Finance and Stoch manuscript No.
(will be inserted by the editor)

Generalized stochastic target problems for the pricing and partial hedging under loss constraints - Application in optimal book liquidation

Bruno Bouchard · Ngoc-Minh Dang

Received: date / Accepted: date

Abstract We consider a singular with state constraints version of the stochastic target problems studied in [22], [23] and more recently [6], among others. This provides a general framework for the pricing of contingent claims under risk constraints. Our extended version perfectly suits to market models with proportional transaction costs and to order book liquidation issues. Our main result is a direct PDE characterization of the associated pricing function. As an example of application, we discuss the evaluation of VWAP-guaranteed type book liquidation contracts, for a general class of risk functions.

Keywords Stochastic target problems · State constraints · Pricing under risk constraint · Book liquidation

Mathematics Subject Classification (2010) 49L25 · 60J60 · 91G80

1 Introduction

Stochastic target technics have been originally introduced in mathematical finance by Soner and Touzi [21] in order to provide a PDE characterization of the super-hedging price of an European claim under gamma constraints.
The classical super-hedging problem takes the general form: find the minimal $Y^{\phi}(0)$ such that there exists a control ϕ, in a suitable admissibility set $\mathcal{A}$,

B. Bouchard
CEREMADE, Université Paris Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France
CREST, 3 Avenue Pierre Larousse, 92240 Malakoff, France
E-mail: bouchard@ceremade.dauphine.fr

N.M. Dang
CEREMADE, Université Paris Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France
CA Cheuvreux, 9 quai Paul Doumer, 92400 Courbevoie, France
E-mail: dang@ceremade.dauphine.fr

satisfying $Y^\phi(T) \geq g(X^\phi(T))$ $\mathbb{P}-$a.s., where g is the payoff function of an European claim, ϕ stands for the financial strategy, Y^ϕ for the wealth process and X^ϕ for the stock price process, which may be influenced by the financial strategy, as in large investor models for instance. In general, such a problem is treated in mathematical finance via the dual formulation approach which allows one to relate the minimal $Y^\phi(0)$ to a stochastic control problem in standard form. However, this approach heavily relies on the fact that the wealth dynamics is linear in the control and that the stocks prices are not influenced by the trading strategy. In particular, it does not apply to large investor models or to more general dynamics or constraints, such as gamma constraints. This was the motivation of Soner and Touzi for introducing the so-called stochastic target approach.

Their main discovery is a dynamic programming principle which is directly written on the associated stochastic target problem, and therefore does not appeal to any form of dual formulation, see Theorem 1 below. It turns out to be sufficient to provide a PDE characterization for the associated value function. This approach led to a series of papers providing a direct way to characterize super-hedging prices, see e.g. [8], [11], [24] and [25].

Up to the recent work of Bouchard, Elie and Touzi [6], this approach was however limited to super-hedging problems which in turn typically lead to high prices which are not reasonable in practice, see e.g. [13] and [12]. Apart from technical improvements, the main result of Bouchard, Elie and Touzi [6] is that pricing problems under risk constraints of the form: find the minimal $Y^\phi(0)$ such that there exists a control $\phi \in \mathcal{A}$ satisfying $\mathbb{E}\left[\ell(Y^\phi(T) - g(X^\phi(T)))\right] \geq p$, for some "loss function" ℓ and a threshold p, can actually be treated via the stochastic target approach of Soner and Touzi [21] and [22]. For ℓ of the form $\ell(r) = \mathbf{1}_{r\geq 0}$ and $p \in (0,1)$, one retrieves the quantile hedging problem of Follmer and Leukert [15]. When ℓ stands for a utility function and $p := \sup\{\mathbb{E}\left[\ell(Y^\phi(T))\right] \ : \ Y^\phi(0) = y_0, \ \phi \in \mathcal{A}\}$, this corresponds to a utility indifference pricing problem. More generally, one can treat risk constraints of the form $\mathbb{E}\left[\Psi(X^\phi(T), Y^\phi(T))\right] \geq p$, for a general class of "risk functions" $-\Psi$. The success ratio hedging problem of Follmer and Leukert [15] enters into this framework. Finally, American type constraints can be introduced, see Bouchard and Vu [9]. This provides a general framework for a direct characterization of risk based prices of contingent contracts.

