Discrete time approximation for continuously and discretely reflected BSDEs

Bruno Bouchard
Université Paris VI, PMA,
and CREST
Paris, France
bouchard@ccr.jussieu.fr

Jean-François Chassagneux
Université Paris VII, PMA,
and CREST
Paris, France
chassagneux@ensae.fr

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Abstract

We study the discrete time approximation of the solution (Y, Z, K) of a reflected BSDE. As in Ma and Zhang (2005), we consider a Markovian setting with a reflecting barrier of the form $h(X)$ where X solves a forward SDE. We first focus on the discretely reflected case. Based on a representation for the Z component in terms of the next reflection time, we retrieve the convergence result of Ma and Zhang (2005) without their uniform ellipticity condition on X. These results are then extended to the case where the reflection operates continuously. We also improve the bound on the convergence rate when $h \in C_b^2$ with Lipschitz second derivative.

Key words: Reflected BSDEs, discrete-time approximation schemes, regularity.

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1 Introduction

In this paper, we consider the solution (Y, Z, K) of a decoupled Forward-Backward SDE with reflection

$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s \ , \\ Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T (Z_s)'\mathrm{d}W_s + K_T - K_t \ , \\ Y_t &\geq h(X_t) \ , \ t \leq T \ \text{ and } \ \int_0^T (\, Y_t - h(X_t)\,)\mathrm{d}K_t \, = \, 0 \ , \end{aligned}$$

where b, σ, f, g and h are Lipschitz-continuous functions. Such equations appear naturally in finance in the pricing and hedging of American contingent claims, see [7]. They are more generally related to semilinear parabolic PDEs with free boundary, see [9].
We study a discrete-time approximation scheme of the form

$$\begin{aligned} \bar{Y}^\pi_T &= g(X^\pi_T) \ , \\ \bar{Z}^\pi_{t_i} &= (t_{i+1} - t_i)^{-1}\, \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \tilde{Y}^\pi_{t_i} &= \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\ \bar{Y}^\pi_{t_i} &= \tilde{Y}^\pi_{t_i} \vee h(X^\pi_{t_i}) \ , \ i \leq N-1 \ , \end{aligned}$$

where $\pi = \{t_0 = 0 < t_1 < \ldots < t_N = T\}$ is a partition of the time interval $[0, T]$ with modulus $|\pi|$, and X^π is the Euler scheme of X.

In the non-reflected case, such approximations have been studied by [3] and [16], see also [2] and [6] for BSDEs with jumps. In all these analysis, it appears that the approximation error

$$\max_{i \leq N-1} \mathbb{E}\left[\sup_{t \in (t_i, t_{i+1}]} |\bar{Y}^\pi_{t_{i+1}} - Y_t|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |\bar{Z}^\pi_{t_i} - Z_t|^2 \mathrm{d}t\right]^{\frac{1}{2}}$$

is intimately related to a regularity property on Z. More, precisely, the above error is controlled by

$$|\pi|^{\frac{1}{2}} + \mathbb{E}\left[\int_0^T |Z_t - \bar{Z}_t|^2 \mathrm{d}t\right]^{\frac{1}{2}}$$

where $\bar{Z}$ is defined on $[t_i, t_{i+1})$ by $\bar{Z}_t = (t_{i+1} - t_i)^{-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right]$. It is shown in [15] that, in the non-reflected case, the last term is bounded by $C|\pi|^{\frac{1}{2}}$. This provides the expected rate of convergence for the discrete-time approximation scheme. This result is remarkable since it does not require any ellipticity condition on σ and the coefficients are only assumed to be Lipschitz.

The reflected case is more difficult to handle except when f is independent of Z as in [1] and [3]. In this case, there is no need to control Z and the error on Y is still bounded by $C|\pi|^{\frac{1}{2}}$. It can even be improved when h is semi-convex, see [1].
The general case was studied in [11]. When b, σ are C_b^1 and h is C_b^2, they prove that $\mathbb{E}\left[\int_0^T |Z_t - \bar{Z}_t|^2 \mathrm{d}t\right]^{\frac{1}{2}}$ is bounded by $C|\pi|^{\frac{1}{4}}$. This can be viewed as a weak regularity result on the "gradient" of the solution of the related obstacle problem and is of own interest, see [9]. This also allows to show that the discrete-time scheme converges at least at a rate $|\pi|^{\frac{1}{4}}$.
Their proof relies on a particular representation of Z obtained by means of an integration by parts argument, in the Malliavin sense. It generalizes a result of [5] obtained in the non-reflected case with $f = 0$. The main drawback of this approach is that it requires some uniform ellipticity condition on σ, an assumption which was not used in the non-reflected case.

The aim of this paper is to improve this result by removing the ellipticity condition on σ. Our approach is slightly different from [11]. We first study the solution $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ of a discretely reflected BSDE. We provide a new representation result for $Z^{\mathrm{d}\Re}$ in terms of the next reflection time. This allows us to prove that $\mathbb{E}\left[\int_0^T |Z_t^{\mathrm{d}\Re} - \bar{Z}_t^{\mathrm{d}\Re}|^2 \mathrm{d}t\right]^{\frac{1}{2}}$ is controlled by $|\pi|^{\frac{1}{4}}$ without any ellipticity condition on σ. By using a standard approximation argument, we then extend this property to Z. As a consequence, we show that the discrete-time scheme approaches both continuously- and discretely-reflected BSDEs at least at a rate $|\pi|^{\frac{1}{4}}$. We only assume that all the functions are Lipschitz-continuous and that h is C_b^1 with Lipschitz-continuous derivatives. When $\sigma \in C_b^1$ with Lipschitz-continuous first derivative and h is C_b^2 with Lipschitz-continuous second derivatives, this result is improved and the error on Y is shown to be bounded by $C|\pi|^{\frac{1}{2}}$ as in the non-reflected case. The error on Z can also be improved when X^π is replaced by an order one scheme.

To conclude this introduction, we would like to observe that the above discrete time scheme can not be directly implemented in practice and requires the estimation of conditional expectations. The global numerical error can therefore be decomposed as the sum of two terms: the first one, which we study here, is the discrete-time approximation error; the second one is related to the numerical approximation of the involved conditional expectations. Different techniques for computing these conditional expectations are discussed in [1], [3], [4] and [6], see also the references therein, and can be easily adapted to our context without any further analysis. Since the global error is the sum of these two terms, the impact of our results on the precision of the numerical approximation is clear. It would be too long to describe here these different methods and we refer to the above papers for a complete presentation.

The rest of the paper is organized as follows. In Section 2 and Section 3, we study the approximation of the discretely reflected BSDE. The representation and the regularity property of $Z^{\mathrm{d}\Re}$ are proved in Section 5. The continuously reflected case is studied in Section 4.

2 The forward process

Let $T > 0$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis supporting a d-dimensional Brownian motion W. We assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \le T}$ generated by W satisfies the usual assumptions and that $\mathcal{F}_T = \mathcal{F}$.
Let X be the solution on $[0, T]$ of the stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$$

where $X_0 \in \mathbb{R}^d$, and, $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto \mathbb{M}^d$ are assumed to be $L-$Lipschitz, i.e.

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le L|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^d \; . \tag{2.1}$$

Here $\mathbb{M}^d$ is the space of d-dimensional matrices, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ or $\mathbb{M}^d$ and all elements of $\mathbb{R}^d$ are viewed as column vectors.
By convention, we assume that $|X_0| + T + |b(0)| + |\sigma(0)| \le L$. In the following, we shall denote by C_L a generic positive constant which depends only on L (but may take different values). We write C_L^p if it depends on an extra parameter $p > 0$.

For later use, we recall the well-known consequence of (2.1):

$$\| \sup_{t \le T} |X_t| \, \|_{L^p} \quad \le \quad C_L^p \; , \tag{2.2}$$

where, for a random variable ξ, we write $\|\xi\|_{L^p} := \mathbb{E}\left[|\xi|^p\right]^{\frac{1}{p}}$.

Remark 2.1. Importantly, we shall not make any ellipticity assumption on σ. We can therefore consider cases where some lines or columns of σ are equal to zero. This allows to embed situations where X and the effective driving Brownian motion have different dimensions and/or the coefficients of the SDE are time dependent. In the later case, one component of X corresponds to the time variable.

The discrete-time approximation of X has been widely studied in the literature, see e.g. [10]. When $(X_{t_i})_{i \le N}$ cannot be perfectly simulated, we use the standard Euler scheme X^π defined for a partition $\pi := \{0 = t_0 < t_1 < \ldots < t_N = T\}$ of $[0, T]$, $N \ge 1$, by

$$\begin{cases} X_0^\pi &= X_0 \\ X_{t_{i+1}}^\pi &= X_{t_i}^\pi + b(X_{t_i}^\pi)(t_{i+1} - t_i) + \sigma(X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i}) \;\;, \; i \le N-1 \; . \end{cases}$$

In the sequel, we shall denote by $|\pi| := \max_{i\le N-1}(t_{i+1}-t_i)$ the modulus of π and assume that

$$N\,|\pi| \quad \le \quad L$$

which holds with $L \ge 1$ when the grid π is regular, i.e. $(t_{i+1}-t_i) = |\pi|$ for all $i \le N-1$.

As usual, we define a continuous-time version of X^π by setting

$$X^\pi_t \;=\; X^\pi_{t_i} + b(X^\pi_{t_i})(t-t_i) + \sigma(X^\pi_{t_i})(W_t - W_{t_i}) \;\;,\; t\in[t_i,t_{i+1})\;,\; i\le N-1\;. \tag{2.3}$$

Remark 2.2. It is well known that under (2.1)

$$\|\sup_{t\le T}|X_t - X^\pi_t|\,\|_{L^p} + \max_{i<N}\,\|\sup_{t\in[t_i,t_{i+1}]}|X_t - X^\pi_{t_i}|\,\|_{L^p} \;\le\; C^p_L\,|\pi|^{\frac{1}{2}}\;,\; p\ge 1\;. \tag{2.4}$$

Using standards arguments, one can also obtain a conditional version of this result:

$$\mathbb{E}_{t_i}\left[|X_{t_{i+1}} - X^\pi_{t_{i+1}}|^2\right] \le e^{C_L|\pi|}|X_{t_i} - X^\pi_{t_i}|^2 + C_L|\pi|^2\mathbb{E}_{t_i}\left[(X^*_T)^2\right] \quad i\le N-1\;, \tag{2.5}$$

where $\mathbb{E}_{t_i}[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot \mid \mathcal{F}_{t_i}]$, $i\le N$, and $X^*_T := \max_{t\le T}|X_t|$.

3 Approximation scheme for discretely reflected BSDEs

In this section, we concentrate on the approximation of "discretely reflected BSDEs", i.e. BSDEs for which the reflection operates only on a finite set of times. The reason for looking at such equations is twofold. First, they provide a good approximation for (continuously) reflected BSDEs, see below. Second, they are related to optimal stopping problems where the stopping times can only take a finite number of different values. For instance, they are related to Bermudan options in finance, see e.g. [14] and the references therein. They are therefore interesting in their own.

