Super-hedging with partial transaction costs

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• Cvitanić, Pham and Touzi (1999), “A closed-form solution to the problem of super-replication under transaction costs”.

• Levental and Skorohod (1997), “On the possibility of hedging options in the presence of transaction costs”.

• Soner, Shreve and Cvitanić (1995), “There is no nontrivial hedging portfolio for option pricing with transaction costs”.

• Kabanov (1999), “Hedging and liquidation under transaction costs in currency markets”.
The Market

- A numéraire: $Q^0 \equiv 1$
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- $P^1, \ldots, P^m$: freely exchangeable (e.g. stocks in the domestic market).
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• $Q^1, \ldots, Q^d$: subject to transaction costs (e.g. foreign currencies).
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**Bid-Ask spread price** of $Q^j$ in terms of $Q^i$: $[\pi^{ij-}, \pi^{ij+}]$

$$
\pi^{ij-} := \frac{Q^j}{Q^i} (1 + \lambda^{ji})^{-1} \quad \text{and} \quad \pi^{ij+} := \frac{Q^j}{Q^i} (1 + \lambda^{ij})
$$
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Efficient frictions: $\lambda^{ij} + \lambda^{ji} > 0 \iff \pi^{ij-} \neq \pi^{ij+}$
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USUAL CASE: $m=0$

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- $Q^1, \ldots, Q^d$: subject to transaction costs (e.g. foreign currencies).

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$$\pi^{ij-} := \frac{Q^j}{Q^i} (1 + \lambda^{ji})^{-1} \quad \text{and} \quad \pi^{ij+} := \frac{Q^j}{Q^i} (1 + \lambda^{ij})$$

Efficient frictions: $\lambda^{ij} + \lambda^{ji} > 0 \iff \pi^{ij-} \neq \pi^{ij+}$
Dynamics

- $S = (P, Q)$ solves: $dS_t = \text{diag}[S_t]\sigma(t, S_t)dW_t$. 
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\[
\begin{align*}
dP_t &= \text{diag}[P_t] \sigma_P(t, P_t) dW_t \\
dQ_t &= \text{diag}[Q_t] \sigma_Q(t, P_t, Q_t) dW_t
\end{align*}
\]
Dynamics

• $S = (P, Q)$ solves: $dS_t = \text{diag}[S_t] \sigma(t, S_t) dW_t$.

• Portfolio: $x^i = \text{initial amount in } Q^i$, $X_t^i = \text{amount in } Q^i \text{ at } t$. 
Dynamics

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- Portfolio: $x^i = \text{initial amount in } Q^i$, $X_t^i = \text{amount in } Q^i \text{ at } t$.

- $\phi^i$: quantity of $P^i$ in the portfolio.
Dynamics

- $S = (P, Q)$ solves: $dS_t = \text{diag}[S_t] \sigma(t, S_t) dW_t$.

- Portfolio: $x^i = \text{initial amount in } Q^i$, $X^i_t = \text{amount in } Q^i \text{ at } t$.

- $\phi^i$: quantity of $P^i$ in the portfolio.

- If no exchange with the $Q^i$'s:

\[
X^0_t = x^0 + \int_0^t \phi_r \cdot dP_r
\]
\[
X^i_t = x^i + \int_0^t \frac{X^i_r}{Q^i_r} \, dQ^i_r \quad \text{for } i > 0.
\]
Dynamics

• \( S = (P, Q) \) solves: \( dS_t = \text{diag}[S_t] \sigma(t, S_t) dW_t \).

• Portfolio: \( x^i = \) initial amount in \( Q^i \), \( X^i_t = \) amount in \( Q^i \) at \( t \).

• \( L^i_{ij} \): cumulated amount transferred to \( X^j \) by selling units of \( Q^i \).
Dynamics

- $S = (P, Q)$ solves: $dS_t = \text{diag}[S_t] \sigma(t, S_t) dW_t$.

- Portfolio: $x^i = \text{initial amount in } Q^i$, $X^i_t = \text{amount in } Q^i \text{ at } t$.

- $L_{ij}^t$: cumulated amount transferred to $X^j$ by selling units of $Q^i$.

- $dL_{ij}^t$: amount transferred to $X^j$ by selling units of $Q^i$,

  $$\left( X^i_t, X^j_t \right) \implies \left( X^i_t - (1 + \lambda_{ij}^t) dL_{ij}^t, X^j_t + dL_{ij}^t \right)$$
Dynamics

- $S = (P, Q)$ solves: $dS_t = \text{diag}[S_t] \sigma(t, S_t) dW_t$.