In Bouchard, Elie and Touzi [6], the authors restrict to dynamics given by Brownian SDEs in which only the drift and the volatility coefficients are controlled. In this paper, we show how their results can be extended to the case where the dynamics are controlled by processes with bounded variations and state constraints have to be satisfied. This extension is mainly motivated by the pricing of a VWAP-type[1] book liquidation contract, however the domain of application is vast, in particular it perfectly suits to partial hedging problems under proportional transaction costs, see Example 3 below.

[1] VWAP means Volume Weighted Average Price, see Sect. 4 for a detailed presentation.

We therefore first consider a general abstract formulation that could be used in many different practical situations/models. It is presented in Sect. 2 together with examples of application. The associated general PDE characterization is provided in Sect. 3. The pricing problem of a VWAP-type book liquidation contract is fully discussed in Sect. 4. The proofs of our abstract results are collected in Sect. 5.

Notations: We denote by x^i the i-th component of a vector $x \in \mathbb{R}^d$, which will always be viewed as a column vector, with transposed vector $x^\top$, and Euclidean norm $|x|$. The element $e_i \in \mathbb{R}^d$ is the i-th unit vector: $e_i^j = \mathbf{1}_{i=j}$, $i, j \leq d$. The set $\mathbb{M}^d$ is the collection of d-dimensional square matrices M with coordinates M^{ij}, and norm $|M|$ defined by viewing M as an element of $\mathbb{R}^{d\times d}$. We denote by $\mathbb{S}^d$ the subset of elements of $\mathbb{M}^d$ that are symmetric. For a subset O of $\mathbb{R}^d$, we denote by $\bar{O}$ its closure, by $\mathrm{int}(O)$ its interior, by ∂O its boundary, and by $\mathbf{dist}(x, O)$ the Euclidean distance from x to O with the convention $\mathbf{dist}(x, \emptyset) = \infty$. We denote by $B_r(x)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. If $B = [s, t] \times O$ for $s \leq t$ and $O \subset \mathbb{R}^d$, we write $\partial_p B := ([s, t) \times \partial O) \cup (\{t\} \times \bar{O})$ for its parabolic boundary. Given a smooth function $\varphi : (t, x_1, \ldots, x_k) \in \mathbb{R}_+ \times \mathbb{R}^{kd} \to \mathbb{R}$, we denote by $\partial_t \varphi$ its derivative with respect to its first variable, we write $D\varphi$ and $D^2\varphi$ for the Jacobian and Hessian matrix with respect to $(x_1, \ldots, x_k)$, and $D_{x_i}\varphi$ and $D^2_{x_i}\varphi$ the Jacobian and Hessian matrix with respect to x_i, $i \geq 1$. Any inequality or inclusion involving random variables has to be taken in the a.s. sense. For a process L with bounded variations, we write $|L|.$ to denote its total variation.

2 Abstract formulation and dynamic programming

2.1 The general singular stochastic target problem with state constraints

We first describe the abstract model. We refer to Sect. 2.2 for examples of typical dynamics in finance, and to Sect. 4 for a full discussion of its application to the pricing of VWAP-type book liquidation contracts.

From now on, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a d-dimensional Brownian motion W, $d \geq 1$, $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ denote the right-continuous completed filtration generated by W, and $T > 0$ be a finite time horizon.

The abstract stochastic target problem is defined as follows.

Our set of controls is $\mathcal{U} \times \mathcal{L}$, where $\mathcal{U}$ stands for the set of all progressively measurable process ν in $L^2([0, T] \times \Omega)$ taking values in a given closed subset U of $\mathbb{R}^d$, and $\mathcal{L}$ denotes the set of continuous $\mathbb{R}^d$-valued adapted processes L which are non-decreasing (component by component) and such that $\mathbb{E}\left[|L|_T^2\right] < \infty$.

