Optimal control under uncertainty and Bayesian parameters adjustments: Application to trading algorithms

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April 13, 2016

Abstract

We propose a general framework for the optimal control/design of trading algorithms in situations where market conditions or impact parameters are uncertain. Given a prior on the distribution of the unknown parameters, we explain how it should evolve according to the classical Bayesian rule after each sequence of trades. Taking these progressive prior-adjustments into account, we characterize the optimal policy through a quasi-variational parabolic equation, which can be solved numerically. From the mathematical point of view, we indeed treat a quite general impulse control problem with unknown parameters, and the derivation of the dynamic programming equation seems to be new in this context. The main difficulty lies in the nature of the set of controls which depends in a non trivial way on the initial data through the filtration itself. Typical examples of application are discussed.

Key words: Optimal control, uncertainty, optimal trading.

MSC 2010: 49L, 91G

1 Introduction

When trading at a high frequency level, several market parameters become of major importance. It can be the nature of the market impact of aggressive orders, or the time to be executed when entering a book order queue, see e.g. [15] and the references therein. However, the knowledge of these execution conditions is in general not perfect. One can try to estimate

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them but they remain random and can change from one market/platform to another one, or depending on the current market conditions. Most importantly, they can only be estimated by actually acting on the market. We therefore face the typical problem of estimating a reaction parameters (impact/execution time) while actually controlling a system (trading) that depends on these parameters.

Such problems have been widely studied in the discrete time stochastic optimal control literature, see e.g. [10] for references. One fixes a certain prior distribution on the unknown parameter, and re-evaluate it each time an action is taken, by applying the standard Bayesian rule to the observed reactions. The optimal strategy generically results from a compromise between acting on the system, to get more information, and being not too aggressive, because of the uncertainty on the real value of the parameters. If the support of the initial prior contains the true value of the parameters, one can expect (under natural identification conditions) that the sequence of updated priors actually converges to it in the long range.

It is a-priori much more difficult to handle in a continuous time framework with continuous time monitoring, as it leads to a filtering problem, leaving on an infinite dimensional space. However, optimal trading under market impact can very naturally be considered in the impulse form, as robots send orders in a discrete time manner. In a sense, we are back to a discrete time problem which dimension can be finite (depending on the nature of the uncertainty), although interventions on the system may occur at any time.

In this paper, we thus consider a general impulse control problem with an unknown parameter, under which an initial prior law is set. Given this prior, we aim at maximizing a certain gain functional. We show that the corresponding value function can be characterized as the unique viscosity solution (in a suitable class) of a quasi-variational parabolic equation, for which a convergent numerical scheme is constructed. To better fit with market practices, we allow for (possibly) not observing immediately the effect of an impulse. This applies for instance to trading robots that are launched for a certain time period and whose impact will be observed only at the end of this period, or to dark pools in which nothing is observed but the execution time.

The study of such non-classical impulse control problems seems to be new in the literature. From the mathematical point of view, the main difficulty consists in establishing a dynamic programming principle. The principal reason lies in the choice of the filtration. Because of the uncertainty on the parameter driving the dynamics, the only natural filtration to which the control policy should be adapted is the one generated by the controlled process himself. This implies in particular that the set of admissible controls depends heavily (and in a very non trivial way) on the initial state of the system at the starting time of the strategy. Hence, no a priori regularity nor good measurability properties can be expected to construct explicitly measurable almost optimal controls, see e.g. [5], or to apply a measurable selection theorem, see e.g. [16]. We therefore proceed differently. The (usually considered as) easy part of the dynamic programming can actually be proved, as it only requires a conditioning argument. It leads as usual to a sub-solution characterization. We surround the difficulty in proving the second (difficult) part by considering a discrete time version of our initial continuous time control problem. When the time step goes to 0, it provides a super-solution of the targeted dynamic programming equation. Using comparison and the natural ordering on the value functions associated to the continuous and the discrete time model, we show that the two coincide at the limit.

We consider two examples of applications. In the first one, aggressive orders are send in a model with immediate and resilient impact. The unknown are the parameters of the impact and liquidity costs functions. In the second one, we only consider limit orders. The unknown is the distribution of the time to be executed. In both situations, the problems can be solved numerically without much difficulties and we provide numerical illustrations showing the dependence of the optimal strategies on the current priors.

The rest of the paper is organized as follows. The model is described in Section 2. In Section 3, we provide the PDE characterization of the value function and an example of numerical scheme. Proofs are collected in Section 5. Section 4 is dedicated to two examples of application.

2 The impulse problem with parameters adjustment

All over this paper, $C([0,T],\mathbb{R}^d)$ is the space of continuous functions from $[0,T]$ into $\mathbb{R}^d$ which start at 0 at the origin. Recall that it is a Polish space for the sup-norm topology. We denote by $W(\omega)=\omega$ the canonical process and let $\mathbb{P}$ be the Wiener measure.
We also consider a Polish space $(\mathrm{U},\mathcal{B}(\mathrm{U}))$ that will support an unknown parameter υ. We denote by $\mathbf{M}$ a locally compact subset of the set of Borel probability measures on U endowed with the topology of weak convergence. In particular, it is Polish. A prior on the unknown parameter υ will be an element $m\in\mathbf{M}$.
To allow additional randomness in the measurement of the effects of actions on the system, we consider another Polish space E on which is defined a family $(\epsilon_i)_{i\geq 0}$ of i.i.d. random variables with common measure $\mathbb{P}_\epsilon$ on E. On the product space $\Omega:=C([0,T],\mathbb{R}^d)\times\mathrm{U}\times\mathrm{E}^{\mathbb{N}}$, we consider the family of measures $\{\mathbb{P}\times m\times\mathbb{P}_\epsilon^{\otimes\mathbb{N}}:m\in\mathbf{M}\}$ and denote by $\mathbb{P}_m$ an element of this family whenever $m\in\mathbf{M}$ is fixed. The operator $\mathbb{E}_m$ is the expectation associated to $\mathbb{P}_m$. Note that W, υ and $(\epsilon_i)_{i\geq 0}$ are independent under each $\mathbb{P}_m$.
For $m\in\mathbf{M}$ given, we let $\mathbb{F}^m=(\mathcal{F}_t^m)_{t\geq 0}$ denote the $\mathbb{P}_m$-augmentation of the filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t=\sigma((W_s)_{s\leq t},\upsilon,(\epsilon_i)_{i\geq 0})$ for $t\geq 0$. Hereafter, all the random variables are considered with respect to the probability space $(\Omega,\mathcal{F}_T^m)$ with $m\in\mathbf{M}$ given by the context, and where T is a fixed time horizon.

2.1 The controlled system

Let $\mathbf{A}\subset[0,T]\times\mathbb{R}^d$ be a (non-empty) compact set. Given $N\in\mathbb{N}$ and $m\in\mathbf{M}$, we denote by $\Phi_N^{\circ,m}$ the collection of sequences of random variables $\phi=(\tau_i,\alpha_i)_{i\geq 1}$ on $(\Omega,\mathcal{F}_T^m)$ with values in $\mathbb{R}_+\times\mathbf{A}$ such that $(\tau_i)_{i\geq 1}$ is a non-decreasing sequence of $\mathbb{F}^m$-stopping times satisfying $\tau_j>T$ $\mathbb{P}_m-$a.s. for $j>N$. We set

$$\Phi^{\circ,m}:=\bigcup_{N\geq 1}\Phi_N^{\circ,m}.$$

An element $\phi = (\tau_i, \alpha_i)_{1\le i\le N} \in \Phi^{\circ,m}$ will be our impulse control and we write α_i in the form

$$\alpha_i = (\ell_i, \beta_i) \text{ with } \ell_i \in [0,T] \text{ and } \beta_i \in \mathbb{R}^d \; \mathbb{P}_m - \text{a.s.}$$

More precisely, the τ_i's will be the times at which an impulse is made on the system (e.g. a trading robot is launched), β_i will model the nature of the order send at time τ_i (e.g. the parameters used for the trading robot), and ℓ_i will stand for the maximal time length during which no new intervention on the system can be made (e.g. the time prescribed to the robot to send orders on the market). Later on we shall impose more precise non-anticipativity conditions.

From now on, we shall always use the notation $(\tau_i^\phi, \alpha_i^\phi)_{i\ge1}$ with $\alpha_i^\phi = (\ell_i^\phi, \beta_i^\phi)$ to refer to a control $\phi \in \Phi^{\circ,m}$.

We allow for not observing nor being able to act on the system before a random time ϑ_i^ϕ defined by

$$\vartheta_i^\phi := \varpi(\tau_i^\phi, X^\phi_{\tau_i^\phi -}, \alpha_i^\phi, \upsilon, \epsilon_i),$$

where X^ϕ is the controlled state process that will be described below, and

$$\varpi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{A} \times \mathrm{U} \times \mathrm{E} \to [0,T] \text{ is measurable, such that } \varpi(t,\cdot) \ge t \text{ for all } t \ge 0. \quad (2.1)$$

In the case where the actions consist in launching a trading robot at τ_i^ϕ during a certain time ℓ_i^ϕ, we can naturally take $\vartheta_i^\phi = \tau_i^\phi + \ell_i^\phi$. If the action consists in placing a limit order during a maximal duration ℓ_i^ϕ, ϑ_i^ϕ is the time at which the limit order is executed if it is less than $\tau_i^\phi + \ell_i^\phi$, and $\tau_i^\phi + \ell_i^\phi$ otherwise.

We say that $\phi \in \Phi^{\circ,m}$ belongs to Φ^m if

$$\vartheta_i^\phi \le \tau_{i+1}^\phi \text{ and } \tau_i^\phi < \tau_{i+1}^\phi \quad \mathbb{P}_m\text{-a.s. for all } i \ge 1,$$

and define

$$\mathcal{N}^\phi := \left[\cup_{i\ge1}[\tau_i^\phi, \vartheta_i^\phi)\right]^c. \quad (2.2)$$

We are now in position to describe our controlled state process. Given some initial data $z := (t,x) \in \mathbf{Z} := [0,T] \times \mathbb{R}^d$, and $\phi \in \Phi^m$, we let $X^{z,\phi}$ be the unique strong solution on $[t, 2T]$ of

$$\begin{aligned} X = x &+ \left(\int_t^\cdot \mathbf{1}_{\mathcal{N}^\phi}(s)\mu(s, X_s)\, ds + \int_t^\cdot \mathbf{1}_{\mathcal{N}^\phi}(s)\sigma(s, X_s)\, dW_s \right) \\ &+ \sum_{i\ge1} \mathbf{1}_{\{t \le \vartheta_i^\phi \le \cdot\}} [F(\tau_i^\phi, X_{\tau_i^\phi -}, \alpha_i^\phi, \upsilon, \epsilon_i) - X_{\tau_i^\phi -}]. \end{aligned} \quad (2.3)$$

In the above, the function

$$\begin{gathered} (\mu, \sigma, F) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{A} \times \mathrm{U} \times \mathrm{E} \mapsto \mathbb{R}^d \times \mathbb{M}^d \times \mathbb{R}^d \text{ is measurable.} \\ \text{The map } (\mu, \sigma) \text{ is continuous, and Lipschitz with linear growth} \\ \text{in its second argument, uniformly in the first one.} \end{gathered} \quad (2.4)$$

In the above, $\mathbb{M}^d$ stands for the set of $d \times d$ matrices. This dynamics means the following. When no action is currently made on the system, i.e. on the intervals in $\mathcal{N}^\phi$, the system evolves according to a stochastic differential equation driven by the Brownian motion W:

$$dX_s = \mu\left(s, X_s\right) ds + \sigma\left(s, X_s\right) dW_s \quad \text{on } \mathcal{N}^\phi.$$

When an impulse is made at τ_i^ϕ, we freeze the dynamics up to the end of the action at time ϑ_i^ϕ. This amounts to saying that we do not observe the current evolution up to ϑ_i^ϕ. At the end of the action, the state process takes a new value

$$X_{\vartheta_i^\phi} = F(\tau_i^\phi, X_{\tau_i^\phi -}, \alpha_i^\phi, \upsilon, \epsilon_i), \; i \geq 1.$$

The fact that F depends on the unknown parameter υ and the additional noise ϵ_i models the fact the correct model is not known with certainty, and that the exact value of the unknown parameter υ can (possibly) not be measured precisely just by observing $(\vartheta_i^\phi - \tau_i^\phi, X_{\vartheta_i^\phi} - X_{\tau_i^\phi -})$.

In order to simplify the notations, we shall now write:

$$Z^{z,\phi} := (\cdot, X^{z,\phi}) \quad \text{and} \quad Z^{z,\circ} := (\cdot, X^{z,\circ}) \tag{2.5}$$

in which $X^{z,\circ}$ denotes the solution of (2.3) for ϕ such that $\tau_1^\phi > T$ and satisfying $X_t^{z,\circ} = x$. This corresponds to the stochastic differential equation (2.3) in the absence of impulse. Note in particular that

$$Z^{z,\phi}_{\vartheta_1^\phi} = \mathrm{z}'(Z^{z,\circ}_{\tau_1^\phi -}, \alpha_1^\phi, \upsilon, \epsilon_1) \quad \text{on } \{\tau_1^\phi \geq t\}, \tag{2.6}$$

in which

$$\mathrm{z}' := (\varpi, F). \tag{2.7}$$

From now on, we denote by $\mathbb{F}^{z,m,\phi} = (\mathcal{F}_s^{z,m,\phi})_{t \leq s \leq 2T}$ the $\mathbb{P}_m$-augmentation of the filtration generated by $(X^{z,\phi}, \sum_{i \geq 1} \mathbf{1}_{[\vartheta_i^\phi, \infty)})$ on $[t, 2T]$. We say that $\phi \in \Phi^m$ belongs to $\Phi^{z,m}$ if $(\tau_i^\phi)_{i \geq 1}$ is a sequence of $\mathbb{F}^{z,m,\phi}$-stopping times and α_i^ϕ is $\mathcal{F}^{z,m,\phi}_{\tau_i^\phi}$-measurable, for each $i \geq 1$. Hereafter an admissible control will be an element of $\Phi^{z,m}$.

2.2 Bayesian updates

Obviously, the prior m will evolve with time, as the value of the unknown parameter is partially revealed through the observation of the impacts of the actions on the system: at time t, one has observed $\{\mathrm{z}'(Z^{z,\phi}_{\tau_i^\phi -}, \alpha_i^\phi, \upsilon, \epsilon_i) : i \geq 1, \vartheta_i^\phi \leq t\}$.
It should therefore be considered as a state variable, in any case, as his dynamics will naturally appear in any dynamic programming principle related to the optimal control of $X^{z,\phi}$, see Proposition 5.2 below.

Moreover, its evolution can be of interest in itself. One can for instance be interested by the precision of our (updated) prior at the end of the control period, as it can serve as a new prior for another control problem.

In this section, we describe how it is updated with time, according to the usual Bayesian procedure.
Given $z = (t, x) \in \mathbf{Z}$, $u \in \mathrm{U}$ and $a \in \mathbf{A}$, we assume that the law of $\mathrm{z}'[z, a, u, \epsilon_1]$, recall (2.7), is given by

$$\mathrm{q}(\cdot|z, a, u)d\mathrm{Q}(\cdot|z, a),$$

in which $\mathrm{q}(\cdot|\cdot)$ is a Borel measurable map and $\mathrm{Q}(\cdot|z, a)$ is a dominating measure on $\mathbf{Z}$ for each $(z, a) \in \mathbf{Z} \times \mathbf{A}$.
For $z = (t, x) \in \mathbf{Z}$, $m \in \mathbf{M}$ and $\phi \in \Phi^{z,m}$, let $M^{z,m,\phi}$ be the process defined by

$$M_s^{z,m,\phi}[C] := \mathbb{P}_m[\upsilon \in C|\mathcal{F}_s^{z,m,\phi}], \;\; C \in \mathcal{B}(\mathrm{U}), \; s \geq t. \tag{2.8}$$

As no new information is revealed in between the end of an action and the start of the next one, the prior should remain constant on these time intervals:

$$M^{z,m,\phi} = M^{z,m,\phi}_{\vartheta_i^\phi} \text{ on } [\vartheta_i^\phi, \tau_{i+1}^\phi) \;, \;\; i \geq 0, \tag{2.9}$$

with the conventions $\vartheta_0^\phi = 0$ and $M_0^{z,m,\phi} = m$. But, $M^{z,m,\phi}$ should jump from each τ_i to each ϑ_i^ϕ, $i \geq 1$, according to the Bayes rule:

$$M^{z,m,\phi}_{\vartheta_i^\phi} = \mathfrak{M}(M^{z,m,\phi}_{\tau_i^\phi -}; Z^{z,\phi}_{\vartheta_i^\phi}, Z^{z,\phi}_{\tau_i^\phi -}, \alpha_i^\phi), \;\; i \geq 1, \tag{2.10}$$

in which

$$\mathfrak{M}(m_o; z_o', z_o, a_o)[C] := \frac{\int_C \mathrm{q}(z_o'|z_o, a_o, u)dm_o(u)}{\int_{\mathrm{U}} \mathrm{q}(z_o'|z_o, a_o, u)dm_o(u)}, \tag{2.11}$$

for almost all $(z_o, z_o', a_o, m_o) \in \mathbf{Z}^2 \times \mathbf{A} \times \mathbf{M}$ and $C \in \mathcal{B}(\mathrm{U})$.