- Portfolio: $x^i = \text{initial amount in } Q^i$, $X^i_t = \text{amount in } Q^i \text{ at } t$.

- Portfolio dynamic:

$$
X^0_t = x^0 + \int_0^t \phi_r \cdot dP_r + \sum_{j=1}^d \int_0^t \left[ dL^{j0}_r - (1 + \lambda^{0j}) dL^{0j}_r \right]
$$

$$
X^i_t = x^i + \int_0^t \frac{X^i_r}{Q^i_r} dQ^i_r + \sum_{j=0}^d \int_0^t \left[ dL^{ij}_r - (1 + \lambda^{ij}) dL^{ij}_r \right] \text{ for } i > 0 .
$$
Hedging Problem

- **Contingent claim:** \( g(S_T) = (g^0(S_T), \ldots, g^d(S_T)) \)

- \( g^i(S_T) \): amounts of \( Q_i \) to be delivered.
Hedging Problem

- Contingent claim: \( g(S_T) = (g^0(S_T), \ldots, g^d(S_T)) \)

- Remark: The solvency region is

\[
K := \left\{ x \in \mathbb{R}^{1+d} : \exists a \in \mathbb{M}^{1+d}, \ x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda_{ij})a^{ij}) \geq 0 \ \forall \ 0 \leq i \leq d \right\}
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Hedging Problem

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- Remark: *The solvency region is*

\[
K := \left\{ x \in \mathbb{R}^{1+d} : \exists a \in M_{1+d}, x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda^{ij})a^{ij}) \geq 0 \ \forall \ 0 \leq i \leq d \right\}
\]

- \( X_{t^-} \in K \iff \exists \Delta L_t \) such that

\[
X_t^i = X_{t^-}^i + \sum_{j=0}^{d} \left[ \Delta L_t^{ji} - (1 + \lambda^{ij})\Delta L_t^{ij} \right] \geq 0
\]
Hedging Problem

- Contingent claim: $g(S_T) = (g^0(S_T), \ldots, g^d(S_T))$

- Remark: *The solvency region is*

  $$K := \left\{ x \in \mathbb{R}^{1+d} : \exists a \in \mathbb{M}^{1+d}_+, x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda^{ij})a^{ij}) \geq 0 \ \forall \ 0 \leq i \leq d \right\}$$

- $X_T$ hedges $g(S_T) \Leftrightarrow \exists \Delta L_T$ such that

  $$X_T^i = X_{T-}^i + \sum_{j=0}^{d} \left[ \Delta L_T^{ji} - (1 + \lambda^{ij})\Delta L_T^{ij} \right] \geq g^i(S_T)$$
Hedging Problem

• Contingent claim: \( g(S_T) = (g^0(S_T), \ldots, g^d(S_T)) \)

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\]

• \( X_T \) hedges \( g(S_T) \) \( \Leftrightarrow \) \( \exists \Delta L_T \) such that

\[
X^i_T = X^i_{T-} + \sum_{j=0}^{d} \left[ \Delta L^{ji}_T - (1 + \lambda^{ij})\Delta L^{ij}_T \right] \geq g^i(S_T)
\]

• \( X_T \) hedges \( g(S_T) \) \( \Leftrightarrow \) \( X_T - g(S_T) \in K \)
Hedging Problem

- Contingent claim: \( g(S_T) = (g^0(S_T), \ldots, g^d(S_T)) \)

- Remark: *The solvency region is*

\[
K := \left\{ x \in \mathbb{R}^{1+d} : \exists a \in M_+^{1+d}, \ x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda^{ij})a^{ij}) \geq 0 \ \forall \ 0 \leq i \leq d \right\}
\]

- *Super-replication price:*

\[
p(0, S_0) := \inf \left\{ w \in \mathbb{R} : \exists (\phi, L) \in \mathcal{A}, \ X_T^{\phi, L} - g(S_T) \in K \text{ with } x = w 1_0 \right\},
\]

where \( w 1_0 = (w, 0, \ldots, 0) \in \mathbb{R}^{1+d} \).
Closed form solution

Theorem: (Efficient friction case) If \( m = 0, \ S = Q \), then

\[
p(0, Q_0) = \min \left\{ w \in \mathbb{R} : \exists \ L \in A^{BH}, \ X_T^L - g(Q_T) \in K \text{ with } x = w_1 \right\},
\]

where

\[
A^{BH} := \{ L \in A : \ L_t = L_0 \text{ for all } 0 \leq t \leq T \}.
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Closed form solution