3.1 Definition

In this section, we define a discretely reflected BSDE. The reflection operates only at the times

$$0 < r_1 < \cdots < r_{\kappa-1} < T$$

for some $\kappa \ge 1$. We set $\Re = \{r_j,\; 0\le j\le\kappa\}$ where by convention $r_0 := 0$ and $r_\kappa := T$. The solution of the discretely reflected BSDE is a pair $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ satisfying

$$Y^{\mathrm{d}\Re}_T \;=\; \tilde{Y}^{\mathrm{d}\Re}_T := g(X_T)$$

and, for $j \leq \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\left\{ \begin{array}{lcl} \tilde{Y}_t^{\mathrm{d}\Re} & = & Y_{r_{j+1}}^{\mathrm{d}\Re} + \int_t^{r_{j+1}} f(\Theta_s^{\mathrm{d}\Re})\mathrm{d}s - \int_t^{r_{j+1}} (Z_s^{\mathrm{d}\Re})'\mathrm{d}W_s \ , \\ Y_t^{\mathrm{d}\Re} & = & \mathcal{R}\left(t \ , \ X_t \ , \ \tilde{Y}_t^{\mathrm{d}\Re}\right) \ . \end{array} \right. \tag{3.1}$$

Here, $g : \mathbb{R}^d \mapsto \mathbb{R}$, $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$, $\Theta^{\mathrm{d}\Re} := (X, \tilde{Y}^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$, $(Z^{\mathrm{d}\Re})'$ is the transposed vector of $Z^{\mathrm{d}\Re}$, and

$$\mathcal{R}(t,x,y) \ \ := \ \ y + [h(x) - y]^+ \mathbf{1}_{\{t \in \Re \setminus \{0,T\}\}} \ \ , \ \ (t,x,y) \in [0,T] \times \mathbb{R}^{d+1} \ ,$$

for some $h : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying $g \geq h$ on $\mathbb{R}^d$.
By a solution, we mean an adapted process $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re}) \in \mathcal{S}^2 \times \mathcal{H}^2$ where, for $p \geq 1$, $\mathcal{S}^p$ is the set of real valued progressively measurable U such that

$$||U||_{\mathcal{S}^p} \ \ := \ \ \| \sup_{t \leq T} |U_t| \, \|_{L^p} \ < \ \infty \ ,$$

and $\mathcal{H}^p$ is the set of progressively measurable $\mathbb{R}^d$-valued processes V satisfying

$$||V||_{\mathcal{H}^p} \ \ := \ \ \| \left(\int_0^T |V_r|^2 \mathrm{d}r \right)^{\frac{1}{2}} \|_{L^p} \ < \ \infty \ .$$

In the following, we shall extend the definition of $|| \cdot ||_{\mathcal{S}^p}$ and $|| \cdot ||_{\mathcal{H}^p}$ to processes with values in $\mathbb{R}^d$ or $\mathbb{M}^d$, these extensions being defined in a straightforward way.

Observe that the solution of (3.1) can be constructed piecewise. Assuming that g, h and f are L-Lipschitz:

$$|g(x_1) - g(x_2)| + |h(x_1) - h(x_2)| + |f(\theta_1) - f(\theta_2)| \leq L \left(|x_1 - x_2| + |\theta_1 - \theta_2| \right)$$

for all $x_1, x_2 \in \mathbb{R}^d$ and $\theta_1, \theta_2 \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, the existence and uniqueness of the solution follow from [13]. By convention, we assume that $|g(0)| + |h(0)| + |f(0)| \leq L$.

Remark 3.1. For later use, observe that (3.1) can be written as

$$\tilde{Y}_t^{\mathrm{d}\Re} = g(X_T) + \int_t^T f(X_u, \tilde{Y}_u^{\mathrm{d}\Re}, Z_u^{\mathrm{d}\Re})\mathrm{d}u - \int_t^T (Z_u^{\mathrm{d}\Re})'\mathrm{d}W_u + \tilde{K}_T^{\mathrm{d}\Re} - \tilde{K}_t^{\mathrm{d}\Re} \ \ , \ t \leq T \ , \tag{3.2}$$

with

$$\tilde{K}_t^{\mathrm{d}\Re} \ \ := \ \ \sum_{j=1}^{\kappa - 1} \left[h(X_{r_j}) - \tilde{Y}_{r_j}^{\mathrm{d}\Re} \right]^+ \mathbf{1}_{\{r_j \leq t\}} \ .$$

By repeating the arguments of the proof of Proposition 3.5 in [9], we then easily check that

$$||\tilde{Y}^{\mathrm{d}\Re}||_{\mathcal{S}^2} + ||Y^{\mathrm{d}\Re}||_{\mathcal{S}^2} + ||Z^{\mathrm{d}\Re}||_{\mathcal{H}^2} + \|\tilde{K}_T^{\mathrm{d}\Re}\|_{L^2} \ \ \leq \ \ C_L \ . \tag{3.3}$$

Recall that $C_L > 0$ is a constant independent of $\Re$.

We conclude this section with a regularity result on $Y^{\mathrm{d}\Re}$ whose proof is given at the end of Section 5.3.

Proposition 3.1. *We have*

$$\max_{i \leq N-1} \mathbb{E}\left[\sup_{t \in (t_i, t_{i+1}]} |Y^{\mathrm{d}\Re}_{t_{i+1}} - Y^{\mathrm{d}\Re}_t|^2\right] \;\leq\; C_L\, |\pi|\ .$$

3.2 Discrete-time approximation

From now on, we assume that $\Re \subset \pi$, i.e. the reflection times are included in the partition defining the Euler scheme of the forward process X.
We approximate $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ by the piecewise constant process $(\bar{Y}^\pi, \bar{Z}^\pi)$ defined by induction by

$$\left\{\begin{array}{rcl} \bar{Z}^\pi_{t_i} & = & (t_{i+1} - t_i)^{-1}\, \mathbb{E}_{t_i}\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})\right] \\ \tilde{Y}^\pi_{t_i} & = & \mathbb{E}_{t_i}\left[\bar{Y}^\pi_{t_{i+1}}\right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\ \bar{Y}^\pi_{t_i} & = & \mathcal{R}\left(t_i\ ,\ X^\pi_{t_i}\ ,\ \tilde{Y}^\pi_{t_i}\right)\ ,\ i \leq N-1\ , \end{array}\right. \tag{3.4}$$

and by the terminal condition

$$\bar{Y}^\pi_T \;=\; \tilde{Y}^\pi_T := g(X^\pi_T)\ .$$

Recall that $\mathbb{E}_{t_i}[\cdot]$ stands for $\mathbb{E}[\cdot \mid \mathcal{F}_{t_i}]$. For ease of notations, we set

$$(\bar{Y}^\pi_t, \bar{Z}^\pi_t) = (\bar{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \quad \text{for } t \in [t_i, t_{i+1})\ ,\ i \leq N-1\ . \tag{3.5}$$

Using an induction argument and the Lipschitz-continuity assumption on g, h and f, one easily checks that the above processes are square integrable. It follows that the conditional expectations are well defined at each step of the algorithm.

Remark 3.2. Observe that $\tilde{Y}^\pi$ is defined implicitly as the solution of a fixed point problem. Since f is Lipschitz-continuous, it is defined with no ambiguity. Moreover, for small values of $|\pi|$ it can be estimated numerically in a very fast and accurate way, if not explicit. We refer to [2] for a discussion on the difference between implicit and explicit schemes.

For later use, let us introduce the continuous time scheme associated to $(\bar{Y}^\pi, \bar{Z}^\pi)$. By the martingale representation theorem, there exists $Z^\pi \in \mathcal{H}^2$ such that

$$\bar{Y}^\pi_{t_{i+1}} = \mathbb{E}_{t_i}\left[\bar{Y}^\pi_{t_{i+1}}\right] + \int_{t_i}^{t_{i+1}} (Z^\pi_u)' \mathrm{d}W_u\ \ ,\ \ i \leq N-1\ .$$

We can then define $\tilde{Y}^\pi$ on $[t_i, t_{i+1})$ by

$$\tilde{Y}^\pi_t \;=\; \bar{Y}^\pi_{t_{i+1}} + (t_{i+1} - t) f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_{i+1}}) - \int_t^{t_{i+1}} (Z^\pi_u)' \mathrm{d}W_u\ , \tag{3.6}$$

and set

$$Y_t^\pi \quad := \quad \mathcal{R}(t, X_t^\pi, \tilde{Y}_t^\pi) \ \text{ for } \ t \le T \ ,$$

so that

$$Y^\pi \ = \ \bar{Y}^\pi \text{ on } \pi \qquad \text{and} \qquad Y^\pi \ = \ \tilde{Y}^\pi \text{ on } [0,T] \setminus \Re \ . \tag{3.7}$$

Remark 3.3. It follows from the Itô isometry that

$$\bar{Z}_t^\pi \quad = \quad (t_{i+1} - t_i)^{-1} \mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} Z_u^\pi \mathrm{d}u \right] \ , \forall \, t \in [t_i, t_{i+1}) \ , \ i \le N-1 \ , \tag{3.8}$$

recall (3.5).

3.3 Convergence results

In order to state our first result, we need to introduce the process $\bar{Z}^{\mathrm{d}\Re}$ defined on each interval $[t_i, t_{i+1})$ by

$$\bar{Z}_t^{\mathrm{d}\Re} \quad := \quad (t_{i+1} - t_i)^{-1} \mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} Z_u^{\mathrm{d}\Re} \mathrm{d}u \right] \ . \tag{3.9}$$

Remark 3.4. For later use, observe that, by (3.8) and Jensen's inequality,

$$\mathbb{E}\left[|\bar{Z}_t^{\mathrm{d}\Re} - \bar{Z}_t^\pi|^2 \right] \quad \le \quad (t_{i+1} - t_i)^{-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_u^{\mathrm{d}\Re} - Z_u^\pi|^2 \right] \mathrm{d}u \ , \tag{3.10}$$

which implies

$$||\bar{Z}^{\mathrm{d}\Re} - \bar{Z}^\pi||_{\mathcal{H}^2} \quad \le \quad ||Z^{\mathrm{d}\Re} - Z^\pi||_{\mathcal{H}^2} \ . \tag{3.11}$$

The following result shows that the approximation error is intimately related to the $\mathcal{H}^2$ norm of $Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}$. A similar property holds in the non-reflected case, see [2], [3], [15] and [16].

Proposition 3.2. *The following holds:*

$$\max_{j \le \kappa - 1} \| \sup_{t \in [r_j, r_{j+1}]} |Y_t^\pi - Y_t^{\mathrm{d}\Re}| \, \|_{L^2} \quad \le \quad C_L \left(|\pi|^{\frac{1}{2}} + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||_{\mathcal{H}^2} \right) \ ,$$

and

$$||Z^\pi - Z^{\mathrm{d}\Re}||_{\mathcal{H}^2} \quad \le \quad C_L \left(\kappa^{\frac{1}{2}} |\pi|^{\frac{1}{2}} + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||_{\mathcal{H}^2} \right) \ .$$

The proof essentially follows the arguments of [3] and is provided in the Appendix.

Remark 3.5. Observing that $\bar{Z}^{\mathrm{d}\Re}$ is the best $L^2(\Omega \times [0,T])$-approximation of $Z^{\mathrm{d}\Re}$ by adapted processes which are constant on each interval $[t_i, t_{i+1})$, we deduce that $||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}$ goes to 0 as $|\pi|$ goes to 0. Thus, the above proposition actually shows that our discrete-time scheme is convergent. This also implies that

$$||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2} \quad \leq \quad \sum_{i=0}^{N-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z^{\mathrm{d}\Re}_t - Z^{\mathrm{d}\Re}_{t_i}|^2 \mathrm{d}t\right] .$$

In order to get a bound on the convergence rate, it remains to control $||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}$. Such a control will be obtained under one of the following additional assumptions.

$(\mathbf{H}1)$: $h \in C^1_b$ with L-Lipschitz derivative,

or

$(\mathbf{H}2)$: $\sigma \in C^1_b$ with L-Lipschitz derivative, and $h \in C^2_b$ with L-Lipschitz first and second derivatives.

Proposition 3.3. *Let* $(\mathbf{H}1)$ *hold. Then,*

$$||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||_{\mathcal{H}^2} \leq C_L \left(\alpha(\kappa)\, |\pi|^{\frac{1}{2}} + \epsilon(\pi)\right) ,$$

where $(\alpha(\kappa), \epsilon(\pi)) = (\kappa^{\frac{1}{4}}, |\pi|^{\frac{1}{4}})$ *under* $(\mathbf{H}1)$*, and* $(\alpha(\kappa), \epsilon(\pi)) = (1, |\pi|^{\frac{1}{2}})$ *under* $(\mathbf{H}2)$*.*

The proof will be provided in Section 5.

Combining the above propositions, we obtain the main result of this section.