Note that we did not explicit $M^{z,m,\phi}$ on each $[\tau_i, \vartheta_i)$ since the controller must wait the time ϑ_i before being able to make another action. A partial information on υ through ϑ_i is known as a right-censored observation of ϑ_i is revealed through the interval $[\tau_i, \vartheta_i)$.
In order to ensure that $M^{z,m,\phi}$ remains in $\mathbf{M}$ whenever $m \in \mathbf{M}$, we assume that

$$\mathfrak{M}(\mathbf{M}; \cdot) \subset \mathbf{M}.$$

We formalize this in the next proposition.

Proposition 2.1. *For all $z = (t, x) \in \mathbf{Z}$, $m \in \mathbf{M}$ and $\phi \in \Phi^{z,m}$, the process $M^{z,m,\phi}$ is $\mathbf{M}$ valued and follows the dynamics* (2.9)-(2.10) *on* $[t, 2T]$.

Proof. We fix $i \geq 0$. Let C be a Borel set of U and φ be a Borel bounded function on the Skorohod space $\mathrm{D}^{\mathrm{d}+1}$ of càdlàg[1] functions with values in $\mathbb{R}^{d+1}$. Set $\xi^\phi := \sum_{i\geq 1} \mathbf{1}_{[\vartheta_i^\phi,\infty)}$ and set $\delta X^i := X^{z,\phi}_{\cdot\vee\vartheta_i^\phi} - X^{z,\phi}_{\vartheta_i^\phi}$. One can find a Borel measurable map $\bar\varphi$ on $\mathrm{D}^{2\mathrm{d}+1}$ such that

$$\varphi(X^{z,\phi}_{\cdot\wedge s}, \xi^\phi_{\cdot\wedge s})\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} = \bar\varphi(X^{z,\phi}_{\cdot\wedge\vartheta_i^\phi}, \delta X^i_{\cdot\wedge s}, \xi^\phi_{\cdot\wedge\vartheta_i^\phi})\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}.$$

Then, the independence of υ with respect to $\sigma(W_{\cdot\vee\vartheta_i^\phi} - W_{\vartheta_i^\phi})$ given $\mathcal{F}^{z,m,\phi}_{\vartheta_i^\phi}$, and the fact that τ_{i+1}^ϕ is measurable with respect to the sigma-algebra generated by $\sigma(W_{\cdot\vee\vartheta_i^\phi} - W_{\vartheta_i^\phi})$ and $\mathcal{F}^{z,m,\phi}_{\vartheta_i^\phi}$ imply that, for $s \geq 0$,

$$\begin{aligned}
\mathbb{E}_m\left[\mathbf{1}_{\{\upsilon\in C\}}\varphi(X^{z,\phi}_{\cdot\wedge s}, \xi^\phi_{\cdot\wedge s})\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}\right] &= \mathbb{E}_m\left[\mathbf{1}_{\{\upsilon\in C\}}\bar\varphi(X^{z,\phi}_{\cdot\wedge\vartheta_i^\phi}, \delta X^i_{\cdot\wedge s}, \xi^\phi_{\cdot\wedge\vartheta_i^\phi})\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}\right] \\
&= \mathbb{E}_m\left[M^{z,m,\phi}_{\vartheta_i^\phi}[C]\bar\varphi(X^{z,\phi}_{\cdot\wedge\vartheta_i^\phi}, \delta X^i_{\cdot\wedge s}, \xi^\phi_{\cdot\wedge\vartheta_i^\phi})\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}\right] \\
&= \mathbb{E}_m\left[M^{z,m,\phi}_{\vartheta_i^\phi}[C]\varphi(X^{z,\phi}_{\cdot\wedge s}, \xi^\phi_{\cdot\wedge s})\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}\right].
\end{aligned}$$

This shows that $M^{z,m,\phi}_s[C]\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} = M^{z,m,\phi}_{\vartheta_i^\phi}[C]\mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}$ $\mathbb{P}_m$ − a.s.
It remains to compute $M^{z,m,\phi}_{\vartheta_i^\phi}$. Note that (2.3) implies that $(X^{z,\phi}_{\tau_i^\phi-}, \xi^\phi_{\tau_i^\phi-}) = (X^{z,\phi}_{\vartheta_i^\phi-}, \xi^\phi_{\vartheta_i^\phi-})$. Let φ be as above, and let $\bar\varphi$ be a Borel measurable map on $\mathrm{D}^{\mathrm{d}+1} \times \mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\begin{aligned}
\varphi(X^{z,\phi}_{\cdot\wedge\vartheta_i^\phi}, \xi^\phi_{\cdot\wedge\vartheta_i^\phi}) &= \bar\varphi(X^{z,\phi}_{\cdot\wedge\tau_i^\phi-}, \xi^\phi_{\cdot\wedge\tau_i^\phi-}, \vartheta_i^\phi, X^{z,\phi}_{\vartheta_i^\phi}) \\
&= \bar\varphi(X^{z,\phi}_{\cdot\wedge\tau_i^\phi-}, \xi^\phi_{\cdot\wedge\tau_i^\phi-}, \mathrm{z}'[\tau_i^\phi, X^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi, \upsilon, \epsilon_i]).
\end{aligned}$$

Then, since ϵ_i is independent of $\mathcal{F}^{z,m,\phi}_{\tau_i^\phi}$ and has the same law as ϵ_1,

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{\{\upsilon\in C\}}\varphi(X^{z,\phi}_{\cdot\wedge\vartheta_i^\phi}, \xi^\phi_{\cdot\wedge\vartheta_i^\phi})\right] \\
&= \mathbb{E}_m\left[\mathbf{1}_{\{\upsilon\in C\}}\bar\varphi(X^{z,\phi}_{\cdot\wedge\tau_i^\phi-}, \xi^\phi_{\cdot\wedge\tau_i^\phi-}, \mathrm{z}'[\tau_i^\phi, X^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi, \upsilon, \epsilon_i])\right] \\
&= \mathbb{E}_m\left[\int \mathbf{1}_{\{\upsilon\in C\}}\bar\varphi(X^{z,\phi}_{\cdot\wedge\tau_i^\phi-}, \xi^\phi_{\cdot\wedge\tau_i^\phi-}, z')\mathrm{q}(z'|Z^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi, \upsilon)d\mathrm{Q}(z'|Z^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi))\right] \\
&= \mathbb{E}_m\left[\int \bar\varphi(X^{z,\phi}_{\cdot\wedge\tau_i^\phi-}, \xi^\phi_{\cdot\wedge\tau_i^\phi-}, z')\left(\int_C \mathrm{q}(z'|Z^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi, u)dM^{z,m,\phi}_{\tau_i^\phi-}(u)\right)d\mathrm{Q}(z'|Z^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi))\right].
\end{aligned}$$

Let us now introduce the notation

$$\mathfrak{M}_i[C](z') := \mathfrak{M}(M^{z,m,\phi}_{\tau_i^\phi-}; z', Z^{z,\phi}_{\tau_i^\phi-}, \alpha_i^\phi).$$

[1]continue à droite et limitée à gauche (right continuous with left limits)

Then,

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{\{\upsilon\in C\}}\varphi(X^{z,\phi}_{\cdot\wedge\vartheta^\phi_i},\xi^\phi_{\cdot\wedge\vartheta^\phi_i})\right]\\
&=\mathbb{E}_m\left[\int\bar\varphi(X^{z,\phi}_{\cdot\wedge\tau^\phi_i-},\xi^\phi_{\cdot\wedge\tau^\phi_i-},z')\mathfrak{M}_i[C](z')\mathrm{q}(z'|Z^{z,\phi}_{\tau^\phi_i-},\alpha^\phi_i,\upsilon)d\mathrm{Q}(z'|Z^{z,\phi}_{\tau^\phi_i-},\alpha^\phi_i))\right]\\
&=\mathbb{E}_m\left[\varphi(X^{z,\phi}_{\cdot\wedge\vartheta^\phi_i},\xi^\phi_{\cdot\wedge\vartheta^\phi_i})\mathfrak{M}_i[C](Z^{z,\phi}_{\vartheta^\phi_i})\right].
\end{aligned}$$

This concludes the proof. □

Remark 2.1. *Note from* (2.11) *that* $M^{z,m,\phi}$ *remains absolutely continuous with respect to* m *over time.*

Remark 2.2. *For later use, note that the above provides the joint conditional distribution of* $(Z^{z,\phi}_{\vartheta^\phi_i}, M^{z,m,\phi}_{\vartheta^\phi_i})$ *given* $\mathcal{F}^{z,m,\phi}_{\tau_i}$*. Namely, for Borel sets* $B\in\mathcal{B}([t,T],\mathbb{R}^d)$ *and* $D\in\mathcal{B}(\mathbf{M})$*, a simple application of Fubini's Lemma implies that*

$$\mathbb{P}[(Z^{z,\phi}_{\vartheta^\phi_i}, M^{z,m,\phi}_{\vartheta^\phi_i})\in B\times D|\mathcal{F}^{z,m,\phi}_{\tau^\phi_i-}]=\mathrm{k}(B\times D|Z^{z,\phi}_{\tau^\phi_i-}, M^{z,m\phi}_{\tau^\phi_i-},\alpha^\phi_i) \tag{2.12}$$

in which

$$\mathrm{k}(B\times D|z_o,m_o,a_o):=\int_{\mathrm{U}}\int_B \mathbf{1}_D(\mathfrak{M}(m_o;z',z_o,a_o))\mathrm{q}(z'|z_o,a_o,u)d\mathrm{Q}(z'|z,a)dm_o(u), \tag{2.13}$$

for $(z_o,m_o,a_o)\in\mathbf{Z}\times\mathbf{M}\times\mathbf{A}$.

2.3 Gain function

Given $z=(t,x)\in\mathbf{Z}$ and $m\in\mathbf{M}$, the aim of the controller is to maximize the expected value of the gain functional

$$\phi\in\Phi^{z,m}\mapsto G^{z,m}(\phi):=g(Z^{z,\phi}_{\mathrm{T}[\phi]},M^{z,m,\phi}_{\mathrm{T}[\phi]},\upsilon,\epsilon_0),$$

in which $\mathrm{T}[\phi]$ is the end of the last action after T:

$$\mathrm{T}[\phi]:=\sup\{\vartheta^\phi_i : i\ge 1,\ \tau^\phi_i\le T\}\vee T.$$

As suggested earlier, the gain may not only depend on the value of the original time-space state process $Z^{z,\phi}_{\mathrm{T}[\phi]}$ but also on $M^{z,m,\phi}_{\mathrm{T}[\phi]}$, to model the fact that we are also interested by the precision of the estimation made on υ at the final time. One also allows for terminating the last action after T. However, since g can depend on $\mathrm{T}[\phi]$ through $Z^{z,\phi}_{\mathrm{T}[\phi]}$, one can penalize the actions that actually terminates strictly after T.

Hereafter, the function g is assumed to be measurable and bounded[2] on $\mathbf{Z}\times\mathbf{M}\times\mathrm{U}\times\mathrm{E}$.

[2]Boundedness is just for sake of simplicity. Much more general frameworks could easily be considered.

Given $\phi \in \Phi^{z,m}$, the expected gain is

$$J(z,m;\phi) := \mathbb{E}_m\left[G^{z,m}(\phi)\right],$$

and

$$\mathrm{v}(z,m) := \sup_{\phi\in\Phi^{z,m}} J(z,m;\phi)\mathbf{1}_{\{t\le T\}} + \mathbf{1}_{\{t>T\}}\mathbb{E}_m\left[g(z,m,\upsilon,\epsilon_0)\right] \tag{2.14}$$

is the corresponding value function. Note that v depends on m through the set of admissible controls $\Phi^{z,m}$ and the expectation operator $\mathbb{E}_m$, even if g does not depend on $M^{z,m,\phi}_{\mathrm{T}[\phi]}$.

3 Value function characterization and numerical approximation

3.1 The dynamic programming quasi-variational equation

The aim of this section is to provide a characterization of the value function v. As usual, it should be related to a dynamic programming principle. In our setting, it should read as follows: Given $z = (t,x) \in \mathbf{Z}$ and $m \in \mathbf{M}$, then

$$\mathrm{v}(z,m) = \sup_{\phi\in\Phi^{z,m}} \mathbb{E}_m[\mathrm{v}(Z^{z,\phi}_{\theta^\phi}, M^{z,m,\phi}_{\theta^\phi})], \tag{3.1}$$

for all collection $(\theta^\phi, \phi \in \Phi^{z,m})$ of $\mathbb{F}^{z,m,\phi}$-stopping times with values in $[t, 2T]$ such that

$$\theta^\phi \in \mathcal{N}^\phi \cap [t, \mathrm{T}[\phi]]\ \mathbb{P}_m - \text{a.s.}\ ,$$

recall the definition of $\mathcal{N}^\phi$ in (2.2).
Let us comment this. First, one should restrict to stopping times such that $\theta^\phi \in \mathcal{N}^\phi$. The reason is that no new impulse can be made outside of $\mathcal{N}^\phi$, each interval $[\tau^\phi_i, \vartheta^\phi_i)$ is a latency period. Second, the terminal gain is evaluated at $\mathrm{T}[\phi]$, which in general is different from T. Hence, the fact that θ^ϕ is only bounded by $\mathrm{T}[\phi]$.
A partial version of (3.1) will be proved in Proposition 5.2 below and will be used to provide a sub-solution property. As already mentioned in the introduction, we are not able to prove a full version (3.1). The reason is that the value function v depends on $z = (t,x) \in \mathbf{Z}$ and $m \in \mathbf{M}$ through the set of admissible controls $\Phi^{z,m}$, and more precisely through the choice of the filtration $\mathbb{F}^{z,m,\phi}$, which even depends on ϕ itself. This makes this dependence highly singular and we are neither in position to play with any a-priori smoothness, see e.g. [5], nor to apply a measurable selection theorem, see e.g. [16].

We continue our discussion, assuming that (3.1) holds and that v is sufficiently smooth. Then, it should in particular satisfy

$$\mathrm{v}(z,m) \ge \mathbb{E}_m[\mathrm{v}(Z^{z,\circ}_{t+h}, m)]$$

whenever $z = (t, x) \in [0, T) \times \mathbb{R}^d$ and $0 < h \leq T - t$ ($Z^{z,\circ}$ is defined after (2.5)). This corresponds to the sub-optimality of the control consisting in making no impulse on $[t, t+h]$. Applying Itô's lemma, dividing by h and letting h go to 0, we obtain

$$-\mathcal{L}\mathrm{v}(z, m) \geq 0$$

in which $\mathcal{L}$ is the Dynkin operator associated to $X^{z,\circ}$,

$$\mathcal{L}\varphi := \partial_t \varphi + \langle \mu, D\varphi \rangle + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top D^2\varphi].$$

On the other hand, it follows from (3.1) and Remark 2.2 that

$$\mathrm{v}(z, m) \geq \sup_{a \in \mathbf{A}} \mathbb{E}_m[\mathrm{v}(\mathrm{z}'[z, a, \upsilon, \epsilon_1], \mathfrak{M}(m; \mathrm{z}'[z, a, \upsilon, \epsilon_1], z, a))] = \mathcal{K}\mathrm{v}(z, m)$$

where

$$\mathcal{K}\varphi := \sup_{a \in \mathbf{A}} \mathcal{K}^a \varphi \quad \text{with} \quad \mathcal{K}^a \varphi := \int \varphi(z', m') d\mathrm{k}(z', m' | \cdot, a) \text{ for } a \in \mathbf{A}. \tag{3.2}$$

As for the time-T boundary condition, the same reasoning as above implies

$$\mathrm{v}(T, \cdot) \geq \mathcal{K}_T g \quad \text{and} \quad \mathrm{v}(T, \cdot) \geq \mathcal{K}\mathrm{v}(T, \cdot),$$

in which

$$\mathcal{K}_T g(\cdot, m) = \int_{\mathrm{U}} \int_{\mathrm{E}} g(\cdot, m, u, e) d\mathbb{P}_\epsilon(e) dm(u). \tag{3.3}$$

By optimality, v should therefore solve the quasi-variational equations

$$\min\{-\mathcal{L}\varphi\ ,\ \varphi - \mathcal{K}\varphi\} = 0 \ \text{ on } [0, T) \times \mathbb{R}^d \times \mathbf{M} \tag{3.4}$$

$$\min\{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}\varphi\} = 0 \ \text{ on } \{T\} \times \mathbb{R}^d \times \mathbf{M}, \tag{3.5}$$

in the sense of the following definition (given for sake of clarity).

