Optimal control under uncertainty and Bayesian parameters adjustments

N. Baradel[*], B. Bouchard[†], N. M. Dang[‡]

First version: January 2017, Revised version: September 2017

Abstract

We propose a general framework for studying optimal impulse control problem in the presence of uncertainty on the parameters. Given a prior on the distribution of the unknown parameters, we explain how it should evolve according to the classical Bayesian rule after each impulse. Taking these progressive prior-adjustments into account, we characterize the optimal policy through a quasi-variational parabolic equation, which can be solved numerically. The derivation of the dynamic programming equation seems to be new in this context. The main difficulty lies in the nature of the set of controls which depends in a non trivial way on the initial data through the filtration itself.

Key words: Optimal control, uncertainty, Bayesian filtering.

MSC 2010: 49L20, 49L25

1 Introduction

We consider a general optimal impulse control problem under parameter uncertainty. This work is motivated by optimal trading problems. In this domain, several market parameters are of major importance. It can be the nature of the market impact of aggressive orders, or the time to be executed when entering a book order queue, see e.g. [13] and the references therein. However, the knowledge of these execution conditions is in general not perfect. One can try to estimate them but they remain random and can change from one market/platform to another one, or depending on the current market conditions. Most importantly, they can

only be estimated by actually acting on the market. We therefore face the typical problem of estimating a reaction parameter (impact/execution time) while actually controlling a system (trading) that depends on these parameters.

Such problems have been widely studied in the discrete time stochastic optimal control literature, see e.g. [10, 12] for references. One fixes a certain prior distribution on the unknown parameter, and re-evaluate it each time an action is taken, by applying the standard Bayesian rule to the observed reactions. The optimal strategy generically results from a compromise between acting on the system, to get more information, and being not too aggressive, because of the uncertainty on the real value of the parameters. If the support of the initial prior contains the true value of the parameters, one can expect (under natural identification conditions) that the sequence of updated priors actually converges to it in the long range.

It is a-priori much more difficult to handle in a continuous time framework with continuous time monitoring, as it leads to a filtering problem, leaving on an infinite dimensional space. However, optimal trading can very naturally be considered in the impulse form, as orders are sent in a discrete time manner. In a sense, we are back to a discrete time problem whose dimension can be finite (depending on the nature of the uncertainty), although interventions on the system may occur at any time.

In this paper, we thus consider a general impulse control problem with an unknown parameter, under which an initial prior law is set. Given this prior, we aim at maximizing a certain gain functional. We show that the corresponding value function can be characterized as the unique viscosity solution (in a suitable class) of a quasi-variational parabolic equation. We also allow for (possibly) not observing immediately the effect of an impulse. This applies to any situations in which the effect of an impulse is observed only with delay, e.g. nothing is observed but the execution time when an order is sent to a dark pool.

The study of such non-classical impulse control problems seems to be new in the literature. From the mathematical point of view, the main difficulty consists in establishing a dynamic programming principle. The principal reason lies in the choice of the filtration. Because of the uncertainty on the parameter driving the dynamics, the only natural filtration to which the control policy should be adapted is the one generated by the controlled process himself. This implies in particular that the set of admissible controls depends heavily (and in a very non trivial way) on the initial state of the system at the starting time of the strategy. Hence, no a priori regularity nor good measurability properties can be expected to construct explicitly measurable almost optimal controls, see e.g. [6], or to apply a measurable selection theorem, see e.g. [4]. We therefore proceed differently. The (usually considered as) easy part of the dynamic programming can actually be proved, as it only requires a conditioning argument. It leads as usual to a sub-solution characterization. We surround the difficulty in proving the second (difficult) part by considering a discrete time version of our initial continuous time control problem. When the time step goes to 0, it provides a super-solution of the targeted dynamic programming equation. Using comparison and the natural ordering on the value functions associated to the continuous and the discrete time model, we show that the two coincide at the limit.

Applications to optimal trading and an example of numerical scheme are provided in the application paper [1].

The rest of the paper is organized as follows. The model is described in Section 2. In Section 3, we provide the PDE characterization of the value function. Proofs are collected in Section 4. A sufficient condition for comparison to hold is provided in Section 5.

2 The impulse problem with parameters adjustment

All over this paper, $C([0,T], \mathbb{R}^d)$ is the space of continuous functions from $[0,T]$ into $\mathbb{R}^d$ which start at 0 at the origin. Recall that it is a Polish space for the sup-norm topology. We denote by $W(\omega) = \omega$ the canonical process and let $\mathbb{P}$ be the Wiener measure. We also consider a Polish space $(\mathrm{U}, \mathcal{B}(\mathrm{U}))$ that will support an unknown parameter υ. We denote by $\mathbf{M}$ a locally compact subset[1] of the set of Borel probability measures on U endowed with the topology of weak convergence. In particular, it is Polish. A prior on the unknown parameter υ will be an element $m \in \mathbf{M}$. To allow for additional randomness in the measurement of the effects of actions on the system, we consider another Polish space E on which is defined a family $(\epsilon_i)_{i \geq 0}$ of i.i.d. random variables with common measure $\mathbb{P}_\epsilon$ on E. On the product space $\Omega := C([0,T], \mathbb{R}^d) \times \mathrm{U} \times \mathrm{E}^{\mathbb{N}}$, we consider the family of measures $\{\mathbb{P} \times m \times \mathbb{P}_\epsilon^{\otimes \mathbb{N}} : m \in \mathbf{M}\}$ and denote by $\mathbb{P}_m$ an element of this family whenever $m \in \mathbf{M}$ is fixed. The operator $\mathbb{E}_m$ is the expectation associated to $\mathbb{P}_m$. Note that W, υ and $(\epsilon_i)_{i \geq 0}$ are independent under each $\mathbb{P}_m$. For $m \in \mathbf{M}$ given, we let $\mathbb{F}^m = (\mathcal{F}_t^m)_{t \geq 0}$ denote the $\mathbb{P}_m$-augmentation of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ defined by $\mathcal{F}_t = \sigma((W_s)_{s \leq t}, \upsilon, (\epsilon_i)_{i \geq 0})$ for $t \geq 0$. Hereafter, all the random variables are considered with respect to the probability space $(\Omega, \mathcal{F}_T^m)$ with $m \in \mathbf{M}$ given by the context, and where T is a fixed time horizon.

2.1 The controlled system

Let $\mathbf{A} \subset [0,T] \times \mathbb{R}^d$ be a (non-empty) compact set. Given $N \in \mathbb{N}$ and $m \in \mathbf{M}$, we denote by $\Phi_N^{\circ, m}$ the collection of sequences of random variables $\phi = (\tau_i, \alpha_i)_{i \geq 1}$ on $(\Omega, \mathcal{F}_T^m)$ with values in $\mathbb{R}_+ \times \mathbf{A}$ such that $(\tau_i)_{i \geq 1}$ is a non-decreasing sequence of $\mathbb{F}^m$-stopping times satisfying $\tau_j > T$ $\mathbb{P}_m - $ a.s. for $j > N$. We set

$$\Phi^{\circ, m} := \bigcup_{N \geq 1} \Phi_N^{\circ, m}.$$

An element $\phi = (\tau_i, \alpha_i)_{1 \leq i \leq N} \in \Phi^{\circ, m}$ will be our impulse control and we write α_i in the form

$$\alpha_i = (\ell_i, \beta_i) \text{ with } \ell_i \in [0,T] \text{ and } \beta_i \in \mathbb{R}^d \ \mathbb{P}_m - \text{a.s.}$$

More precisely, the τ_i's will be the times at which an impulse is made on the system (e.g. a trading robot is launched), β_i will model the nature of the order send at time τ_i (e.g. the parameters used for the trading robot), and ℓ_i will stand for the maximal time length during which no new intervention on the system can be made (e.g. the time prescribed to the robot to send orders on the market). Later on we shall impose more precise non-anticipativity conditions.

---

[1] In many situations, the family of probability measures of interest will in fact be parameterized or be the set of measures on a compact metrizable space, see Remark 2.1 below.

From now on, we shall always use the notation $(\tau_i^\phi, \alpha_i^\phi)_{i \geq 1}$ with $\alpha_i^\phi = (\ell_i^\phi, \beta_i^\phi)$ to refer to a control $\phi \in \Phi^{\circ,m}$.

We allow for not observing nor being able to act on the system before a random time ϑ_i^ϕ defined by

$$\vartheta_i^\phi := \varpi(\tau_i^\phi, X_{\tau_i^\phi-}^\phi, \alpha_i^\phi, \upsilon, \epsilon_i),$$

where X^ϕ is the controlled state process that will be described below, and

$$\varpi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{A} \times \mathrm{U} \times \mathrm{E} \to [0, T] \quad \text{is measurable, such that } \varpi(t, \cdot) \geq t \text{ for all } t \geq 0. \quad (2.1)$$

In the case where the actions consist in launching a trading robot at τ_i^ϕ during a certain time ℓ_i^ϕ, we can naturally take $\vartheta_i^\phi = \tau_i^\phi + \ell_i^\phi$. If the action consists in placing a limit order during a maximal duration ℓ_i^ϕ, ϑ_i^ϕ is the time at which the limit order is executed if it is less than $\tau_i^\phi + \ell_i^\phi$, and $\tau_i^\phi + \ell_i^\phi$ otherwise.

We say that $\phi \in \Phi^{\circ,m}$ belongs to Φ^m if $\vartheta_i^\phi \leq \tau_{i+1}^\phi$ and $\tau_i^\phi < \tau_{i+1}^\phi$ $\mathbb{P}_m$-a.s. for all $i \geq 1$, and define

$$\mathcal{N}^\phi := \left[\cup_{i \geq 1} [\tau_i^\phi, \vartheta_i^\phi) \right]^c. \quad (2.2)$$

We are now in a position to describe our controlled state process. Given some initial data $z := (t, x) \in \mathbf{Z} := [0, T] \times \mathbb{R}^d$, and $\phi \in \Phi^m$, we let $X^{z,\phi}$ be the unique strong solution on $[t, 2T]$ of

$$X = x + \left(\int_t^\cdot \mathbf{1}_{\mathcal{N}^\phi}(s) \mu(s, X_s)\, ds + \int_t^\cdot \mathbf{1}_{\mathcal{N}^\phi}(s) \sigma(s, X_s)\, dW_s \right)$$
$$+ \sum_{i \geq 1} \mathbf{1}_{\{t \leq \vartheta_i^\phi \leq \cdot\}} [F(\tau_i^\phi, X_{\tau_i^\phi-}, \alpha_i^\phi, \upsilon, \epsilon_i) - X_{\tau_i^\phi-}]. \quad (2.3)$$

In the above, the function

$$(\mu, \sigma, F) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{A} \times \mathrm{U} \times \mathrm{E} \mapsto \mathbb{R}^d \times \mathbb{M}^d \times \mathbb{R}^d \quad \text{is measurable.}$$
$$\text{The map } (\mu, \sigma) \text{ is continuous, and Lipschitz with linear growth} \quad (2.4)$$
$$\text{in its second argument, uniformly in the first one,}$$

with $\mathbb{M}^d$ defined as the set of $d \times d$ matrices. This dynamics means the following. When no action is currently made on the system, i.e. on the intervals in $\mathcal{N}^\phi$, the system evolves according to a stochastic differential equation driven by the Brownian motion W:

$$dX_s = \mu(s, X_s)\, ds + \sigma(s, X_s)\, dW_s \quad \text{on } \mathcal{N}^\phi.$$

When an impulse is made at τ_i^ϕ, we freeze the dynamics up to the end of the action at time ϑ_i^ϕ. This amounts to saying that we do not observe the current evolution up to ϑ_i^ϕ. At the end of the action, the state process takes a new value $X_{\vartheta_i^\phi} = F(\tau_i^\phi, X_{\tau_i^\phi-}, \alpha_i^\phi, \upsilon, \epsilon_i)$. The fact that F depends on the unknown parameter υ and the additional noise ϵ_i models the fact the correct

4

model is not known with certainty, and that the exact value of the unknown parameter υ can (possibly) not be measured precisely just by observing $(\vartheta_i^\phi - \tau_i^\phi, X_{\vartheta_i^\phi} - X_{\tau_i^\phi -})$.

