous semimartingales with values in $\bar{\mathcal{O}}$ and assume that (Y', L') satisfies the same properties as (Y, L) with X' in place of X. Then, there is a contant $C > 0$ such that

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\le t}|Y(s)-Y'(s)|^2 + \int_0^t \left|Y(s)-Y'(s)\right|^2 d(L+L')(s)\right]\\ &\le C\left(|X(0)-X'(0)|^2 + \int_0^t \mathbb{E}\left[\sup_{0\le s\le u}|X(s)-X'(s)|^2\right]du\right) \quad,\;\; t\le T\;.\end{aligned}$$

...Existence of the controlled reflected SDE

• Corollary (Dupuis and Ishii) Fix $(t, x) \in [0, T] \times \bar{\mathcal{O}}$. Then, there exists a unique continuous adapted process (X, L) such that $L \in$ $\mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ and

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We fix an open bounded set $\mathcal{O} \subset \mathbb{R}^d$ and $\gamma \in C(\mathbb{R}^{d+\ell}, \mathbb{R})$ such that

$$\gamma \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^d) \ , \ |\gamma| = 1$$

$$\exists\, r \in (0,1) \text{ s.t. } \bigcup_{0 \le \lambda \le r} B\left(x - \lambda\gamma(x,e), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } (x,e) \in \partial\mathcal{O} \times \mathbb{R}^\ell \ .$$

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...Existence of the controlled reflected SDE

Theorem Let the above conditions hold. Fix $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ and $\beta = (\alpha, \epsilon) \in \mathcal{A} \times \mathcal{E}$. Then, there exists a unique continuous adapted process (X, L) such that L is non-decreasing, belongs to $\mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ and

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\begin{aligned}
X(s) &= x + \int_t^s \mu(X(r), \beta_r)dr + \int_t^s \sigma(X(r), \beta_r)dW(r) + \int_t^s \gamma(X(r), \epsilon_r)dL(r) \\
L(s) &= \int_t^s \mathbf{1}_{\{X(r) \in \partial\mathcal{O}\}} \, d|L|(r) \ , \quad t \le s \le T \ .
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Remark: If Y is an Ito process with values in a compact then existence also holds for

$$\begin{aligned} X(s) &= x + \int_t^s \mu(X(r),\beta_r)dr + \int_t^s \sigma(X(r),\beta_r)dW(r) + \int_t^s \gamma(X(r),\epsilon_r,Y_r)dL \\ L(s) &= \int_t^s \mathbf{1}_{\{X(r)\in\partial\mathcal{O}\}}\, d|L|(r) \;, \quad t \le s \le T \;. \end{aligned}$$

Problem formulation (bis)

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Control problem: $u(t,x) := \sup_{(\alpha,\epsilon)} J(t,x;\alpha,\epsilon)$ where

$$
\begin{aligned}
J(t,x;\alpha,\epsilon) &:= \mathbb{E}\left[\beta^{\alpha,\epsilon}_{t,x}(T) g\left(X^{\alpha,\epsilon}_{t,x}(T)\right) + \int_t^T \beta^{\alpha,\epsilon}_{t,x}(s) f\left(X^{\alpha,\epsilon}_{t,x}(s), \alpha(s), \epsilon(s)\right) ds\right] \\
\beta^{\alpha,\epsilon}_{t,x}(s) &:= e^{-\int_t^s \rho(X^{\alpha,\epsilon}_{t,x}(r),\epsilon(r))dL^{\alpha,\epsilon}_{t,x}(r)} \,,
\end{aligned}
$$

with ρ, g, f are continuous, $\rho \ge 0$, ρ is C^1 with Lipschitz first derivative in its first variable, uniformly in the second one, and Lipschitz in its second variable, uniformly in the first one.