For $t \in [0, T]$, $z := (x, y) \in \mathbb{R}^d \times \mathbb{R}$ and $\phi := (\nu, L) \in \mathcal{A} := \mathcal{U} \times \mathcal{L}$, the controlled process $Z^\phi_{t,z} := (X^\phi_{t,x}, Y^\phi_{t,x,y})$ is defined as the $\mathbb{R}^d \times \mathbb{R}$-valued unique

strong solution of the stochastic differential equation

$$\begin{aligned} X^{\phi}_{t,x}(s) &= x + \int_t^s \mu_X(X^{\phi}_{t,x}(r),\nu_r)dr + \int_t^s \beta_X(X^{\phi}_{t,x}(r))dL_r \\ &\quad + \int_t^s \sigma_X(X^{\phi}_{t,x}(r),\nu_r)dW_r \\ Y^{\phi}_{t,x,y}(s) &= y + \int_t^s \mu_Y(Z^{\phi}_{t,x,y}(r),\nu_r)dr + \int_t^s \beta_Y(Z^{\phi}_{t,x,y}(r))^{\top}dL_r \\ &\quad + \int_t^s \sigma_Y(Z^{\phi}_{t,x,y}(r),\nu_r)^{\top}dW_r \ , \end{aligned} \tag{1}$$

where $(\mu_X,\sigma_X):(x,u)\in\mathbb{R}^d\times U\mapsto\mathbb{R}^d\times\mathbb{M}^d$, $(\mu_Y,\sigma_Y):(z,u)\in\mathbb{R}^{d+1}\times U\mapsto\mathbb{R}\times\mathbb{R}^d$, $\beta_X\in C^2(\mathbb{R}^d,\mathbb{M}^d)$ and $\beta_Y\in C^2(\mathbb{R}^{d+1},\mathbb{R}^d)$ are assumed to be Lipschitz continuous.

Given a family of non-empty Borel subsets $(O(t))_{t\leq T}$ of $\mathbb{R}^{d+1}$, the stochastic target problem consists in characterizing the value function

$$(t,x)\in[0,T]\times\mathbb{R}^d\mapsto v(t,x):=\inf\{y\in\mathbb{R}\ :\ (x,y)\in V(t)\}\ , \tag{2}$$

where the set valued map V is defined as

$$t\in[0,T]\mapsto V(t):=\{z\in\mathbb{R}^{d+1}\ :\ \mathcal{A}_{t,z}\neq\emptyset\}\ , \tag{3}$$

and

$$\mathcal{A}_{t,z}\ :=\ \{\phi\in\mathcal{A}\ :\ Z^{\phi}_{t,z}(s)\in O(s)\ \forall s\in[t,T]\ \mathbb{P}-\text{a.s.}\}\ . \tag{4}$$

In order to fully characterize the set valued map V in terms of the value function v, we shall assume all over this paper the following:

Standing Assumption 1: For all $(t,x)\in[0,T]\times\mathbb{R}^d$: $(x,y)\in O(t)$ and $y'\geq y\Rightarrow(x,y')\in O(t)$.

Remark 1 It follows from Standing Assumption 1 and standard comparison arguments for stochastic differential equations that $\mathcal{A}_{t,x,y'}\subset\mathcal{A}_{t,x,y}$ for $y'\geq y$. In particular, $(x,y')\in V(t)$ whenever $(x,y)\in V(t)$ and $y'\geq y$, so that $V(t)$ can be (at least when the infimum in the definition of v is achieved) identified to $\{(x,y)\in\mathbb{R}^d\times\mathbb{R}\ :\ y\geq v(t,x)\}$. More precisely, the following holds:

$$(x,y)\in V(t)\Rightarrow y\geq v(t,x)\text{ and }y>v(t,x)\Rightarrow(x,y)\in V(t)).$$

In order to give a sense to the following discussions, we also assume that:

Standing Assumption 2: v is locally bounded on $\bar{D}_Y$ where

$$D_Y:=\{(t,x)\in[0,T)\times\mathbb{R}^d\ :\ \exists\, y\in\mathbb{R}\text{ s.t. }(x,y)\in O(t)\}.$$

Remark 2 Obviously, the fact that all the relevant quantities take values in $\mathbb{R}^d$ is used to save notations. One could without difficulty restrict to the case where some components of X only take positive values, which is typically the case for prices of stocks or bonds. By putting to 0 part of the coefficients, one can also retrieve situations where W, L and X do not have the same effective dimension. One could similarly add a time dependence in the coefficients, e.g. by considering the first component of X as a time parameter.