In order to simplify the notations, we shall now write:

$$Z^{z,\phi} := (\cdot, X^{z,\phi}) \quad \text{and} \quad Z^{z,\circ} := (\cdot, X^{z,\circ}) \tag{2.5}$$

in which $X^{z,\circ}$ denotes the solution of (2.3) for ϕ such that $\tau_1^\phi > T$ and satisfying $X_t^{z,\circ} = x$. This corresponds to the stochastic differential equation (2.3) in the absence of impulse. Note in particular that

$$Z_{\vartheta_1^\phi}^{z,\phi} = \mathrm{z}'(Z_{\tau_1^\phi-}^{z,\circ}, \alpha_1^\phi, \upsilon, \epsilon_1) \quad \text{on } \{\tau_1^\phi \geq t\}, \quad \text{with} \quad \mathrm{z}' := (\varpi, F). \tag{2.6}$$

From now on, we denote by $\mathbb{F}^{z,m,\phi} = (\mathcal{F}_s^{z,m,\phi})_{t \leq s \leq 2T}$ the $\mathbb{P}_m$-augmentation of the filtration generated by $(X^{z,\phi}, \sum_{i \geq 1} \mathbf{1}_{[\vartheta_i^\phi, \infty)})$ on $[t, 2T]$. We say that $\phi \in \Phi^m$ belongs to $\Phi^{z,m}$ if $(\tau_i^\phi)_{i \geq 1}$ is a sequence of $\mathbb{F}^{z,m,\phi}$-stopping times and α_i^ϕ is $\mathcal{F}_{\tau_i^\phi}^{z,m,\phi}$-measurable, for each $i \geq 1$. Hereafter an admissible control will be an element of $\Phi^{z,m}$.

2.2 Bayesian updates

Obviously, the prior m will evolve with time, as the value of the unknown parameter is partially revealed through the observation of the impacts of the actions on the system: at time t, one has observed $\{\mathrm{z}'(Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, \upsilon, \epsilon_i) : i \geq 1, \vartheta_i^\phi \leq t\}$. It should therefore be considered as a state variable, in any case, as its dynamics will naturally appear in any dynamic programming principle related to the optimal control of $X^{z,\phi}$, see Proposition 4.2 below. Moreover, its evolution can be of interest in itself. One can for instance be interested by the precision of our (updated) prior at the end of the control period, as it can serve as a new prior for another control problem.

In this section, we describe how it is updated with time, according to the usual Bayesian procedure. Given $z = (t, x) \in \mathbf{Z}$, $u \in \mathrm{U}$ and $a \in \mathbf{A}$, we assume that the law under $\mathbb{P}_\epsilon$ of $\mathrm{z}'[z, a, u, \epsilon_1]$, recall (2.6), is given by $\mathrm{q}(\cdot|z, a, u)d\mathrm{Q}(\cdot|z, a)$, in which $\mathrm{q}(\cdot|\cdot)$ is a Borel measurable map and $\mathrm{Q}(\cdot|z, a)$ is a dominating measure on $\mathbf{Z}$ for each $(z, a) \in \mathbf{Z} \times \mathbf{A}$. For $z = (t, x) \in \mathbf{Z}$, $m \in \mathbf{M}$ and $\phi \in \Phi^{z,m}$, let $M^{z,m,\phi}$ be the process defined by

$$M_s^{z,m,\phi}[C] := \mathbb{P}_m[\upsilon \in C|\mathcal{F}_s^{z,m,\phi}], \quad C \in \mathcal{B}(\mathrm{U}), \ s \geq t. \tag{2.7}$$

As no new information is revealed in between the end of an action and the start of the next one, the prior should remain constant on these time intervals:

$$M^{z,m,\phi} = M_{\vartheta_i^\phi}^{z,m,\phi} \text{ on } [\vartheta_i^\phi, \tau_{i+1}^\phi), \ i \geq 0, \tag{2.8}$$

with the conventions $\vartheta_0^\phi = 0$ and $M_0^{z,m,\phi} = m$. But, $M^{z,m,\phi}$ should jump from each τ_i^ϕ to each ϑ_i^ϕ, $i \geq 1$, according to the Bayes rule:

$$M_{\vartheta_i^\phi}^{z,m,\phi} = \mathfrak{M}(M_{\tau_i^\phi-}^{z,m,\phi}; Z_{\vartheta_i^\phi}^{z,\phi}, Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi), \ i \geq 1, \tag{2.9}$$

5

in which

$$\mathfrak{M}(m_o; z'_o, z_o, a_o)[C] := \frac{\int_C \mathsf{q}(z'_o | z_o, a_o, u) dm_o(u)}{\int_{\mathrm{U}} \mathsf{q}(z'_o | z_o, a_o, u) dm_o(u)}, \qquad (2.10)$$

for almost all $(z_o, z'_o, a_o, m_o) \in \mathbf{Z}^2 \times \mathbf{A} \times \mathbf{M}$ and $C \in \mathcal{B}(\mathrm{U})$.

Note that we did not specify $M^{z,m,\phi}$ on each $[\tau_i^\phi, \vartheta_i^\phi)$ since the controller must wait until ϑ_i^ϕ before being able to make another action. A partial information on υ through ϑ_i^ϕ is known as a right-censored observation of ϑ_i^ϕ is revealed through the interval $[\tau_i^\phi, \vartheta_i^\phi)$.

In order to ensure that $M^{z,m,\phi}$ remains in $\mathbf{M}$ whenever $m \in \mathbf{M}$, we need the following standing assumption:

Assumption 2.1 (Standing Assumption).

$$\mathfrak{M}(\mathbf{M}; \cdot) \subset \mathbf{M}.$$

Remark 2.1. *The above assumption means that we have to define a locally compact space $\mathbf{M}$ such the initial prior belongs to $\mathbf{M}$, and that is stable under the operator $\mathfrak{M}$. It is important for the use of viscosity solutions. This is clearly a limitation of our approach, from a theoretical point of view. An alternative would be to lift $\mathbf{M}$ to the space of square integrable random variables, and then use the methodologies developped in the context of mean-field games (see e.g. [7, Section 6]). We prevent from doing this for sake of clarity. On the other hand, our assumptions are satisfied in many pratical applications where $\mathbf{M}$ is either a set of measures defined on a metrizable compact space, see e.g. [4, Proposition 7.22 p130], or a parameterized family (which needs to be the case eventually if a numerical resolution is performed). If it is a parameterized family, it suffices to find an homeomorphism f from an open set of $\mathbb{R}^k$, $k \geq 1$, to $\mathbf{M}$ to ensure that $\mathbf{M}$ is locally compact. On the other hand, the stability of $\mathbf{M}$ with respect to $\mathfrak{M}$ can be ensured by using conjugate families, as explained in e.g. [3, Chapter 5.2]. The simplest example being the convex hull of a family of Dirac masses. See [1] for examples of applications.*

We formalize the dynamics of $M^{z,m,\phi}$ in the next proposition.

Proposition 2.1. *For all $z = (t, x) \in \mathbf{Z}$, $m \in \mathbf{M}$ and $\phi \in \Phi^{z,m}$, the process $M^{z,m,\phi}$ is $\mathbf{M}$ valued and follows the dynamics (2.8)-(2.9) on $[t, 2T]$.*

Proof. Let C be a Borel set of U and φ be a Borel bounded function on the Skorohod space D^{d+1} of càdlàg[2] functions with values in $\mathbb{R}^{d+1}$. Set $\xi^\phi := \sum_{i \geq 1} \mathbf{1}_{[\vartheta_i^\phi, \infty)}$ and set $\delta X^i := X^{z,\phi}_{\cdot \vee \vartheta_i^\phi} - X^{z,\phi}_{\vartheta_i^\phi}$. One can find a Borel measurable map $\bar{\varphi}$ on D^{2d+1} such that

$$\varphi(X^{z,\phi}_{\cdot \wedge s}, \xi^\phi_{\cdot \wedge s}) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} = \bar{\varphi}(X^{z,\phi}_{\cdot \wedge \vartheta_i^\phi}, \delta X^i_{\cdot \wedge s}, \xi^\phi_{\cdot \wedge \vartheta_i^\phi}) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}.$$

---

[2]continue à droite et limitée à gauche (right continuous with left limits)

Then, the independence of υ with respect to $\sigma(W_{\cdot \vee \vartheta_i^\phi} - W_{\vartheta_i^\phi})$ given $\mathcal{F}_{\vartheta_i^\phi}^{z,m,\phi}$, and the fact that τ_{i+1}^ϕ is measurable with respect to the sigma-algebra generated by $\sigma(W_{\cdot \vee \vartheta_i^\phi} - W_{\vartheta_i^\phi})$ and $\mathcal{F}_{\vartheta_i^\phi}^{z,m,\phi}$ imply that, for $s \geq 0$,

$$
\begin{aligned}
\mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \varphi(X_{\cdot \wedge s}^{z,\phi}, \xi_{\cdot \wedge s}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right] &= \mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \bar\varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \delta X_{\cdot \wedge s}^i, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right] \\
&= \mathbb{E}_m \left[M_{\vartheta_i^\phi}^{z,m,\phi}[C] \bar\varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \delta X_{\cdot \wedge s}^i, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right] \\
&= \mathbb{E}_m \left[M_{\vartheta_i^\phi}^{z,m,\phi}[C] \varphi(X_{\cdot \wedge s}^{z,\phi}, \xi_{\cdot \wedge s}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right].
\end{aligned}
$$

This shows that $M_s^{z,m,\phi}[C] \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} = M_{\vartheta_i^\phi}^{z,m,\phi}[C] \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}$ $\mathbb{P}_m$ − a.s.