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where

$$A^{BH} := \{ L \in \mathcal{A} : L_t = L_0 \text{ for all } 0 \leq t \leq T \}.$$

- $p(0, Q_0) = G(Q_0)$ where $G$ is related to the concave envelope of $g$
Closed form solution

**Theorem: (General case)** If \( m \geq 1 \), then

\[
p(0, S_0) = \min \left\{ w \in \mathbb{R} : \exists (\phi, L) \in A^{BH}, \ X_T^{\phi,L} - g(S_T) \in K \text{ with } x = w1_0 \right\}
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where

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A^{BH} := \left\{ (\phi, L) \in A : L_t = L_0 \text{ for all } 0 \leq t \leq T \right\}
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* Similar solution: buy-and-hold on $Q$ + dynamical trading on $P$. 
Closed form solution

**Theorem: (General case)** If $m \geq 1$, then

$$p(0, S_0) = \min \left\{ w \in \mathbb{R} : \exists (\phi, L) \in A^{BH}, X_T^{\phi, L} - g(S_T) \in K \text{ with } x = w1_0 \right\}$$

where

$$A^{BH} := \{ (\phi, L) \in A : L_t = L_0 \text{ for all } 0 \leq t \leq T \}.$$

- Closed form solution

$$p(0, S_0) = \mathbb{E} \left[ C(P_T; \hat{\Delta}) \right] + \sup_{\xi \in \Lambda} \xi \cdot \text{diag}[\hat{\Delta}] Q_0$$

where $C(P_T; \Delta) := \sup_{z \in (0, \infty)^d} G(P_T, z) - \Delta \cdot z$. 
Closed form solution

- If we forget transaction costs at 0 and $T$:

$$p(0, S_0) = \mathbb{E} \left[ C(P_T; \Delta) \right] + \Delta \cdot Q_0$$

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and \( G \) is the concave enveloppe of \( g \) with respect to \( Q \).

• Optimal strategy:

- \( L = \hat{\Delta}, \hat{\Delta}^i = \) quantity of \( Q^i \) held.
Closed form solution

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- $L = \hat{\Delta}$, $\hat{\Delta}^i = $ quantity of $Q^i$ held.

$$\Rightarrow \text{It remains to hedge } g(P_T, Q_T) - \hat{\Delta} \cdot Q_T$$
Closed form solution

- If we forget transaction costs at 0 and $T$:

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and $G$ is the concave envelope of $g$ with respect to $Q$.

- Optimal strategy:

  - $L = \hat{\Delta}$, $\hat{\Delta}^i$ = quantity of $Q^i$ held.

  $$\Rightarrow$$ It remains to hedge $g(P_T, Q_T) - \hat{\Delta} \cdot Q_T$

  - $\phi$ chosen to hedge $C(P_T; \hat{\Delta}) \geq g(P_T, Q_T) - \hat{\Delta} \cdot Q_T$ with $P$. 
Idea of the proof (forgetting transaction costs at 0 and $T$)

1. Show that $p(0, S_0) \geq \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ g(P_T, Z_{T_T}^\mu) \right] =: v(0, P_0, Q_0)$

where $Z_t^\mu = Q_0 + \int_0^t \text{diag}[Z_s^\mu] \mu_s \, dW_s$. 
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where $Z_t^\mu = Q_0 + \int_0^t \text{diag}[Z_s^\mu] \mu_s \, dW_s$.

2. Show that $v$ is concave in $Q_0$

$$
\inf_{\mu} -v_t - \frac{1}{2} p^2 (\sigma_P)^2 v_{pp} - \frac{1}{2} z^2 (\mu)^2 v_{zz} - pz \sigma_P \mu v_{pz} \geq 0
$$

$v(T, s, z) \geq g(s, z) \Rightarrow v(T, s, z) \geq G(s, z)$
Idea of the proof (forgetting transaction costs at 0 and $T$)

• 1. Show that $p(0, S_0) \geq \sup_{\mu \in \mathcal{U}} \mathbb{E}[g(P_T, Z_T^\mu)] =: v(0, P_0, Q_0)$ where $Z_T^\mu = Q_0 + \int_0^t \text{diag}[Z_s^\mu] \mu_s \, dW_s$.