2.2 Examples of application

Before to go further in the general treatment, let us immediately discuss some typical examples of application that motivate this work (see also Sect. 4 for an application to optimal book liquidation that will be studied in details).

Example 1 Let us first consider the case where $(\beta_X, \beta_Y) = 0$,

$$\mu_X(x,u) = \mathrm{diag}[x]\mu \ , \ \sigma_X(x,u) = \mathrm{diag}[x]\sigma$$

and

$$\mu_Y(x,y,u) = u^\top \mathrm{diag}[x]\mu \ , \ \sigma_Y(x,y,u) = u^\top \mathrm{diag}[x]\sigma$$

where $\mathrm{diag}[x]$ stands for the diagonal matrix with x^i as the i-th diagonal element, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{M}^d$. The dynamics (1) then read, for $s \in [t,T]$:

$$\begin{aligned} X_{t,x}(s) &= x + \int_t^s \mathrm{diag}\left[X_{t,x}(r)\right]\mu dr + \int_t^s \mathrm{diag}\left[X_{t,x}(r)\right]\sigma dW_r \\ Y^\nu_{t,x,y}(s) &= y + \int_t^s \nu_r^\top dX_{t,x,}(r) \ , \end{aligned}$$

where we only write X for $X^{\nu,L}$ and Y^ν for $Y^{\nu,L}$ because X is not affected by the control and Y depends on (ν, L) only through ν.

Restricting to initial condition $x \in (0,\infty)^d$, this corresponds to the d-dimensional Black and Scholes model: X^i models the dynamics of a financial asset, the risk free interest rate is 0, ν^i_t stands for the number of units of X^i held in a financial portfolio at time t, and Y is the associated wealth process starting from the initial endowment y.

If we now take O of the form:

$$O(t) = (0,\infty)^d \times \mathbb{R}\mathbf{1}_{t<T} + \mathbf{1}_{t=T}\left\{(x,y) \in (0,\infty)^d \times \mathbb{R} \ : \ y \geq g(x)\right\}, \ \ t \leq T \ ,$$

for some measurable map $g : \mathbb{R}^d \to \mathbb{R}$, the value function can be written as

$$v(t,x) := \inf\left\{y \in \mathbb{R} \ : \ Y^\nu_{t,x,y}(T) \geq g\left(X_{t,x}(T)\right) \ \text{ for some } \nu \in \mathcal{U}\right\} \ .$$

This corresponds to the usual definition of the super-hedging price of an European option of payoff function g.

For O of the form

$$O(t) = \left\{(x,y) \in (0,\infty)^d \times \mathbb{R} \ : \ y \geq g(x)\right\}, \ \ t \leq T \ ,$$

this corresponds to the super-hedging price of an American option.

Example 2 Let us now consider a two-dimensional model $d = 2$ with the following parameters

$$\mu^1_X(x,u) = x^1\mu \ , \ \mu^2_X(t,u) = x^2\mu \ , \ \sigma^{11}_X(x,u) = x^1\sigma \ , \ \sigma^{21}_X(x,u) = x^2\sigma$$

and

$$\beta^{21}_X(x) = -1 \ , \ \beta^{22}_X(x) = 1 \ , \ \beta_Y(x,y) = (1-\lambda, -1-\lambda) \ ,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\lambda \in (0,1)$, and the other parameters are equal to 0. The dynamics (1) then read, for $s \in [t,T]$:

$$\begin{aligned} X^1_{t,x}(s) &= x^1 + \int_t^s X^1_{t,x}(r)\mu dr + \int_t^s X^1_{t,x}(r)\sigma dW^1_r \\ X^{2,L}_{t,x}(s) &= x^2 + \int_t^s \frac{X^{2,L}_{t,x}(r)}{X^1_{t,x}(r)} dX^1_{t,x}(r) - \int_t^s dL^1_r + \int_t^s dL^2_r \\ Y^L_{t,y}(s) &= y + \int_t^s (1-\lambda) dL^1_r - \int_t^s (1+\lambda) dL^2_r \,. \end{aligned}$$

This corresponds to the one-dimensional model with proportional transaction costs studied in [13]. More precisely, there is only one risky asset, X^1, with a Black and Scholes type dynamics. When buying or selling this risky asset, the investor pays a proportional transaction costs $\lambda \in (0,1)$. The process $\bar{L} := L^2 - L^1$ stands for the cumulative net amount of money invested in the risky asset from time 0, i.e. L^2_r (resp. L^1_r) is the cumulated value of bought (resp. sold) shares of X^1. Each time a buying or selling operation $d\bar{L}$ is done, the investor pays, in money, a proportional transaction cost $\lambda d|\bar{L}|$. The wealth process is described by the two dimensional process (Y, X^2) where Y models the evolution of the cash account, and X^2 corresponds to the value of the part of the portfolio invested in X^1, when taking into account the transaction costs.