It remains to compute $M_{\vartheta_i^\phi}^{z,m,\phi}$. Note that (2.3) implies that $(X_{\tau_i^\phi-}^{z,\phi}, \xi_{\tau_i^\phi-}^\phi) = (X_{\vartheta_i^\phi-}^{z,\phi}, \xi_{\vartheta_i^\phi-}^\phi)$. Let φ be as above, and let $\bar\varphi$ be a Borel measurable map on $\mathrm{D}^{d+1} \times \mathbb{R}_+ \times \mathbb{R}^d$ such that

$$
\varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) = \bar\varphi(X_{\cdot \wedge \tau_i^\phi-}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi-}^\phi, \vartheta_i^\phi, X_{\vartheta_i^\phi}^{z,\phi}) = \bar\varphi(X_{\cdot \wedge \tau_i^\phi-}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi-}^\phi, \mathrm{z}'[\tau_i^\phi, X_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, \upsilon, \epsilon_i]).
$$

Then, since ϵ_i is independent of $\mathcal{F}_{\tau_i^\phi}^{z,m,\phi}$ and has the same law as ϵ_1,

$$
\begin{aligned}
&\mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \right] \\
&= \mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \bar\varphi(X_{\cdot \wedge \tau_i^\phi-}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi-}^\phi, \mathrm{z}'[\tau_i^\phi, X_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, \upsilon, \epsilon_i]) \right] \\
&= \mathbb{E}_m \left[\int \mathbf{1}_{\{v \in C\}} \bar\varphi(X_{\cdot \wedge \tau_i^\phi-}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi-}^\phi, z') \mathrm{q}(z'|Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, \upsilon) d\mathrm{Q}(z'|Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi)) \right] \\
&= \mathbb{E}_m \left[\int \bar\varphi(X_{\cdot \wedge \tau_i^\phi-}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi-}^\phi, z') \left(\int_C \mathrm{q}(z'|Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, u) dM_{\tau_i^\phi-}^{z,m,\phi}(u) \right) d\mathrm{Q}(z'|Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi)) \right].
\end{aligned}
$$

Let us now introduce the notation $\mathfrak{M}_i[C](z') := \mathfrak{M}(M_{\tau_i^\phi-}^{z,m,\phi}; z', Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi)$. Then,

$$
\begin{aligned}
&\mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \right] \\
&= \mathbb{E}_m \left[\int \bar\varphi(X_{\cdot \wedge \tau_i^\phi-}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi-}^\phi, z') \mathfrak{M}_i[C](z') \mathrm{q}(z'|Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, \upsilon) d\mathrm{Q}(z'|Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi)) \right] \\
&= \mathbb{E}_m \left[\varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathfrak{M}_i[C](Z_{\vartheta_i^\phi}^{z,\phi}) \right].
\end{aligned}
$$

This concludes the proof. $\qquad \square$

Remark 2.2. *For later use, note that the above provides the joint conditional distribution of $(Z_{\vartheta_i^\phi}^{z,\phi}, M_{\vartheta_i^\phi}^{z,m,\phi})$ given $\mathcal{F}_{\tau_i-}^{z,m,\phi}$. Namely, for Borel sets $B \in \mathcal{B}([t,T] \times \mathbb{R}^d)$ and $D \in \mathcal{B}(\mathbf{M})$, a simple application of Fubini's Lemma implies that*

$$
\mathbb{P}[(Z_{\vartheta_i^\phi}^{z,\phi}, M_{\vartheta_i^\phi}^{z,m,\phi}) \in B \times D | \mathcal{F}_{\tau_i^\phi-}^{z,m,\phi}] = \mathrm{k}(B \times D | Z_{\tau_i^\phi-}^{z,\phi}, M_{\tau_i^\phi-}^{z,m\phi}, \alpha_i^\phi) \tag{2.11}
$$

in which

$$\mathrm{k}(B \times D | z_o, m_o, a_o) := \int_{\mathrm{U}} \int_B \mathbf{1}_D(\mathfrak{M}(m_o; z', z_o, a_o)) \mathrm{q}(z'|z_o, a_o, u) d\mathrm{Q}(z'|z, a) dm_o(u), \quad (2.12)$$

for $(z_o, m_o, a_o) \in \mathbf{Z} \times \mathbf{M} \times \mathbf{A}$.

2.3 Gain function

Given $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$, the aim of the controller is to maximize the expected value of the gain functional

$$\phi \in \Phi^{z,m} \mapsto G^{z,m}(\phi) := g(Z^{z,\phi}_{\mathrm{T}[\phi]}, M^{z,m,\phi}_{\mathrm{T}[\phi]}, \upsilon, \epsilon_0),$$

in which $\mathrm{T}[\phi]$ is the end of the last action after T:

$$\mathrm{T}[\phi] := \sup\{\vartheta^\phi_i : i \geq 1, \ \tau^\phi_i \leq T\} \vee T.$$

As suggested earlier, the gain may not only depend on the value of the original time-space state process $Z^{z,\phi}_{\mathrm{T}[\phi]}$ but also on $M^{z,m,\phi}_{\mathrm{T}[\phi]}$, to model the fact that we are also interested by the precision of the estimation made on υ at the final time. One also allows for terminating the last action after T. However, since g can depend on $\mathrm{T}[\phi]$ through $Z^{z,\phi}_{\mathrm{T}[\phi]}$, one can penalize the actions that actually terminates strictly after T.

Hereafter, the function g is assumed to be measurable and bounded[3] on $\mathbf{Z} \times \mathbf{M} \times \mathrm{U} \times \mathrm{E}$.

Given $\phi \in \Phi^{z,m}$, the expected gain is

$$J(z, m; \phi) := \mathbb{E}_m \left[G^{z,m}(\phi) \right],$$

and

$$\mathrm{v}(z, m) := \sup_{\phi \in \Phi^{z,m}} J(z, m; \phi) \mathbf{1}_{\{t \leq T\}} + \mathbf{1}_{\{t > T\}} \mathbb{E}_m \left[g(z, m, \upsilon, \epsilon_0) \right] \quad (2.13)$$

is the corresponding value function. Note that v depends on m through the set of admissible controls $\Phi^{z,m}$ and the expectation operator $\mathbb{E}_m$, even if g does not depend on $M^{z,m,\phi}_{\mathrm{T}[\phi]}$.

Remark 2.3. *Note that a running gain term could be added without any difficulty. One usually reduces to a Mayer formulation by adding a component to the space process and by modifying the terminal reward accordingly. Here, if this running gain only covers the period $[0, T]$, it should be added explicitly because of the modified time horizon $\mathrm{T}[\phi]$ at which the terminal gain is computed.*

---

[3]Boundedness is just for sake of simplicity. Much more general frameworks could easily be considered.

3 Value function characterization

The aim of this section is to provide a characterization of the value function v. As usual, it should be related to a dynamic programming principle. In our setting, it corresponds to: Given $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$, then

$$v(z, m) = \sup_{\phi \in \Phi^{z,m}} \mathbb{E}_m[v(Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi})], \tag{3.1}$$

for all collection $(\theta^\phi, \phi \in \Phi^{z,m})$ of $\mathbb{F}^{z,m,\phi}$-stopping times with values in $[t, 2T]$ such that $\theta^\phi \in \mathcal{N}^\phi \cap [t, \mathrm{T}[\phi]] \; \mathbb{P}_m - $ a.s., recall the definition of $\mathcal{N}^\phi$ in (2.2).

Let us comment this. First, one should restrict to stopping times such that $\theta^\phi \in \mathcal{N}^\phi$. The reason is that no new impulse can be made outside of $\mathcal{N}^\phi$, each interval $[\tau_i^\phi, \vartheta_i^\phi)$ is a latency period. Second, the terminal gain is evaluated at $\mathrm{T}[\phi]$, which in general is different from T. Hence, the fact that θ^ϕ is only bounded by $\mathrm{T}[\phi]$.

A partial version of (3.1) will be proved in Proposition 4.2 below and will be used to provide a sub-solution property. As already mentioned in the introduction, we are not able to prove a full version (3.1). The reason is that the value function v depends on $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$ through the set of admissible controls $\Phi^{z,m}$, and more precisely through the choice of the filtration $\mathbb{F}^{z,m,\phi}$, which even depends on ϕ itself. This makes this dependence highly singular and we are neither in position to play with any a-priori smoothness, see e.g. [6], nor to apply a measurable selection theorem, see e.g. [4].

We continue our discussion, assuming that (3.1) holds and that v is sufficiently smooth. Then, it should in particular satisfy $v(z, m) \geq \mathbb{E}_m[v(Z_{t+h}^{z,\circ}, m)]$ whenever $z = (t, x) \in [0, T) \times \mathbb{R}^d$ and $0 < h \leq T - t$ ($Z^{z,\circ}$ is defined after (2.5)). This corresponds to the sub-optimality of the control consisting in making no impulse on $[t, t + h]$. Applying Itô's lemma, dividing by h and letting h go to 0, we obtain $-\mathcal{L}v(z, m) \geq 0$ in which $\mathcal{L}$ is the Dynkin operator associated to $X^{z,\circ}$,

$$\mathcal{L}\varphi := \partial_t \varphi + \langle \mu, D\varphi \rangle + \frac{1}{2}\mathrm{Tr}[\sigma \sigma^\top D^2 \varphi].$$

On the other hand, it follows from (3.1) and Remark 2.2 that

$$v(z, m) \geq \sup_{a \in \mathbf{A}} \mathbb{E}_m[v(z'[z, a, \upsilon, \epsilon_1], \mathfrak{M}(m; z'[z, a, \upsilon, \epsilon_1], z, a))] = \mathcal{K}v(z, m)$$

where $\mathcal{K}\varphi := \sup_{a \in \mathbf{A}} \mathcal{K}^a \varphi$ with $\mathcal{K}^a \varphi := \int \varphi(z', m') dk(z', m'|\cdot, a)$ for $a \in \mathbf{A}$. (3.2)

As for the time-T boundary condition, the same reasoning as above implies $v(T, \cdot) \geq \mathcal{K}_T g$ and $v(T, \cdot) \geq \mathcal{K}v(T, \cdot)$, in which

$$\mathcal{K}_T g(\cdot, m) = \int_{\mathrm{U}} \int_{\mathrm{E}} g(\cdot, m, u, e) d\mathbb{P}_\epsilon(e) dm(u). \tag{3.3}$$

By optimality, v should therefore solve the quasi-variational equations

$$\min\{-\mathcal{L}\varphi, \; \varphi - \mathcal{K}\varphi\} = 0 \;\; \text{on } [0, T) \times \mathbb{R}^d \times \mathbf{M} \tag{3.4}$$

$$\min\{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}\varphi\} = 0 \;\; \text{on } \{T\} \times \mathbb{R}^d \times \mathbf{M}, \tag{3.5}$$

in the sense of the following definition (given for sake of clarity).