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Proposition For all $(\alpha, \epsilon) \in \mathcal{A} \times \mathcal{E}^b$, there is some constant $C > 0$ such that, for all $t \leq t' \leq T$ and $x, x' \in \bar{\mathcal{O}}$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t'\leq s\leq T} |X^{\alpha,\epsilon}_{t,x}(s) - X^{\alpha,\epsilon}_{t',x'}(s)|^2\right] &\leq C\left(|x-x'|^2 + |t'-t|\right), \\
\mathbb{E}\left[\int_{t'}^T |X^{\alpha,\epsilon}_{t,x}(s) - X^{\alpha,\epsilon}_{t',x'}(s)|^2 d(L^{\alpha,\epsilon}_{t',x'}(s) + L^{\alpha,\epsilon}_{t,x}(s))\right] &\leq C\left(|x-x'|^2 + |t'-t|\right), \\
\mathbb{E}\left[\sup_{t\leq s\leq t'} |X(s) - x|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[L^{\alpha,\epsilon}_{t,x}(t')\right] &\leq C\,|t'-t|^{\frac{1}{2}}, \\
\mathbb{E}\left[\sup_{t'\leq s\leq T} |\ln(\beta^{\alpha,\epsilon}_{t,x}(s)) - \ln(\beta^{\alpha,\epsilon}_{t',x'}(s))|\right] &\leq C\left(|x-x'|^2 + |t'-t|\right)^{\frac{1}{2}}.
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Lemma The following holds.

(i) $J(\cdot;\alpha,\epsilon)$ is continuous on $[0,T]\times\bar{\mathcal{O}}$ for all $(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}^b$.

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Lemma Fix $(t,x)\in[0,T)\times\bar{\mathcal{O}}$. For all $[t,T]$-valued stopping time θ, we have

$$u(t,x)=\sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}}\mathbb{E}\left[\beta_{t,x}^{\alpha,\epsilon}(\theta)u\left(\theta,X_{t,x}^{\alpha,\epsilon}(\theta)\right)+\int_t^\theta\beta_{t,x}^{\alpha,\epsilon}(s)f\left(X_{t,x}^{\alpha,\epsilon}(s),\alpha(s),\epsilon(s)\right)ds\right]$$

PDE characterization

- Set

$$\mathcal{L}^{a,e}\varphi := \frac{\partial}{\partial t}\varphi + \langle \mu(\cdot,a,e), D\varphi\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma(\cdot,a,e)\sigma(\cdot,a,e)' D^2\varphi\right] + f(\cdot,a,e)$$

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- Define

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\mathcal{K}_+\varphi \ := \ \begin{cases} \min\limits_{(a,e)\in A\times E} \left(-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e)\right) & \text{on} \quad [0,T)\times\mathcal{O} \\ \min\limits_{(a,e)\in A\times E} \max\left\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e)\ ,\ \mathcal{H}^{e}\varphi\right\} & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - g & \text{on} \quad \{T\}\times\bar{\mathcal{O}} \end{cases}
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and

$$\mathcal{K}_-\varphi := \begin{cases} \min_{(a,e)\in A\times E} \left(-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e)\right) & \text{on} \quad [0,T)\times\mathcal{O} \\ \min_{(a,e)\in A\times E} \min\left\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e)\ ,\ \mathcal{H}^{e}\varphi\right\} & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - g & \text{on} \quad \{T\}\times\mathcal{O} \\ \min\left\{\varphi - g\ ,\ \mathcal{H}^{e}\varphi\right\} & \text{on} \quad \{T\}\times\partial\mathcal{O}\ . \end{cases}$$

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Proposition Let v (resp. w) be a bounded upper-semicontinuous viscosity subsolution (resp. lower-semicontinuous viscosity supersolution) of $\mathcal{K}\varphi = 0$. Assume that $v \geq 0$ on $[0, T] \times \partial\mathcal{O}$ and

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Remark Works also if $w \geq 0$ on $[0, T] \times \partial\mathcal{O}$ or if there exists a non-negative subsolution (in particular if $f, g \geq 0$).