Definition 3.1. *We say that a lower-semicontinuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity super-solution of* (3.4)-(3.5) *if for any $z_\circ = (t_\circ, x_\circ) \in \mathbf{Z}$, $m_\circ \in \mathbf{M}$, and $\varphi \in C^{1,2,0}([0, T] \times \mathbb{R}^d \times \mathbf{M})$ such that $\min_{\mathbf{Z} \times \mathbf{M}}(U - \varphi) = (U - \varphi)(z_\circ, m_\circ) = 0$ we have*

$$\left[\min\{-\mathcal{L}\varphi\ ,\ \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_\circ < T\}} + \min\{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_\circ = T\}}\right](z_\circ, m_\circ) \geq 0.$$

We say that a upper-semicontinuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity sub-solution of (3.4)-(3.5) *if for any $z_\circ = (t_\circ, x_\circ) \in \mathbf{Z}$, $m_\circ \in \mathbf{M}$ and $\varphi \in C^{1,2,0}([0, T] \times \mathbb{R}^d \times \mathbf{M})$ such that $\max_{\mathbf{Z} \times \mathbf{M}}(U - \varphi) = (U - \varphi)(z_\circ, m_\circ) = 0$ we have*

$$\left[\min\{-\mathcal{L}\varphi\ ,\ \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_\circ < T\}} + \min\{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_\circ = T\}}\right](z_\circ, m_\circ) \leq 0.$$

We say that a continuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity solution of (3.4)-(3.5) *if it is a super- and a sub-solution.*

To ensure that the above operator is continuous, we assume from now on that, on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$,

$$\begin{aligned} &\mathcal{K}_T g \text{ is continuous, and } \mathcal{K}\varphi \text{ is upper- (resp. lower-) semicontinuous,} \\ &\text{for all upper- (resp. lower-) semicontinuous bounded function } \varphi. \end{aligned} \tag{3.6}$$

A sufficient condition for (3.6) to hold is that k defined in (2.13) is a continuous stochastic kernel, see [16, Proposition 7.31 and 7.32 page 148].
Finally, we assume that comparison holds for (3.4)-(3.5).

Assumption 3.1. *Let U (resp. V) be a upper- (resp. lower-) semicontinuous bounded viscosity sub- (resp. super-) solution of* (3.4)-(3.5)*. Assume further that $U \le V$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Then, $U \le V$ on $\mathbf{Z} \times \mathbf{M}$.*

See Proposition 3.1 below for a sufficient condition. We are now in position to state the main result of this paper.

Theorem 3.1. *Let Assumption 3.1 hold. Then,* v *is continuous on $\mathbf{Z} \times \mathbf{M}$ and is the unique bounded viscosity solution of* (3.4)-(3.5)*.*

The proof is postponed to Section 5.

Proposition 3.1. *A sufficient condition for Assumption 3.1 to hold is: There exists a continuous function Ψ on $[0, 2T] \times \mathbb{R}^d \times \mathbf{M}$ satisfying:*

(i) *$\Psi(., m) \in C^{1,2}([0, T) \times \mathbb{R}^d)$, for all $m \in \mathbf{M}$.*

(ii) *$\varrho\Psi \ge \mathcal{L}\Psi$ on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ for some constant $\varrho > 0$,*

(iii) *$\Psi - \mathcal{K}\Psi \ge \delta$ on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ for some $\delta > 0$,*

(iv) *$\Psi \ge \mathcal{K}_T[\tilde{g}]$ on $[T, \infty) \times \mathbb{R}^d \times \mathbf{M}$ with $\tilde{g}(t, .) := e^{\varrho t} g(t, .)$ and ϱ is defined in* (ii)*,*

(v) *Ψ^- is bounded.*

The idea of the proof is the same as in [4, Proposition 4.12]. Note that their condition H2 (v) is not required here because we only consider bounded sub and super-solutions and we take a different approach. To avoid it, we slighlty reinforce the hypothesis H2 (iii) and asked for Ψ^- to be bounded.
The proof is postponed to Section 6.

3.2 An example of numerical scheme

When the comparison result of Assumption 3.1 holds, one can easily derive a convergent finite different scheme for (3.4)-(3.5).

We consider here a simple explicit scheme based on [6, 7]. We let h_0 be a time-discretization step so that T/h_0 is an integer, and set $\mathbf{T}^{h_0} := \{t_j^{h_0} := jh_0, j \le T/h_0\}$. The space $\mathbb{R}^d$

is discretized with a space step h_1 on a rectangle $[-c,c]^d$, containing $N^x_{h_1}$ points on each direction. The corresponding finite set is denoted by $\mathbf{X}^{h_1}_c$.
The first order derivatives $\partial_t\varphi$ and $(\partial\varphi/\partial x^i)_{i\le d}$ are approximated by using the standard up-wind approximations:

$$\Delta_t^{h_0}\varphi(t,x,m) := h_0^{-1}(\varphi(t+h_0,x,m)-\varphi(t,x,m))$$
$$\Delta_{h_1,i}^{h_0}\varphi(t,x,m) := \begin{cases} h_1^{-1}(\varphi(t+h_0,x+e_ih_1,m)-\varphi(t,x,m)) & \text{if} \quad \mu^i(x)\ge 0 \\ h_1^{-1}(\varphi(t,x,m)-\varphi(t+h_0,x-e_ih_1,m)) & \text{if} \quad \mu^i(x)<0, \end{cases}$$

in which e_i is i-th unit vector of $\mathbb{R}^d$.
As for the second order term, we use the fact that each point $x\in\mathbb{R}^d$ can be approximated as a weighted combination

$$x = \sum_{x'\in C_{h_1}(x)} x'\omega(x'|x)$$

of the points x' lying on the corners $C_{h_1}(x)$ of the cube formed by the partition of $\mathbb{R}^d$ it belongs too. Then, given another small parameter $h_2>0$, we approximate $\mathrm{Tr}[\sigma(x)\sigma(x)^\top D^2\varphi(t,x,m)]$ by $\mathrm{T}^{h_2}_{h_0,h_1}[\varphi](t,x,m)$ defined as

$$(h_2d)^{-1}\sum_{i=1}^{d}[\varphi]_{h_1}(t+h_0,x+\sqrt{h_2}\sigma^i(x),m)+[\varphi]_{h_1}(t+h_0,x-\sqrt{h_2}\sigma^i(x),m)$$
$$-2h_2^{-1}\varphi(t,x,m)$$

in which σ^i is the i-th column of σ and

$$[\varphi]_{h_1}(t,x,m) := \sum_{x'\in C_{h_1}(x)} \omega(x'|x)\varphi(t,x',m)$$

is a piecewise linear approximation of φ. In the case where only the first row $\sigma^{1\cdot}$ of σ is not identically equal to 0, one can use the usual simpler approximation

$$(h_1)^{-1}\|\sigma^{1\cdot}\|^2\left(\varphi(t+h_0,x+\sqrt{h_1}e_1,m)+\varphi(t+h_0,x-\sqrt{h_1}e_1,m)\right)$$
$$-2(h_1)^{-1}\|\sigma^{1\cdot}\|^2\varphi(t,x,m).$$

Similarly, we approximate $\mathcal{K}\varphi$ by

$$\mathcal{K}_{h_0,h_1}\varphi(t,x,m) := \sup_{a\in\mathbf{A}}\int[\varphi]_{h_1}(\max(t+h_0,t'),x',m')\mathrm{dk}(t',x',m'|t,x,m,a).$$

Letting $h:=(h_0,h_1,h_2)$, and setting

$$\mathcal{L}^h\varphi := \Delta_t^{h_0}\varphi+\sum_{i\le d}\mu^i\Delta_{h_1,i}^{h_0}\varphi+\frac{1}{2}\mathrm{T}^{h_2}_{h_0,h_1}[\varphi], \tag{3.7}$$

our numerical scheme consists in solving

$$\min\left\{-\mathcal{L}^h\varphi\ ,\ \varphi-\mathcal{K}_{h_1}\varphi\right\}=0\ \text{ on } (\mathbf{T}^{h_0}\setminus\{T\})\times(\mathbf{X}_c^{h_1}\setminus\partial\mathbf{X}_c^{h_1})\times\mathbf{M}, \tag{3.8}$$

$$\min\{\varphi-\mathcal{K}_T g\ ,\ \varphi-\mathcal{K}_{h_1}\varphi\}=0\ \text{ on } \{T\}\times(\mathbf{X}_c^{h_1}\setminus\partial\mathbf{X}_c^{h_1})\times\mathbf{M}, \tag{3.9}$$

$$\varphi-\mathcal{K}_T g:=0\ \text{ on } ([0,T]\times\partial\mathbf{X}_c^{h_1}\times\mathbf{M})\cup((T,2T]\times\mathbb{R}^d\times\mathbf{M}). \tag{3.10}$$

We specify here a precise boundary condition on $\partial\mathbf{X}_c^{h_1}$ but any other (bounded) boundary condition could be used.
This scheme is always convergent as $(h_2,h_1/h_2,h_0/h_1)\to 0$ and $c\to\infty$. The proof requires the following additional lemma.

Lemma 3.1. *If $(u_n)_{n\geq 1}$ is a bounded sequence of functions on $\mathbf{Z}\times\mathbf{M}$ and $(z_n,m_n)_{n\geq 1}$ is a sequence in $\mathbf{Z}\times\mathbf{M}$ that converges to $(z_\circ,m_\circ)$, then*

$$\liminf_{\substack{n\to\infty\\(h_0,h_1)\to(0,0)}}\ \mathcal{K}_{h_0,h_1}u_n(z_n,m_n)\geq\mathcal{K}u_\circ(z_\circ,m_\circ)\quad,\text{where } u_\circ:=\liminf_{\substack{n\to\infty\\(z',m')\to\cdot}}\ u_n(z',m'),$$

and

$$\limsup_{\substack{n\to\infty\\(h_0,h_1)\to(0,0)}}\ \mathcal{K}_{h_0,h_1}u_n(z_n,m_n)\leq\mathcal{K}u^\circ(z_\circ,m_\circ)\quad,\text{where } u^\circ:=\limsup_{\substack{n\to\infty\\(z',m')\to\cdot}}\ u_n(z',m').$$

Proof. We first rewrite

$$\mathcal{K}_{h_0,h_1}u_n(z_n,m_n)=\sup_{a\in\mathbf{A}}\int u_{n,h}(z',m')dk(z',m'|z_n,m_n,a) \tag{3.11}$$

where $u_{n,h}(z',m'):=[u_n]_{h_1}(\max(t_n+h_0,t'),x',m')$. Let $\bar{u}_{n_\circ,h_\circ}$ be the lower-semicontinuous envelope of $\inf_{n\geq n_\circ,h\leq h_\circ}u_{n,h}$. From (3.11), we get, for $n\geq n_\circ$ and $h\leq h_\circ$,

$$\mathcal{K}u_{n,h}(z_n,m_n)\geq\mathcal{K}\bar{u}_{n_\circ,h_\circ}(z_n,m_n),$$

and, by (3.6), passing to the limit inf as $(n,h)\to(+\infty,0)$ leads to

$$\liminf_{(n,h)\to(+\infty,0)}\mathcal{K}u_{n,h}(z_n,m_n)\geq\mathcal{K}\bar{u}_{n_\circ,h_\circ}(z_\circ,m_\circ).$$

Moreover, $\bar{u}_{n_\circ,h_\circ}\uparrow u_\circ$ point-wise. The required result is then obtained by monotone convergence.

Proposition 3.2. *Let v_h denote the solution of* (3.8)-(3.9)-(3.10). *If Assumptions 3.1 holds, then $\mathrm{v}_h\to\mathrm{v}$ as $(h_2,h_1/h_2,h_0/h_1)\to 0$.*

Proof. Using Lemma 3.1, one easily checks that our scheme satisfies the conditions of [3, Theorem 2.1.]. In particular, $|\mathrm{v}_h|\leq\sup|g|<\infty$. Then, the convergence holds by the same arguments as in [3, Theorem 2.1.], it suffices to replace their assertion (2.7) by Lemma 3.2 stated below. □

Remark 3.1. *We did not discuss in the above the problem of the discrete approximation of* $\mathbf{M}$*. Applications will typically be based on a parameterized family* $\mathbf{M} = \{m_\theta, \theta \in \Theta\}$*, for a subset* Θ *of a finite dimensional space. We can then further approximate* Θ *by a sequence of finite sets to build up a numerical scheme.*

We conclude this section with the technical lemma that was used in the above proof.

Lemma 3.2. *Let* $(u_n)_{n\geq 1}$ *be a sequence of lower semi-continuous maps on* $\mathbf{Z}\times\mathbf{M}$ *and define* $u_\circ := \liminf_{(z',m',n)\to(\cdot,\infty)} u_n(z',m')$ *on* $\mathbf{Z}\times\mathbf{M}$*. Assume that* $u_\circ$ *is locally bounded. Let* φ *be a continuous map and assume that* $(z_\circ, m_\circ)$ *is a strict minimal point of* $u_\circ - \varphi$ *on* $\mathbf{Z}\times\mathbf{M}$*. Then, one can find a bounded open set* B *of* $[0,T]\times\mathbb{R}^d$ *and a sequence* $(z_k, m_k, n_k)_{n\geq 1} \subset B\times\mathbf{M}\times\mathbb{N}$ *such that* $n_k\to\infty$*,* (z_k, m_k) *is a minimum point of* $u_{n_k} - \varphi$ *on* $B\times\mathbf{M}$ *and* $(z_k, m_k, u_{n_k}(z_k, m_k)) \to (z_o, m_\circ, u_\circ(z_o, m_o))$*.*

Proof. Since $\mathbf{M}$ is assumed to be locally compact, it suffices to repeat the arguments in the proof of [2, p80, Proof of Lemma 6.1]. □

4 Applications to optimal trading

This section is devoted to the study of two examples of application. Each of them corresponds to an idealized model, the aim here is not to come up with a good model but rather to show the flexibility of our approach, and to illustrate numerically the behavior of our backward algorithm.