Theorem 3.1. *Let* $(\mathbf{H}1)$ *hold. Then,*

$$\max_{j \leq \kappa-1} \| \sup_{t \in [r_j, r_{j+1}]} |Y^\pi_t - Y^{\mathrm{d}\Re}_t| \|_{L^2} \quad \leq \quad C_L \left(\alpha_Y(\kappa)\, |\pi|^{\frac{1}{2}} + \epsilon(\pi)\right)$$

and

$$||Z^\pi - Z^{\mathrm{d}\Re}||_{\mathcal{H}^2} \quad \leq \quad C_L \left(\alpha_Z(\kappa)\, |\pi|^{\frac{1}{2}} + \epsilon(\pi)\right)$$

with $(\alpha_Y(\kappa), \alpha_Z(\kappa), \epsilon(\pi)) = (\kappa^{\frac{1}{4}}, \kappa^{\frac{1}{2}}, |\pi|^{\frac{1}{4}})$ *under* $(\mathbf{H}1)$*, and* $(\alpha_Y(\kappa), \alpha_Z(\kappa), \epsilon(\pi)) = (1, \kappa^{\frac{1}{2}}, |\pi|^{\frac{1}{2}})$ *under* $(\mathbf{H}2)$*.*

Recalling (3.7), (3.11) and combining Proposition 3.1 with Theorem 3.1, we finally obtain a bound on the error due to the approximation of $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ by the piecewise constant process $(\bar{Y}^\pi, \bar{Z}^\pi)$ which can actually be estimated numerically, see the end of the introduction.

Corollary 3.1. *Let* $(\mathbf{H}1)$ *hold. Then,*

$$\max_{i \leq N-1} \| \, |\bar{Y}^\pi_{t_i} - Y^{\mathrm{d}\Re}_{t_i}| + \sup_{t \in (t_i, t_{i+1}]} |\bar{Y}^\pi_{t_{i+1}} - Y^{\mathrm{d}\Re}_t| \, \|_{L^2} \; \leq \; C_L \left(\alpha_Y(\kappa)\, |\pi|^{\frac{1}{2}} + \epsilon(\pi)\right)$$

and

$$||\bar{Z}^{\pi} - Z^{\mathrm{d}\Re}||_{\mathcal{H}^2} \;\; \leq \;\; C_L \, \left(\alpha_Z(\kappa)\, |\pi|^{\frac{1}{2}} + \epsilon(\pi)\right)$$

with $(\alpha_Y(\kappa), \alpha_Z(\kappa), \epsilon(\pi)) = (\kappa^{\frac{1}{4}}, \kappa^{\frac{1}{2}}, |\pi|^{\frac{1}{4}})$ *under* (**H**1)*, and* $(\alpha_Y(\kappa), \alpha_Z(\kappa), \epsilon(\pi)) = (1, \kappa^{\frac{1}{2}}, |\pi|^{\frac{1}{2}})$ *under* (**H**2)*.*

Remark 3.6. It was shown in [11] that the results of Proposition 3.3 and Theorem 3.1 hold with the bound $C_L|\pi|^{\frac{1}{4}}$ when $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ is replaced by the solution (Y, Z) of a continuously reflected BSDE, see (4.1) below. Their proof is based on a particular representation of Z obtained by an integration by parts argument. However, it requires an uniform ellipticity condition on σ. Our approach is completely different. It is based on a representation for $Z^{\mathrm{d}\Re}$ in terms of the next reflection time, see Section 5 below. This allows us to get rid of the inversibility condition on σ. The above results will be extended to the continuously reflected case in Section 4 below.

Remark 3.7. For sake of simplicity, we restrict ourselves to the case where X is approximated by its Euler scheme. However, it would be natural to wonder what happens if X is approximated by an order one scheme, i.e. such that:

$$\max_{i \leq N} \mathbb{E}\left[|X_{t_i} - X^{\pi}_{t_i}|^2\right] \;\; \leq \;\; C_L|\pi|^2 \, .$$

This would be the case if X can be perfectly simulated on the grid π or if we can use a Milshtein's scheme. In this case, the proof of Proposition 3.2 can be easily adapted, see Remark A.1 in the Appendix, to obtain

$$||Z^{\pi} - Z^{\mathrm{d}\Re}||_{\mathcal{H}^2} \;\; \leq \;\; C_L \left(|\pi|^{\frac{1}{2}} + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||_{\mathcal{H}^2}\right) \, .$$

The bounds of Theorem 3.1 and Corollary 3.1 then hold with $\alpha_Z(\kappa) = \kappa^{\frac{1}{4}}$ under (**H**1), and $\alpha_Z(\kappa) = 1$ under (**H**2).

3.4 Discretely reflected BSDE constructed with the Euler scheme

In this subsection, we introduce the solution $(Y^{\mathrm{d}\Re,e}, Z^{\mathrm{d}\Re,e})$ of a discretely reflected BSDE defined similarly as $(Y^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ but with X^{π} instead of X, i.e.

$$Y^{\mathrm{d}\Re,e}_T \;\; = \;\; \tilde{Y}^{\mathrm{d}\Re,e}_T := g(X^{\pi}_T)$$

and, for $j \leq \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\left\{ \begin{array}{lcl} \tilde{Y}^{\mathrm{d}\Re,e}_t & = & Y^{\mathrm{d}\Re,e}_{r_{j+1}} + \int_t^{r_{j+1}} f(\Theta^{\mathrm{d}\Re,e}_u)\mathrm{d}s - \int_t^{r_{j+1}} (Z^{\mathrm{d}\Re,e}_s)'\mathrm{d}W_s \;\; , \\ Y^{\mathrm{d}\Re,e}_t & = & \mathcal{R}\left(t \; , \; X^{\pi}_t \; , \; \tilde{Y}^{\mathrm{d}\Re,e}_t\right) \; . \end{array} \right. \tag{3.12}$$

with $\Theta^{\mathrm{d}\Re,e} := (X^{\pi}, \tilde{Y}^{\mathrm{d}\Re,e}, Z^{\mathrm{d}\Re,e})$.

This construction will be useful to extend the results of the previous section to the continuously reflected case.

Observe that

$$\tilde{Y}_t^{\mathrm{d}\Re,e} = g(X_T^\pi) + \int_t^T f(\Theta_u^{\mathrm{d}\Re,e})\mathrm{d}u - \int_t^T (Z_u^{\mathrm{d}\Re,e})'\mathrm{d}W_u + \tilde{K}_T^{\mathrm{d}\Re,e} - \tilde{K}_t^{\mathrm{d}\Re,e} \ , \ t \le T \ ,$$

with

$$\tilde{K}_t^{\mathrm{d}\Re,e} \ := \ \sum_{j=1}^{\kappa-1} \left[h(X_{r_j}^\pi) - \tilde{Y}_{r_j}^{\mathrm{d}\Re,e} \right]^+ \mathbf{1}_{r_j \le t} \ .$$

Moreover, it follows from the same arguments as in the proof of Proposition 3.2, see Remark A.1 after the proof in the Appendix, that

$$||Z^\pi - Z^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} \ \le \ C_L \left(|\pi|^{\frac{1}{2}} + ||Z^{\mathrm{d}\Re,e} - \bar{Z}^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} \right) \ , \tag{3.13}$$

where $\bar{Z}^{\mathrm{d}\Re,e}$ is defined similarly as $\bar{Z}^{\mathrm{d}\Re}$, i.e.

$$\bar{Z}_t^{\mathrm{d}\Re,e} \ := \ (t_{i+1} - t_i)^{-1} \, \mathbb{E}_{t_i} \left[\int_{t_i}^{t_{i+1}} Z_s^{\mathrm{d}\Re,e} \mathrm{d}s \right] \ , \ t \in [t_i, t_{i+1}), \ i \le N-1 \ .$$

We shall also prove in Section 5 that the result of Proposition 3.3 can be extended to $Z^{\mathrm{d}\Re,e}$.

Proposition 3.4. *Let* $(\mathbf{H}1)$ *hold. Then,*

$$||Z^{\mathrm{d}\Re,e} - \bar{Z}^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} \le C_L \ \left(\kappa^{\frac{1}{4}} \ |\pi|^{\frac{1}{2}} + |\pi|^{\frac{1}{4}} \right) \ .$$

4 Extension to the continuously reflected case

Let (Y, Z, K) be the $\mathbb{F}$-progressively measurable process satisfying

$$\begin{aligned} Y_t \ &= \ g(X_T) + \int_t^T f(X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T (Z_s)'\mathrm{d}W_s + K_T - K_t \ , \\ Y_t \ &\ge \ h(X_t) \ , \ 0 \le t \le T \end{aligned} \tag{4.1}$$

with K continuous, non-decreasing, such that $K_0 = 0$ and

$$\int_0^T (Y_t - h(X_t))\mathrm{d}K_t \ = \ 0 \ . \tag{4.2}$$

Existence and uniqueness of a solution $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}^2$ follows from Theorem 5.2 in [9], recall that g, h and f are Lipschitz-continuous.

As in Section 3.4, we also define (Y^e, Z^e, K^e) as the solution of (4.1) with X^π in place of X, i.e.

$$\begin{aligned} Y_t^e \ &= \ g(X_T^\pi) + \int_t^T f(X_s^\pi, Y_s^e, Z_s^e)\mathrm{d}s - \int_t^T (Z_s^e)'\mathrm{d}W_s + K_T^e - K_t^e \ , \\ Y_t^e \ &\ge \ h(X_t^\pi) \ , \ 0 \le t \le T \ , \end{aligned}$$

where K^e is continuous and non-decreasing, $K^e_0 = 0$ and $\int_0^T (Y^e_t - h(X^\pi_t))\mathrm{d}K^e_t = 0$.

Our first result is standard and we omit the proof, see e.g. [1]. It shows that (Y, Z) and (Y^e, Z^e) can be approximated by the solutions of discretely reflected BSDEs at a speed $|\Re|^{\frac{1}{2}}$ under the assumption:

(**H**3): There exists $\rho_1 : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\rho_2 : \mathbb{R}^d \mapsto \mathbb{R}_+$ such that

$$
\begin{aligned}
|\rho_1(x)| + |\rho_2(x)| &\leq C_L(1 + |x|^{C_L}) \\
h(x) - h(y) &\leq \rho_1(x)'(y - x) + \rho_2(x)|x - y|^2 \ , \ \forall\, x, y \in \mathbb{R}^d \ .
\end{aligned}
$$

This condition is slightly weaker than the semi-convexity assumption of Definition 1 in [1] which is satisfied whenever (**H**1) or (**H**2) hold.

Proposition 4.1. *Assume that* (**H**3) *holds. Then,*

$$
\sup_{t \in [0,T]} \|Y_t - Y^{\mathrm{d}\Re}_t\|_{L^2} + ||Z - Z^{\mathrm{d}\Re}||_{\mathcal{H}^2} \quad \leq \quad C_L\ |\Re|^{\frac{1}{2}}
$$

and

$$
\sup_{t \in [0,T]} ||Y^e_t - Y^{\mathrm{d}\Re,e}_t\|_{L^2} + ||Z^e - Z^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} \quad \leq \quad C_L\ |\Re|^{\frac{1}{2}} \ .
$$

If moreover (**H**1) *holds, then*

$$
\max_{j \leq \kappa - 1} \left(\| \sup_{t \in [r_j, r_{j+1}]} |Y_t - Y^{\mathrm{d}\Re}_t| \ \|_{L^2} + \| \sup_{t \in [r_j, r_{j+1}]} |Y^e_t - Y^{\mathrm{d}\Re,e}_t| \ \|_{L^2} \right) \quad \leq \quad C_L\ |\Re|^{\frac{1}{2}} \ .
$$

We can now extend the convergence results of the previous section to the continuously reflected case.

Theorem 4.1. *Let* (**H**1) *hold, then*

$$
\begin{aligned}
&\max_{i \leq N-1} \| \sup_{t \in (t_i, t_{i+1}]} |\bar{Y}^\pi_{t_{i+1}} - Y_t| + \sup_{t \in [t_i, t_{i+1}]} |Y^\pi_t - Y_t| \ \|_{L^2} \ \leq \ C_L\ \alpha(\pi) \\
&\textit{and } ||\bar{Z}^\pi - Z||_{\mathcal{H}^2} + ||Z^\pi - Z||_{\mathcal{H}^2} \ \leq \ C_L\ |\pi|^{\frac{1}{4}} \ ,
\end{aligned}
$$

with $\alpha(\pi) = |\pi|^{\frac{1}{4}}$ *under* (**H**1) *and* $\alpha(\pi) = |\pi|^{\frac{1}{2}}$ *under* (**H**2).