Definition 3.1. *We say that a lower-semicontinuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity super-solution of* (3.4)-(3.5) *if for any $z_\circ = (t_\circ, x_\circ) \in \mathbf{Z}$, $m_\circ \in \mathbf{M}$, and $\varphi \in C^{1,2,0}([0,T] \times \mathbb{R}^d \times \mathbf{M})$ such that $\min_{\mathbf{Z} \times \mathbf{M}}(U - \varphi) = (U - \varphi)(z_\circ, m_\circ) = 0$ we have*

$$\left[\min\left\{ -\mathcal{L}\varphi \,,\, \varphi - \mathcal{K}U \right\} \mathbf{1}_{\{t_\circ < T\}} + \min\left\{ \varphi - \mathcal{K}_T g, \varphi - \mathcal{K}U \right\} \mathbf{1}_{\{t_\circ = T\}} \right] (z_\circ, m_\circ) \geq 0.$$

We say that a upper-semicontinuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity sub-solution of (3.4)-(3.5) *if for any $z_\circ = (t_\circ, x_\circ) \in \mathbf{Z}$, $m_\circ \in \mathbf{M}$ and $\varphi \in C^{1,2,0}([0,T] \times \mathbb{R}^d \times \mathbf{M})$ such that $\max_{\mathbf{Z} \times \mathbf{M}}(U - \varphi) = (U - \varphi)(z_\circ, m_\circ) = 0$ we have*

$$\left[\min\left\{ -\mathcal{L}\varphi \,,\, \varphi - \mathcal{K}U \right\} \mathbf{1}_{\{t_\circ < T\}} + \min\left\{ \varphi - \mathcal{K}_T g, \varphi - \mathcal{K}U \right\} \mathbf{1}_{\{t_\circ = T\}} \right] (z_\circ, m_\circ) \leq 0.$$

We say that a continuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity solution of (3.4)-(3.5) *if it is a super- and a sub-solution.*

To ensure that the above operator is continuous, we assume from now on that, on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$,

$$\mathcal{K}_T g \text{ is continuous, and } \mathcal{K}\varphi \text{ is upper- (resp. lower-) semicontinuous,} \tag{3.6}$$
$$\text{for all upper- (resp. lower-) semicontinuous bounded function } \varphi.$$

A sufficient condition for (3.6) to hold is that k defined in (2.12) is a continuous stochastic kernel, see [4, Proposition 7.31 and 7.32 page 148].

Finally, we assume that comparison holds for (3.4)-(3.5).

Assumption 3.1. *Let U (resp. V) be a upper- (resp. lower-) semicontinuous bounded viscosity sub- (resp. super-) solution of* (3.4)-(3.5). *Assume further that $U \leq V$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Then, $U \leq V$ on $\mathbf{Z} \times \mathbf{M}$.*

See Proposition 5.1 below for a sufficient condition. We are now in position to state the main result of this paper. The proof is provided in the next section.

Theorem 3.1. *Let Assumption 3.1 (or the conditions of Proposition 5.1 below) hold. Then, v is continuous on $\mathbf{Z} \times \mathbf{M}$ and is the unique bounded viscosity solution of* (3.4)-(3.5).

Remark 3.1. *We do not discuss here the issue of existence of an optimal control. We refer to the application paper [1] for an example of numerical scheme allowing to construct approximately optimal controls. Note also that the construction of Section 4.2 below produces an almost optimal control as the arguments of Section 4.4 show that the sequence of value functions $(v_n)_{n \geq 1}$ actually converges to v.*

4 Viscosity solution properties

This part is dedicated to the proof of the viscosity solution characterization of Theorem 3.1. We start with the sub-solution property, which is the more classical part. As for the super-solution property, we shall later on introduce a discrete time version of the model that will provide a natural lower bound. We will then show that the sequence of corresponding value functions converges to a super-solution of our quasi-variational equation as the time step goes to 0. By comparison, we will finally identify this (limit) lower bound to the original value function, thus showing that the later is also a super-solution.

4.1 Sub-solution property

We start with the sub-solution property and show that it is satisfied by the upper-semicontinuous enveloppe of v defined in (2.13):

$$v^*(z, m) := \limsup_{(z', m') \to (z, m)} v(z', m') \ , \quad (z, m) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}.$$

Proposition 4.1. v^* *is a viscosity subsolution of* (3.4)-(3.5).

The proof is rather standard. As usual, it is based on the partial dynamic programming principle contained in Proposition 4.2 below, that can be established by adapting standard lines of arguments, see e.g. [6]. For this part, the dependency of the filtration on the initial data is not problematic as it only requires a conditioning argument. Before to state it, let us make an observation.

Remark 4.1. *Note that, given* $z = (t, x) \in \mathbf{Z}$, *the process* $X^{z, \circ}$ *defined in* (2.5) *is predictable with respect to the* $\mathbb{P}$-*augmentation of the raw filtration* $\mathbb{F}^{t, W}$ *generated by* $(W_{\cdot \vee t} - W_t)$. *By [9, Lemma 7, Appendix I], it is indistinguishable from a* $\mathbb{F}^{t, W}$-*predictable process. Using this identification,* $X_s^{z, \circ}(\omega) = X_s^{z, \circ}(\omega^{t, s})$ *for* $s \geq t$, *with* $\omega^{t, s} := \omega_{t \vee \cdot \wedge s} - \omega_t$. *Similarly,* τ_1^ϕ *and* α_1^ϕ *can be identified to Borel measurable maps on* $C([0, T]; \mathbb{R}^d)$ *that depends only on* $\omega^{t, \tau_1^\phi(\omega^{t, T})}$ *so that* $(Z_{\vartheta_1^\phi}^{z, \phi}, M_{\vartheta_1^\phi}^{z, m, \phi})$ *can be seen as a Borel map on* $C([0, T]; \mathbb{R}^d) \times \mathrm{U} \times \mathrm{E}$, *while* $(Z_{\tau_1^\phi -}^{z, \phi}, M_{\tau_1^\phi -}^{z, m, \phi})$ *can be seen as a Borel map on* $C([0, T]; \mathbb{R}^d)$ *that only depends on* $\omega^{t, \tau_1^\phi(\omega^{t, T})}$, *recall* (2.6), (2.8) *and* (2.9). *Iterating this argument, we also obtain that* $(Z_{\mathrm{T}[\phi]}^{z, \phi}, M_{\mathrm{T}[\phi]}^{z, m, \phi})$ *is equal, up to* $\mathbb{P}_m$-*null sets, to a Borel map on* $C([0, T]; \mathbb{R}^d) \times \mathrm{U} \times \mathrm{E}^N$, *for some* $N \geq 1$ *that depends on* ϕ.

We use the notations introduced in (2.5), (3.2) and (3.3) in the following.

Proposition 4.2. *Fix* $(z, m) \in \mathbf{Z} \times \mathbf{M}$, *and let* θ *be the first exit time of* $Z^{z, \circ}$ *from a Borel set* $B \subset \mathbf{Z}$ *containing* (z, m). *Then,*

$$v(z, m) \leq \sup_{\phi \in \Phi_{\geq t}^{z, m}} \mathbb{E}_m[f(Z_\theta^{z, \circ}, m) \mathbf{1}_{\{\theta < \tau_1^\phi\}} + \mathcal{K}^{\alpha_1^\phi} f(Z_{\tau_1^\phi -}^{z, \circ}, m)] \mathbf{1}_{\{\theta \geq \tau_1^\phi\}}] \tag{4.1}$$

in which $z := (t, x)$, $\Phi_{\geq t}^{z, m} := \{\phi \in \Phi^{z, m} : \tau_1^\phi \geq t\}$ *and*

$$f(z', m') := v^*(z', m') \mathbf{1}_{\{t' < T\}} + \mathcal{K}_T g(z', m') \mathbf{1}_{\{t' \geq T\}} \tag{4.2}$$

for $z' = (t', x') \in \mathbf{A}$ *and* $m' \in \mathbf{M}$.

Proof. Let $N \geq 1$ be such that $\tau_i^\phi > T$ for $i \geq N$. By right continuity of $(Z^{z, \phi}, M^{z, m, \phi})$ and upper-semicontinuity of f and $\mathcal{K}f$ on $[0, T) \times \mathbb{R}^d \times \mathbf{M}$, see (3.6), it suffices to prove the result for the projections on the right of θ and τ_1^ϕ on a deterministic time grid. Then, it is enough to consider the case where $(\theta, \tau_1^\phi) \equiv (s, s') \in [t, T]^2$, by arguing as below and conditioning by the values taken by (θ, τ_1^ϕ) on the grid. In the following, we use regular conditional expectation operators. We shall make use of Remark 4.1. In particular, we write $\phi(\omega, u, (e_i)_{i \leq N})$ to

denote the Borel map $(\omega, u, (e_i)_{i \leq N}) \in C([0,T]; \mathbb{R}^d) \times U \times E^N \mapsto \{(\tau_i^\phi, \alpha_i^\phi)(\omega^{t,T}, u, (e_j)_{j \leq i-1}), i \leq N\}$ associated to ϕ. If $s < s'$, we have $\mathbb{P}_m$-a.s.

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_s^{z,m,\phi}](\omega, u, (e_i)_{i \geq 1}) = \mathbb{E}_m[G^{Z_s^{z,\circ}(\omega^{t,s}),m}(\phi_{\omega^{t,s}})]$$
$$= \mathbb{E}_m[\mathcal{K}_T g(X_T^{Z_s^{z,\circ}(\omega^{t,s}),\phi_{\omega^{t,s}}}, M_T^{Z_s^{z,\circ}(\omega^{t,s}),m,\phi_{\omega^{t,s}}})]$$

in which $\mathcal{K}_T$ is defined in (3.3) and

$$\phi_{\omega^{t,s}} : (\omega', u, (e_i)_{i \leq N}) \in C([s,T]; \mathbb{R}^d) \times U \times E^N \mapsto \phi(\omega^{t,s} + \omega'_{\cdot \vee s} - \omega'_s, u, (e_i)_{i \leq N})$$

is an element of $\Phi^{Z_s^{z,\circ}(\omega^{t,s}),m,\phi_{\omega^{t,s}}}$. It follows that $\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_s^{z,m,\phi}]\mathbf{1}_{s<s'} \leq f(Z_s^{z,\circ},m)\mathbf{1}_{s<s'}$ $\mathbb{P}_m$ − a.s. Similarly, if $s \geq s'$, we have $\mathbb{P}_m$-a.s.

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_{s'-}^{z,m,\phi}](\omega, u, (e_i)_{i \leq N}) = \mathbb{E}_m[G^{\xi(\omega^{t,s'}, v, \epsilon_1, \alpha_1^\phi(\omega^{t,s'}))}(\phi_{\omega^{t,s'}})]$$

with

$$\xi(\omega^{t,s'}, v, \epsilon_1, \alpha_1^\phi(\omega^{t,s'})) = \left(\cdot, \mathfrak{M}(m; \cdot, Z_{s'-}^{z,\circ}(\omega^{t,s'}), \alpha_1^\phi(\omega^{t,s'})) \right) \circ z'(Z_{s'-}^{z,\circ}(\omega^{t,s'}), \alpha_1^\phi(\omega^{t,s'}), v, \epsilon_1),$$

recall the notations in (2.6) and (2.10). Hence, $\mathbb{P}_m$-a.s.,

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_{s'-}^{z,m,\phi}](\omega, u, (e_i)_{i \leq N}) \leq \mathbb{E}_m[f(\xi(\omega^{t,s'}, v, \epsilon_1, \alpha_1^\phi(\omega^{t,s'})))] = \mathcal{K}^{\alpha_1^\phi(\omega^{t,s'})} f(Z_{s'-}^{z,\circ}(\omega^{t,s'}), m),$$

in which $a \in \mathbf{A} \mapsto \mathcal{K}^a$ is defined in (3.2). $\qquad\square$

Proof of Proposition 4.1 As already mentioned, the proof is standard, we provide it for completeness. Let φ be a (bounded) $C^{1,2,0}$ function and fix $(z_\circ, m_\circ) \in \mathbf{Z} \times \mathbf{M}$ such that

$$0 = (\mathrm{v}^* - \varphi)(z_\circ, m_\circ) = \max_{\mathbf{Z} \times \mathbf{M}}(\mathrm{v}^* - \varphi). \tag{4.3}$$

We use the notation $z_\circ = (t_\circ, x_\circ) \in [0,T] \times \mathbb{R}^d$.