4.1 Immediate impact of aggressive orders with dynamic resilience

We consider first a model in which the impact of each single order sent to the market is taken into account. It means that α_i represents the number of shares bought exactly at time τ_i, so that $\ell_i = 0$, for each i. This corresponds to $\mathbf{A} = \{0\}\times\mathbf{B}$ in which $\mathbf{B}\subset\mathbb{R}_+$ is a compact set of values of admissible orders. Therefore, one can identify $\mathbf{A}$ to $\mathbf{B}$ in the following, and we will only write b for $a = (0, b) \in \mathbf{A}$ and β_i for $\alpha_i = (\ell_i, \beta_i)$.
Our model can be viewed as a scheduling model or as a model for illiquid market. The first component of X represents the stock price. We consider a simple linear impact: when a trade of size β_i occurs at τ_i, the stock price jumps by

$$X^1_{\vartheta_i} = X^1_{\tau_i -} + \beta_i(\upsilon + \epsilon_i)/2$$

in which $\upsilon \in \mathbb{R}$ is the unknown linear impact parameter, $(\epsilon_i)_{i\geq 1}$ is a sequence of independent noises following a centered Gaussian distribution with standard deviation σ_ϵ. The coefficient $1/2$ in the dynamics of X^1 stands for a 50% proportion of immediate resilience.
It evolves according to a Brownian diffusion between two trades and has a residual resilience effect:

$$dX^1_t = \sigma dW^1_t + dX^4_t \text{ and } dX^4_t = -\rho X^4_t dt, \tag{4.1}$$

where $\sigma, \rho > 0$ and $X^1_0 \in \mathbb{R}$ are constants. The process X^4 represents the drift of X^1 due to the non immediate resilience and $X^4_0 = 0$. When a trade occurs, it jumps according to

$$X^4_{\vartheta_i} = X^4_{\tau_i-} + \beta_i(\upsilon + \epsilon_i)/2.$$

We call it spread hereafter. This is part of the deviation from the un-impacted dynamic. The third component, which describes the total cost, evolves as

$$X^2_{\vartheta_i} = X^2_{\tau_i-} + X^1_{\tau_i-}\beta_i + (\upsilon + \epsilon_i)\frac{\beta_i^2}{2}.$$

Finally, the last component is used to keep track of the cumulative number of shares bought:

$$X^3_{\vartheta_i} = X^3_{\tau_i-} + \beta_i.$$

We are interest in the cost of buying N shares, and maximize the criteria

$$-\mathbb{E}_m[e^{\eta L(X_T,\upsilon)} \wedge C]$$

where $\eta > 0$ is a risk aversion parameter, $C > 0$, and

$$L(X_T, \upsilon) := X^2_T + X^1_T(N - X^3_T) + (\upsilon + \epsilon_0)\frac{(N - X^3_T)^2}{2}$$

represents the total cost after setting the total number of shares bought to N at T.
If the prior law m on υ is a Gaussian distribution, then $\mathrm{q}(\cdot|t, x, b, u)$ is a Gaussian density with respect to

$$d\mathrm{Q}(x'|t, x, b) = dx^{1'} d\delta_{x^2+bx^{1'}}(x^{2'}) d\delta_{x^3+b}(x^{3'}) d\delta_{x^4+(x^{1'}-x^1)}(x^{4'})$$

and the transition map

$$\mathfrak{M}(m; t', x', t, x, b)[C] = \frac{\int_C \mathrm{q}(x'|t, x, b, u)dm(u)}{\int_{\mathbb{R}} \mathrm{q}(x'|t, x, b, u)dm(u)},$$

maps Gaussian distributions into Gaussian distributions, which, in practice, enables us to restrict $\mathbf{M}$ to the set of Gaussian distributions. More precisely, if $(m_\upsilon(\tau_i-), \sigma_\upsilon(\tau_i-))$ are the mean and the standard deviation of M_{τ_i-}, then the values corresponding to the posterior distribution M_{ϑ_i} are

$$\sigma_\upsilon(\vartheta_i) = \mathbf{1}_{\{\sigma_\upsilon(\tau_i-)\neq 0\}} \left(\frac{1}{\sigma_\upsilon(\tau_i-)^2} + \frac{1}{\sigma^2_\epsilon}\right)^{-\frac{1}{2}},$$
$$m_\upsilon(\vartheta_i) = m_\upsilon(\tau_i-)\mathbf{1}_{\{\sigma_\upsilon(\tau_i-)=0\}} + \left(\frac{X^1_{\vartheta_i} - X^1_{\tau_i-}}{\sigma^2_\epsilon} + \frac{m_\upsilon(\tau_i-)}{\sigma_\upsilon(\tau_i-)^2}\right)\mathbf{1}_{\{\sigma_\upsilon(\tau_i-)\neq 0\}}.$$

Comparing to the general result of the previous section, we add a boundary condition $\mathrm{v}(t, x^1, x^2, N, x^4) = 1$ and restrict the domain of X^3 to be $\{0, \ldots, N\}$. Since this parameter x^3 is discrete this does not change the nature of our general results.

Note also that the map $\Psi(t,x,m) = N - x^3$ defined on $[0,T] \times \mathbb{R}^2 \times \{0,\ldots,N\} \times \mathbb{R} \times \mathbf{M}$ actually satisfies the conditions of Proposition 3.1.

We now discuss a numerical illustration. We consider 30 seconds of trading and $N = 25$ shares to buy. We take $\eta = 1$, $x_0 = 100$ and $\sigma = 0.4x_0$ which corresponds to a volatility of 40% in annual terms. The trading period is divided into intervals of 1 second-length. The size of an order β_i ranges in $\{1,2,3,4,5\}$. We take $\sigma_\varepsilon = 10^{-4}$ and ρ such that the spread X^4 is divided by 3 every second if no new order is sent. We start with a prior given by a Gaussian distribution with mean $m_\upsilon(0)$ and standard deviation $\sigma_\upsilon(0)$. Finally, we take $C = 10^{200}$ which makes this threshold parameter essentially inefficient while still ensuring that the terminal condition is bounded.
In Figure 1, we plot the optimal strategy for $\sigma_\upsilon(0) = 5.10^{-4}$ and $m_\upsilon(0) = 5.10^{-2}$ in terms of (X^2, X^3). Clearly, the level of spread X^4 has a significant impact: when it is large, it is better to wait for it to decrease before sending a new order. This can also be observed in Figure 2 which provides a simulated path corresponding to an initial prior $(m_\upsilon(0) = 2.10^{-2}, \sigma_\upsilon(0) = 10^{-3})$: after 15 seconds the algorithm alternates between sending an order and doing nothing, i.e. waiting for the spread to be reduced at the next time step. On the top right graph, we can also observe that the low mean of the initial prior combined with a zero initial resilience leads to sending an order of size 3 at first, then the mean of the prior is quickly adjusted to a higher level and the algorithm slows down immediately.

4.2 Random execution times: application to strategies using limit-orders

In this section, we consider a limit-order trading model. X^1 now represents a mid-price (of reference) and, between two trades, has the dynamic

$$dX_t^1 = \sigma dW_t^1. \tag{4.2}$$

An order is of the form (ℓ, β) in which ℓ is the maximal time we are ready to wait before being executed, while β is the price at which the limit order is sent[3]. For simplicity, each order corresponds to buying one share.
We assume that the time θ it takes to be executed follows an exponential distribution of parameter $\rho(\upsilon, X_\tau^1 - \beta)$, given the information at time τ. One can send a new order only after $\vartheta := \tau + \ell \wedge \theta$.
Hence, given a flow of orders $\phi = (\tau_i, \ell_i, \beta_i)_{i \geq 1}$, the number X^3 of shares bought evolves according to

$$X^3 = X^3_{\vartheta_i} \quad \text{on } [\vartheta_i, \tau_{i+1})$$
$$X^3_{\vartheta_i} = X^3_{\tau_i-} + \mathbf{1}_{\{\theta_i \leq \ell_i\}},$$

in which $\vartheta_i := \tau_i + \ell_i \wedge \theta_i$. Each θ_i follows an exponential distribution of parameter $\rho(\upsilon, X^1_{\tau_i} - \beta_i)$ given $\mathcal{F}^{z,m,\phi}_{\tau_i-}$. As in the previous model, X^3 is restricted to $\{0,\ldots,N\}$. The total cost X^2 of

[3] Dark pool strategies could be considered similarly, in this case, β would rather describe the choice of the trading plateform

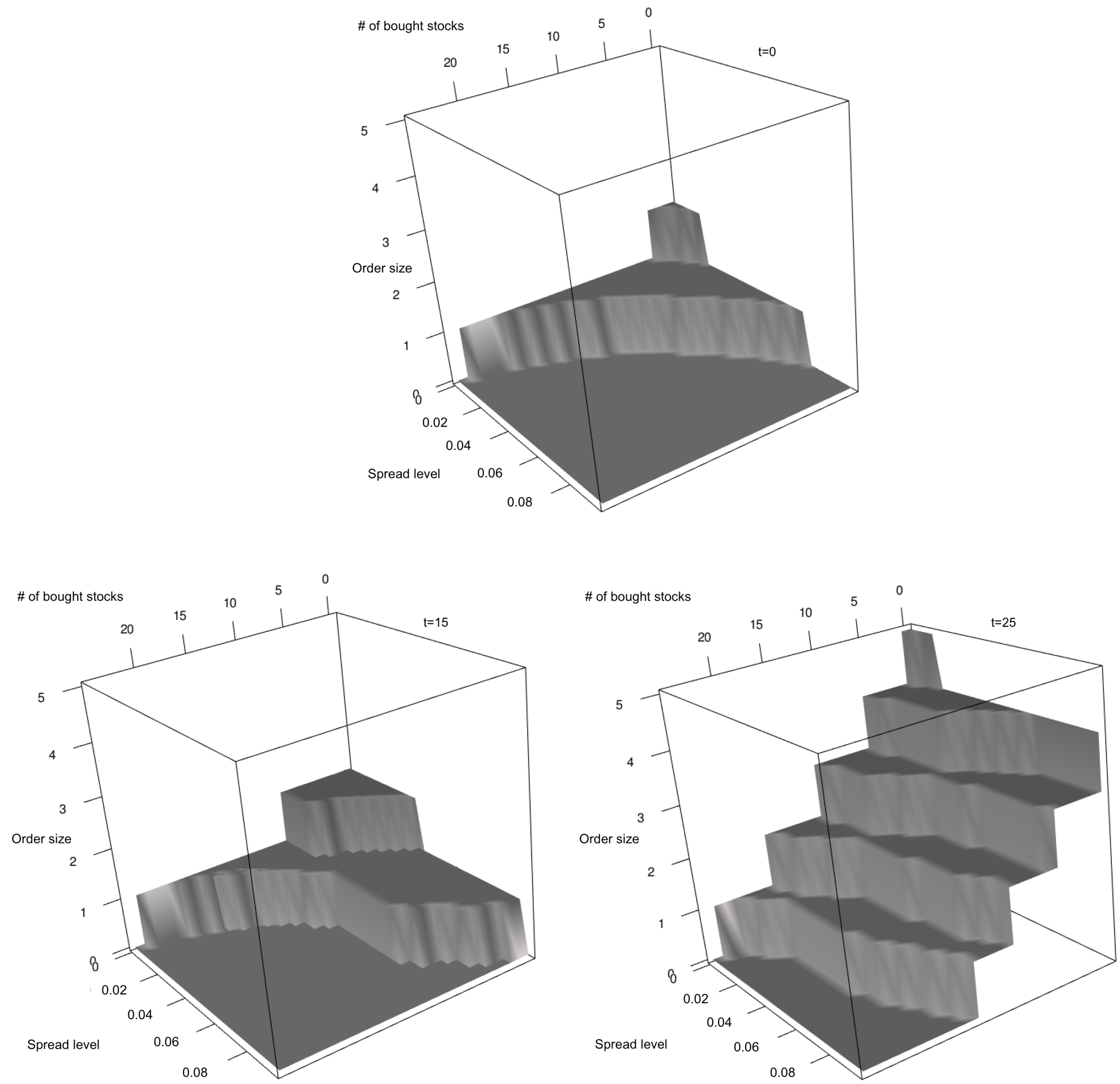

Figure 1: Evolution of β in terms of (X^3, X^4) at time 0s (top), 15s (left) and 25s (right), for $(m_\upsilon, \sigma_\upsilon) = (5.10^{-2}, 5.10^{-4})$.

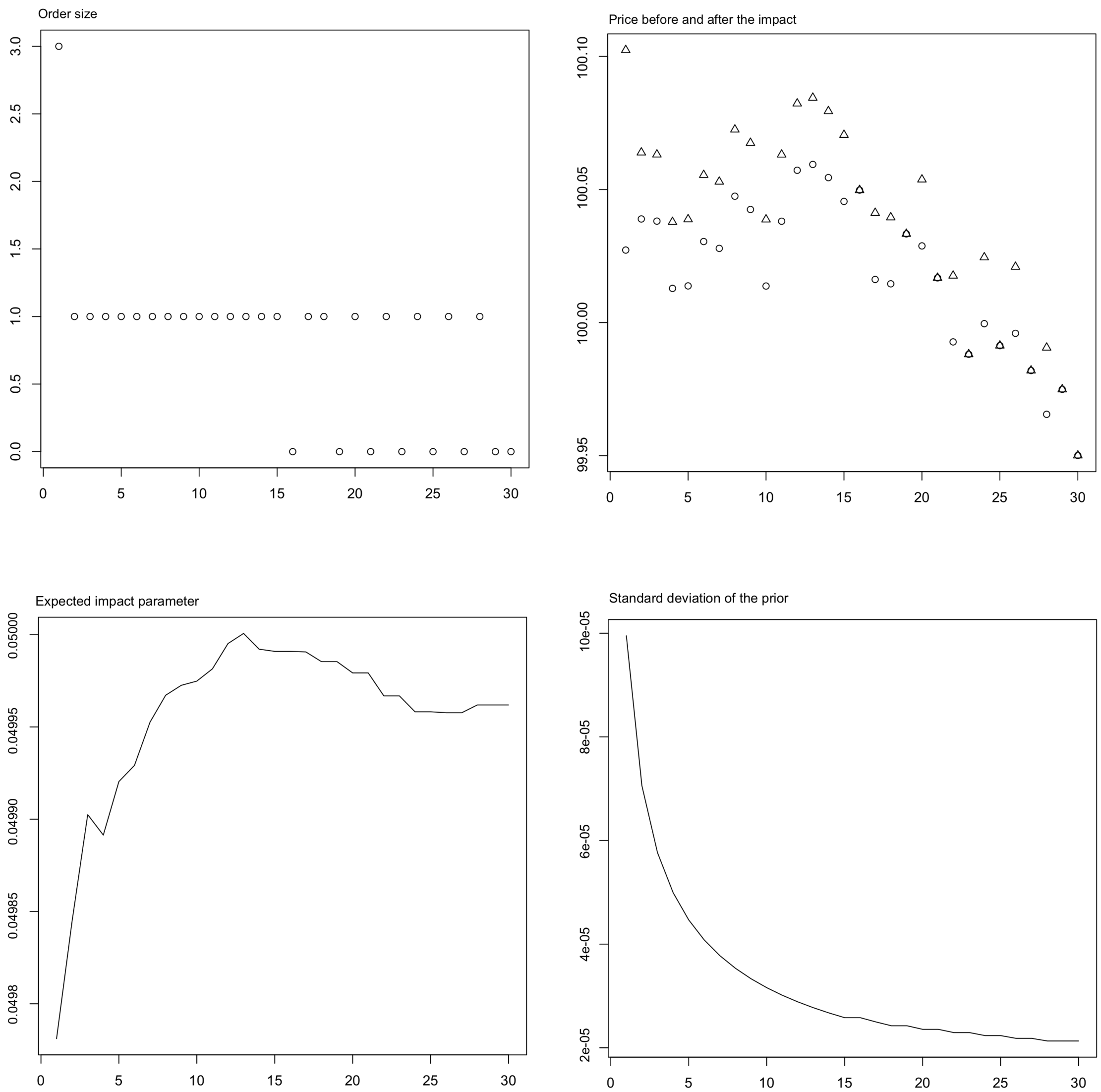

Figure 2: Evolution of β (top left), price before (circles) and after (triangles) the impact (top right), m_υ (bottom left), σ_υ (bottom right), with time in second. The true value of υ is 5.10^{-2}. x-axis: time in seconds.

buying the shares has the dynamics

$$X^2 = X^2_{\vartheta_i} \text{ on } [\vartheta_i, \tau_{i+1})$$
$$X^2_{\vartheta_i} = X^2_{\tau_i -} + \beta_i \mathbf{1}_{\{\theta_i \le \ell_i\}}.$$

We want to maximize

$$-\mathbb{E}\left[e^{X^2_{\mathrm{T}[\phi]}+1.02(N-X^3_{\mathrm{T}[\phi]})+\frac{5.10^2}{2}(N-X^3_{\mathrm{T}[\phi]})^2} \wedge C\right],$$

in which 1.02 is the best ask (kept constant) and 5.10^2 is an impact coefficient. This corresponds to the cost of liquidating instantaneously the remaining shares $(N - x^3)^+$ at T. This model is a version of [1], [12], [14], see also [13].

Direct computations show that the prior process M evolves according to

$$M = M_{\vartheta_i} \quad \text{on } [\vartheta_i, \tau_{i+1})$$
$$M_{\vartheta_i} = \mathfrak{M}_1(M_{\tau_i-}; Z_{\vartheta_i}, Z_{\tau_i-}, \alpha_i)\mathbf{1}_{\{\theta_i \le \ell_i\}} + \mathfrak{M}_2(M_{\tau_i-}; Z_{\vartheta_i}, Z_{\tau_i-}, \alpha_i)\mathbf{1}_{\{\theta_i > \ell_i\}}$$

in which

$$\mathfrak{M}_1(m; t', x', t, x, l, b)[B] := \frac{\int_B \rho(u, x^1 - b)e^{-\rho(u,x^1-b)t'} dm(u)}{\int_{\mathbb{R}^+} \rho(u, x^1 - b)e^{-\rho(u,x^1-b)t'} dm(u)}$$

and

$$\mathfrak{M}_2(m; t', x', t, x, l, b)[B] := \frac{\int_B e^{-\rho(u,x^1-b)l} dm(u)}{\int_{\mathbb{R}^+} e^{-\rho(u,x^1-b)l} dm(u)}$$

for all Borel set B.
In the case where $\mathbf{M}$ is the convex hull of a finite number of Dirac masses, then the weights associated to M can be computed explicitly.

Here again, the map $\Psi(t, x, m) = N - x^3$ satisfies the conditions of Proposition 3.1.