Proof. 1.The error on Y follows from Proposition 4.1, Corollary 3.1 and Theorem 3.1 applied with $\Re = \pi$.
2. The estimate for Z is a bit more involved. We first approximate (Y, Z) by (Y^e, Z^e). It follows from Proposition 3.6 in [9], our Lipschitz-continuity assumptions, (2.2) and (2.4) that $||Z - Z^e||^2_{\mathcal{H}^2} \leq C_L\ |\pi|^{\frac{1}{2}}$. Then, we approximate (Y^e, Z^e) by $(Y^{\mathrm{d}\Re,e}, Z^{\mathrm{d}\Re,e})$ defined in Section 3.3. By Proposition 4.1, $||Z^e - Z^{\mathrm{d}\Re,e}||^2_{\mathcal{H}^2} \leq C_L\ |\pi|$. Finally, it follows from (3.13) that $||Z^\pi - Z^{\mathrm{d}\Re,e}||^2_{\mathcal{H}^2} \leq C_L \left(|\pi| + ||Z^{\mathrm{d}\Re,e} - \bar{Z}^{\mathrm{d}\Re,e}||^2_{\mathcal{H}^2}\right)$,

where the last term is controlled by Proposition 3.4. To conclude, we deduce from Jensen's inequality that $||\bar{Z}^\pi - Z^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} \leq ||Z^\pi - Z^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} + ||Z^{\mathrm{d}\Re,e} - \bar{Z}^{\mathrm{d}\Re,e}||_{\mathcal{H}^2}$, recall (3.8). □

Remark 4.1. In view of Remark 3.7 and Proposition 4.1 applied with $\Re = \pi$, it is clear that, if the Euler scheme X^π is replaced by an order one scheme, then

$$||\bar{Z}^\pi - Z||_{\mathcal{H}^2} + ||Z^\pi - Z||_{\mathcal{H}^2} \ \leq\ C_L\ |\pi|^{\frac{1}{2}}\ ,$$

whenever $(\mathbf{H}2)$ holds.

As in (3.9), we now define

$$\begin{aligned}\bar{Z}_t &:= (t_{i+1} - t_i)^{-1}\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_u \mathrm{d}u\right]\ ,\\ \bar{Z}^e_t &:= (t_{i+1} - t_i)^{-1}\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z^e_u \mathrm{d}u\right] \quad \text{for}\ \ t \in [t_i, t_{i+1})\ ,\ i \leq N-1\ .\end{aligned}$$

Observe that, by Jensen's inequality,

$$||\bar{Z}^{\mathrm{d}\Re} - \bar{Z}||_{\mathcal{H}^2} \ \leq\ ||Z^{\mathrm{d}\Re} - Z||_{\mathcal{H}^2} \ \text{ and }\ ||\bar{Z}^{\mathrm{d}\Re,e} - \bar{Z}^e||_{\mathcal{H}^2} \ \leq\ ||Z^{\mathrm{d}\Re,e} - Z^e||_{\mathcal{H}^2}\ . \tag{4.3}$$

Combining (4.3), Proposition 4.1, Proposition 3.3 and Proposition 3.4 for $\Re = \pi$, we obtain the following regularity result for Z and Z^e.

Corollary 4.1. *Let* $(\mathbf{H}1)$ *holds, then*

$$||Z - \bar{Z}||_{\mathcal{H}^2} + ||Z^e - \bar{Z}^e||_{\mathcal{H}^2} \leq C_L\ |\pi|^{\frac{1}{4}}\ .$$

If moreover $(\mathbf{H}2)$ *holds, then*

$$||Z - \bar{Z}||_{\mathcal{H}^2} \leq C_L\ |\pi|^{\frac{1}{2}}\ .$$

Remark 4.2. As explained in the previous section, similar results were obtained in [11]. However, their approach requires that σ is uniformly elliptic. Here, we do not need this condition on σ. We also obtain better bounds for $||Z - \bar{Z}||_{\mathcal{H}^2}$ and $\sup_{t\in[0,T]} \|Y^\pi_t - Y_t\|_{L^2}$ under $(\mathbf{H}2)$. This last assumption is slightly stronger than the C^2_b regularity imposed on h by [11].

5 Representation and regularity of $Z^{\mathrm{d}\Re}$ and $Z^{\mathrm{d}\Re,e}$

5.1 Preliminaries

In the sequel, we denote by $\mathbb{D}^{1,2}$ the space of random variable F which are differentiable in the Malliavin sense and such that

$$\|F\|^2_{L^2} + \int_0^T \|D_tF\|^2_{L^2}\mathrm{d}t < \infty\ .$$

Here, D_tF denotes the Malliavin derivative of F at time $t \leq T$, see e.g. [12].
We also introduce the space $\mathbb{L}_a^{1,2}$ of adapted processes V such that, after possibly passing to a suitable version, $V_s \in \mathbb{D}^{1,2}$ for all $s \leq T$ and

$$||V||^2_{\mathcal{H}^2} + \int_0^T ||D_tV||^2_{\mathcal{H}^2} \mathrm{d}t < \infty .$$

In the following, we shall always consider a suitable version if necessary.

In this section, we work under the stronger assumptions:

$(\mathbf{H}')$: b, σ, g and f are C_b^1.

The general case will be obtained by using an approximation argument.

Remark 5.1. It is well known that under the above assumptions $X \in \mathbb{L}_a^{1,2}$, see e.g. [12], and satisfies for $p \geq 2$ and $t, u \leq T$

$$\sup_{s \leq t \wedge u} \|D_sX_t - D_sX_u\|_{L^p} + \| \sup_{t \vee u \leq s \leq T} |D_tX_s - D_uX_s| \|_{L^p} \quad \leq \quad C_L^p \, |t-u|^{\frac{1}{2}} \; . \quad (5.1)$$

Moreover, the first variation process ∇X of X is well defined and solves on $[0, T]$

$$\nabla X_t \quad = \quad I_d + \int_0^t \nabla b(X_r) \nabla X_r \mathrm{d}r + \int_0^t \sum_{j=1}^d \nabla \sigma^j(X_r) \nabla X_r \mathrm{d}W_r^j$$

where I_d is the identity matrix of $\mathbb{M}^d$, σ^j is the j-th column of σ, and ∇b, $\nabla \sigma^j$ the Jacobian matrix of b and σ^j. Its inverse $(\nabla X)^{-1}$ is the solution on $[0, T]$ of

$$\begin{aligned} (\nabla X)_t^{-1} \quad = \quad & I_d - \int_0^t (\nabla X)_r^{-1} \left[\nabla b(X_r) - \sum_{j=1}^d \nabla \sigma^j(X_r) \nabla \sigma^j(X_r) \right] \mathrm{d}r \\ & - \int_0^t \sum_{j=1}^d (\nabla X)_r^{-1} \nabla \sigma^j(X_r) \mathrm{d}W_r^j \; , \end{aligned}$$

and the following standard estimates hold:

$$||\nabla X||_{\mathcal{S}^p} + ||(\nabla X)^{-1}||_{\mathcal{S}^p} \quad \leq \quad C_L^p \; . \quad (5.2)$$

Finally, we recall the well-known relation between ∇X and DX:

$$D_tX_s \quad = \quad \nabla X_s (\nabla X_t)^{-1} \sigma(X_t) \mathbf{1}_{t \leq s} \quad \text{for all } t, s \leq T \; . \quad (5.3)$$

Using the above estimates, (2.2) and the Lipschitz-continuity of σ, we deduce that

$$|| \sup_{s \leq T} |D_sX| \; ||_{\mathcal{S}^p} \quad \leq \quad C_L^p \; . \quad (5.4)$$

Remark 5.2. Observe that X^π also belongs to $\mathbb{L}_a^{1,2}$ under $(\mathbf{H}')$ and satisfies

$$D_s X_t^\pi \;=\; \sigma(X_{\phi_s}^\pi) + \int_s^t \nabla b(X_{\phi_r}^\pi) D_s X_{\phi_r}^\pi \mathrm{d}r + \int_s^t \sum_{j=1}^d \nabla\sigma^j(X_{\phi_r}^\pi) D_s X_{\phi_r}^\pi \mathrm{d}W_r^j$$

for $s \le t$, where $\phi_t = \max\{u \in \pi \;:\; u \le t\}$. Thus, $D_s X_t^\pi$ is given by

$$\left\{ \prod_{k \in N_{s,t}} \left(I_d + \nabla b(X_{t_k}^\pi)(t_{k+1} \wedge t - t_k) + \sum_{j=1}^d \nabla\sigma^j(X_{t_k}^\pi)(W_{t_{k+1}\wedge t}^j - W_{t_k}^j) \right) \right\} \sigma(X_{\phi_s}^\pi) \mathbf{1}_{s \le t}$$

with $N_{s,t} := \{k \le N \;:\; s \le t_k < t\}$. Using the bound on ∇b and $\nabla\sigma^j$, $j \le d$, we obtain

$$\mathbb{E}\left[\sup_{s \le t \le T} |D_s X_t^\pi|^p\right] \;\le\; C_L^p \left(1 + C_L^p |\pi|^{2p}\right)^N \left(1 + \mathbb{E}\left[\sup_{t \le T} |X_t^\pi|^{2p}\right]\right)^{\frac{1}{2}}$$

which leads to

$$\mathbb{E}\left[\sup_{s \le t \le T} |D_s X_t^\pi|^p\right] \;\le\; C_L^p \;,\; p \ge 1 \;. \tag{5.5}$$

By using standard arguments, one also easily checks that the bounds (5.1) can be extended to X^π, uniformly in π:

$$\sup_{s \le t \wedge u} \|D_s X_t - D_s X_u\|_{L^p} + \|\sup_{t \vee u \le s \le T} |D_t X_s - D_u X_s|\|_{L^p} \;\le\; C_L^p \;|t-u|^{\frac{1}{2}} \;. \tag{5.6}$$

5.2 Representation

In order to provide a suitable representation of $Z^{\mathrm{d}\Re}$, we shall appeal to the following easy lemma.

Lemma 5.1. *If $F \in \mathbb{D}^{1,2}$, then $[F]^+ \in \mathbb{D}^{1,2}$ and $D_t[F]^+ = (D_t F)\mathbf{1}_{\{F>0\}}$.*

Proof. By a straightforward adaptation of Proposition 1.2.3 in [12], we observe that $[F]^+$ belongs to $\mathbb{D}^{1,2}$ and $D_t[F]^+ = \alpha(D_t F)$ where α is a random variable bounded by 1 satisfying $\mathbf{1}_{\{F>0\}}\alpha = \mathbf{1}_{\{F>0\}}$. The proof is then concluded by appealing to Proposition 1.3.7 in [12]. □

Recalling that $g \ge h$, using Remark 5.1, Lemma 5.1, Proposition 5.3 in [8] and an induction argument, we easily deduce from (3.1) that $(\tilde{Y}^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ belongs to $\mathbb{L}_a^{1,2}$.

