No-arbitrage Under Model Ambiguity and Fundamental Theorems of Asset Pricing

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Preliminaries

Classical Framework

- Only one reference measure $\mathcal{P} = \{P_o\}$ which fixes the null sets.
- No-Arbitrage NA(P_o) : $(H \bullet S)_T \geq 0$ P_o-a.s. $\Rightarrow (H \bullet S)_T = 0$ P_o-a.s.
- NA(P_o)$\Leftrightarrow \mathcal{Q}(P_o) := \{Q \sim P_o : S \text{ is a } Q\text{-mart.}\} \neq \emptyset$.
- Super-hedging price of f is $\sup\{\mathbb{E}_Q[f],\ Q \in \mathcal{Q}(P_o)\}$.

The non-dominated case

□ $\{P_o\}$ is replaced by a family $\mathcal{P}$ made of (possibly) singular measures P which fix the polar sets : $A \subset A'$ with $P[A'] = 0 \ \forall \ P \in \mathcal{P}$, i.e. $A = \emptyset$ $\mathcal{P}$-q.s.

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□ Questions :

- What is the good notion of arbitrage ? (q.s. or pathwise)
- Which duality do we look for ? (a family of MM with the same polar sets or just one)
- What minimal conditions can we afford ? (try to avoid continuity assumptions)

Discrete time frictionless markets

Joint with M. Nutz

Arbitrage and duality in nondominated discrete-time models

to appear in Annals of Applied Probability.

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□ Different possibilities :

• $(H \bullet S)_T \geq 0$ $\mathcal{P}$-q.s. and $P[(H \bullet S)_T > 0] > 0 \; \forall \; P \in \mathcal{P}$ is impossible.
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Theorem : The following are equivalent :

(i) $\mathrm{NA}(\mathcal{P})$ holds.

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Theorem : Let f be upper semi-analytic. Then,

$$\inf \left\{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|\mathrm{I}|} \text{ s.t. } x + (H \bullet S)_T + h \cdot g \geq f \ \mathcal{P}\text{-q.s.} \right\}$$
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Assumption : Convexity, stability under pasting and measurability of $\mathcal{P}$.

□ One can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $\mathbf{L}^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(\mathbf{L}^1(\mathcal{P}))^*$ (e.g. Nutz 2013).

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Step 2 : Measurable selection + pasting of the one-period results

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□ Again, one can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. But can rely on finite dimensional separation arguments on each period.

Models with proportional transaction costs

Joint with M. Nutz

Consistent Price Systems under Model Uncertainty

arXiv :1408.5510

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No-arbitrage criteria

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Interpretation : Martingale lying in the bid-ask spreads

$$\frac{1}{\pi^{ji}} < \frac{Z^j}{Z^i} < \pi^{ij}$$

As in Jouni and Kallal, or Cvitanic and Karatzas, is a fictitious price process, consistent with the bid-ask spreads, which is a martingale under an equivalent measure.

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We suggest an easier way to go (in a more general framework).

Fundamental theorem of asset pricing under NA_2

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$\forall\ t$, $P \in \mathcal{P}$ and $Y \in L^0_P(\mathcal{F}_t, \mathrm{int} K^*_t)$ $\exists$ a SCPS (Q, Z) s.t.

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Rem : It is the exact counterpart of the frictionless result.

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Rem : It is the exact counterpart of the frictionless result.

Assumptions : Measurablity and stability conditions on $\mathcal{P}$ and

- $K_t(\omega)$ closed, convex cone, contains $\mathbb{R}^d_+$
- $\mathrm{int}K^*_t(\omega) \neq \emptyset$ and $K^*_t(\omega) \cap \partial\mathbb{R}^d_+ = \{0\}$
- $x^j/y^j \leq c(x^i/y^i)$, $1 \leq i, j \leq d$, $x, y \in K^*_t(\omega) \setminus \{0\}$

Extension to continuous time (without friction)

Joint with S. Biagini, C. Kardaras and M. Nutz

Robust Fundamental Theorem for Continuous Processes

arXiv :1410.4962

Main difficulty

Can not rely anymore on one period models...

Robust no-arbitrage of the first kind

Super-hedging with simple strategies :

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$\mathrm{NA}_1(\mathcal{P})$: $\pi^{\mathrm{s}}(f, T) = 0 \iff f = 0$ $\mathcal{P}$-q.s.

Key property : Assume S is continuous $\mathcal{P}$-q.s., then

$$\mathrm{NA}_1(\mathcal{P}) \iff \mathrm{NA}_1(\{P\}) \ \forall \ P \in \mathcal{P}.$$

Fundamental theorem

Probability space with killing time : Ω is the set of path ω on a (Polish) space $E \cup \{\Delta\}$ that are are càdlàg on $[0, \zeta(\omega))$ and constant after

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- $\tau_n < \zeta$ $\forall$ n and $\lim_n \tau_n = \zeta$ Q-a.s.,
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Remark : $\mathcal{Q}^P \neq \emptyset$ for all $P \in \mathcal{P}$ seems stronger than $\mathcal{Q} \sim \mathcal{P}$, but $\sim_\zeta$ is weaker than $\sim$.

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Moreover, $\exists$ a minimal super-hedging strategy (continuous trading).

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Rem : Ongoing by Cheridito, Kupper, and Tangpi, using a different approach (more general but stronger no-arbitrage condition).

Thank you for your attention

Related talks :

- J. Obloj, Friday 11am,
- Robust Hedging and Pricing under Model Uncertainty, Friday 3pm and Saturday 8.30am