Step 1. We first assume that $t_\circ < T$. Let us suppose that $\min\{-\mathcal{L}\varphi \,, \, \varphi - \mathcal{K}\mathrm{v}^*\}(z_\circ, m_\circ) > 0$, and work towards a contradiction to Proposition 4.2. Let $\mathrm{d}_\mathbf{M}$ be a metric compatible with the weak topology and let $\|\cdot\|_\mathbf{Z}$ be the Euclidean norm on $\mathbf{Z}$. We define

$$\bar\varphi(z', m') := \varphi(z', m') + \|z' - z_\circ\|_\mathbf{Z}^4 + \mathrm{d}_\mathbf{M}(m', m_\circ).$$

If the above holds, then $\min\{-\mathcal{L}\bar\varphi \,, \, \bar\varphi - \mathcal{K}\mathrm{v}^*\}(z_\circ, m_\circ) > 0$. By our continuity assumption (3.6), we can find $\iota, \eta > 0$, such that

$$\min\{-\mathcal{L}\bar\varphi \,, \, \bar\varphi - \mathcal{K}\mathrm{v}^*\} \geq \eta \quad \text{on } B_\iota, \tag{4.4}$$

in which

$$B_\iota := \{(z', m') \in \mathbf{Z} \times \mathbf{M} : \|z' - z_\circ\|_\mathbf{Z}^4 + \mathrm{d}_\mathbf{M}(m', m_\circ) < \iota\} \subset [0,T) \times \mathbb{R}^d \times \mathbf{M}.$$

Note that, after possibly changing $\eta > 0$, we can assume that

$$(\mathrm{v}^* - \bar\varphi) \leq -\eta \quad \text{on } (B_\iota)^c. \tag{4.5}$$

In the following, we let $(z, m) \in B_\iota$ be such that

$$|\mathrm{v}(z, m) - \bar\varphi(z, m)| \leq \eta/2, \tag{4.6}$$

recall (4.3). As above, we write $z = (t, x) \in [0, T] \times \mathbb{R}^d$. Fix $\phi \in \Phi^{z,m}$. We write $(\tau_i, \alpha_i, \vartheta_i)_{i \geq 1}$, Z and M for $(\tau_i^\phi, \alpha_i^\phi, \vartheta_i^\phi)_{i \geq 1}$, $Z^{z,\phi}$ and $M^{z,m,\phi}$. Let θ be the first time when (Z, M) exits B_ι. Without loss of generality, one can assume that $\tau_1 \geq t$. Define $\chi := \theta \mathbf{1}_{\{\theta < \tau_1\}} + \mathbf{1}_{\{\theta \geq \tau_1\}} \vartheta_1$. In view of (4.4), (4.5) and (4.6),

$$\begin{aligned}
\mathbb{E}_m[\mathrm{v}^*(Z_\chi, M_\chi)] &= \mathbb{E}_m[\mathrm{v}^*(Z_{\vartheta_1}, M_{\vartheta_1}) \mathbf{1}_{\{\chi \neq \theta\}} + \mathrm{v}^*(Z_\theta, M_\theta) \mathbf{1}_{\{\chi = \theta\}}] \\
&\leq \mathbb{E}_m[\mathcal{K}\mathrm{v}^*(Z_{\tau_1-}, M_{\tau_1-}) \mathbf{1}_{\{\chi \neq \theta\}} + \mathrm{v}^*(Z_\theta, M_\theta) \mathbf{1}_{\{\chi = \theta\}}] \\
&\leq \mathbb{E}_m[\bar\varphi(Z_{\theta \wedge \tau_1-}, M_{\theta \wedge \tau_1-})] - \eta \\
&\leq \bar\varphi(z, m) - \eta \\
&\leq \mathrm{v}(z, m) - \eta/2.
\end{aligned}$$

Since $\chi < T$, this contradicts Proposition 4.2 by arbitrariness of ϕ.

Step 2. We now consider the case $t_\circ = T$. We assume that $\min\{\varphi - \mathcal{K}\mathrm{v}^*, \varphi - \mathcal{K}_T g\}(z_\circ, m_\circ) > 0$, and work toward a contradiction. Let us define

$$\bar\varphi(t', x', m') := \bar\varphi(t', x', m') + C(T - t') + \|(t', x') - z_\circ\|_{\mathbf{Z}}^4 + \mathrm{d}_{\mathbf{M}}(m', m_\circ)$$

and note that, for C large enough, $\min\{-\mathcal{L}\bar\varphi, \bar\varphi - \mathcal{K}\mathrm{v}^*, \bar\varphi - \mathcal{K}_T g\}(z_\circ, m_\circ) > 0$. Then, as in Step 1, we can find $\iota, \eta > 0$, such that

$$\min\{-\mathcal{L}\bar\varphi, \bar\varphi - \mathcal{K}\mathrm{v}^*, \bar\varphi - \mathcal{K}_T g\} \geq \eta \quad \text{on } B_\iota,$$

in which

$$B_\iota := \{(t', x', m') \in (T - \iota, T] \times \mathbf{M} : \|x' - x_\circ\|_{\mathbb{R}^d}^4 + \mathrm{d}_{\mathbf{M}}(m', m_\circ) < \iota\}.$$

After possibly changing $\eta > 0$, one can assume that

$$(\mathrm{v}^* - \bar\varphi) \leq -\eta \quad \text{on } (B_\iota)^c.$$

Let $(t, x, m) \in B_\iota$ be such that

$$|\mathrm{v}(t, x, m) - \bar\varphi(t, x, m)| \leq \eta/2.$$

One can assume that $t < T$. Otherwise, this would mean that

$$\mathrm{v}^*(z_\circ, m_\circ) = \limsup_{(T, x', m') \to (z_\circ, m_\circ)} \mathrm{v}(T, x', m') = \limsup_{(T, x', m') \to (z_\circ, m_\circ)} \mathcal{K}_T(T, x', m') = \mathcal{K}_T g(z_\circ, m_\circ),$$

recall (3.6), and there is nothing to prove. Given $\phi \in \Phi^{z,m}$, with $z := (t, x)$, let $(\tau_1, \vartheta_1, Z = (\cdot, X), M)$ be defined as in Step 1 with respect to ϕ and (z, m), and consider $\chi := \theta \mathbf{1}_{\{\theta < \tau_1\}} + \mathbf{1}_{\{\theta \geq \tau_1\}} \vartheta_1$, where θ is the first exit time of (X, M) from $\{(x', m') \in \mathbb{R}^d \times \mathbf{M} : \|x' - x_\circ\|_{\mathbb{R}^d}^4 + \mathrm{d}_{\mathbf{M}}(m', m_\circ) < \iota\}$. As in Step 1, the above implies that $\mathbb{E}_m[\mathrm{v}^*(Z_\chi, M_\chi)] \leq \mathrm{v}(z, m) - \eta/2$, which contradicts Proposition 4.2 by arbitrariness of ϕ. $\qquad\square$

13

4.2 Discrete time approximation and dynamic programming

In this part, we prepare for the proof of the super-solution property. As already mentioned above, we could not provide the opposite inequality in (4.1), with v^* replaced by the lower-semicontinuous envelope of v, because of the non-trivial dependence of $\mathbb{F}^{z,m,\phi}$ with respect to the initial data. Instead, we use the natural idea of approximating our continuous time control problem by a sequence of discrete time counterparts defined on a sequence of time grids. In discrete time, the dynamic programming principle can be proved along the lines of [4] for the corresponding value functions $(v_n)_{n \geq 1}$. Passing to the limit as the time mesh vanishes provides a super-solution $v_\circ$ of (3.4)-(3.5). As v^* is a sub-solution of the same equation, Assumption 3.1 will imply that $v_\circ \geq v^*$, while the opposite will hold by construction. Then, we will conclude that v is a actually a super-solution, and is even continuous. This approach is similar to the one used in [11] in the context of differential games.

We first construct the sequence of discrete time optimal control problems. For $n \geq 1$, let $\pi_n := \{t_j^n, j \leq 2^n\}$ with $t_j^n := jT/2^n$, and let $\Phi_n^{z,m}$ be the set of controls $\phi = (\tau_i^\phi, \alpha_i^\phi)_{i \geq 1}$ in $\Phi^{z,m}$ such that $(\tau_i^\phi)_{i \geq 1}$ takes values in $\pi_n \cup \{t\} \cup [T, \infty)$, if $z = (t, x)$. The corresponding value function is

$$v_n(z, m) = \sup_{\phi \in \Phi_n^{z,m}} J(z, m, \phi), \quad (z, m) \in \mathbf{Z} \times \mathbf{M}.$$

We extend v_n by setting

$$v_n := \mathcal{K}_T g, \quad \text{on } (T, \infty) \times \mathbf{M}, \tag{4.7}$$

Remark 4.2. *Note that $v_n \leq v \leq v^*$ by construction.*

We first prove that v_n satisfies a dynamic programming principle. This requires additional notations. We first define the next time on the grid at which a new action can be made, given that a is plaid:

$$s^{n,a}[t, x] := \min\{s \in \pi_n \cup [T, \infty) : s \geq \varpi(t, x, a, v, \epsilon_j) \text{ and } s > t\}.$$

Let ∂ denote a cemetery point that does not belong to $\mathbf{A}$. Given $a \in \mathbf{A} \cup \{\partial\}$, we make a slight abuse of notation by denoting by $(Z^{(t,x),a}, M^{(t,x),m,a})$ the process defined as $(Z^{(t,x),\phi}, M^{(t,x),m,\phi})$ for ϕ such that

$$(\tau_1^\phi, \alpha_1^\phi) = (t, a)\mathbf{1}_{\{a \neq \partial\}} + (T + 1, a_\star)\mathbf{1}_{\{a = \partial\}}$$

in which $a_\star \in \mathbf{A}$ and $\tau_i^\phi > T + 1$ for $i > 1$. Then, we set

$$\bar{J}(T, \cdot; a) := \mathcal{K}_T \mathcal{K}^a g, \; \bar{v}_n(T, \cdot) := \sup_{a \in \mathbf{A} \cup \{\partial\}} \bar{J}(T, \cdot; a) \; \text{ on } \mathbb{R}^d \times \mathbf{M} \times (\mathbf{A} \cup \{\partial\}),$$

with the convention that $\mathcal{K}^\partial$ is the identity, and define by backward induction on the intervals $[t_j^n, T)$, $j = n - 1, \cdots, 0$,

$$\bar{J}(z, m; a) := \mathbb{E}_m[\bar{v}_n(Z_{s^{n,a}[z]}^{z,a}, M_{s^{n,a}[z]}^{z,m,a})] , \; \bar{v}_n := \sup_{a \in \mathbf{A} \cup \{\partial\}} \bar{J}(\cdot; a),$$

together with the extension

$$\bar{v}_n := \mathcal{K}_T g \; \text{ on } (T, \infty) \times \mathbb{R}^d \times \mathbf{M}.$$