Proposition 5.1. *Let $(\mathbf{H}')$ hold. Then, the process $(\tilde{Y}^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ belongs to $\mathbb{L}_a^{1,2}$ and, for all $t \le T$, $D_t(\tilde{Y}^{\mathrm{d}\Re}, Z^{\mathrm{d}\Re})$ solves on $[r_j, r_{j+1})$, $j \le \kappa - 1$,*

$$\begin{aligned} D_t \tilde{Y}_s^{\mathrm{d}\Re} \;&=\; (D_t h(X_{r_{j+1}}) - D_t \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re}) \mathbf{1}_{\{h(X_{r_{j+1}}) > \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re}\}} \\ &+\; D_t \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re} + \int_s^{r_{j+1}} \nabla f(\Theta_u^{\mathrm{d}\Re}) D_t \Theta_u^{\mathrm{d}\Re} \mathrm{d}u - \int_s^{r_{j+1}} D_t Z_s^{\mathrm{d}\Re} \mathrm{d}W_s \;. \end{aligned} \tag{5.7}$$

In order to get rid of the indicator functions appearing in (5.7), we now define the following sequence of stopping times

$$\tau_j = \inf\{t \in \Re \mid t \geq r_{j+1},\ h(X_t) > \tilde{Y}_t^{\mathrm{d}\Re}\} \wedge T \ \ , \ j \leq \kappa - 1 \ . \tag{5.8}$$

Following [15] , we also define, for $s \leq t \leq T$,

$$\Lambda_t^s \ := \ \exp\left\{\int_s^t \nabla_z f(\Theta_u^{\mathrm{d}\Re})' \mathrm{d}W_u - \int_s^t \left(\frac{1}{2}|\nabla_z f(\Theta_u^{\mathrm{d}\Re})|^2 - \nabla_y f(\Theta_u^{\mathrm{d}\Re})\right)\mathrm{d}u\right\} ,$$

where $\nabla_y f$ denote the partial derivative of f with respect to its second variable y, and $\nabla_x f$ and $\nabla_z f$ the gradient of f with respect to its first and last variable.

Remark 5.3. The following estimates are standard:

$$\|\sup_{s\leq t\leq T} \Lambda_t^s\|_{L^p} \ \leq \ C_L^p , \tag{5.9}$$

$$\|\sup_{u\leq t\wedge s} |\Lambda_t^u - \Lambda_s^u|\ \|_{L^p} \ \leq \ C_L^p |t-s|^{\frac{1}{2}} \ , \ t,s \leq T . \tag{5.10}$$

Using (5.1), we deduce that

$$\|\sup_{u\vee t\leq s\leq T} |\Lambda_s^t D_t X_s - \Lambda_s^u D_u X_s|\ \|_{L^p} \ \leq \ C_L^p\ |t-u|^{\frac{1}{2}} \ , \ u,t \leq T . \tag{5.11}$$

We can now state the main result of this section which provides a representation for $Z^{\mathrm{d}\Re}$.

Corollary 5.1. *Let* $(\mathbf{H}')$ *hold. Then, there is a version of* $Z^{\mathrm{d}\Re}$ *such that for each* $j \leq \kappa - 1$ *and* $t \in [r_j, r_{j+1})$*:*

$$\begin{aligned}(Z_t^{\mathrm{d}\Re})' \ &= \ \mathbb{E}\left[\nabla g(X_T)(\Lambda^t D_t X)_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \mid \mathcal{F}_t\right] \\ &+ \ \mathbb{E}\left[\int_t^{\tau_j} \nabla_x f(\Theta_u^{\mathrm{d}\Re})(\Lambda^t D_t X)_u \mathrm{d}u \mid \mathcal{F}_t\right] .\end{aligned}$$

Proof. 1. It follows from Proposition 5.1 and the assumption $g \geq h$ that, for all $t \leq T$, $j \leq \kappa - 1$ and $s \in [r_j, r_{j+1})$, we have

$$\begin{aligned} D_t \tilde{Y}_s^{\mathrm{d}\Re} \ &= \ \left(\nabla h(X_{r_{j+1}}) D_t X_{r_{j+1}} - D_t \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re}\right) \mathbf{1}_{\{h(X_{r_{j+1}}) > \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re}\}} \\ &+ \ D_t \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re} + \int_s^{r_{j+1}} \nabla f(\Theta_u^{\mathrm{d}\Re}) D_t \Theta_u^{\mathrm{d}\Re} \mathrm{d}u - \int_s^{r_{j+1}} D_t Z_u^{\mathrm{d}\Re} \mathrm{d}W_u .\end{aligned}$$

In particular,

$$\begin{aligned} D_t \tilde{Y}_{r_j}^{\mathrm{d}\Re} \ &= \ \left(\nabla h(X_{r_{j+1}}) D_t X_{r_{j+1}} - D_t \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re}\right) \mathbf{1}_{\{h(X_{r_{j+1}}) > \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re}\}} \\ &+ \ D_t \tilde{Y}_{r_{j+1}}^{\mathrm{d}\Re} + \int_{r_j}^{r_{j+1}} \nabla f(\Theta_u^{\mathrm{d}\Re}) D_t \Theta_u^{\mathrm{d}\Re} \mathrm{d}u - \int_{r_j}^{r_{j+1}} D_t Z_u^{\mathrm{d}\Re} \mathrm{d}W_u .\end{aligned}$$

Since $\tilde{Y}^{\mathrm{d}\Re}_{r_\kappa} = g(X_T)$, it follows that $D_t\tilde{Y}^{\mathrm{d}\Re}_{r_\kappa} = \nabla g(X_T)D_tX_T$. Recalling that $g \geq h$, it then results from a simple induction that for $s \in [r_j, r_{j+1})$

$$\begin{aligned} D_t\tilde{Y}^{\mathrm{d}\Re}_s &= \nabla g(X_T)D_tX_T\mathbf{1}_{\{\tau_j=T\}} + \nabla h(X_{\tau_j})(D_tX)_{\tau_j}\mathbf{1}_{\{\tau_j<T\}} \\ &+ \int_s^{\tau_j} \nabla f(\Theta^{\mathrm{d}\Re}_u)D_t\Theta^{\mathrm{d}\Re}_u \mathrm{d}u - \int_s^{\tau_j} D_tZ^{\mathrm{d}\Re}_u \mathrm{d}W_u \,. \end{aligned}$$

By the same arguments as in Proposition 5.3 in [8], we have $D_t\tilde{Y}^{\mathrm{d}\Re}_t = D_tY^{\mathrm{d}\Re}_t = (Z^{\mathrm{d}\Re}_t)'$ on (r_j, r_{j+1}). The result then follows from the previous equation, Itô's formula and by considering a suitable version. □

Remark 5.4. Assume that $(\mathbf{H}')$ holds. Then, it follows from (5.4), (5.9) and Corollary 5.1 that $||Z^{\mathrm{d}\Re}||_{\mathcal{S}^p} \leq C^p_L$.

Remark 5.5. Let $(\mathbf{H}')$ hold. We deduce from the same arguments as in the proof of Corollary 5.1 that there is a version of $Z^{\mathrm{d}\Re,e}$ such that for each $t \in [r_j, r_{j+1})$, $j \leq \kappa - 1$:

$$\begin{aligned} (Z^{\mathrm{d}\Re,e}_t)' &= \mathbb{E}\left[\nabla g(X_T)D_tX^\pi_T\mathbf{1}_{\{\tau^e_j=T\}} + \nabla h(X^\pi_{\tau^e_j})(\Lambda^{e,t}D_tX^\pi)_{\tau_j}\mathbf{1}_{\{\tau^e_j<T\}} \mid \mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau^e_j} \nabla_x f(\Theta^{\mathrm{d}\Re,e}_u)(\Lambda^{e,t}D_tX^\pi)_u \mathrm{d}u \mid \mathcal{F}_t\right] \ , \ t \leq T \ , \end{aligned}$$

where

$$\tau^e_j = \inf\{t \in \Re \mid t \geq r_{j+1},\ h(X^\pi_t) > \tilde{Y}^{\mathrm{d}\Re,e}_t\} \wedge T \ \ , \ j \leq \kappa - 1 \,.$$

and $\Lambda^{e,s}_t$ is defined, for $s \leq t \leq T$, by

$$\Lambda^{e,s}_t := \exp\left\{\int_s^t \nabla_z f(\Theta^{\mathrm{d}\Re,e}_u)'\mathrm{d}W_u - \int_s^t \left(\frac{1}{2}|\nabla_z f(\Theta^{\mathrm{d}\Re,e}_u)|^2 - \nabla_y f(\Theta^{\mathrm{d}\Re,e}_u)\right)\mathrm{d}u\right\} \,.$$

The following estimates are standard:

$$\|\sup_{s\leq t\leq T} \Lambda^{e,s}_t\|_{L^p} \leq C^p_L \,, \tag{5.12}$$

$$\|\sup_{u\leq t\wedge s} |\Lambda^{e,u}_t - \Lambda^{e,u}_s|\,\|_{L^p} \leq C^p_L|t-s|^{\frac{1}{2}} \ , \ t,s \leq T \,. \tag{5.13}$$

Using (5.6), we deduce that

$$\|\sup_{t\vee u\leq s\leq T} |\Lambda^{e,t}_s D_tX^\pi_s - \Lambda^{e,u}_s D_uX^\pi_s|\,\|_{L^p} \leq C^p_L\,|t-u|^{\frac{1}{2}} \ \ , \ u,t \leq T \,. \tag{5.14}$$

5.3 Regularity

In this section, we replace $(\mathbf{H}2)$ by the stronger assumption:

$(\mathbf{H}2')$: $\sigma \in C^2_b$ with derivatives up to order two bounded by L, and $h \in C^3_b$ with derivatives up to order three bounded by L.

The extension of the following results to $(\mathbf{H}2)$ will be obtained by using an approximation argument.

Proposition 5.2. *Let* ($\mathbf{H}1$)-($\mathbf{H}'$) *hold. Then*

$$||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||_{\mathcal{H}^2} \leq C_L \left(\alpha(\kappa)\, |\pi|^{\frac{1}{2}} + \epsilon(\pi)\right) ,$$

where $(\alpha(\kappa), \epsilon(\pi)) = (\kappa^{\frac{1}{4}}, |\pi|^{\frac{1}{4}})$ *under* ($\mathbf{H}1$), *and* $(\alpha(\kappa), \epsilon(\pi)) = (1, |\pi|^{\frac{1}{2}})$ *under* ($\mathbf{H}2'$).

The following remark prepares for the proof.

Remark 5.6. Set

$$\beta \;\; := \;\; \left(1 + \sup_{s\leq t\leq T} |D_s X_t| + \sup_{t\leq T} |X_t| + \sup_{s\leq t\leq T} |\Lambda^s_t|\right)^4 ,$$

and observe that, by (2.2), (5.4) and (5.9),

$$\|\beta\|_{L^p} \;\; \leq \;\; C^p_L \; , \; p \geq 2 \; . \tag{5.15}$$

Fix $t \leq T$ and let θ_1 and θ_2 be two stopping times such that $t \leq \theta_1 \leq \theta_2 \leq T$ $\mathbb{P}-$a.s. By the Lipschitz-continuity assumption on b and σ, we have

$$\mathbb{E}\left[|X_{\theta_1} - X_{\theta_2}|^2 \mid \mathcal{F}_{\theta_1}\right] \;\; \leq \;\; C_L\, \mathbb{E}\left[\beta(\theta_2 - \theta_1) \mid \mathcal{F}_{\theta_1}\right] \; . \tag{5.16}$$

Under ($\mathbf{H}2'$), we deduce from Itô's Lemma that

$$\left|\mathbb{E}\left[\nabla h(X_{\theta_2})\Lambda^t_{\theta_2}(D_t X)_{\theta_2} - \nabla h(X_{\theta_1})\Lambda^t_{\theta_1}(D_t X)_{\theta_1} \mid \mathcal{F}_{\theta_1}\right]\right| \leq C_L\, \mathbb{E}\left[\beta(\theta_2 - \theta_1) \mid \mathcal{F}_{\theta_1}\right] \; . \tag{5.17}$$

When ($\mathbf{H}1$) holds, we can use the bound $|\nabla h| \leq L$ to obtain

$$\begin{aligned} \left|\nabla h(X_{\theta_2})\Lambda^t_{\theta_2}(D_t X)_{\theta_2} - \nabla h(X_{\theta_1})\Lambda^t_{\theta_1}(D_t X)_{\theta_1}\right| \;\; &\leq \;\; \beta\, |\nabla h(X_{\theta_2}) - \nabla h(X_{\theta_1})| \\ &+ \;\; C_L \left|\Lambda^t_{\theta_2}(D_t X)_{\theta_2} - \Lambda^t_{\theta_1}(D_t X)_{\theta_1}\right| , \end{aligned}$$

which, by Lipschitz-continuity of ∇h, Itô's Lemma and the Cauchy-Schwartz inequality, implies

$$\mathbb{E}\left[\left|\nabla h(X_{\theta_2})\Lambda^t_{\theta_2}(D_t X)_{\theta_2} - \nabla h(X_{\theta_1})\Lambda^t_{\theta_1}(D_t X)_{\theta_1}\right| \mid \mathcal{F}_{\theta_1}\right] \leq C_L \left(\bar{\beta}\, \mathbb{E}\left[\beta(\theta_2 - \theta_1) \mid \mathcal{F}_{\theta_1}\right]\right)^{\frac{1}{2}} \tag{5.18}$$

where

$$\bar{\beta} := \sup_{t\leq T} \mathbb{E}\left[\beta^2 \mid \mathcal{F}_t\right] \quad \text{satisfies} \quad \|\bar{\beta}\|_{L^p} \leq C^p_L \; , \; p \geq 2 \; , \tag{5.19}$$

recall (5.15).