• 2. Show that $v$ is concave in $Q_0$

• 3. Show that $v(0, P_0, Q_0) = \sup_{\mu \in \mathcal{U}} \mathbb{E}[G(P_T, Z_T^\mu)]$

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$v(T, s, z) \geq G(s, z)$
Idea of the proof (forgetting transaction costs at 0 and \( T \))

1. Show that \( p(0, S_0) \geq \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ g(P_T, Z^\mu_T) \right] =: v(0, P_0, Q_0) \)

where \( Z^\mu_t = Q_0 + \int_0^t \text{diag}[Z^\mu_s] \mu_s \, dW_s \).

2. Show that \( v \) is concave in \( Q_0 \)

3. Show that \( v(0, P_0, Q_0) = \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z^\mu_T) \right] \)

4. \( v(0, P_0, Q_0) \geq \sup_z v(0, P_0, z) - \Delta \cdot (z - Q_0), \) with \( \Delta \in v'_z(0, P_0, Q_0) \)
Idea of the proof (forgetting transaction costs at 0 and \( T \))

1. Show that
\[
p(0, S_0) \geq \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ g(P_T, Z_T^{\mu}) \right] =: v(0, P_0, Q_0)
\]
where
\[
Z_t^{\mu} = Q_0 + \int_0^t \text{diag}[Z_s^{\mu}] \mu_s \, dW_s.
\]

2. Show that \( v \) is concave in \( Q_0 \)

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5. \( p(0, S_0) \geq \sup_z \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z_T^{\mu}) \right] - \Delta \cdot (z - Q_0) \)
Idea of the proof (forgetting transaction costs at 0 and $T$)

1. Show that $p(0, S_0) \geq \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ g(P_T, Z^\mu_T) \right] =: v(0, P_0, Q_0)$
where $Z^\mu_t = Q_0 + \int_0^t \text{diag}[Z^\mu_s] \mu_s \, dW_s$.

2. Show that $v$ is concave in $Q_0$

3. Show that $v(0, P_0, Q_0) = \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z^\mu_T) \right]$.

4. $v(0, P_0, Q_0) \geq \sup_z v(0, P_0, z) - \Delta \cdot (z - Q_0)$, with $\Delta \in v'_z(0, P_0, Q_0)$

5. $p(0, S_0) \geq \sup_z \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z^{\tilde{z}}_T, \mu) \right] - \Delta \cdot (z - Q_0)$

6. $p(0, S_0) \geq \sup_z \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z^{\tilde{z}}_T, \mu) - \Delta \cdot Z^{\tilde{z}}_T, \mu \right] + \Delta \cdot Q_0$. 
Idea of the proof (forgetting transaction costs at 0 and $T$)

1. Show that $p(0, S_0) \geq \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ g(P_T, Z_T^\mu) \right] =: v(0, P_0, Q_0)$

where $Z_t^\mu = Q_0 + \int_0^t \text{diag}[Z_s^\mu] \mu_s \, dW_s$.

2. Show that $v$ is concave in $Q_0$

3. Show that $v(0, P_0, Q_0) = \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z_T^\mu) \right]$

4. $v(0, P_0, Q_0) \geq \sup_z v(0, P_0, z) - \Delta \cdot (z - Q_0)$, with $\Delta \in v_z'(0, P_0, Q_0)$

5. $p(0, S_0) \geq \sup_z \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z_T^z, \mu) \right] - \Delta \cdot (z - Q_0)$

6. $p(0, S_0) \geq \sup_z \sup_{\mu \in \mathcal{U}} \mathbb{E} \left[ G(P_T, Z_T^z, \mu) - \Delta \cdot Z_T^z, \mu \right] + \Delta \cdot Q_0$.

7. $p(0, S_0) \geq \mathbb{E} \left[ \sup_z G(P_T, z) - \Delta \cdot z \right] + \Delta \cdot Q_0$  ⊙
Closed form solution

- If we forget transaction costs at 0 and $T$:

$$p(0, S_0) = \mathbb{E}[C(P_T; \Delta)] + \Delta \cdot Q_0$$

where $C(P_T; \Delta) := \sup_{z \in (0, \infty)^d} G(P_T, z) - \Delta \cdot z$.

and $G$ is the concave envelope of $g$ with respect to $Q$. 
Example

- $m = d = 1$, $\sigma(t, s) = \sigma \in M^2$ invertible, $g^0(P_T, Q_T) = ([P_T - K]^+1_{Q_T > \hat{K}})$
Example