We now consider a numerical illustration. We take $C = 10^{200}$. The time horizon is $T = 15$ minutes. To simplify, we fix the reference mid-price to be $X^1 \equiv 1$ (i.e. $\sigma = 0$) and restrict to $\ell = 1$, i.e. an order is sent each minute. We take $N = 10$. One can send limit buy orders in the range $B := \{0.90, 0.92, 0.94, 0.96, 0.98\}$.
As for the intensity of the execution time, we use an exponential form as in [12]: $\rho(u, x^1 - b) = \lambda(u)e^{-20(0.98-b)}$ in which $\lambda(u) = -\ln(1 - u)$. This means that the probability to be executed at the price 0.98 within one minute is u. Orders are sent each minute, but we use a finner time grid in order to take into account that it can be executed before this maximal time-length. The original prior is supported by two Dirac masses at $u = 0.3$ and $u = 0.8$. The corresponding probabilities of being executed within one minute are plotted in Figure 3.
Our time step corresponds to 15 seconds, so that every 15 seconds the controller can launch a new order if the previous one has been executed before the maximal 1 minute time-length. In Figure 4, we plot the difference, in logarithms, between the value functions obtained in the latter case and for a time step of 1 minute (in which case a new order cannot be

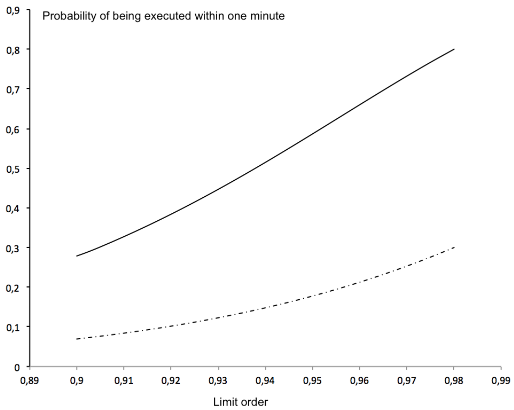

Figure 3: Solid: $u = 0.8$. Dashed: $u = 0.3$

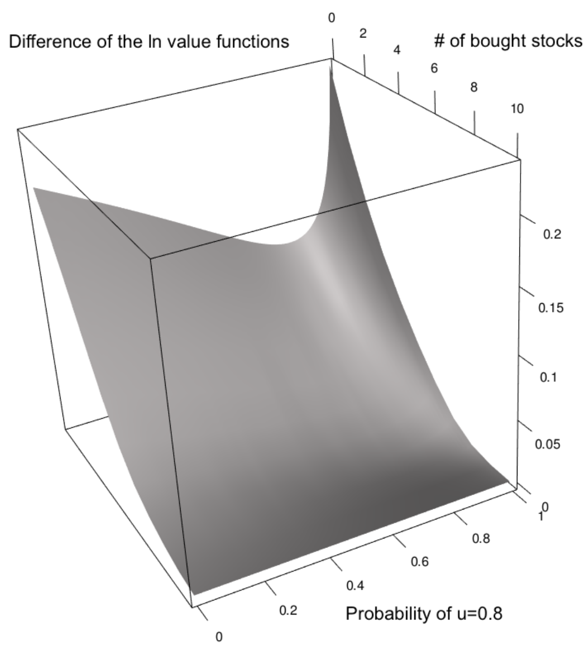

Figure 4

launched before one minute). Clearly, the possibility of launching new orders in advance is an advantage.
In Figure 5, we plot the optimal policy at time $t = 0$ and $t = 7.5$ minutes. As expected, the algorithm is more aggressive when the probability of having $\upsilon = 0.8$ is higher.
In Figure 6, we plot a simulated path. The red and black lines and points correspond to the

same realization of the random variables at hand, but for different values of the real value of υ. Black corresponds to the most favorable case $\upsilon = 0.8$, while red corresponds to $\upsilon = 0.8$ for the first 7.5 minutes and $\upsilon = 0.3$ for the remaining time. The initial prior is $\mathbb{P}[\upsilon = 0.8] = 9\%$. Again, the algorithm adapts pretty well to this shock on the true parameter. We also see that it is more aggressive when the prior probability of being in the favorable case is high. The difference in total cost is not important because of the number of shares already bought before the shock, only two shares remain to be bought at T in the less favorable case.

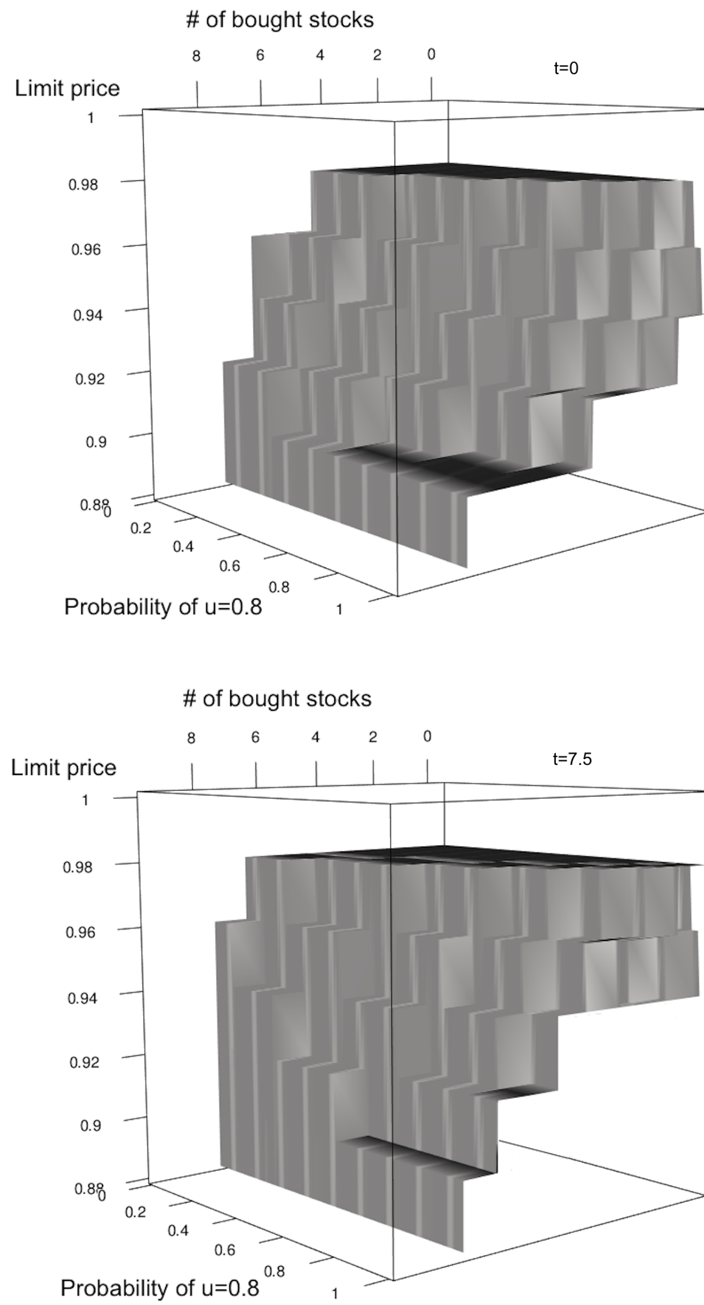

Figure 5: Top: $t = 0$. Bottom: $t = 7.5$ minutes

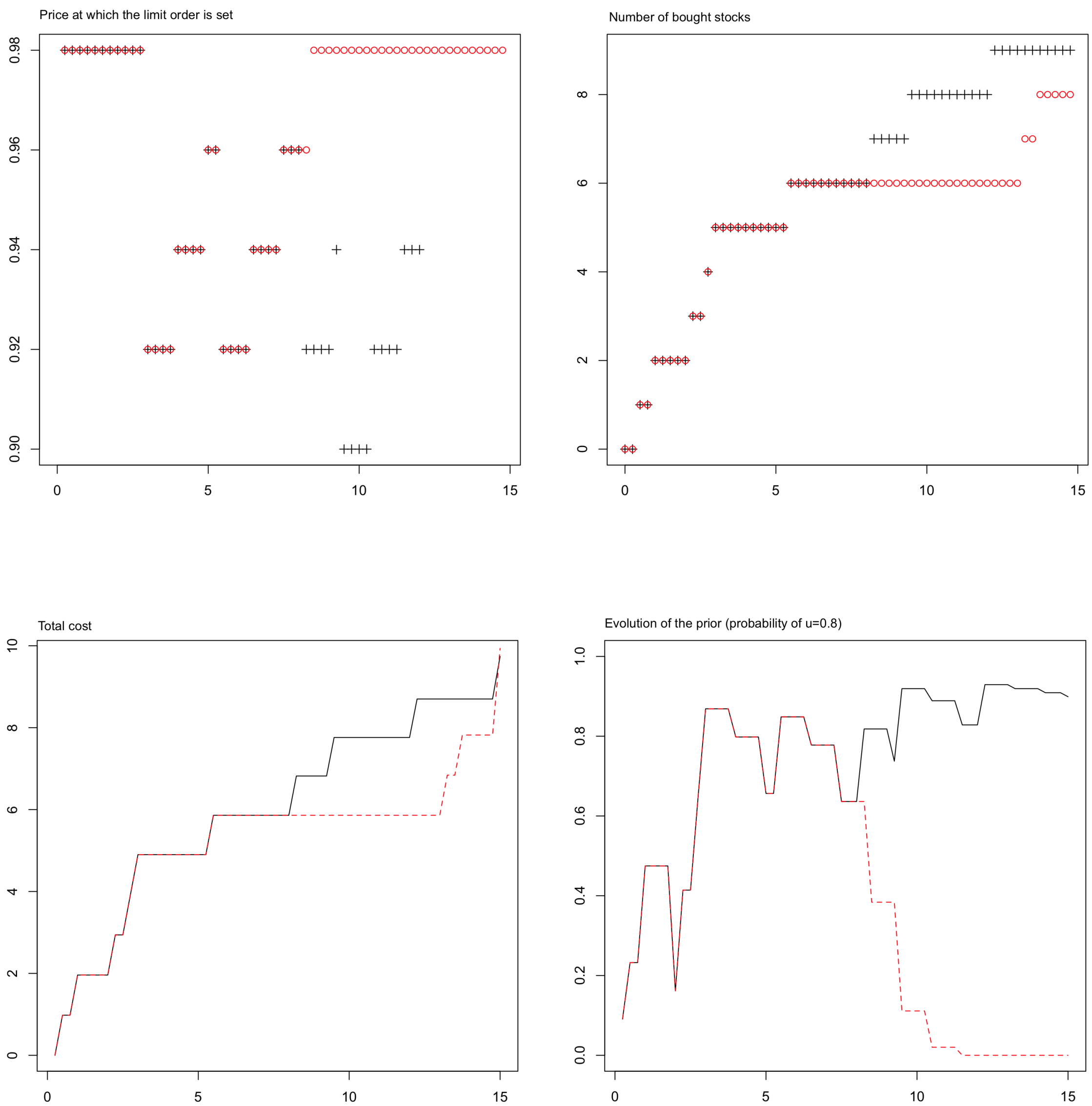

Figure 6: Black crosses and solid lines: $\upsilon = 0.8$. Red circles and dashed lines: $\upsilon = 0.8$ before $t = 7.5$ minutes and $\upsilon = 0.3$ after. x-axis= time in minutes.

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5 Proof of the viscosity solution characterization

This part is dedicated to the proof of the viscosity solution characterization of Theorem 3.1. We start with the sub-solution property, which is the more classical part. As for the super-solution property, we shall later on introduce a discrete time version of the model that will provide a natural lower bound. We will then show that the sequence of corresponding value functions converges to a super-solution of our quasi-variational equation as the time step goes to 0. By comparison, we will finally identify this (limit) lower bound to the original value function, thus showing that the later is also a super-solution.

5.1 Sub-solution property

We start with the sub-solution property and show that it is satisfied by the upper-semicontinuous enveloppe

$$\mathrm{v}^*(z,m) := \limsup_{(z',m')\to(z,m)} \mathrm{v}(z',m') \ , \ \ (z,m) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M},$$

recall (2.14).

Proposition 5.1. v^* *is a viscosity subsolution of* (3.4)-(3.5)*.*

The proof is rather standard. As usual, it is based on the partial dynamic programming principle contained in Proposition 5.2 below, that can be established by adapting standard lines of arguments, see e.g. [5]. For this part, the dependency of the filtration on the initial data is not problematic as it only requires a conditioning argument. Before to state it, let us make an observation.

Remark 5.1. *Note that, given $z=(t,x)\in\mathbf{Z}$, the process $X^{z,\circ}$ defined in* (2.5) *is predictable with respect to the $\mathbb{P}$-augmentation of the raw filtration $\mathbb{F}^{t,W}$ generated by $(W_{\cdot\vee t}-W_t)$. By [9, Lemma 7, Appendix I], it is indistinguishable from a $\mathbb{F}^{t,W}$-predictable process. Using this identification, $X^{z,\circ}_s(\omega)=X^{z,\circ}_s(\omega^{t,s})$ for $s\ge t$, with $\omega^{t,s}:=\omega_{t\vee\cdot\wedge s}-\omega_t$. Similarly, τ^ϕ_1 and α^ϕ_1 can be identified to Borel measurable maps on $C([0,T];\mathbb{R}^d)$ that depends only on $\omega^{t,\tau^\phi_1(\omega^{t,T})}$ so that $(Z^{z,\phi}_{\vartheta^\phi_1},M^{z,m,\phi}_{\vartheta^\phi_1})$ can be seen as a Borel map on $C([0,T];\mathbb{R}^d)\times\mathrm{U}\times\mathrm{E}$, while $(Z^{z,\phi}_{\tau^\phi_1-},M^{z,m,\phi}_{\tau^\phi_1-})$ can be seen as a Borel map on $C([0,T];\mathbb{R}^d)$ that only depends on $\omega^{t,\tau^\phi_1(\omega^{t,T})}$, recall* (2.6), (2.9) *and* (2.10)*. Iterating this argument, we also obtain that $(Z^{z,\phi}_{\mathrm{T}[\phi]},M^{z,m,\phi}_{\mathrm{T}[\phi]})$ is equal, up to $\mathbb{P}_m$-null sets, to a Borel map on $C([0,T];\mathbb{R}^d)\times\mathrm{U}\times\mathrm{E}^N$, for some $N\ge1$ that depends on ϕ.*

We use the notations introduced in (2.5), (3.2) and (3.3) in the following.

Proposition 5.2. *Fix $(z,m)\in\mathbf{Z}\times\mathbf{M}$, and let θ be the first exit time of $Z^{z,\circ}$ from an Borel set $B\subset\mathbf{Z}$ containing (z,m). Then,*

$$\mathrm{v}(z,m) \le \sup_{\phi\in\Phi^{z,m}_{\ge t}} \mathbb{E}_m[f(Z^{z,\circ}_\theta,m)\mathbf{1}_{\{\theta<\tau^\phi_1\}} + \mathcal{K}^{\alpha^\phi_1} f(Z^{z,\circ}_{\tau^\phi_1-},m)]\mathbf{1}_{\{\theta\ge\tau^\phi_1\}}] \tag{5.1}$$

in which $z := (t,x)$, $\Phi^{z,m}_{\geq t} := \{\phi \in \Phi^{z,m} : \tau^\phi_1 \geq t\}$ *and*

$$f(z',m') := \mathrm{v}^*(z',m')\mathbf{1}_{\{t'<T\}} + \mathcal{K}_T g(z',m')\mathbf{1}_{\{t'\geq T\}} \tag{5.2}$$

for $z' = (t',x') \in \mathbf{A}$ *and* $m' \in \mathbf{M}$.