Proof of Proposition 5.2.
1. It follows from Corollary 5.1 that, after passing to a suitable version,

$$(Z^{\mathrm{d}\Re}_t)' = V^{j,t}_t \quad , \; r_j \leq t < r_{j+1} \; , \; j \leq \kappa - 1 \; ,$$

where, for $j \le \kappa - 1$,

$$
\begin{aligned}
V_t^{j,s} \;&:=\; \mathbb{E}\left[\nabla g(X_T)(\Lambda^s D_s X)_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(\Lambda^s D_s X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \mid \mathcal{F}_t\right] \\
&+\; \mathbb{E}\left[\int_s^{\tau_j} \nabla_x f(\Theta_u^{\mathrm{d}\Re})(\Lambda^s D_s X)_u \mathrm{d}u \mid \mathcal{F}_t\right] \;,\;\; s \le t \;.
\end{aligned}
$$

Also observe from (5.3) and (5.8) that

$$
V_t^{j,t} = A_t^j \eta_t \quad \text{for } t \le r_{j+1} \tag{5.20}
$$

where

$$
\begin{aligned}
A_t^j \;&:=\; \mathbb{E}\left[\nabla g(X_T)\Lambda_T^0 \nabla X_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(\Lambda^0 \nabla X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \mid \mathcal{F}_t\right] \\
&+ \mathbb{E}\left[\int_t^{\tau_j} \nabla_x f(\Theta_u^{\mathrm{d}\Re})(\Lambda^0 \nabla X)_u \mathrm{d}u \mid \mathcal{F}_t\right] \;,\quad t \le T
\end{aligned}
$$

and

$$
\eta_t \;:=\; (\Lambda_t^0 \nabla X_t)^{-1} \sigma(X_t) \;,\;\; t \le T \;.
$$

It follows that

$$
|Z_t^{\mathrm{d}\Re} - Z_{t_i}^{\mathrm{d}\Re}|^2 \;\le\; C_L \sum_{\ell_1=1}^{d} \sum_{\ell_2=1}^{d} |(A_t^j)^{\ell_1} (\eta_t)^{\ell_1,\ell_2} - (A_{t_i}^j)^{\ell_1} (\eta_{t_i})^{\ell_1,\ell_2}|^2
$$

where the superscript ℓ_1 and ℓ_1, ℓ_2 denote the components of the vector A^j and matrix η. In order to avoid too complicated notations, we shall now restrict to the case $d = 1$. The general case is obtained by the same argument, by working on each term $|(A_t^j)^{\ell_1}(\eta_t)^{\ell_1,\ell_2} - (A_{t_i}^j)^{\ell_1}(\eta_{t_i})^{\ell_1,\ell_2}|^2$ separately.

2. We first deduce from the definition of $V_t^{j,s}$ that, for $t \in [t_i, t_{i+1}) \subset [r_j, r_{j+1})$,

$$
|Z_t^{\mathrm{d}\Re} - Z_{t_i}^{\mathrm{d}\Re}| \le |V_t^{j,t} - V_t^{j,t_i}| + |V_t^{j,t_i} - V_{t_i}^{j,t_i}| \;, \tag{5.21}
$$

where, by (5.11),

$$
\|V_t^{j,t} - V_t^{j,t_i}\|_{L^2}^2 \;\le\; C_L \, |\pi| \;. \tag{5.22}
$$

Moreover, the martingale property of V^{j,t_i} on $[t_i, t_{i+1}]$, (5.4) and (5.20) imply that

$$
\begin{aligned}
\mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2\right] \;&\le\; \mathbb{E}\left[|V_{t_{i+1}}^{j,t_i}|^2 - |V_{t_i}^{j,t_i}|^2\right] \\
&\le\; \mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}}|^2 - |V_{t_i}^{j,t_i}|^2 + (|A_{t_{i+1}}^j \eta_{t_i}|^2 - |A_{t_{i+1}}^j \eta_{t_{i+1}}|^2)\right] + C_L|\pi|.
\end{aligned} \tag{5.23}
$$

3. In this part, we study the first term in the right-hand side of (5.23). Define i_j through $t_{i_j} = r_j$, $j \leq \kappa$, and observe that

$$
\begin{aligned}
\Sigma &:= \sum_{j=0}^{\kappa-1} \sum_{k=i_j}^{i_{j+1}-1} \mathbb{E}\left[|V^{j,t_{k+1}}_{t_{k+1}}|^2 - |V^{j,t_k}_{t_k}|^2\right] \\
&= \sum_{j=0}^{\kappa-1} \mathbb{E}\left[|V^{j,r_{j+1}}_{r_{j+1}}|^2 - |V^{j,r_j}_{r_j}|^2\right] \\
&\leq \mathbb{E}\left[|V^{\kappa-1,r_\kappa}_{r_\kappa}|^2 - |V^{0,r_0}_{r_0}|^2\right] + \sum_{j=1}^{\kappa-1} \mathbb{E}\left[|V^{j-1,r_j}_{r_j}|^2 - |V^{j,r_j}_{r_j}|^2\right] \\
&\leq C_L \left(1 + \sum_{j=1}^{\kappa-1} \mathbb{E}\left[|V^{j-1,r_j}_{r_j}|^2 - |V^{j,r_j}_{r_j}|^2\right]\right) \qquad (5.24)
\end{aligned}
$$

where the last inequality follows from (5.15).

3.a. For ease of notations, we now write $\mathbb{E}_{r_j}[\cdot]$ for $\mathbb{E}\left[\cdot \mid \mathcal{F}_{r_j}\right]$. By Cauchy-Schwartz inequality,

$$
\begin{aligned}
|V^{j-1,r_j}_{r_j}|^2 - |V^{j,r_j}_{r_j}|^2 &\leq |V^{j-1,r_j}_{r_j} - V^{j,r_j}_{r_j}|\,|V^{j-1,r_j}_{r_j} + V^{j,r_j}_{r_j}| \\
&\leq C_L\,\mathbb{E}_{r_j}[\beta]\;|V^{j-1,r_j}_{r_j} - V^{j,r_j}_{r_j}|\,, \qquad (5.25)
\end{aligned}
$$

where β is defined in Remark 5.6.

Recalling that ∇g, ∇h are bounded by L and that $\tau_{j-1} \leq \tau_j \leq T$, we observe that

$$
\begin{aligned}
&\nabla g(X_T) D_t X_T \mathbf{1}_{\{\tau_j = T\}} + \nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} \mathbf{1}_{\{\tau_j < T\}} \\
&-\nabla g(X_T) D_t X_T \mathbf{1}_{\{\tau_{j-1} = T\}} - \nabla h(X_{\tau_{j-1}})(\Lambda^t D_t X)_{\tau_{j-1}} \mathbf{1}_{\{\tau_{j-1} < T\}} \\
&= \left(\nabla g(X_T) D_t X_T - \nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j}\right) \mathbf{1}_{\{\tau_j = T\}} \\
&- \left(\nabla g(X_T) D_t X_T - \nabla h(X_{\tau_{j-1}})(\Lambda^t D_t X)_{\tau_{j-1}}\right) \mathbf{1}_{\{\tau_{j-1} = T\}} \\
&+\nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} - \nabla h(X_{\tau_{j-1}})(\Lambda^t D_t X)_{\tau_{j-1}} \\
&\leq \beta \mathbf{1}_{\{\tau_{j-1} < \tau_j = T\}} + \left(\nabla h(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} - \nabla h(X_{\tau_{j-1}})(\Lambda^t D_t X)_{\tau_{j-1}}\right)\,.
\end{aligned}
$$

When (**H**1) holds, it then follows from (5.4), (5.9) and (5.18) that

$$
\begin{aligned}
|V^{j-1,r_j}_{r_j} - V^{j,r_j}_{r_j}| &\leq C_L\,\mathbb{E}_{r_j}\left[\mathbf{1}_{\{\tau_{j-1} < \tau_j = T\}}\right] \\
&\quad + C_L \left(\mathbb{E}_{r_j}\left[\beta(\tau_j - \tau_{j-1})\right] + \bar\beta^{\frac{1}{2}}\,\mathbb{E}_{r_j}\left[\beta(\tau_{j+1} - \tau_j)\right]^{\frac{1}{2}}\right)\,.
\end{aligned}
$$

Since $\sum_{j=1}^{\kappa-1} \mathbf{1}_{\{\tau_{j-1} < \tau_j = T\}} \leq 1$, the above inequality combined with (5.24) and (5.25) implies

$$
\begin{aligned}
\Sigma &\leq C_L\,\mathbb{E}\left[1 + \sum_{j=1}^{\kappa-1} \mathbb{E}_{r_j}[\beta]\left(\mathbb{E}_{r_j}\left[\beta(\tau_j - \tau_{j-1})\right] + \bar\beta^{\frac{1}{2}}\,\mathbb{E}_{r_j}\left[\beta(\tau_j - \tau_{j-1})\right]^{\frac{1}{2}}\right)\right] \\
&\leq C_L \left\{1 + \sum_{j=1}^{\kappa-1}\left(\mathbb{E}\left[\bar\beta\beta(\tau_j - \tau_{j-1})\right] + \mathbb{E}\left[\beta(\tau_j - \tau_{j-1})\right]^{\frac{1}{2}}\right)\right\}
\end{aligned}
$$

where we used Cauchy-Schwartz inequality and (5.19). By (5.19) again, this shows that

$$\begin{aligned}\Sigma &\leq C_L \left\{1+\mathbb{E}\left[\bar\beta\beta(\tau_{\kappa-1}-\tau_0)\right]+\sqrt{\kappa}\,\mathbb{E}\left[\beta(\tau_{\kappa-1}-\tau_0)\right]^{\frac12}\right\}\\ &\leq C_L\left(1+\sqrt{\kappa}\right)\ . \end{aligned}\tag{5.26}$$

3.b. Under ($\mathbf{H}2'$), we use exactly the same arguments except that we appeal to (5.17) instead of (5.18). This leads to

$$\Sigma \ \leq\ C_L\left\{1+\sum_{j=1}^{\kappa-1}\mathbb{E}\left[\bar\beta\beta(\tau_j-\tau_{j-1})\right]\right\}\ \leq\ C_L\ . \tag{5.27}$$

4. We now study the second term in the right-hand side of (5.23).
4.a. Using Cauchy-Schwartz inequality, (2.2), (5.2), (5.9), the Lipschitz continuity of σ and standard estimates, we first observe that

$$\begin{aligned}\mathbb{E}\left[|A^j_{t_{i+1}}\eta_{t_i}|^2-|A^j_{t_{i+1}}\eta_{t_{i+1}}|^2\right] &\leq C_L\,\mathbb{E}\left[|\eta_{t_i}-\eta_{t_{i+1}}|^4\right]^{\frac14}\\ &\leq C_L\,|\pi|^{\frac12}\ .\end{aligned}$$

It follows that

$$\Sigma' \ :=\ \sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\mathbb{E}\left[|A^j_{t_{k+1}}\eta_{t_k}|^2-|A^j_{t_{k+1}}\eta_{t_{k+1}}|^2\right]\leq C_L\,|\pi|^{-\frac12}\ . \tag{5.28}$$