Part IV: Dual formulation for the hedging price

BS model

- Up-and-out barrier option with shortselling constraints

$$
\begin{aligned}
S_{t,x}(s) &= x + \int_t^s \mathrm{diag}\left[S_{t,x}(r)\right] \Sigma \, dW(r) \\
K &:= \prod_{i=1}^{d} [-m^i, \infty) \\
\mathcal{O}^* &:= \mathcal{O} \cap (0,\infty)^d = \left\{ x \in (0,\infty)^d \ : \ \sum_{i=1}^{d} x^i < \kappa \right\} \ , \ \kappa > 0 \ .
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Proposition If there exists a "smooth" solution ψ to

$$-\mathcal{L}\psi(t,x) = 0 \text{ on } [0,T) \times \mathcal{O}^*$$

$$\min_{e \in \tilde{K}_1} \left(\delta(e)\psi(t,x) - \langle e, \text{diag}\,[x]\, D\psi(t,x)\rangle\right) = 0 \text{ on } [0,T) \times \partial\mathcal{O}^*$$

$$\psi = \hat{g} \text{ on } \{T\} \times \bar{\mathcal{O}}^*$$

such that $\lim_{\substack{(t',x') \to (T,x) \\ (t',x') \in [0,T) \times \mathcal{O}^*}} D\psi(t',x') = D\hat{g}(x)$ almost everywhere on $\bar{\mathcal{O}}^*$,

then $\psi = v$.

Dual formulation for the BS model

- Set $E_n := \{e \in \tilde{K}_1 \ : \ e^i \leq -n^{-1} \, \forall \, i \leq d\}$ and $\mathcal{E}_0 := \cup_{n \geq 1} \, \mathcal{E}_n$.

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For $\varepsilon \in \mathcal{E}_0$, we can define the solution $(X^\varepsilon, L^\varepsilon)$ of

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\begin{aligned}
X(s) &= x + \int_t^s X(r)\Sigma dW(r) + \int_t^s \gamma(X(r), \epsilon_r) dL(r) \\
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$$
v(t,x) \;\; := \;\; \sup_{\epsilon \in \mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s), \epsilon(s)) dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T) \right) \right] \;\; , \; (t,x) \in [0,T] \times \bar{\mathcal{O}}^* \, .
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Dual formulation for the BS model

- **Optimal reflection problem:**

$$u(0, X_0) \;:=\; \sup_{\epsilon} \mathbb{E}\left[\beta_T^{\epsilon} \hat{g}\left(X_T^{\epsilon}\right)\right]$$

solves

$$0 \;=\; \begin{cases} -\mathcal{L}\varphi & \text{on} \quad [0,T) \times \mathcal{O} \\ \min_{e \in E} \rho(\cdot, e)\varphi - \gamma(\cdot, e)' D\varphi & \text{on} \quad [0,T) \times \partial\mathcal{O} \\ \varphi - \hat{g} & \text{on} \quad \{T\} \times \bar{\mathcal{O}} \end{cases}$$

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Super-hedging problem:

$$v(0, X_0) \ := \ \inf\left\{y \in \mathbb{R} \ : \ Y_\tau^{y,\phi} \geq g(X_\tau)\mathbf{1}_{\tau < T} \ \text{ for some } \phi \in \mathcal{K}\right\}$$

solves

$$0 \ = \ \begin{cases} -\mathcal{L}\varphi & \text{on} \quad [0,T) \times \mathcal{O} \\ \inf_{\rho} \delta(\rho)\varphi - \rho' \text{diag}\,[x]\, D\varphi & \text{on} \quad [0,T) \times \partial\mathcal{O} \\ \varphi - \hat{g} & \text{on} \quad \{T\} \times \bar{\mathcal{O}} \end{cases}$$

Dual formulation for the BS model

- **Equality of the value functions:**

$$\text{inf}\left\{y \in \mathbb{R} \ : \ Y_{\tau}^{y,\phi} \geq g(X_{\tau})\mathbf{1}_{\tau<T} \quad \text{for some } \phi \in \mathcal{K}\right\} \ = \ \sup_{\epsilon} \mathbb{E}\left[\beta_T^{\epsilon} \widehat{g}\left(X_T^{\epsilon}\right)\right]$$

where

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