Proof. Let $N \geq 1$ be such that $\tau^\phi_i > T$ for $i \geq N$. By right continuity of $(Z^{z,\phi}, M^{z,m,\phi})$ and upper-semicontinuity of f and $\mathcal{K}f$ on $[0,T) \times \mathbb{R}^d \times \mathbf{M}$, see (3.6), it suffices to prove the result for the projections on the right of θ and τ^ϕ_1 on a deterministic time grid. Then, it is enough to consider the case where $(\theta, \tau^\phi_1) \equiv (s,s') \in [t,T]^2$, by arguing as below and conditioning by the values taken by (θ,τ^ϕ_1) on the grid. In the following, we use regular conditional expectation operators. We shall make use of Remark 5.1. In particular, we write $\phi(\omega, u, (e_i)_{i\leq N})$ to denote the Borel map $(\omega,u,(e_i)_{i\leq N}) \in C([0,T];\mathbb{R}^d) \times \mathrm{U} \times \mathrm{E}^N \mapsto \{(\tau^\phi_i, \alpha^\phi_i)(\omega^{t,T}, u, (e_j)_{j\leq i-1}),$ $i \leq N\}$ associated to ϕ. If $s < s'$, we have $\mathbb{P}_m$-a.s.

$$\begin{aligned}\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}^{z,m,\phi}_s](\omega,u,(e_i)_{i\geq 1}) &= \mathbb{E}_m[G^{Z^{z,\circ}_s(\omega^{t,s}),m}(\phi_{\omega^{t,s}})] \\ &= \mathbb{E}_m[\mathcal{K}_T g(X_T^{Z^{z,\circ}_s(\omega^{t,s}),\phi_{\omega^{t,s}}}, M_T^{Z^{z,\circ}_s(\omega^{t,s}),m,\phi_{\omega^{t,s}}})]\end{aligned}$$

in which $\mathcal{K}_T$ is defined in (3.3) and

$$\phi_{\omega^{t,s}} : (\omega',u,(e_i)_{i\leq N}) \in C([s,T];\mathbb{R}^d) \times \mathrm{U} \times \mathrm{E}^N \mapsto \phi(\omega^{t,s} + \omega'_{\cdot\vee s} - \omega'_s, u, (e_i)_{i\leq N})$$

is an element of $\Phi^{Z^{z,\circ}_s(\omega^{t,s}),m,\phi_{\omega^{t,s}}}$. It follows that

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}^{z,m,\phi}_s]\mathbf{1}_{s<s'} \leq f(Z^{z,\circ}_s, m)\mathbf{1}_{s<s'} \;\; \mathbb{P}_m - \text{a.s.}$$

Similarly, if $s \geq s'$, we have $\mathbb{P}_m$-a.s.

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}^{z,m,\phi}_{s'-}](\omega,u,(e_i)_{i\leq N}) = \mathbb{E}_m[G^{\xi(\omega^{t,s'},\upsilon,\epsilon_1,\alpha^\phi_1(\omega^{t,s'}))}(\phi_{\omega^{t,s'}})]$$

with

$$\xi(\omega^{t,s'},\upsilon,\epsilon_1,\alpha^\phi_1(\omega^{t,s'})) = \Big(\cdot, \mathfrak{M}(m;\cdot, Z^{z,\circ}_{s'-}(\omega^{t,s'}), \alpha^\phi_1(\omega^{t,s'}))\Big) \circ \mathrm{z}'(Z^{z,\circ}_{s'-}(\omega^{t,s'}), \alpha^\phi_1(\omega^{t,s'}), \upsilon, \epsilon_1),$$

recall the notations in (2.7) and (2.11). Hence, $\mathbb{P}_m$-a.s.,

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}^{z,m,\phi}_{s'-}](\omega,u,(e_i)_{i\leq N}) \leq \mathbb{E}_m[f(\xi(\omega^{t,s'},\upsilon,\epsilon_1,\alpha^\phi_1(\omega^{t,s'})))] = \mathcal{K}^{\alpha^\phi_1(\omega^{t,s'})} f(Z^{z,\circ}_{s'-}(\omega^{t,s'}), m),$$

in which $a \in \mathbf{A} \mapsto \mathcal{K}^a$ is defined in (3.2). □

Proof of Proposition 5.1 As already mentioned, the proof is standard, we provide it for completeness. Let φ be a (bounded) $C^{1,2,0}$ function and fix $(z_\circ, m_\circ) \in \mathbf{Z} \times \mathbf{M}$ such that

$$0 = (\mathrm{v}^* - \varphi)(z_\circ, m_\circ) = \max_{\mathbf{Z}\times\mathbf{M}}(\mathrm{v}^* - \varphi). \tag{5.3}$$

We use the notation $z_\circ = (t_\circ, x_\circ) \in [0,T] \times \mathbb{R}^d$.

Step 1. We first assume that $t_\circ < T$. Let us suppose that

$$\min\left\{-\mathcal{L}\varphi\ ,\ \varphi - \mathcal{K}\mathrm{v}^*\right\}(z_\circ, m_\circ) > 0,$$

and work towards a contradiction to Proposition 5.2. Let $\mathrm{d}_{\mathbf{M}}$ be a metric compatible with the weak topology and let $\|\cdot\|_{\mathbf{Z}}$ be the Euclidean norm on $\mathbf{Z}$. We define

$$\bar\varphi(z', m') := \varphi(z', m') + \|z' - z_\circ\|^4_{\mathbf{Z}} + \mathrm{d}_{\mathbf{M}}(m', m_\circ).$$

If the above holds, then

$$\min\left\{-\mathcal{L}\bar\varphi\ ,\ \bar\varphi - \mathcal{K}\mathrm{v}^*\right\}(z_\circ, m_\circ) > 0.$$

By our continuity assumption (3.6), we can find $\iota, \eta > 0$, such that

$$\min\left\{-\mathcal{L}\bar\varphi\ ,\ \bar\varphi - \mathcal{K}\mathrm{v}^*\right\} \geq \eta \quad \text{on } B_\iota, \tag{5.4}$$

in which

$$B_\iota := \{(z', m') \in \mathbf{Z} \times \mathbf{M} : \|z' - z_\circ\|^4_{\mathbf{Z}} + \mathrm{d}_{\mathbf{M}}(m', m_\circ) < \iota\} \subset [0, T) \times \mathbb{R}^d \times \mathbf{M}.$$

Note that, after possibly changing $\eta > 0$, we can assume that

$$(\mathrm{v}^* - \bar\varphi) \leq -\eta \text{ on } (B_\iota)^c. \tag{5.5}$$

In the following, we let $(z, m) \in B_\iota$ be such that

$$|\mathrm{v}(z, m) - \bar\varphi(z, m)| \leq \eta/2, \tag{5.6}$$

recall (5.3). As above, we write $z = (t, x) \in [0, T] \times \mathbb{R}^d$. Fix $\phi \in \Phi^{z,m}$. We write $(\tau_i, \alpha_i, \vartheta_i)_{i\geq 1}$, Z and M for $(\tau_i^\phi, \alpha_i^\phi, \vartheta_i^\phi)_{i\geq 1}$, $Z^{z,\phi}$ and $M^{z,m,\phi}$. Let θ be the first time when (Z, M) exits B_ι. Without loss of generality, one can assume that $\tau_1 \geq t$. Define

$$\chi := \theta \mathbf{1}_{\{\theta < \tau_1\}} + \mathbf{1}_{\{\theta \geq \tau_1\}} \vartheta_1.$$

In view of (5.4), (5.5) and (5.6),

$$\begin{aligned}
\mathbb{E}_m[\mathrm{v}^*(Z_\chi, M_\chi)] &= \mathbb{E}_m[\mathrm{v}^*(Z_{\vartheta_1}, M_{\vartheta_1})\mathbf{1}_{\{\chi \neq \theta\}} + \mathrm{v}^*(Z_\theta, M_\theta)\mathbf{1}_{\{\chi = \theta\}}] \\
&\leq \mathbb{E}_m[\mathcal{K}\mathrm{v}^*(Z_{\tau_1-}, M_{\tau_1-})\mathbf{1}_{\{\chi \neq \theta\}} + \mathrm{v}^*(Z_\theta, M_\theta)\mathbf{1}_{\{\chi = \theta\}}] \\
&\leq \mathbb{E}_m[\bar\varphi(Z_{\theta\wedge\tau_1-}, M_{\theta\wedge\tau_1-})] - \eta \\
&\leq \bar\varphi(z, m) - \eta \\
&\leq \mathrm{v}(z, m) - \eta/2.
\end{aligned}$$

Since $\chi < T$, this contradicts Proposition 5.2 by arbitrariness of ϕ.
Step 2. We now consider the case $t_\circ = T$. We assume that

$$\min\left\{\varphi - \mathcal{K}\mathrm{v}^*\ ,\ \varphi - \mathcal{K}_T g\right\}(z_\circ, m_\circ) > 0,$$

and work toward a contradiction. Let us define

$$\bar{\varphi}(t',x',m') := \bar{\varphi}(t',x',m') + C(T-t') + \|(t',x') - z_\circ\|^4_{\mathbf{Z}} + \mathrm{d}_{\mathbf{M}}(m',m_\circ)$$

and note that, for C large enough,

$$\min\left\{-\mathcal{L}\bar{\varphi}\ ,\ \bar{\varphi} - \mathcal{K}\mathrm{v}^*\ ,\ \bar{\varphi} - \mathcal{K}_T g\right\}(z_\circ, m_\circ) > 0.$$

Then, as in Step 1, we can find $\iota, \eta > 0$, such that

$$\min\left\{-\mathcal{L}\bar{\varphi}\ ,\ \bar{\varphi} - \mathcal{K}\mathrm{v}^*\ ,\ \bar{\varphi} - \mathcal{K}_T g\right\} \geq \eta \quad \text{on } B_\iota,$$

in which

$$B_\iota := \{(t',x',m') \in (T-\iota, T] \times \mathbf{M} : \|x' - x_\circ\|^4_{\mathbb{R}^d} + \mathrm{d}_{\mathbf{M}}(m', m_\circ) < \iota\}.$$

After possibly changing $\eta > 0$, one can assume that

$$(\mathrm{v}^* - \bar{\varphi}) \leq -\eta \text{ on } (B_\iota)^c.$$

Let $(t,x,m) \in B_\iota$ be such that

$$|\mathrm{v}(t,x,m) - \bar{\varphi}(t,x,m)| \leq \eta/2.$$

One can assume that $t < T$. Otherwise, this would mean that

$$\mathrm{v}^*(z_\circ, m_\circ) = \limsup_{(T,x',m') \to (z_\circ, m_\circ)} \mathrm{v}(T,x',m') = \limsup_{(T,x',m') \to (z_\circ, m_\circ)} \mathcal{K}_T(T,x',m') = \mathcal{K}_T g(z_\circ, m_\circ),$$

recall (3.6), and there is nothing to prove.
Given $\phi \in \Phi^{z,m}$, with $z := (t,x)$, let $(\tau_1, \vartheta_1, Z = (\cdot, X), M)$ be defined as in Step 1 with respect to ϕ and (z,m), and consider

$$\chi := \theta \mathbf{1}_{\{\theta < \tau_1\}} + \mathbf{1}_{\{\theta \geq \tau_1\}} \vartheta_1,$$

where θ is the first exit time of (X,M) from $\{(x',m') \in \mathbb{R}^d \times \mathbf{M} : \|x' - x_\circ\|^4_{\mathbb{R}^d} + \mathrm{d}_{\mathbf{M}}(m', m_\circ) < \iota\}$. As in Step 1, the above implies that

$$\mathbb{E}_m[\mathrm{v}^*(Z_\chi, M_\chi)] \leq \mathrm{v}(z,m) - \eta/2,$$

which contradicts Proposition 5.2 by arbitrariness of ϕ. □

5.2 Discrete time approximation and dynamic programming

In this part, we prepare for the proof of the super-solution property. As already mentioned above, we could not provide the opposite inequality in (5.1), with v^* replaced by the lower-semicontinuous envelope of v, because of the non-trivial dependence of $\mathbb{F}^{z,m,\phi}$ with respect to the initial data.

Instead, we use the natural idea of approximating our continuous time control problem by a sequence of discrete time counterparts defined on a sequence of time grids. In discrete time, the dynamic programming principle can be proved along the lines of [16] for the corresponding value functions $(\mathrm{v}_n)_{n\geq 1}$. Passing to the limit as the time mesh vanishes provides a super-solution $\mathrm{v}_\circ$ of (3.4)-(3.5). As v^* is a sub-solution of the same equation, Assumption 3.1 will imply that $\mathrm{v}_\circ \geq \mathrm{v}^*$, while the opposite will hold by construction. Then, we will conclude that v is a actually a super-solution, and is even continuous. This approach is similar to the one used in [11] in the context of differential games.

We first construct the sequence of discrete time optimal control problems. For $n \geq 1$, let $\pi_n := \{t_j^n, j \leq 2^n\}$ with $t_j^n := jT/2^n$, and let $\Phi_n^{z,m}$ be the set of controls $\phi = (\tau_i^\phi, \alpha_i^\phi)_{i\geq 1}$ in $\Phi^{z,m}$ such that $(\tau_i^\phi)_{i\geq 1}$ takes values in $\pi_n \cup \{t\} \cup [T, \infty)$, if $z = (t, x)$. The corresponding value function is

$$\mathrm{v}_n(z, m) = \sup_{\phi\in\Phi_n^{z,m}} J(z, m, \phi), \quad (z, m) \in \mathbf{Z} \times \mathbf{M}.$$

We extend v_n by setting

$$\mathrm{v}_n := \mathcal{K}_T g, \quad \text{on } (T, \infty) \times \mathbf{M}, \tag{5.7}$$

Remark 5.2. *Note that* $\mathrm{v}_n \leq \mathrm{v} \leq \mathrm{v}^*$ *by construction.*

We first prove that v_n satisfies a dynamic programming principle. This requires additional notations. We first define the next time on the grid at which a new action can be made, given that a is plaid:

$$s^{n,a}[t, x] := \min\{s \in \pi_n \cup [T, \infty) : s \geq \varpi(t, x, a, \upsilon, \epsilon_j) \text{ and } s > t\}.$$

Let ∂ denote a cemetery point that does not belong to $\mathbf{A}$. Given $a \in \mathbf{A}\cup\{\partial\}$, we make a slight abuse of notation by denoting by $(Z^{(t,x),a}, M^{(t,x),m,a})$ the process defined as $(Z^{(t,x),\phi}, M^{(t,x),m,\phi})$ for ϕ such that

$$(\tau_1^\phi, \alpha_1^\phi) = \begin{cases} (t, a) & \text{if } a \neq \partial \\ (T + 1, a_\star) & \text{if } a = \partial \end{cases}$$

in which $a_\star \in \mathbf{A}$ and $\tau_i^\phi > T + 1$ for $i > 1$. Then, we set

$$\bar{J}(T, \cdot; a) := \mathcal{K}_T\mathcal{K}^a g\ ,\ \bar{\mathrm{v}}_n(T, \cdot) := \sup_{a\in\mathbf{A}\cup\{\partial\}} \bar{J}(T, \cdot; a) \ \text{ on } \mathbb{R}^d \times \mathbf{M} \times (\mathbf{A} \cup \{\partial\}),$$

with the convention that $\mathcal{K}^\partial$ is the identity, and define by backward induction on the intervals $[t_j^n, T)$, $j = n - 1, \cdots, 0$,

$$\bar{J}(z, m; a) := \mathbb{E}_m[\bar{\mathrm{v}}_n(Z^{z,a}_{s^{n,a}[z]}, M^{z,m,a}_{s^{n,a}[z]})]$$
$$\bar{\mathrm{v}}_n := \sup_{a\in\mathbf{A}\cup\{\partial\}} \bar{J}(\cdot; a),$$

together with the extension

$$\bar{\mathrm{v}}_n := \mathcal{K}_T g \ \text{ on } (T, \infty) \times \mathbb{R}^d \times \mathbf{M}.$$

Lemma 5.1. *Fix $\iota > 0$. Then, there exists a universally measurable map $(z, m) \in \mathbf{Z} \times \mathbf{M} \mapsto \hat{a}^{n,\iota}[z, m] \in \mathbf{A} \cup \{\partial\}$ such that*

$$\bar{J}(\cdot; \hat{a}^{n,\iota}[\cdot]) \geq \bar{\mathrm{v}}_n - \iota, \quad \text{on } \mathbf{Z} \times \mathbf{M}.$$

Moreover, the map $\bar{\mathrm{v}}_n$ is upper semi-analitic.

Proof. Since $\mathcal{K}_T g$ is assumed to be upper semi-analytic (indeed continuous), it follows from [16, Proposition 7.48 page 180] that $\bar{J}$ is upper semi-analytic on $[t^n_{n-1}, T] \times \mathbb{R}^d \times \mathbf{M} \times (\mathbf{A} \cup \{\partial\})$. Then, the required result holds on $[t^n_{n-2}, T] \times \mathbb{R}^d \times \mathbf{M}$ by [16, Proposition 7.50 page 184]. It is then extended to $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ by a backward induction. □

Proposition 5.3. $\bar{\mathrm{v}}_n = \mathrm{v}_n$ *on $\mathbf{Z} \times \mathbf{M}$. Moreover, given a random variable (ζ, μ) with values in $\mathbf{Z} \times \mathbf{M}$ and $\iota > 0$, there exists a measurable map $(z, m) \mapsto \phi^\iota[z, m]$ such that*

$$J(\zeta, \mu; \phi^\iota[\zeta, \mu]) \geq \mathrm{v}_n(\zeta, \mu) - \iota \ \mathbb{P}_m - \text{a.s.}$$

Proof. The proof proceeds by induction. Our claim follows from definitions on $[t^n_n, T] \times \mathbb{R}^d \times \mathbf{M}$. Assume that it holds on $[t^n_{j+1}, T] \times \mathbb{R}^d \times \mathbf{M}$ for some $j \leq n - 1$. For the following, we fix $z = (t, x) \in \mathbf{Z}$ with $t \in [t^n_j, t^n_{j+1})$ and $m \in \mathbf{M}$.