4.b. We now work under ($\mathbf{H}2'$). We first observe that

$$\begin{aligned}\mathbb{E}\left[|A^j_{t_{i+1}}\eta_{t_i}|^2-|A^j_{t_{i+1}}\eta_{t_{i+1}}|^2\right] &\leq \mathbb{E}\left[|A^j_{t_i}|^2\left(|\eta_{t_i}|^2-|\eta_{t_{i+1}}|^2\right)\right]\\ &+\ \mathbb{E}\left[(|A^j_{t_{i+1}}|^2-|A^j_{t_i}|^2)\left(|\eta_{t_i}|^2-|\eta_{t_{i+1}}|^2\right)\right]\ .\end{aligned}$$

Since ($\mathbf{H}2'$) is in force, we can apply Itô's Lemma on $|\eta|^2$ between t_i and t_{i+1}. In view of (2.2), (5.2), (5.9), this leads to

$$\mathbb{E}\left[|A^j_{t_i}|^2\left(|\eta_{t_i}|^2-|\eta_{t_{i+1}}|^2\right)\right]\ \leq\ C_L\,|\pi|\ .$$

On the other hand, Cauchy-Schwartz inequality, Itô's Lemma applied to $|\eta|^2$ and (2.2), (5.2), (5.9) imply

$$\begin{aligned}&\mathbb{E}\left[(|A^j_{t_{i+1}}|^2-|A^j_{t_i}|^2)\left(|\eta_{t_i}|^2-|\eta_{t_{i+1}}|^2\right)\right]\\ &\leq \mathbb{E}\left[(|A^j_{t_{i+1}}|^2-|A^j_{t_i}|^2)^2\right]^{\frac12}\mathbb{E}\left[\left(|\eta_{t_i}|^2-|\eta_{t_{i+1}}|^2\right)^2\right]^{\frac12}\\ &\leq C_L\,|\pi|^{\frac12}\mathbb{E}\left[(|A^j_{t_{i+1}}|^2-|A^j_{t_i}|^2)^2\right]^{\frac12}\ .\end{aligned}$$

Moreover, Jensen's inequality, the bound on $\nabla_x f$ and (5.2), (5.9) show that

$$\mathbb{E}_{t_i}\left[|A^j_{t_{i+1}}|^2\right]\ \geq\ |\mathbb{E}_{t_i}\left[A^j_{t_{i+1}}\right]|^2\geq|A^j_{t_i}|^2-C_L\,\beta\,|\pi|\ ,$$

which implies

$$\mathbb{E}\left[(|A^j_{t_{i+1}}|^2 - |A^j_{t_i}|^2)^2\right] \;=\; \mathbb{E}\left[|A^j_{t_{i+1}}|^4 + |A^j_{t_i}|^4 - 2|A^j_{t_{i+1}}|^2|A^j_{t_i}|^2\right]$$
$$\leq\; \mathbb{E}\left[|A^j_{t_{i+1}}|^4 - |A^j_{t_i}|^4\right] + C_L\,|\pi|\;.$$

Thus, combining the above estimates and using Jensen's inequality again, we obtain

$$\begin{aligned}\Sigma' &= \sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\mathbb{E}\left[|A^j_{t_{k+1}}\eta_{t_k}|^2 - |A^j_{t_{k+1}}\eta_{t_{k+1}}|^2\right]\\ &\leq C_L\left(1 + |\pi|^{\frac12}\sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\mathbb{E}\left[|A^j_{t_{k+1}}|^4 - |A^j_{t_k}|^4\right]^{\frac12}\right)\\ &\leq C_L\left(1 + \left(\mathbb{E}\left[\sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}|A^j_{t_{k+1}}|^4 - |A^j_{t_k}|^4\right]\right)^{\frac12}\right)\end{aligned}$$

where the right-hand side term can be bounded by a straightforward adaptation of the arguments used in 3. under ($\mathbf{H}2'$). This shows that

$$\Sigma' \;\leq\; C_L\;. \tag{5.29}$$

5. By (5.21), (5.22), (5.23), the definition of Σ and Σ' in (5.24) and (5.28)

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[|Z^{\mathrm{d}\Re}_t - Z^{\mathrm{d}\Re}_{t_i}|^2\right]\mathrm{d}t \;\leq\; C_L\,|\pi|\,\left(1+\Sigma+\Sigma'\right)\;.$$

The proof is then concluded by appealing to (5.26) and (5.28) under ($\mathbf{H}1$), and to (5.27) and (5.29), under ($\mathbf{H}2'$), and by using Remark 3.5. □

Proof of Proposition 3.3 Let f_n be defined by :

$$f_n(x,y,z) \;=\; \int_{\mathbb{R}^{2d+1}}\phi_n(x-\xi, y-\upsilon, z-\zeta)f(\xi,\upsilon,\zeta)\mathrm{d}\xi\mathrm{d}\upsilon\mathrm{d}\zeta\;,$$

with $\phi_n(x,y,z) = n^{2d+1}\phi(n(x,y,z))$ and ϕ a compactly supported smooth probability density function on $\mathbb{R}^{2d+1}$. Since f is L-Lipschitz, so is f_n and moreover:

$$||f-f_n||_\infty \;\leq\; \frac{C_L}{n}\;,$$

for some $C>0$. Let σ_n, b_n, g_n, h_n be defined similarly for σ, b, g, h so that we have:

$$||\sigma-\sigma_n||_\infty + ||b-b_n||_\infty + ||g-g_n||_\infty + ||h-h_n||_\infty \;\leq\; \frac{C_L}{n}\;.$$

Let X^n be the forward diffusion associated to b_n and σ_n and let $(Y^{\mathrm{d}\Re,n}, Z^{\mathrm{d}\Re,n}, K^{\mathrm{d}\Re,n})$ be the solution of the discretely reflected BSDE (3.1) associated to X^n, f_n and g_n. Arguing as in Proposition 3.6 of [9], we get

$$||Z^{\mathrm{d}\Re} - Z^{\mathrm{d}\Re,n}||^2_{\mathcal{H}^2} \leq \frac{C_L}{n}\;. \tag{5.30}$$

Since, by Jensen's inequality,

$$\begin{aligned}||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||_{\mathcal{H}^2} &\leq ||\bar{Z}^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re,n}||_{\mathcal{H}^2} + ||Z^{\mathrm{d}\Re} - Z^{\mathrm{d}\Re,n}||_{\mathcal{H}^2} + ||Z^{\mathrm{d}\Re,n} - \bar{Z}^{\mathrm{d}\Re,n}||_{\mathcal{H}^2} \\ &\leq 2\,||Z^{\mathrm{d}\Re} - Z^{\mathrm{d}\Re,n}||_{\mathcal{H}^2} + ||Z^{\mathrm{d}\Re,n} - \bar{Z}^{\mathrm{d}\Re,n}||_{\mathcal{H}^2}\ ,\end{aligned}$$

the proof is concluded by applying Proposition 5.2 to $Z^{\mathrm{d}\Re,n}$, using (5.30) and letting n go to infinity. □

We now consider the case where the forward diffusion is approximated by its Euler scheme.

Proposition 5.3. *If* (**H**1)-(**H**′) *hold, then*

$$||Z^{\mathrm{d}\Re,e} - \bar{Z}^{\mathrm{d}\Re,e}||_{\mathcal{H}^2} \leq C_L\left(\kappa^{\frac{1}{4}}\,|\pi|^{\frac{1}{2}} + |\pi|^{\frac{1}{4}}\right)\ .$$

Proof. In view of Remark 5.2 and Remark 5.5, we can follow line by line the arguments of the proof of Proposition 5.2, after replacing the corresponding quantities in the definitions of β and $\bar{\beta}$, and re-defining, for $j \leq \kappa - 1$,

$$\begin{aligned}V_t^{j,s} &:= \mathbb{E}\left[\nabla g(X_T^\pi)(\Lambda^{e,s}D_sX^\pi)_T\mathbf{1}_{\{\tau_j^e=T\}} + \nabla h(X_{\tau_j^e})(\Lambda^{e,s}D_sX^\pi)_{\tau_j^e}\mathbf{1}_{\{\tau_j^e<T\}} \mid \mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\int_s^{\tau_j^e}\nabla_x f(\Theta_u^{\mathrm{d}\Re,e})(\Lambda^{e,s}D_sX^\pi)_u\mathrm{d}u \mid \mathcal{F}_t\right]\ ,\ s\leq t\ . \qquad (5.31)\end{aligned}$$

The only difference appears in step 2. Instead of using a relation like (5.3) for X^π (which does not hold), we use the martingale property of V^{j,t_i} on $[t_i, t_{i+1})$ and write

$$\begin{aligned}\mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2\right] &\leq \mathbb{E}\left[|V_{t_{i+1}}^{j,t_i}|^2 - |V_{t_i}^{j,t_i}|^2\right] \\ &\leq \mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}}|^2 - |V_{t_i}^{j,t_i}|^2 + |V_{t_{i+1}}^{j,t_{i+1}} - V_{t_{i+1}}^{j,t_i}|\,|V_{t_{i+1}}^{j,t_{i+1}} + V_{t_{i+1}}^{j,t_i}|\right]\ ,\end{aligned}$$

where by (5.5), (5.12), (5.14) and Cauchy-Schwartz inequality

$$\mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}} - V_{t_{i+1}}^{j,t_i}|\,|V_{t_{i+1}}^{j,t_{i+1}} + V_{t_{i+1}}^{j,t_i}|\right] \leq C_L\sqrt{|\pi|}\ .$$

The inequality (5.23) then becomes

$$\mathbb{E}\left[|V_t^{j,t_i} - V_{t_i}^{j,t_i}|^2\right] \leq \mathbb{E}\left[|V_{t_{i+1}}^{j,t_{i+1}}|^2 - |V_{t_i}^{j,t_i}|^2\right] + C_L\sqrt{|\pi|}\ .$$

□

Proof of Proposition 3.4 The required result follows from Proposition 5.3 and by arguing as in the proof of Proposition 3.3. □

We conclude this section with the proof of Proposition 3.1.
Proof of Proposition 3.1. Assume that (**H**′) holds. By Remark 5.4, we have

$$\mathbb{E}\left[\int_{t_i}^{t_{i+1}}|Z_s^{\mathrm{d}\Re}|^2ds\right] \leq C_L\,|\pi|\ .$$

Arguing as in the proof of Proposition 3.3, we obtain that the above bound holds without $(\mathbf{H}')$. The required result then follows from Itô's Lemma, the Lipschitz-continuity of f, (2.2), the bound on $Y^{\Re}$ given in (3.3) and Burkholder-Davis-Gundy's inequality, recall (3.1). □