14

Lemma 4.1. *Fix $\iota > 0$. Then, there exists a universally measurable map $(z,m) \in \mathbf{Z} \times \mathbf{M} \mapsto \hat{a}^{n,\iota}[z,m] \in \mathbf{A} \cup \{\partial\}$ such that $\bar{J}(\cdot; \hat{a}^{n,\iota}[\cdot]) \geq \bar{v}_n - \iota$ on $\mathbf{Z} \times \mathbf{M}$. Moreover, the map $\bar{v}_n$ is upper semi-analytic.*

Proof. Since $\mathcal{K}_T g$ is assumed to be upper semi-analytic (indeed continuous), it follows from [4, Proposition 7.48 page 180] that $\bar{J}$ is upper semi-analytic on $[t^n_{n-1}, T] \times \mathbb{R}^d \times \mathbf{M} \times (\mathbf{A} \cup \{\partial\})$. Then, the required result holds on $[t^n_{n-2}, T] \times \mathbb{R}^d \times \mathbf{M}$ by [4, Proposition 7.50 page 184]. It is then extended to $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ by a backward induction. $\qquad\square$

Proposition 4.3. $\bar{v}_n = v_n$ *on $\mathbf{Z} \times \mathbf{M}$. Moreover, given a random variable (ζ, μ) with values in $\mathbf{Z} \times \mathbf{M}$ and $\iota > 0$, there exists a measurable map $(z,m) \mapsto \phi^\iota[z,m]$ such that*

$$J(\zeta, \mu; \phi^\iota[\zeta, \mu]) \geq v_n(\zeta, \mu) - \iota \quad \mathbb{P}_m - \text{a.s.}$$

Proof. The proof proceeds by induction. Our claim follows from definitions on $[t^n_n, T] \times \mathbb{R}^d \times \mathbf{M}$. Assume that it holds on $[t^n_{j+1}, T] \times \mathbb{R}^d \times \mathbf{M}$ for some $j \leq n - 1$. For the following, we fix $z = (t, x) \in \mathbf{Z}$ with $t \in [t^n_j, t^n_{j+1})$ and $m \in \mathbf{M}$.
Step 1: In this step, we first construct a suitable candidate to be an almost-optimal control. Fix $\varepsilon_1, \ldots, \varepsilon_n > 0$, $\varepsilon_0 := 0$, and set $\varepsilon(i) := (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_i)$. Let $(\hat{a}^{n,\iota})_{\iota > 0}$ be as in Lemma 4.1, and consider its extension defined by $\hat{a}^{n,\iota} = a_\star$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Define $r_1^{\varepsilon(0)} := t$ and $\phi_1^{\varepsilon(1)} \in \Phi_n^{z,m}$ by

$$(\tau_i^{\phi_1^{\varepsilon(1)}}, \alpha_i^{\phi_1^{\varepsilon(1)}}) = (r_1^{\varepsilon(0)}, \tilde{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m])\mathbf{1}_{\{i=1\}} + \mathbf{1}_{\{i>1\}}(T+i, a_\star), \quad i \geq 1.$$

where

$$\tilde{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m] := \hat{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m].$$

We then set

$$r_2^{\varepsilon(1)} := \min \pi_n \cap [\vartheta_1^{\phi_1^{\varepsilon(1)}}, 2T] \cap (r_1^{\varepsilon(0)}, \infty).$$

By Lemma 4.1 and [4, Lemma 7.27 page 173] applied to the pull-back measure of $(Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z,m,\phi_1^{\varepsilon(1)}})$, we can find a Borel measurable map $(t', x', m') \in \mathbf{Z} \times \mathbf{M} \mapsto \tilde{a}_2^{n,\varepsilon_2}[t', x', m'] \in \mathbf{A} \cup \{\partial\}$ such that

$$\tilde{a}^{n,\varepsilon_2}[Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z,m,\phi_1^{\varepsilon(1)}}] = \hat{a}^{n,\varepsilon_2}[Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z,m,\phi_1^{\varepsilon(1)}}] \quad \mathbb{P}_m - \text{a.s.}$$

We define $\phi_2^{\varepsilon(2)}$ by

$$(\tau_i^{\phi_2^{\varepsilon(2)}}, \alpha_i^{\phi_2^{\varepsilon(2)}}) = (r_2^{\varepsilon(1)}, \tilde{a}^{n,\varepsilon_2}[Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z,m,\phi_1^{\varepsilon(1)}}])\mathbf{1}_{\{i=2, r_2^{\varepsilon(1)} \leq T\}} + (\tau_i^{\phi_1^{\varepsilon(1)}}, \alpha_i^{\phi_1^{\varepsilon(1)}})\mathbf{1}_{\{i \neq 2\} \cup \{r_2^{\varepsilon(1)} > T\}},$$

for $i \geq 1$. We then define recursively for $k \geq 2$

$$r_{k+1}^{\varepsilon(k)} := \inf \pi_n \cap [\vartheta_k^{\phi_k^{\varepsilon(k)}}, 2T] \cap (r_k^{\varepsilon(k-1)}, \infty)$$

$$(\tau_i^{\phi_{k+1}^{\varepsilon(k+1)}}, \alpha_i^{\phi_{k+1}^{\varepsilon(k+1)}}) = (r_{k+1}^{\varepsilon(k)}, \tilde{a}_{k+1}^{n,\varepsilon_{k+1}}[Z_{r_{k+1}^{\varepsilon(k)}}^{z,\phi_k^{\varepsilon(k)}}, M_{r_{k+1}^{\varepsilon(k)}}^{z,m,\phi_k^{\varepsilon(k)}}]) \mathbf{1}_{\{i=k+1, r_{k+1}^{\varepsilon(k)} \leq T\}}$$

$$+ (\tau_i^{\phi_k^{\varepsilon(k)}}, \alpha_i^{\phi_k^{\varepsilon(k)}}) \mathbf{1}_{\{i \neq k+1\} \cup \{r_{k+1}^{\varepsilon(k)} > T\}},$$

for $i \geq 1$, in which $(t', x', m') \in \mathbf{Z} \times \mathbf{M} \mapsto \tilde{a}_{k+1}^{n,\varepsilon_{k+1}}[t', x', m'] \in \mathbf{A} \cup \{\partial\}$ is a Borel measurable map such that

$$\tilde{a}^{n,\varepsilon_{k+1}}[Z_{r_{k+1}^{\varepsilon(k)}}^{z,\phi_k^{\varepsilon(k)}}, M_{r_{k+1}^{\varepsilon(k)}}^{z,m,\phi_k^{\varepsilon(k)}}] = \hat{a}^{n,\varepsilon_{k+1}}[Z_{r_{k+1}^{\varepsilon(k)}}^{z,\phi_k^{\varepsilon(k)}}, M_{r_{k+1}^{\varepsilon(k)}}^{z,m,\phi_k^{\varepsilon(k)}}] \quad \mathbb{P}_m - \text{a.s.}$$

We finally set

$$\phi^\varepsilon := (\tau_i^{\phi_i^{\varepsilon(i)}}, \alpha_i^{\phi_i^{\varepsilon(i)}})_{i \geq 1} \in \Phi_n^{z,m}.$$

Step 2: We now prove that $\bar{v}_n(z,m) \geq v_n(z,m)$. By the above construction and Lemma 4.1,

$$\bar{v}_n(z,m) \geq \bar{J}(z,m; \alpha_1^{\phi_1^{\varepsilon(1)}}) \geq \bar{v}_n(z,m) - \varepsilon_1.$$

Since $v_n(t_k, \cdot) = \bar{v}_n(t_k, \cdot)$ for $k > j$ by our induction hypothesis, we obtain

$$\bar{v}_n(z,m) \geq \sup_{a \in \mathbf{A} \cup \{\partial\}} \mathbb{E}_m[v_n(Z_{r_2^{\varepsilon(1)}}^{z,a}, M_{r_2^{\varepsilon(1)}}^{z,m,a})] - \varepsilon_1 \geq v_n(z,m) - \varepsilon_1,$$

in which the last inequality follows from a simple conditioning argument as in the proof of Proposition 4.2. By arbitrariness of $\varepsilon_1 > 0$, this implies that $\bar{v}_n(z,m) \geq v_n(z,m)$.

Step 3: It remains to prove that $\bar{v}_n(z,m) \leq v_n(z,m)$. Define

$$Y_i^{\varepsilon(i-1)} := (Z_{r_i^{\varepsilon(i-1)}}^{z,\phi^\varepsilon}, M_{r_i^{\varepsilon(i-1)}}^{z,m,\phi^\varepsilon}), \ i \geq 1,$$

with $Y_0^{\varepsilon(-1)} := (z,m)$, and observe that $Y_i^{\varepsilon(i-1)}$ and $\mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z,m,\phi^\varepsilon}$ only depend on $\varepsilon(i-1)$. Then, for each $i \geq 0$,

$$\bar{v}_n(Y_i^{\varepsilon(i-1)}) = \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\bar{v}_n(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}) | \mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z,m,\phi^\varepsilon}]]$$

$$= \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} \leq T\}} \bar{v}_n(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}) | \mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z,m,\phi^\varepsilon}]$$

$$+ \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} > T\}} g(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, \upsilon, \epsilon_0) | \mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z,m,\phi^\varepsilon}] \quad \mathbb{P}_m - \text{a.s.}$$

on $\{r_i^{\varepsilon(i-1)} \leq T\}$. Since g is bounded, so is $\bar{v}_n$. The above combined with the dominated convergence theorem then implies

$$\bar{v}_n(z,m) = \lim_{\varepsilon_1 \downarrow 0} \cdots \lim_{\varepsilon_n \downarrow 0} \mathbb{E}_m[\sum_{i=0}^n \mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} > T \geq r_i^{\varepsilon(i-1)}\}} g(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, \upsilon, \epsilon_0)]$$

$$= \lim_{\varepsilon_1 \downarrow 0} \cdots \lim_{\varepsilon_n \downarrow 0} J(z,m; \phi^\varepsilon) \leq v_n(z,m),$$

16

which concludes the proof that $\bar{\mathrm{v}}_n = \mathrm{v}_n$.

Step 4. The second assertion of the proposition is obtained by observing that, given a random variable (ζ, μ) with values in $\mathbf{Z} \times \mathbf{M}$, one can choose $\tilde{a}^{n,\varepsilon_1}$ Borel measurable such that $\tilde{a}^{n,\varepsilon_1}[\zeta, \mu] = \hat{a}^{n,\varepsilon_1}[\zeta, \mu]$ $\mathbb{P}_m$ − a.s. $\qquad\square$

We are now in position to conclude that v_n satisfies a dynamic programming principle.