Step 1: In this step, we first construct a suitable candidate to be an almost-optimal control. Fix $\varepsilon_1, \ldots, \varepsilon_n > 0$, $\varepsilon_0 := 0$, and set $\varepsilon(i) := (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_i)$. Let $(\hat{a}^{n,\iota})_{\iota>0}$ be as in Lemma 5.1, and consider its extension defined by $\hat{a}^{n,\iota} = a_\star$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Define $r_1^{\varepsilon(0)} := t$ and $\phi_1^{\varepsilon(1)} \in \Phi_n^{z,m}$ by

$$(\tau_i^{\phi_1^{\varepsilon(1)}}, \alpha_i^{\phi_1^{\varepsilon(1)}}) = (r_1^{\varepsilon(0)}, \tilde{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m])\mathbf{1}_{\{i=1\}} + \mathbf{1}_{\{i>1\}}(T + i, a_\star) \ , \ \ i \geq 1.$$

where

$$\tilde{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m] := \hat{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m].$$

We then set

$$r_2^{\varepsilon(1)} := \min \pi_n \cap [\vartheta_1^{\phi_1^{\varepsilon(1)}}, 2T] \cap (r_1^{\varepsilon(0)}, \infty).$$

By Lemma 5.1 and [16, Lemma 7.27 page 173] applied to the pull-back measure of $(Z^{z,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}, M^{z,m,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}})$, we can find a Borel measurable map $(t', x', m') \in \mathbf{Z} \times \mathbf{M} \mapsto \tilde{a}_2^{n,\varepsilon_2}[t', x', m'] \in \mathbf{A} \cup \{\partial\}$ such that

$$\tilde{a}^{n,\varepsilon_2}[Z^{z,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}, M^{z,m,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}] = \hat{a}^{n,\varepsilon_2}[Z^{z,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}, M^{z,m,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}] \ \ \mathbb{P}_m - \text{a.s.}$$

We define $\phi_2^{\varepsilon(2)}$ by

$$(\tau_i^{\phi_2^{\varepsilon(2)}}, \alpha_i^{\phi_2^{\varepsilon(2)}}) = (r_2^{\varepsilon(1)}, \tilde{a}^{n,\varepsilon_2}[Z^{z,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}, M^{z,m,\phi_1^{\varepsilon(1)}}_{r_2^{\varepsilon(1)}}])\mathbf{1}_{\{i=2, r_2^{\varepsilon(1)} \leq T\}} + (\tau_i^{\phi_1^{\varepsilon(1)}}, \alpha_i^{\phi_1^{\varepsilon(1)}})\mathbf{1}_{\{i \neq 2\} \cup \{r_2^{\varepsilon(1)} > T\}},$$

for $i \geq 1$. We then define recursively for $k \geq 2$

$$\begin{aligned}
r_{k+1}^{\varepsilon(k)} :=& \inf \pi_n \cap [\vartheta_k^{\phi_k^{\varepsilon(k)}}, 2T] \cap (r_k^{\varepsilon(k-1)}, \infty) \\
(\tau_i^{\phi_{k+1}^{\varepsilon(k+1)}}, \alpha_i^{\phi_{k+1}^{\varepsilon(k+1)}}) =& (r_{k+1}^{\varepsilon(k)}, \tilde{a}^{n,\varepsilon_{k+1}}[Z^{z,\phi_k^{\varepsilon(k)}}_{r_{k+1}^{\varepsilon(k)}}, M^{z,m,\phi_k^{\varepsilon(k)}}_{r_{k+1}^{\varepsilon(k)}}])\mathbf{1}_{\{i=k+1, r_{k+1}^{\varepsilon(k)} \leq T\}} \\
&+ (\tau_i^{\phi_k^{\varepsilon(k)}}, \alpha_i^{\phi_k^{\varepsilon(k)}})\mathbf{1}_{\{i \neq k+1\} \cup \{r_{k+1}^{\varepsilon(k)} > T\}},
\end{aligned}$$

for $i \geq 1$, in which $(t', x', m') \in \mathbf{Z} \times \mathbf{M} \mapsto \tilde{a}_{k+1}^{n,\varepsilon_{k+1}}[t', x', m'] \in \mathbf{A} \cup \{\partial\}$ is a Borel measurable map such that

$$\tilde{a}^{n,\varepsilon_{k+1}}[Z^{z,\phi_k^{\varepsilon(k)}}_{r_{k+1}^{\varepsilon(k)}}, M^{z,m,\phi_k^{\varepsilon(k)}}_{r_{k+1}^{\varepsilon(k)}}] = \hat{a}^{n,\varepsilon_{k+1}}[Z^{z,\phi_k^{\varepsilon(k)}}_{r_{k+1}^{\varepsilon(k)}}, M^{z,m,\phi_k^{\varepsilon(k)}}_{r_{k+1}^{\varepsilon(k)}}] \quad \mathbb{P}_m - \text{a.s.}$$

We finally set

$$\phi^\varepsilon := (\tau_i^{\phi_i^{\varepsilon(i)}}, \alpha_i^{\phi_i^{\varepsilon(i)}})_{i \geq 1} \in \Phi_n^{z,m}.$$

Step 2: We now prove that $\bar{\mathrm{v}}_n(z,m) \geq \mathrm{v}_n(z,m)$. By the above construction and Lemma 5.1,

$$\bar{\mathrm{v}}_n(z,m) \geq \bar{J}(z,m; \alpha_1^{\phi_1^{\varepsilon(1)}}) \geq \bar{\mathrm{v}}_n(z,m) - \varepsilon_1.$$

Since $\mathrm{v}_n(t_k, \cdot) = \bar{\mathrm{v}}_n(t_k, \cdot)$ for $k > j$ by our induction hypothesis, we obtain

$$\begin{aligned}
\bar{\mathrm{v}}_n(z,m) &\geq \sup_{a \in \mathbf{A} \cup \{\partial\}} \mathbb{E}_m[\mathrm{v}_n(Z^{z,a}_{r_2^{\varepsilon(1)}}, M^{z,m,a}_{r_2^{\varepsilon(1)}})] - \varepsilon_1 \\
&\geq \mathrm{v}_n(z,m) - \varepsilon_1,
\end{aligned}$$

in which the last inequality follows from a simple conditioning argument as in the proof of Proposition 5.2. By arbitrariness of $\varepsilon_1 > 0$, this implies that $\bar{\mathrm{v}}_n(z,m) \geq \mathrm{v}_n(z,m)$.
Step 3: It remains to prove that $\bar{\mathrm{v}}_n(z,m) \leq \mathrm{v}_n(z,m)$. Define

$$Y_i^{\varepsilon(i-1)} := (Z^{z,\phi^\varepsilon}_{r_i^{\varepsilon(i-1)}}, M^{z,m,\phi^\varepsilon}_{r_i^{\varepsilon(i-1)}}),\ i \geq 1,$$

with $Y_0^{\varepsilon(-1)} := (z,m)$, and observe that $Y_i^{\varepsilon(i-1)}$ and $\mathcal{F}^{z,m,\phi^\varepsilon}_{r_i^{\varepsilon(i-1)}}$ only depend on $\varepsilon(i-1)$. Then, for each $i \geq 0$,

$$\begin{aligned}
\bar{\mathrm{v}}_n(Y_i^{\varepsilon(i-1)}) &= \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\bar{\mathrm{v}}_n(Z^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}_{r_{i+1}^{\varepsilon(i)}}, M^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}_{r_{i+1}^{\varepsilon(i)}}) | \mathcal{F}^{z,m,\phi^\varepsilon}_{r_i^{\varepsilon(i-1)}}]] \\
&= \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} \leq T\}} \bar{\mathrm{v}}_n(Z^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}_{r_{i+1}^{\varepsilon(i)}}, M^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}_{r_{i+1}^{\varepsilon(i)}}) | \mathcal{F}^{z,m,\phi^\varepsilon}_{r_i^{\varepsilon(i-1)}}] \\
&+ \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} > T\}} g(Z^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}_{r_{i+1}^{\varepsilon(i)}}, M^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}_{r_{i+1}^{\varepsilon(i)}}, \upsilon, \epsilon_0) | \mathcal{F}^{z,m,\phi^\varepsilon}_{r_i^{\varepsilon(i-1)}}] \quad \mathbb{P}_m - \text{a.s.}
\end{aligned}$$

on $\{r_i^{\varepsilon(i-1)} \le T\}$. Since g is bounded, so is $\bar{\mathrm{v}}_n$. The above combined with the dominated convergence theorem then implies

$$\begin{aligned}\bar{\mathrm{v}}_n(z,m) &= \lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_n\downarrow 0}\mathbb{E}_m[\sum_{i=0}^{n}\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)}>T\ge r_i^{\varepsilon(i-1)}\}}g(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)},\phi_i^{\varepsilon(i)}},M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)},\phi_i^{\varepsilon(i)}},\upsilon,\epsilon_0)]\\ &= \lim_{\varepsilon_1\downarrow 0}\cdots\lim_{\varepsilon_n\downarrow 0}J(z,m;\phi^{\varepsilon})\\ &\le \mathrm{v}_n(z,m),\end{aligned}$$

which concludes the proof that $\bar{\mathrm{v}}_n = \mathrm{v}_n$.
Step 4. The second assertion of the proposition is obtained by observing that, given a random variable (ζ,μ) with values in $\mathbf{Z}\times\mathbf{M}$, one can choose $\tilde{a}^{n,\varepsilon_1}$ Borel measurable such that

$$\tilde{a}^{n,\varepsilon_1}[\zeta,\mu] = \hat{a}^{n,\varepsilon_1}[\zeta,\mu]\ \mathbb{P}_m - \text{a.s.}$$

□

We are now in position to conclude that v_n satisfies a dynamic programming principle.

Corollary 5.1. *Fix $z=(t,x)\in\mathbf{Z}$ and $m\in\mathbf{M}$. Let $(\theta^{\phi},\phi\in\Phi_n^{z,m})$ be such that each θ^{ϕ} is a $\mathbb{F}^{z,m,\phi}$-stopping time with values in $[t,2T]\cap(\pi_n\cup[T,\infty))$ such that*

$$\theta^{\phi}\in\mathcal{N}^{\phi}\cap[t,\mathrm{T}[\phi]]\ \mathbb{P}_m - \text{a.s.}$$

for $\phi\in\Phi_n^{z,m}$. Then,

$$\mathrm{v}_n(z,m) = \sup_{\phi\in\Phi_n^{z,m}}\mathbb{E}_m[\mathrm{v}_n(Z_{\theta^{\phi}}^{z,\phi},M_{\theta^{\phi}}^{z,m,\phi})].$$

Proof. The inequality $\le$ can be obtained trivially by a conditioning argument. Fix $\phi\in\Phi_n^{z,m}$. By Proposition 5.3, we can find a Borel measurable map $(z',m')\mapsto\phi^{\iota}[z',m']$ such that

$$J(Z_{\theta^{\phi}}^{z,\phi},M_{\theta^{\phi}}^{z,m,\phi};\phi^{\iota}[Z_{\theta^{\phi}}^{z,\phi},M_{\theta^{\phi}}^{z,m,\phi}])\ge\mathrm{v}_n(Z_{\theta^{\phi}}^{z,\phi},M_{\theta^{\phi}}^{z,m,\phi})-\iota.$$

Let us now simply write ϕ^{ι} for $\phi^{\iota}[Z_{\theta^{\phi}}^{z,\phi},M_{\theta^{\phi}}^{z,m,\phi}]$. Without loss of generality, one can assume that $\tau_1^{\phi}\ge t$ and that $\tau_1^{\phi^{\iota}}\ge\theta^{\phi}$. Let $I:=\mathrm{card}\{i\ge 1:\tau_i^{\phi}<\theta^{\phi}\}$. Then,

$$J(z,m;\tilde{\phi}^{\iota})\ge\mathbb{E}_m[\mathrm{v}_n(Z_{\theta^{\phi}}^{z,\phi},M_{\theta^{\phi}}^{z,m,\phi})]-\iota$$

in which

$$(\tau_i^{\tilde{\phi}^{\iota}},\alpha_i^{\tilde{\phi}^{\iota}})=\mathbf{1}_{i\le I}(\tau_i^{\phi},\alpha_i^{\phi})+\mathbf{1}_{i>I}(\tau_{i-I}^{\phi^{\iota}},\alpha_{i-I}^{\phi^{\iota}}),\ i\ge 1.$$

Send $\iota\to 0$ leads to the required result. □

5.3 Super-solution property as the time step vanishes

We now consider the limit $n\to\infty$. Let us set

$$\mathrm{v}_{\circ}(z,m):=\liminf_{(t',x',m',n)\to(z,m,\infty)}\mathrm{v}_n(t',x',m'),$$

for $(z,m)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbf{M}$.

Remark 5.3. *Note that* (5.7) *and* (3.6) *implies that* $\mathrm{v}_\circ = \mathcal{K}_T g$ *on* $(T,\infty)\times\mathbb{R}^d\times\mathbf{M}$.

Proposition 5.4. *The function* $\mathrm{v}_\circ$ *is a viscosity super-solution of* (3.4)-(3.5).

Proof. Let $n_k\to\infty$ and $(z_k,m_k)\to(z_o,m_\circ)$ be such that $\mathrm{v}_{n_k}(z_k,m_k)\to\mathrm{v}_\circ(z_o,m_o)$.
Step 1. We first show that $\mathrm{v}_\circ(z_\circ,m_\circ)\ge\mathcal{K}\mathrm{v}_\circ(z_\circ,m_\circ)$. By Corollary 5.1 applied to v_{n_k} with a control ϕ^k defined by

$$(\tau_i^k,\alpha_i^k)=(t_k,a_k)\mathbf{1}_{\{i=1\}}+\sum_{j>1}(T+j,a_\star)\mathbf{1}_{\{i=j\}},\ i\ge 1,$$

with $a_k\in\mathbf{A}$, we obtain

$$\mathrm{v}_{n_k}(z_k,m_k)\ge\sup_{a_k\in\mathbf{A}}\int\mathbb{E}[\mathrm{v}_{n_k}(Z^{z',\circ}_{s_+^{n_k}[z']},m')]\mathrm{dk}(z',m'|z_k,m_k,a_k)]=\mathcal{K}\mathbb{E}[\mathrm{v}_{n_k}(Z^{\cdot,\circ}_{s_+^{n_k}[\cdot]},\cdot)](z_k,m_k),$$

in which $s_+^{n_k}[t,x]:=\min\pi_{n_k}\cap[t,\infty)$. Let $\varphi_{k_\circ}$ be the lower-semicontinuous enveloppe of $\inf\{\mathbb{E}[\mathrm{v}_{n_k}(Z^{\cdot,\circ}_{s_+^{n_k}[\cdot]},\cdot)],k\ge k_\circ\}$. Then, for $k\ge k_\circ$,

$$\mathrm{v}_{n_k}(z_k,m_k)\ge\int\varphi_{k_\circ}(z',m')\mathrm{dk}(z',m'|z_k,m_k,a_k),$$

and, by (3.6), passing to the limit $k\to\infty$ leads to

$$\mathrm{v}_\circ(z_\circ,m_\circ)\ge\int\varphi_{k_\circ}(z',m')\mathrm{dk}(z',m'|z_\circ,m_\circ,a_\circ).$$

We shall prove in step 3 that $\lim_{k_\circ\to\infty}\varphi_{k_\circ}\ge\mathrm{v}_\circ$. These maps are bounded, since g is. Dominated convergence then implies that

$$\mathrm{v}_\circ(z_\circ,m_\circ)\ge\int\mathrm{v}_\circ(z',m')\mathrm{dk}(z',m'|z_\circ,m_\circ,a_\circ).$$