A Appendix

Proof of Proposition 3.2

1. Set $\delta X = X - X^\pi$, $\delta Y = Y^{\mathrm{d}\Re} - Y^\pi$, $\delta\tilde{Y} = \tilde{Y}^{\mathrm{d}\Re} - \tilde{Y}^\pi$, $\delta Z = Z^{\mathrm{d}\Re} - Z^\pi$, $\delta f_s = f(X_s, \tilde{Y}^{\mathrm{d}\Re}_s, Z^{\mathrm{d}\Re}_s) - f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i})$ for $s \in [t_i, t_{i+1})$. Recalling (3.2), (3.6), (3.7), the fact that $\Re \subset \pi$ and using Itô's Lemma, we compute that for $t \in [t_i, t_{i+1})$

$$A^i_t \;:=\; \mathbb{E}_{t_i}\left[|\delta\tilde{Y}_t|^2 + \int_t^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s - \left|\delta Y_{t_{i+1}}\right|^2\right] = \mathbb{E}_{t_i}\left[\int_t^{t_{i+1}} 2\delta\tilde{Y}_s \delta f_s \mathrm{d}s\right] ,$$

recall that $\mathbb{E}_{t_i}[\cdot]$ stands for $\mathbb{E}[\cdot \mid \mathcal{F}_{t_i}]$. By (3.10), the Lipschitz-continuity of f and the inequality $xy \le cx^2 + c^{-1}y^2$, for $x, y \in \mathbb{R}_+$ and $c > 0$, we therefore obtain

$$\begin{aligned} A^i_t \;&\le\; \mathbb{E}_{t_i}\left[\int_t^{t_{i+1}} \alpha|\delta\tilde{Y}_s|^2 \mathrm{d}s + \frac{C_L}{\alpha}\left(|\pi|\;|\delta\tilde{Y}_{t_i}|^2 + \int_{t_i}^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s\right)\right] \\ &+\; \frac{C_L}{\alpha}\mathbb{E}_{t_i}\left[\int_t^{t_{i+1}} |X_s - X^\pi_{t_i}|^2 + |\tilde{Y}^{\mathrm{d}\Re}_s - \tilde{Y}^{\Re}_{t_i}|^2 + |Z^{\mathrm{d}\Re}_s - \bar{Z}^{\mathrm{d}\Re}_{t_i}|^2 \mathrm{d}s\right] \end{aligned}$$

where α is a positive parameter to be chosen later on. Using Gronwall's Lemma and taking α large enough, we deduce that, for $|\pi|$ small enough, there is some $\eta > 0$, independent of π, such that

$$\mathbb{E}_{t_i}\left[|\delta\tilde{Y}_{t_i}|^2 + \eta\int_{t_i}^{t_{i+1}} |\delta Z_s|^2 \mathrm{d}s\right] \;\le\; e^{C_L|\pi|}\mathbb{E}_{t_i}\left[|\delta Y_{t_{i+1}}|^2\right] + C_L B_i \tag{A.1}$$

$$\sup_{t\in[t_i,t_{i+1}]} \mathbb{E}_{t_i}\left[|\delta\tilde{Y}_t|^2\right] \;\le\; C_L\left(\mathbb{E}_{t_i}\left[|\delta Y_{t_{i+1}}|^2\right] + |\pi|\;|\delta\tilde{Y}_{t_i}|^2 + B_i\right) \tag{A.2}$$

where

$$B_i \;:=\; \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} \left(|X_s - X^\pi_{t_i}|^2 + |\tilde{Y}^{\mathrm{d}\Re}_s - \tilde{Y}^{\mathrm{d}\Re}_{t_i}|^2 + |Z^{\mathrm{d}\Re}_s - \bar{Z}^{\mathrm{d}\Re}_{t_i}|^2\right) \mathrm{d}s\right] .$$

2. Since $|\delta Y_{t_i}| \le \max\{|\delta\tilde{Y}_{t_i}|; |h(X_{t_i}) - h(X^\pi_{t_i})|\mathbf{1}_{t_i\in\Re}\}$ for $i < N$, see (3.1), (3.4) and (3.7), it follows from (A.1) applied at $t = t_i$ and the Lipschitz-continuity of h that, for $|\pi|$ small enough,

$$|\delta Y_{t_i}|^2 \;\le\; \max\left\{e^{C_L|\pi|}\mathbb{E}_{t_i}\left[|\delta Y_{t_{i+1}}|^2\right] + C_L B_i \;;\; L^2|\delta X_{t_i}|^2\mathbf{1}_{t_i\in\Re}\right\} . \tag{A.3}$$

We claim that, for all $0 \le k \le N$,

$$|\delta Y_{t_{N-k}}|^2 \le \mathcal{P}_k \quad := \quad L^2 e^{2kC_L|\pi|}|\delta X_{t_{N-k}}|^2 + C_L\, L^2|\pi|^2\mathbb{E}_{t_{N-k}}\left[(X_T^*)^2\right]\sum_{j=k}^{2k-1} e^{C_L j|\pi|}$$
$$+\, C_L \sum_{j=1}^{k} e^{C_L|\pi|(k-j)}\mathbb{E}_{t_{N-k}}\left[B_{N-j}\right]\ , \tag{A.4}$$

recall the definition of X^* after (2.5). For $k = 0$, the result follows from the Lipschitz-continuity of g (with the convention $\sum_{\emptyset} = 0$). Assume now that this inequality holds for some $k \le N-1$. Observing that (A.4) and (2.5) implies

$$e^{C_L|\pi|}\mathbb{E}_{t_{N-k-1}}\left[|\delta Y_{t_{N-k}}|^2\right] + C_L B_{N-k-1} \quad \le \quad \mathcal{P}_{k+1}$$

and that $\mathcal{P}_{k+1} \ge L^2|\delta X_{t_{N-(k+1)}}|^2$, we deduce from (A.3) that the inequality $|\delta Y_{t_{N-(k+1)}}|^2 \le \mathcal{P}_{k+1}$ holds too. This proves (A.4) which by (2.4) implies

$$\max_{i\le N}\mathbb{E}\left[|\delta Y_{t_i}|^2\right] \quad \le \quad C_L\ \left(|\pi| + N|\pi|^2 + \bar{B}\right)$$

with

$$\bar{B} := \mathbb{E}\left[\sum_{i=0}^{N-1} B_i\right]\ .$$

Since by assumption $N|\pi| \le L$, this implies

$$\max_{i\le N}\mathbb{E}\left[|\delta Y_{t_i}|^2\right] \quad \le \quad C_L\ \left(|\pi| + \bar{B}\right)\ . \tag{A.5}$$

3. Observing that for $s \in [t_i, t_{i+1})$

$$\mathbb{E}\left[\left|\tilde{Y}^{\mathrm{d}\Re}_s - \tilde{Y}^{\mathrm{d}\Re}_{t_i}\right|^2\right] \quad \le \quad C_L \int_{t_i}^{s}\mathbb{E}\left[|f(\Theta^{\mathrm{d}\Re}_u)|^2 + |Z^{\mathrm{d}\Re}_u|^2\right]\mathrm{d}u$$

it follows from (2.2), (3.3), the Lipschitz-continuity of f and the assumption $N|\pi| \le L$ that

$$\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[|\tilde{Y}^{\mathrm{d}\Re}_s - \tilde{Y}^{\mathrm{d}\Re}_{t_i}|^2\right]ds \quad \le \quad C_L\ |\pi|\ .$$

Combined with (2.4), this implies

$$\bar{B} \quad \le \quad C_L\ \left(|\pi| + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}\right)\ . \tag{A.6}$$

In view of (A.1) and (A.5), this leads to

$$\mathbb{E}\left[|\delta\tilde{Y}_{t_i}|^2 + \eta\int_{t_i}^{t_{i+1}}|\delta Z_s|^2\mathrm{d}s\right] \quad \le \quad (1 + C_L\,|\pi|)\,\mathbb{E}\left[|\delta Y_{t_{i+1}}|^2 + C_L B_i\right]\ , \tag{A.7}$$
$$\le \quad C_L\left(|\pi| + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}\right)$$

which, by (3.1), (3.7), (A.2), (A.5) and (A.6) shows that

$$\sup_{t\le T}\mathbb{E}\left[|\delta Y_t|^2\right]+\sup_{t\le T}\mathbb{E}\left[|\delta\tilde{Y}_t|^2\right] \;\le\; C_L(|\pi|+||Z^{\mathrm{d}\Re}-\bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2})\,. \tag{A.8}$$

Let i_j be defined through $t_{i_j}=r_j$. Using (3.1) and (3.7) again, we deduce from (A.7) that

$$\begin{aligned}\mathbb{E}\left[\int_{r_j}^{r_{j+1}}|\delta Z_s|^2\mathrm{d}s\right] &= \sum_{k=i_j}^{i_{j+1}-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}}|\delta Z_s|^2\mathrm{d}s\right]\\ &\le C_L\,\mathbb{E}\left[|\delta Y_{r_{j+1}}|^2-|\delta\tilde{Y}_{r_j}|^2+\sum_{k=i_j}^{i_{j+1}-1}(B_k+|\pi||\delta Y_{t_{k+1}}|^2)\right]\,.\end{aligned}$$

Since, by the Lipschitz continuity of h and g,

$$|\delta Y_{r_{j+1}}|^2\le|\delta\tilde{Y}_{r_{j+1}}|^2+L^2|\delta X_{r_{j+1}}|^2 \tag{A.9}$$

we obtain

$$\begin{aligned}\mathbb{E}\left[\int_{r_j}^{r_{j+1}}|\delta Z_s|^2\mathrm{d}s\right] &\le C_L\mathbb{E}\left[|\delta\tilde{Y}_{r_{j+1}}|^2-|\delta\tilde{Y}_{r_j}|^2+L^2|\delta X_{r_{j+1}}|^2\right]\\ &+ \mathbb{E}\left[\sum_{k=i_j}^{i_{j+1}-1}(B_k+|\pi|(|\pi|+\bar{B}))\right]\end{aligned} \tag{A.10}$$

where we used (A.5). It then follows from (A.6) and (2.4) that

$$||Z^{\mathrm{d}\Re}-Z^{\pi}||^2_{\mathcal{H}^2} \;=\; \mathbb{E}\left[\sum_{j=0}^{\kappa-1}\int_{r_j}^{r_{j+1}}|\delta Z_s|^2\mathrm{d}s\right]\le\; C_L\left(\kappa\,|\pi|+||Z^{\mathrm{d}\Re}-\bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}\right)\,. \tag{A.11}$$

This proves the second claim of Proposition 3.2.
4. Using Burkholder-Davis-Gundy's inequality and arguing as in the first steps of 1, we now compute that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[r_j,r_{j+1}]}|\delta\tilde{Y}_t|^2\right] &\le \mathbb{E}\left[\sup_{t\in[r_j,r_{j+1})}|\delta\tilde{Y}_t|^2+|\delta\tilde{Y}_{r_{j+1}}|^2\right]\\ &\le C_L\mathbb{E}\left[|\delta Y_{r_{j+1}}|^2+\int_{r_j}^{r_{j+1}}\left(|\delta f_s|^2+|\delta Z_s|^2\right)\mathrm{d}s+|\delta\tilde{Y}_{r_{j+1}}|^2\right]\\ &\le C_L\left(\bar{B}+\mathbb{E}\left[|\delta Y_{r_{j+1}}|^2+\int_{r_j}^{r_{j+1}}|\delta Z_s|^2\mathrm{d}s\right]+\max_{i\le N}\mathbb{E}\left[|\delta\tilde{Y}_{t_i}|^2\right]\right)\\ &\le C_L\left(|\pi|+||Z^{\mathrm{d}\Re}-\bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}\right)\end{aligned}$$

where we used (A.6), (A.8) and (A.10). Since

$$|\delta Y_t| \quad \leq \quad |\delta \tilde{Y}_t| + |h(X_t) - h(X_t^\pi)|$$

the first assertion of Proposition 3.2 follows from the Lipschitz-continuity of h and (2.4). □

Remark A.1. Observe that the inequality (A.3) implies

$$|\delta Y_{t_i}|^2 \quad \leq \quad e^{C_L|\pi|}\mathbb{E}_{t_i}\left[|\delta Y_{t_{i+1}}|^2\right] + C_L B_i + L^2|\delta X_{t_i}|^2 \mathbf{1}_{t_i\in\Re} \; .$$

In the case where the Euler scheme is replaced by an order one scheme X^π satisfying

$$\max_{i\leq N} \mathbb{E}\left[|X_{t_i} - X_{t_i}^\pi|^2\right] \leq C_L|\pi|^2 \; ,$$

the above inequality immediately leads to (A.5). Moreover, the term $\mathbb{E}\left[|\delta X_{r_{j+1}}|^2\right]$ in (A.10) is controlled in $C_L|\pi|^2$. Thus, (A.11) reads

$$\begin{aligned} ||Z^{\mathrm{d}\Re} - Z^\pi||^2_{\mathcal{H}^2} \; = \; \mathbb{E}\left[\sum_{j=0}^{\kappa-1}\int_{r_j}^{r_{j+1}} |\delta Z_s|^2 \mathrm{d}s\right] \quad &\leq \quad C_L\left(\kappa\,|\pi|^2 + |\pi| + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}\right) \\ &\leq \quad C_L\left(|\pi| + ||Z^{\mathrm{d}\Re} - \bar{Z}^{\mathrm{d}\Re}||^2_{\mathcal{H}^2}\right) \; , \end{aligned}$$

since $\kappa|\pi| \leq L$.

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