• $m = d = 1$, $\sigma(t, s) = \sigma \in M^2$ invertible, $g^0(P_T, Q_T) = ([P_T - K]^+ 1_{\{Q_T > \tilde{K}\}})$

• Hedging price: $\min_{\Delta \geq 0} \mathbb{E} \left[ (P_T - K - \Delta \tilde{K})^+ \right] + (1 + \lambda^{01}) \Delta Q_0$

with $\tilde{K} := \hat{K}/(1 + \lambda^{10})$. 
Example

- \( m = d = 1, \sigma(t, s) = \sigma \in M^2 \) invertible, \( g^0(P_T, Q_T) = ([P_T - K]^+ 1_{\{Q_T > \tilde{K}\}} \right)

- **Hedging price:** \( \min_{\Delta \geq 0} \mathbb{E} \left[ [P_T - K - \Delta \tilde{K}]^+ \right] + (1 + \lambda^{01}) \Delta Q_0 \)

  with \( \tilde{K} := \hat{K}/(1 + \lambda^{10}). \)

- **Hedging strategy:**

  1. If \( \mathbb{P} [P_T - K \geq 0] \leq (1 + \lambda^{01}) Q_0 / \tilde{K} \) then \( \hat{\Delta} = 0 \)
Example

- $m = d = 1$, $\sigma(t, s) = \sigma \in \mathcal{M}^2$ invertible, $g^0(P_T, Q_T) = \left( [P_T - K]^+ 1_{\{Q_T > \hat{K}\}} \right)$

- Hedging price: $\min_{\Delta \geq 0} \mathbb{E} \left[ [P_T - K - \Delta \tilde{K}]^+ \right] + (1 + \lambda^{01}) \Delta Q_0$
  with $\tilde{K} := \hat{K} / (1 + \lambda^{10})$.

- Hedging strategy:
  
  1. If $\mathbb{P}[P_T - K \geq 0] \leq (1 + \lambda^{01}) Q_0 / \tilde{K}$ then $\hat{\Delta} = 0$

  $\Rightarrow$ hedges $[P_T - K]^+ : \phi_t = \mathbb{E} \left[ P_T 1_{\{P_T \geq K\}} \mid \mathcal{F}_t \right] / P_t$, $L = 0$. 
Example

- $m = d = 1$, $\sigma(t, s) = \sigma \in \mathcal{M}^2$ invertible, $g^0(P_T, Q_T) = \left( [P_T - K]^+ 1_{\{Q_T > \tilde{K}\}} \right)$

- Hedging price: $\min_{\Delta \geq 0} \mathbb{E} \left[ [P_T - K - \Delta \tilde{K}]^+ \right] + (1 + \lambda^{01}) Q_0$
  with $\tilde{K} := \tilde{K} / (1 + \lambda^{10})$.

- Hedging strategy:

  2. If $\mathbb{P} [P_T - K \geq 0] > (1 + \lambda^{01}) Q_0 / \tilde{K}$ then $\tilde{\Delta}$ solves
     
     $-\tilde{K} \mathbb{E} \left[ 1_{\{P_T - K \geq \Delta \tilde{K}\}} \right] + (1 + \lambda^{01}) Q_0 = 0$
Example

- \( m = d = 1, \sigma(t, s) = \sigma \in M^2 \) invertible, \( g^0(P_T, Q_T) = ([P_T - K]^+1_{\{Q_T > \tilde{K}\}}) \)

- **Hedging price:** \( \min_{\Delta \geq 0} \mathbb{E} \left[ [P_T - K - \Delta \tilde{K}]^+ \right] + (1 + \lambda^{01})\Delta Q_0 \)
  with \( \tilde{K} := \tilde{K}/(1 + \lambda^{10}) \).

- **Hedging strategy:**
  
  2. If \( \mathbb{P} [P_T - K \geq 0] > (1 + \lambda^{01})Q_0/\tilde{K} \) then \( \tilde{\Delta} \) solves
  
  \(-\tilde{K} \mathbb{E} \left[ 1_{\{P_T - K \geq \Delta \tilde{K}\}} \right] + (1 + \lambda^{01})Q_0 = 0 \)
  
  \( \Rightarrow \) hedges \( [P_T - K - \tilde{\Delta} \tilde{K}]^+ : \phi_t = \mathbb{E} \left[ P_T 1_{\{P_T \geq K + \tilde{\Delta} \tilde{K}\}} \mid \mathcal{F}_t \right] / P_t, L = \tilde{\Delta} \).