Step 2. Let φ be a (bounded) $C^{1,2,0}([0,T]\times\mathbb{R}^d\times\mathbf{M})$ function and $(z_\circ,m_\circ)\in[0,T)\times\mathbb{R}^d\times\mathbf{M}$ be a minimal point of $\mathrm{v}_\circ-\varphi$ on $\mathbf{Z}\times\mathbf{M}$. Without loss of generality, one can assume that $(\mathrm{v}_\circ-\varphi)(z_\circ,m_\circ)=0$. Let B and $(z_k,m_k,n_k)_{n\ge1}$ be as in Lemma 3.2. We write $z_k=(t_k,x_k)$, $z_\circ=(t_\circ,x_\circ)\in[0,T]\times\mathbb{R}^d$. On the other hand, by considering the control ϕ^k defined by

$$(\tau_i^k,\alpha_i^k)=(T+i,a_\star),\ i\ge1,$$

we obtain from Corollary 5.1 that

$$\mathrm{v}_{n_k}(z_k,m_k)\ge\mathbb{E}_m[\mathrm{v}_{n_k}(Z^{z_k,\circ}_{t_k+h_k},m)]$$

with $h_k\in T2^{-n_k}(\mathbb{N}\cup\{0\})$ such that $t_k+h_k<T$ if $t_\circ\ne T$ and $t_k+h_k=T$ otherwise.
Let $C>0$ be a common bound for $(\mathrm{v}_n)_{n\ge1}$ and φ. Then we can choose $(h_k)_{k\ge1}$ such that

$$\delta_k:=(\varphi(z_k,m_k)-\mathrm{v}_{n_k}(z_k,m_k)-2C\,\mathbb{P}[Z^{z_k,\circ}_{t_k+h_k}\notin B])/h_k\to 0.$$

This follows from standard estimates on the solution of sde's with Lipschitz coefficients. Then, if $t_\circ < T$,

$$0 \ge h_k^{-1}\mathbb{E}_m[\varphi(Z^{z_k,\circ}_{t_k+h_k}, m_k) - \varphi_{n_k}(z_k, m_k)] + \delta_k = \mathbb{E}_m[h_k^{-1}\int_{t_k}^{t_k+h_k} \mathcal{L}\varphi(Z^{z_k,\circ}_s, m_k)ds] + \delta_k,$$

sending $k \to \infty$ leads to $\mathcal{L}\varphi(z_\circ, m_\circ) \le 0$.
If $t_\circ = T$,

$$\mathrm{v}_{n_k}(z_k, m_k) \ge \mathbb{E}_m[g(Z^{z_k,\circ}_T, m_k, \upsilon, \epsilon_0)] = \mathbb{E}_m[\mathcal{K}_T g(Z^{z_k,\circ}_T, m_k)]$$

and passing to the limit leads to

$$\varphi(z_\circ, m_\circ) \ge \mathcal{K}_T g(z_\circ, m_\circ),$$

recall (3.6). Finally, $\varphi(z_\circ, m_\circ) \ge \mathcal{K}\varphi(z_\circ, m_\circ)$ by Step 1. □
Step 3: It remains to prove the claim used in Step 1. Let us set

$$\bar\varphi_{k_\circ}(z', m') := \inf_{k \ge k_\circ} \left\{ \mathbb{E}\left[\mathrm{v}_{n_k}\left(Z^{z',\circ}_{s_+^{n_k}}[z'], m' \right) \right] \right\},$$

so that $\varphi_{k_\circ}$ is the lower-semicontinuous envelope of $\bar\varphi_{k_\circ}$. Note that $Z^{z',\circ}_{s_+^{n_k}[z']}$ converges a.s. to z as $(z', k) \to (z, \infty)$. Hence, for all $\varepsilon > 0$, there exist open neighborhoods $B_\varepsilon(z, m)$ and $B_{\frac{\varepsilon}{2}}(z, m)$ of (z, m), as well as $k_\varepsilon \in \mathbb{N}$ such that $\mathbb{P}[(Z^{z',\circ}_{s_+^{n_k}[z']}, m') \notin B_\varepsilon(z, m)] \le \varepsilon$ for $k \ge k_\varepsilon$ and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$. One can also choose k_ε and $B_{\frac{\varepsilon}{2}}(z, m)$ such that

$$\inf_{k \ge k_\varepsilon} \mathrm{v}_{n_k}(z', m') \ge \mathrm{v}_\circ(z, m') - \varepsilon$$

for all $k \ge k_\varepsilon$ and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$. Let $C > 0$ be a bound for $(|\mathrm{v}_n|)_{n \ge 1}$ and $|\mathrm{v}_\circ|$, recall that g is bounded. Then, for $k_\circ$ large enough and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$,

$$\bar\varphi_{k_\circ}(z', m') \ge \mathrm{v}_\circ(z, m) - \varepsilon - 2C \sup_{k \ge k_\circ} \mathbb{P}[(Z^{z',\circ}_{s_+^{n_k}[z']}, m') \notin B_\varepsilon(z, m)] \ge \mathrm{v}_\circ(z, m) - \varepsilon(1 + 2C).$$

Hence,

$$\lim_{k_\circ \to \infty} \varphi_{k_\circ}(z, m) = \lim_{k_\circ \to \infty} \liminf_{(z', m') \to (z, m)} \bar\varphi_{k_\circ}(z', m') \ge \mathrm{v}_\circ(z, m),$$

since $\mathrm{v}_\circ$ is lower-semicontinuous. □

5.4 Conclusion of the proof of Theorem 3.1

We can now conclude the proof of Theorem 3.1. We already know from Proposition 5.1 and Proposition 5.4 that v^* and $\mathrm{v}_\circ$ are respectively a bounded viscosity sub- and super-solution of (3.4)-(3.5). By (2.14), Remark 5.3 and (3.6), we also have $\mathrm{v}_\circ \ge \mathrm{v}^*$ on $(0, T) \times \mathbb{R}^d \times \mathbf{M}$. In view of Assumption 3.1 and Remark 5.2, we obtain $\mathrm{v}_\circ \ge \mathrm{v}^* \ge \mathrm{v}_\circ$. Hence, v is continuous on $\mathbf{Z} \times \mathbf{M}$ and is a viscosity solution of (3.4)-(3.5). Uniqueness follows from Assumption 3.1.
□

6 Proof of the sufficient condition for the comparison

We provide here the proof of Proposition 3.1.

Proof of Proposition 3.1. Step 1. As usual, we shall argue by contradiction. We assume that there exists $(z_0, m_0) \in \mathbf{Z} \times \mathbf{M}$ such that $(U - V)(z_0, m_0) > 0$, in which U and V are as in Assumption 3.1. Recall the definition of Ψ, ϱ and $\tilde{g}$ in Proposition 3.1. We set $\tilde{u}(t, x, m) := e^{\varrho t} U(t, x, m)$ and $\tilde{v}(t, x, m) := e^{\varrho t} V(t, x, m)$ for all $(t, x, m) \in \mathbf{Z} \times \mathbf{M}$. Then, there exists $\lambda \in (0, 1)$ such that

$$(\tilde{u} - \tilde{v}^{\lambda})(z_0, m_0) > 0, \tag{6.1}$$

in which $\tilde{v}^{\lambda} := (1 - \lambda)\tilde{v} + \lambda\Psi$. Note that $\tilde{u}$ and $\tilde{v}$ are sub and supersolution on $\mathbf{Z} \times \mathbf{M}$ of

$$\min\{\varrho\varphi - \mathcal{L}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \tag{6.2}$$

associated to the boundary condition

$$\min\{\varphi - \mathcal{K}_T \tilde{g}, \varphi - \mathcal{K}\varphi\} = 0. \tag{6.3}$$

Step 2. Let $\mathrm{d}_{\mathbf{M}}$ be a metric on $\mathbf{M}$ compatible with the topology of weak convergence. For $(t, x, y, m) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$, we set

$$\Gamma_{\varepsilon}(t, x, y, m) := \tilde{u}(t, x, m) - \tilde{v}^{\lambda}(t, y, m) - \varepsilon\left(\|x\|^2 + \|y\|^2 + \mathrm{d}_{\mathbf{M}}(m)\right) \tag{6.4}$$

with $\varepsilon > 0$ small enough such that $\Gamma_{\varepsilon}(t_0, x_0, x_0, m_0) > 0$. Note that the supremum of $(t, x, m) \mapsto \Gamma_{\varepsilon}(t, x, x, m)$ over $\mathbf{Z} \times \mathbf{X} \times \mathbf{M}$ is achieved by some $(t_{\varepsilon}, x_{\varepsilon}, x_{\varepsilon}, m_{\varepsilon})$. This follows from the the upper semi-continuity of Γ_{ε} and the fact that $\tilde{u}, -\tilde{v}, -\Psi$ are bounded from above. Recall that $\mathbf{M}$ is locally compact. For $(t, x, y, m) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$, we set

$$\Theta_{\varepsilon}^{n}(t, x, y, m) := \Gamma_{\varepsilon}(t, x, y, m) - n\|x - y\|^2.$$

Again, there is $(t_n^{\varepsilon}, x_n^{\varepsilon}, y_n^{\varepsilon}, m_n^{\varepsilon}) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$ such that

$$\sup_{\mathbf{Z} \times \mathbf{X} \times \mathbf{M}} \Theta_{\varepsilon}^{n} = \Theta_{\varepsilon}^{n}(t_n^{\varepsilon}, x_n^{\varepsilon}, y_n^{\varepsilon}, m_n^{\varepsilon}).$$

It is standard to show that, after possibly considering a subsequence,

$$\begin{aligned} &(t_n^{\varepsilon}, x_n^{\varepsilon}, y_n^{\varepsilon}, m_n^{\varepsilon}) \to (\hat{t}_{\varepsilon}, \hat{x}_{\varepsilon}, \hat{x}_{\varepsilon}, \hat{m}_{\varepsilon}) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}, \quad n\|x_n^{\varepsilon} - y_n^{\varepsilon}\|^2 \to 0, \\ &\text{and } \Theta_{\varepsilon}^{n}(t_n^{\varepsilon}, x_n^{\varepsilon}, y_n^{\varepsilon}, m_n^{\varepsilon}) \to \Gamma_{\varepsilon}(\hat{t}_{\varepsilon}, \hat{x}_{\varepsilon}, \hat{x}_{\varepsilon}, \hat{m}_{\varepsilon}) = \Gamma_{\varepsilon}(t_{\varepsilon}, x_{\varepsilon}, x_{\varepsilon}, m_{\varepsilon}), \end{aligned} \tag{6.5}$$

see e.g. [8, Lemma 3.1].

Step 3. We first assume that, up to a subsequence,

$$(\tilde{u} - \mathcal{K}\tilde{u})(t_n^{\varepsilon}, x_n^{\varepsilon}, m_n^{\varepsilon}) \le 0, \quad \text{for } n \ge 1.$$

It follows from the supersolution property of $\tilde{v}$ and Condition (iii) of Proposition 3.1 that

$$\tilde{u}(t_n^{\varepsilon}, x_n^{\varepsilon}, m_n^{\varepsilon}) - \tilde{v}^{\lambda}(t_n^{\varepsilon}, y_n^{\varepsilon}, m_n^{\varepsilon}) \le \mathcal{K}\tilde{u}(t_n^{\varepsilon}, x_n^{\varepsilon}, m_n^{\varepsilon}) - \mathcal{K}\tilde{v}^{\lambda}(t_n^{\varepsilon}, y_n^{\varepsilon}, m_n^{\varepsilon}) - \lambda\delta.$$

Passing to the lim sup and using (6.5) and (3.6), we obtain

$$(\tilde{u}-\tilde{v}^{\lambda})(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon})+\lambda\delta \le \mathcal{K}(\tilde{u}-\tilde{v}^{\lambda})(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon}).$$

In particular, by (6.4),

$$\Gamma_{\varepsilon}(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon})+\lambda\delta \le \mathcal{K}(\tilde{u}-\tilde{v}^{\lambda})(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon}).$$

Now let us observe that

$$\begin{aligned}\sup_{\mathbf{Z}\times\mathbf{M}}(\tilde{u}-\tilde{v}^{\lambda}) &= \lim_{\varepsilon\to 0}\sup_{(t,x,m)\in\mathbf{Z}\times\mathbf{M}}\Gamma_{\varepsilon}(t,x,x,m)\\ &= \lim_{\varepsilon\to 0}\Gamma_{\varepsilon}(t_{\varepsilon},x_{\varepsilon},x_{\varepsilon},m_{\varepsilon})\\ &= \lim_{\varepsilon\to 0}\Gamma_{\varepsilon}(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon}),\end{aligned} \tag{6.6}$$

in which the last identity follows from (6.5). Combined with the above inequality, this shows that

$$\sup_{\mathbf{Z}\times\mathbf{M}}(\tilde{u}-\tilde{v}^{\lambda})+\lambda\delta \le \lim_{\varepsilon\to 0}\mathcal{K}(\tilde{u}-\tilde{v}^{\lambda})(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon}),$$

which leads to a contradiction for ε small enough.

Step 4. We now show that there is a subsequence such that $t_n^{\varepsilon} < T$ for all $n \ge 1$. If not, one can assume that $t_n^{\varepsilon} = T$ and it follows from the boundary condition (6.3) and step 3 that $\tilde{u}(T,x_n^{\varepsilon},m_n^{\varepsilon}) \le \mathcal{K}_T\tilde{g}(T,x_n^{\varepsilon},m_n^{\varepsilon})$ for all $n \ge 1$. Since, by (6.3) and Condition (iv) of Proposition 3.1, $\tilde{v}^{\lambda}(T,y_n^{\varepsilon},m_n^{\varepsilon}) \ge \mathcal{K}_T\tilde{g}(T,y_n^{\varepsilon},m_n^{\varepsilon})$, it follows that $\tilde{u}(T,x_n^{\varepsilon},m_n^{\varepsilon})-\tilde{v}^{\lambda}(T,y_n^{\varepsilon},m_n^{\varepsilon}) \le \mathcal{K}_T\tilde{g}(T,x_n^{\varepsilon},m_n^{\varepsilon})-\mathcal{K}_T\tilde{g}(T,y_n^{\varepsilon},m_n^{\varepsilon})$. Hence,

$$\Gamma_{\varepsilon}(T,x_n^{\varepsilon},y_n^{\varepsilon},m_n^{\varepsilon}) \le \mathcal{K}_T\tilde{g}(T,x_n^{\varepsilon},m_n^{\varepsilon})-\mathcal{K}_T\tilde{g}(T,y_n^{\varepsilon},m_n^{\varepsilon}).$$

Combining (3.6), (6.5) and (6.6) as above, we obtain $\sup(\tilde{u}-\tilde{v}^{\lambda}) \le 0$, a contradiction.

Step 5. In view of step 3 and 4, we may assume that

$$t_n^{\varepsilon} < T \quad \text{and} \quad (\tilde{u}-\mathcal{K}\tilde{u})(t_n^{\varepsilon},x_n^{\varepsilon},m_n^{\varepsilon}) > 0 \quad \text{for all } n \ge 1.$$

Using Ishii's Lemma and following standard arguments, see Theorem 8.3 and the discussion after Theorem 3.2 in [8], we deduce from the sub- and supersolution viscosity property of $\tilde{u}$ and $\tilde{v}^{\lambda}$, and the Lipschitz continuity assumptions on μ and σ, that

$$\varrho\left(\tilde{u}(t_n^{\varepsilon},x_n^{\varepsilon},m_n^{\varepsilon})-\tilde{v}^{\lambda}(t_n^{\varepsilon},y_n^{\varepsilon},m_n^{\varepsilon})\right) \le C\left(n\|x_n^{\varepsilon}-y_n^{\varepsilon}\|^2+\varepsilon\left(1+\|x_n^{\varepsilon}\|^2+\|y_n^{\varepsilon}\|^2\right)\right),$$

for some $C > 0$, independent on n and ε. In view of (6.4) and (6.5), we get

$$\varrho\Gamma_{\varepsilon}(\hat{t}_{\varepsilon},\hat{x}_{\varepsilon},\hat{x}_{\varepsilon},\hat{m}_{\varepsilon}) \le 2C\varepsilon\left(1+\|\hat{x}_{\varepsilon}\|^2\right). \tag{6.7}$$

We shall prove in next step that the right-hand side of (6.7) goes to 0 as $\varepsilon \to 0$, up to a subsequence. Combined with (6.6), this leads to a contradiction to (6.1).

Step 6. We conclude the proof by proving that $\varepsilon\|\hat{x}_\varepsilon\|^2 \to 0$ as $\varepsilon \to 0$, up to a subsequence. This is standard. First note that we can always construct a sequence $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon)_{\varepsilon>0}$ such that

$$\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon) \to \sup_{\mathbf{Z}\times\mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) \quad \text{and} \quad \varepsilon(\|\tilde{x}_\varepsilon\|^2 + \mathrm{d}_{\mathbf{M}}(\tilde{m}_\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0.$$

By (6.5), $\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon) \leq \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$. Hence,

$$\sup_{\mathbf{Z}\times\mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) \leq \sup_{\mathbf{Z}\times\mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) - 2\liminf_{\varepsilon\to 0} \varepsilon\|\hat{x}_\varepsilon\|^2.$$

□

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