Corollary 4.1. *Fix $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$. Let $(\theta^\phi, \phi \in \Phi_n^{z,m})$ be such that each θ^ϕ is a $\mathbb{F}^{z,m,\phi}$-stopping time with values in $[t, 2T] \cap (\pi_n \cup [T, \infty))$ such that $\theta^\phi \in \mathcal{N}^\phi \cap [t, \mathrm{T}[\phi]]$ $\mathbb{P}_m$ − a.s. for $\phi \in \Phi_n^{z,m}$. Then,*

$$\mathrm{v}_n(z, m) = \sup_{\phi \in \Phi_n^{z,m}} \mathbb{E}_m[\mathrm{v}_n(Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi})].$$

Proof. The inequality $\leq$ can be obtained trivially by a conditioning argument. Fix $\phi \in \Phi_n^{z,m}$. By Proposition 4.3, we can find a Borel measurable map $(z', m') \mapsto \phi^\iota[z', m']$ such that

$$J(Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi}; \phi^\iota[Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi}]) \geq \mathrm{v}_n(Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi}) - \iota.$$

Let us now simply write ϕ^ι for $\phi^\iota[Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi}]$. Without loss of generality, one can assume that $\tau_1^\phi \geq t$ and that $\tau_1^{\phi^\iota} \geq \theta^\phi$. Let $I := \mathrm{card}\{i \geq 1 : \tau_i^\phi < \theta^\phi\}$. Then, $J(z, m; \tilde{\phi}^\iota) \geq \mathbb{E}_m[\mathrm{v}_n(Z_{\theta^\phi}^{z,\phi}, M_{\theta^\phi}^{z,m,\phi})] - \iota$ in which $(\tau_i^{\tilde{\phi}^\iota}, \alpha_i^{\tilde{\phi}^\iota}) = \mathbf{1}_{i \leq I}(\tau_i^\phi, \alpha_i^\phi) + \mathbf{1}_{i > I}(\tau_{i-I}^{\phi^\iota}, \alpha_{i-I}^{\phi^\iota})$, $i \geq 1$. Sending $\iota \to 0$ leads to the required result. $\qquad\square$

4.3 Super-solution property as the time step vanishes

We now consider the limit $n \to \infty$. Let us set, for $(z, m) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$,

$$\mathrm{v}_\circ(z, m) := \liminf_{(t', x', m', n) \to (z, m, \infty)} \mathrm{v}_n(t', x', m').$$

Remark 4.3. *Note that (4.7) and (3.6) implies that $\mathrm{v}_\circ = \mathcal{K}_T g$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$.*

Proposition 4.4. *The function $\mathrm{v}_\circ$ is a viscosity super-solution of (3.4)-(3.5).*

Proof. Let $n_k \to \infty$ and $(z_k, m_k) \to (z_o, m_o)$ be such that $\mathrm{v}_{n_k}(z_k, m_k) \to \mathrm{v}_\circ(z_o, m_o)$.

Step 1. We first show that $\mathrm{v}_\circ(z_o, m_o) \geq \mathcal{K}\mathrm{v}_\circ(z_o, m_o)$. By Corollary 4.1 applied to v_{n_k} with a control ϕ^k defined by $(\tau_i^k, \alpha_i^k) = (t_k, a_k)\mathbf{1}_{\{i=1\}} + \sum_{j>1}(T + j, a_\star)\mathbf{1}_{\{i=j\}}$, $i \geq 1$, with $a_k \in \mathbf{A}$, we obtain

$$\mathrm{v}_{n_k}(z_k, m_k) \geq \sup_{a_k \in \mathbf{A}} \int \mathbb{E}[\mathrm{v}_{n_k}(Z_{s_+^{n_k}[z']}^{z',\circ}, m')] d\mathrm{k}(z', m' | z_k, m_k, a_k) = \mathcal{K}\mathbb{E}[\mathrm{v}_{n_k}(Z_{s_+^{n_k}[\cdot]}^{\cdot,\circ}, \cdot)](z_k, m_k),$$

in which $s_+^{n_k}[t, x] := \min \pi_{n_k} \cap [t, \infty)$. Let $\varphi_{k_\circ}$ be the lower-semicontinuous enveloppe of $\inf\{\mathbb{E}[\mathrm{v}_{n_k}(Z_{s_+^{n_k}[\cdot]}^{\cdot,\circ}, \cdot)], k \geq k_\circ\}$. Then, for $k \geq k_\circ$, $\mathrm{v}_{n_k}(z_k, m_k) \geq \int \varphi_{k_\circ}(z', m') d\mathrm{k}(z', m' | z_k, m_k, a_k)$, and, by (3.6), passing to the limit $k \to \infty$ leads to $\mathrm{v}_\circ(z_o, m_o) \geq \int \varphi_{k_\circ}(z', m') d\mathrm{k}(z', m' | z_o, m_o, a_o)$.

17

We shall prove in step 3 that $\lim_{k_\circ \to \infty} \varphi_{k_\circ} \geq \mathrm{v}_\circ$. These maps are bounded, since g is. Dominated convergence then implies that $\mathrm{v}_\circ(z_\circ, m_\circ) \geq \int \mathrm{v}_\circ(z', m') d\mathrm{k}(z', m'|z_\circ, m_\circ, a_\circ)$.

Step 2. Let φ be a (bounded) $C^{1,2,0}([0,T] \times \mathbb{R}^d \times \mathbf{M})$ function and $(z_\circ, m_\circ) \in [0,T) \times \mathbb{R}^d \times \mathbf{M}$ be a minimal point of $\mathrm{v}_\circ - \varphi$ on $\mathbf{Z} \times \mathbf{M}$. Without loss of generality, one can assume that $(\mathrm{v}_\circ - \varphi)(z_\circ, m_\circ) = 0$. Let B and $(z_k, m_k, n_k)_{n \geq 1}$ be as in Lemma 4.2 below. We write $z_k = (t_k, x_k), z_\circ = (t_\circ, x_\circ) \in [0,T] \times \mathbb{R}^d$. On the other hand, by considering the control ϕ^k defined by $(\tau_i^k, \alpha_i^k) = (T + i, a_\star)$, $i \geq 1$, we obtain from Corollary 4.1 that

$$\mathrm{v}_{n_k}(z_k, m_k) \geq \mathbb{E}_m[\mathrm{v}_{n_k}(Z_{t_k + h_k}^{z_k, \circ}, m)]$$

with $h_k \in T2^{-n_k}(\mathbb{N} \cup \{0\})$ such that $t_k + h_k < T$ if $t_\circ \neq T$ and $t_k + h_k = T$ otherwise.
Let $C > 0$ be a common bound for $(\mathrm{v}_n)_{n \geq 1}$ and φ. Then we can choose $(h_k)_{k \geq 1}$ such that

$$\delta_k := (\varphi(z_k, m_k) - \mathrm{v}_{n_k}(z_k, m_k) - 2C \, \mathbb{P}[Z_{t_k + h_k}^{z_k, \circ} \notin B])/h_k \to 0.$$

This follows from standard estimates on the solution of sde's with Lipschitz coefficients. Then, if $t_\circ < T$,

$$0 \geq h_k^{-1} \mathbb{E}_m[\varphi(Z_{t_k + h_k}^{z_k, \circ}, m_k) - \varphi_{n_k}(z_k, m_k)] + \delta_k = \mathbb{E}_m[h_k^{-1} \int_{t_k}^{t_k + h_k} \mathcal{L}\varphi(Z_s^{z_k, \circ}, m_k) ds] + \delta_k,$$

sending $k \to \infty$ leads to $\mathcal{L}\varphi(z_\circ, m_\circ) \leq 0$. If $t_\circ = T$, $\mathrm{v}_{n_k}(z_k, m_k) \geq \mathbb{E}_m[g(Z_T^{z_k, \circ}, m_k, \upsilon, \epsilon_0)] = \mathbb{E}_m[\mathcal{K}_T g(Z_T^{z_k, \circ}, m_k)]$ and passing to the limit leads to $\varphi(z_\circ, m_\circ) \geq \mathcal{K}_T g(z_\circ, m_\circ)$, recall (3.6). Finally, $\varphi(z_\circ, m_\circ) \geq \mathcal{K}\varphi(z_\circ, m_\circ)$ by Step 1. $\qquad\square$

Step 3: It remains to prove the claim used in Step 1. Let us set

$$\bar{\varphi}_{k_\circ}(z', m') := \inf_{k \geq k_\circ} \left\{ \mathbb{E}\left[\mathrm{v}_{n_k}\left(Z_{s_+^{n_k}[z']}^{z', \circ}[z'], m'\right) \right] \right\},$$

so that $\varphi_{k_\circ}$ is the lower-semicontinuous envelope of $\bar{\varphi}_{k_\circ}$. Note that $Z_{s_+^{n_k}[z']}^{z', \circ}$ converges a.s. to z as $(z', k) \to (z, \infty)$. Hence, for all $\varepsilon > 0$, there exist open neighborhoods $B_\varepsilon(z, m)$ and $B_{\frac{\varepsilon}{2}}(z, m)$ of (z, m), as well as $k_\varepsilon \in \mathbb{N}$ such that $\mathbb{P}[(Z_{s_+^{n_k}[z']}^{z', \circ}, m') \notin B_\varepsilon(z, m)] \leq \varepsilon$ for $k \geq k_\varepsilon$ and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$. One can also choose k_ε and $B_{\frac{\varepsilon}{2}}(z, m)$ such that $\inf_{k \geq k_\varepsilon} \mathrm{v}_{n_k}(z', m') \geq \mathrm{v}_\circ(z, m') - \varepsilon$ for all $k \geq k_\varepsilon$ and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$. Let $C > 0$ be a bound for $(|\mathrm{v}_n|)_{n \geq 1}$ and $|\mathrm{v}_\circ|$, recall that g is bounded. Then, for $k_\circ$ large enough and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$,

$$\bar{\varphi}_{k_\circ}(z', m') \geq \mathrm{v}_\circ(z, m) - \varepsilon - 2C \sup_{k \geq k_\circ} \mathbb{P}[(Z_{s_+^{n_k}[z']}^{z', \circ}, m') \notin B_\varepsilon(z, m)] \geq \mathrm{v}_\circ(z, m) - \varepsilon(1 + 2C).$$

Hence, since $\mathrm{v}_\circ$ is lower-semicontinuous,

$$\lim_{k_\circ \to \infty} \varphi_{k_\circ}(z, m) = \lim_{k_\circ \to \infty} \liminf_{(z', m') \to (z, m)} \bar{\varphi}_{k_\circ}(z', m') \geq \mathrm{v}_\circ(z, m).$$

$\qquad\square$

We conclude this section with the technical lemma that was used in the above proof.

Lemma 4.2. *Let $(u_n)_{n \geq 1}$ be a sequence of lower semi-continuous maps on $\mathbf{Z} \times \mathbf{M}$ and define $u_\circ := \liminf_{(z',m',n) \to (\cdot, \infty)} u_n(z', m')$ on $\mathbf{Z} \times \mathbf{M}$. Assume that $u_\circ$ is locally bounded. Let φ be a continuous map and assume that $(z_\circ, m_\circ)$ is a strict minimal point of $u_\circ - \varphi$ on $\mathbf{Z} \times \mathbf{M}$. Then, one can find a bounded open set B of $[0, T] \times \mathbb{R}^d$ and a sequence $(z_k, m_k, n_k)_{n \geq 1} \subset B \times \mathbf{M} \times \mathbb{N}$ such that $n_k \to \infty$, (z_k, m_k) is a minimum point of $u_{n_k} - \varphi$ on $B \times \mathbf{M}$ and $(z_k, m_k, u_{n_k}(z_k, m_k)) \to (z_o, m_o, u_\circ(z_o, m_o))$.*

Proof. Since $\mathbf{M}$ is assumed to be locally compact, it suffices to repeat the arguments in the proof of [2, p80, Proof of Lemma 6.1]. $\qquad\square$

4.4 Conclusion of the proof of Theorem 3.1

We already know from Proposition 4.1 and Proposition 4.4 that v^* and $v_\circ$ are respectively a bounded viscosity sub- and super-solution of (3.4)-(3.5). By (2.13), Remark 4.3 and (3.6), we also have $v_\circ \geq v^*$ on $(0, T) \times \mathbb{R}^d \times \mathbf{M}$. In view of Assumption 3.1 and Remark 4.2, v is continuous on $\mathbf{Z} \times \mathbf{M}$ and is the unique bounded viscosity solution of (3.4)-(3.5). $\qquad\square$

Remark 4.4. *The above arguments actually show that $(v_n)_{n \geq 1}$ converges to v.*

5 A sufficient condition for the comparison

In this section, we provide a sufficient condition for Assumption 3.1 to hold. We refer to [1] for examples of application.

Proposition 5.1. *Assumption 3.1 holds whenever there exists a continuous function Ψ on $[0, 2T] \times \mathbb{R}^d \times \mathbf{M}$ satisfying*

(i) $\Psi(., m) \in C^{1,2}([0, T) \times \mathbb{R}^d)$, *for all $m \in \mathbf{M}$.*

(ii) $\varrho \Psi \geq \mathcal{L}\Psi$ *on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ for some constant $\varrho > 0$,*

(iii) $\Psi - \mathcal{K}\Psi \geq \delta$ *on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ for some $\delta > 0$,*

(iv) $\Psi \geq \mathcal{K}_T[\tilde{g}]$ *on $[T, \infty) \times \mathbb{R}^d \times \mathbf{M}$ with $\tilde{g}(t, .) := e^{\varrho t} g(t, .)$ and ϱ is defined in (ii),*

(v) Ψ^- *is bounded.*

The idea of the proof is the same as in [5, Proposition 4.12]. Note that their condition H2 (v) is not required here because we only consider bounded sub and super-solutions and we take a different approach. To avoid it, we slighlty reinforce the hypothesis H2 (iii) and asked for Ψ^- to be bounded.

Proof. Step 1. As usual, we shall argue by contradiction. We assume that there exists $(z_0, m_0) \in \mathbf{Z} \times \mathbf{M}$ such that $(U - V)(z_0, m_0) > 0$, in which U and V are as in Assumption 3.1. Recall the definition of Ψ, ϱ and $\tilde{g}$ in Proposition 5.1. We set $\tilde{u}(t, x, m) := e^{\varrho t} U(t, x, m)$

and $\tilde{v}(t,x,m) := e^{\varrho t} V(t,x,m)$ for all $(t,x,m) \in \mathbf{Z} \times \mathbf{M}$. Then, there exists $\lambda \in (0,1)$ such that

$$(\tilde{u} - \tilde{v}^\lambda)(z_0, m_0) > 0, \tag{5.1}$$

in which $\tilde{v}^\lambda := (1-\lambda)\tilde{v} + \lambda\Psi$. Note that $\tilde{u}$ and $\tilde{v}$ are sub and supersolution on $\mathbf{Z} \times \mathbf{M}$ of

$$\min\{\varrho\varphi - \mathcal{L}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \tag{5.2}$$

associated to the boundary condition

$$\min\{\varphi - \mathcal{K}_T\tilde{g}, \varphi - \mathcal{K}\varphi\} = 0. \tag{5.3}$$

Step 2. Let $\mathrm{d}_\mathbf{M}$ be a metric on $\mathbf{M}$ compatible with the topology of weak convergence. For $(t,x,y,m) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$, we set

$$\Gamma_\varepsilon(t,x,y,m) := \tilde{u}(t,x,m) - \tilde{v}^\lambda(t,y,m) - \varepsilon\left(\|x\|^2 + \|y\|^2 + \mathrm{d}_\mathbf{M}(m)\right) \tag{5.4}$$

with $\varepsilon > 0$ small enough such that $\Gamma_\varepsilon(t_0, x_0, x_0, m_0) > 0$. Note that the supremum of $(t,x,m) \mapsto \Gamma_\varepsilon(t,x,x,m)$ over $\mathbf{Z} \times \mathbf{X} \times \mathbf{M}$ is achieved by some $(t_\varepsilon, x_\varepsilon, x_\varepsilon, m_\varepsilon)$. This follows from the the upper semi-continuity of Γ_ε and the fact that $\tilde{u}, -\tilde{v}, -\Psi$ are bounded from above. Recall that $\mathbf{M}$ is locally compact. For $(t,x,y,m) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$, we set

$$\Theta_\varepsilon^n(t,x,y,m) := \Gamma_\varepsilon(t,x,y,m) - n\|x-y\|^2.$$

Again, there is $(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$ such that $\sup_{\mathbf{Z} \times \mathbf{X} \times \mathbf{M}} \Theta_\varepsilon^n = \Theta_\varepsilon^n(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon)$. It is standard to show that, after possibly considering a subsequence,

$$\begin{aligned}
&(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \to (\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}, \quad n\|x_n^\varepsilon - y_n^\varepsilon\|^2 \to 0, \\
&\text{and } \Theta_\varepsilon^n(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \to \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) = \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, m_\varepsilon),
\end{aligned} \tag{5.5}$$

see e.g. [8, Lemma 3.1].

Step 3. We first assume that, up to a subsequence, $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) \le 0$, for $n \ge 1$. It follows from the supersolution property of $\tilde{v}$ and Condition (iii) of Proposition 5.1 that

$$\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \le \mathcal{K}\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) - \lambda\delta.$$

Passing to the lim sup and using (5.5) and (3.6), we obtain $(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) + \lambda\delta \le \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$. In particular, by (5.4), $\Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) + \lambda\delta \le \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$. Now let us observe that

$$\sup_{\mathbf{Z} \times \mathbf{M}}(\tilde{u} - \tilde{v}^\lambda) = \lim_{\varepsilon \to 0} \sup_{(t,x,m) \in \mathbf{Z} \times \mathbf{M}} \Gamma_\varepsilon(t,x,x,m) = \lim_{\varepsilon \to 0} \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, m_\varepsilon) = \lim_{\varepsilon \to 0} \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon), \tag{5.6}$$

in which the last identity follows from (5.5). Combined with the above inequality, this shows that $\sup_{\mathbf{Z} \times \mathbf{M}}(\tilde{u} - \tilde{v}^\lambda) + \lambda\delta \le \lim_{\varepsilon \to 0} \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$, which leads to a contradiction for ε small enough.

Step 4. We now show that there is a subsequence such that $t_n^\varepsilon < T$ for all $n \geq 1$. If not, one can assume that $t_n^\varepsilon = T$ and it follows from the boundary condition (5.3) and step 3 that $\tilde{u}(T, x_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}_T \tilde{g}(T, x_n^\varepsilon, m_n^\varepsilon)$ for all $n \geq 1$. Since, by (5.3) and Condition (iv) of Proposition 5.1, $\tilde{v}^\lambda(T, y_n^\varepsilon, m_n^\varepsilon) \geq \mathcal{K}_T \tilde{g}(T, y_n^\varepsilon, m_n^\varepsilon)$, it follows that $\tilde{u}(T, x_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(T, y_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}_T \tilde{g}(T, x_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}_T \tilde{g}(T, y_n^\varepsilon, m_n^\varepsilon)$. Hence, $\Gamma_\varepsilon(T, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}_T \tilde{g}(T, x_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}_T \tilde{g}(T, y_n^\varepsilon, m_n^\varepsilon)$. Combining (3.6), (5.5) and (5.6) as above, we obtain $\sup(\tilde{u} - \tilde{v}^\lambda) \leq 0$, a contradiction.

Step 5. In view of step 3 and 4, we may assume that $t_n^\varepsilon < T$ and $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) > 0$ for all $n \geq 1$. Using Ishii's Lemma and following standard arguments, see Theorem 8.3 and the discussion after Theorem 3.2 in [8], we deduce from the sub- and supersolution viscosity property of $\tilde{u}$ and $\tilde{v}^\lambda$, and the Lipschitz continuity assumptions on μ and σ, that

$$\varrho\left(\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon)\right) \leq C\left(n\|x_n^\varepsilon - y_n^\varepsilon\|^2 + \varepsilon\left(1 + \|x_n^\varepsilon\|^2 + \|y_n^\varepsilon\|^2\right)\right),$$

for some $C > 0$, independent on n and ε. In view of (5.4) and (5.5), we get

$$\varrho\Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) \leq 2C\varepsilon\left(1 + \|\hat{x}_\varepsilon\|^2\right). \tag{5.7}$$

We shall prove in next step that the right-hand side of (5.7) goes to 0 as $\varepsilon \to 0$, up to a subsequence. Combined with (5.6), this leads to a contradiction to (5.1).

Step 6. We conclude the proof by proving the claim used above. First note that we can always construct a sequence $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon)_{\varepsilon > 0}$ such that

$$\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon) \to \sup_{\mathbf{Z} \times \mathbf{M}}(\tilde{u} - \tilde{v}^\lambda) \quad \text{and} \quad \varepsilon(\|\tilde{x}_\varepsilon\|^2 + \mathrm{d}_{\mathbf{M}}(\tilde{m}_\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0.$$

By (5.5), $\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon) \leq \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$. Hence, $\sup_{\mathbf{Z} \times \mathbf{M}}(\tilde{u} - \tilde{v}^\lambda) \leq \sup_{\mathbf{Z} \times \mathbf{M}}(\tilde{u} - \tilde{v}^\lambda) - 2\liminf_{\varepsilon \to 0} \varepsilon\|\hat{x}_\varepsilon\|^2$. $\qquad \square$

References

[1] N. Baradel, B. Bouchard and N. M. Dang. Optimal trading with online parameters revisions. *Market Microstructure and Liquidity*, 2(03n04), 2016.

[2] G. Barles. An introduction to the theory of viscosity solutions for first-order hamilton–jacobi equations and applications. In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, pages 49–109. Springer, 2013.

[3] J. Bernardo and A. F. M. Smith. *Bayesian theory*, IOP Publishing, 2001.

[4] D.P. Bertsekas and S.E. Shreve. *Stochastic optimal control: The discrete time case.* Athena Scientific, 1996.

[5] B. Bouchard. A stochastic target formulation for optimal switching problems in finite horizon. *Stochastics: An International Journal of Probability and Stochastics Processes*, 81(2):171–197, 2009.

[6] B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM Journal on Control and Optimization*, 49(3):948–962, 2011.

[7] Pierre Cardaliaguet. Notes on Mean Field Games (from P.-L. Lions' lectures at Collège de France). `https://www.ceremade.dauphine.fr/~cardalia/MFG20130420.pdf`, 2012.

[8] M.G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1):1–67, 1992.

[9] C. Dellacherie and P. A. Meyer. *Probabilities and Potential B*. North Holland, Amsterdam, 1982.

[10] D. Easley and N. M. Kiefer. Controlling a stochastic process with unknown parameters. *Econometrica: Journal of the Econometric Society*, pages 1045–1064, 1988.

[11] W. H. Fleming and P. E. Souganidis. On the existence of value-functions of 2-player, zero-sum stochastic differential-games. *Indiana University Mathematics Journal*, 38(2):293–314, 1989.

[12] O. Hernández-Lerma. *Adaptive Markov control processes*. Springer Science & Business Media, 79, 2012.

[13] C.-A. Lehalle and S. Laruelle. *Market Microstructure in Practice*. World Scientific, 2013.