For O defined for $t \le T$ as

$$O(t) = (0,\infty) \times \mathbb{R}^2 \mathbf{1}_{t<T} + \mathbf{1}_{t=T} \left\{ (x,y) \in (0,\infty) \times \mathbb{R}^2 \ : \ \Lambda(y,x) \ge g(x) \right\} \ ,$$

where

$$\Lambda(y,x) := y + x^2 - \lambda |x^2|$$

provides the value in cash of a terminal position (y, x^2) if the value of the stock is x^1, one retrieves the notion of super-hedging price of an European option with cash delivery, in the financial market model with proportional transaction costs. Obviously, one can consider similarly markets with more than one risky asset, see e.g. [8].

Example 3 As shown in [13] and [8], the super-hedging criteria is much too strict in markets with proportional transaction, as it leads to degenerate strategies of buy-and-hold type which do not reflect the market behavior. It follows that it should be relaxed by using, for instance, quantile or expected loss approaches as studied in [15], [16], for frictionless markets.

The loss function pricing approach consists in choosing a non-decreasing (typically concave) function $\ell : \mathbb{R} \to \mathbb{R}$ and defining the price at time t of an European option of payoff (say with cash delivery) $g\left(X^1_{t,x}(T)\right)$ as $\hat{v}(t, x^1, 0; p)$ with $\hat{v}(t,x;p)$ defined as

$$\inf \left\{ y \in \mathbb{R} : \exists\, L \in \mathcal{L} \text{ s.t. } \mathbb{E}\left[\ell\left(\Lambda(Y^L_{t,y}(T), X^L_{t,x}(T)) - g\left(X^1_{t,x}(T)\right)\right)\right] \ge p \right\} \ ,$$

where p is a given threshold in $\mathbb{R}$, and Λ is defined as in the previous example.

Assuming that $\Xi^L_{t,x,y}(T) := \ell\left(\Lambda(Y^L_{t,y}(T), X^L_{t,x}(T)) - g\left(X^1_{t,x}(T)\right)\right) \in L^2$ for all initial conditions and control L, the arguments of Proposition 3.1 in [6] then show that

$$\hat{v}(t,x;p) = \inf\left\{y \in \mathbb{R} \ : \ \Xi^L_{t,x,y}(T) \geq X^{3,\nu}_{t,p}(T) \text{ for some } (L,\nu) \in \mathcal{L} \times \mathcal{U}\right\} ,$$

with $U = \mathbb{R}^2$ and

$$X^{3,\nu}_{t,p} := p + \int_t^{\cdot} \nu^1_s dW^1_s \ .$$

Indeed, if $\Xi^L_{t,x,y}(T) \geq X^{3,\nu}_{t,p}(T)$ then taking expectation leads to $\mathbb{E}\left[\Xi^L_{t,x,y}(T)\right] \geq p$, while, if $p_0 := \mathbb{E}\left[\Xi^L_{t,x,y}(T)\right] \geq p$, then the martingale representation theorem implies that we can find $\nu \in \mathcal{U}$ such that $\Xi^L_{t,x,y}(T) = X^{3,\nu}_{t,p_0}(T) \geq X^{3,\nu}_{t,p}(T)$.

Hence, this last example enters into our general framework with the dynamics given in Example 2 and an additional controlled process $X^{3,\nu}$ defined as above.

Example 4 Influence of the trading strategies can be incorporated in the previous example without much difficulties. It suffices to consider more general models in which the dynamics of X^1 depends on L. It can for instance take the form

$$\begin{aligned} X^{1,L}_{t,x}(s) = x^1 &+ \int_t^s X^{1,L}_{t,x}(r)\mu dr + \int_t^s X^{1,L}_{t,x}(r)\sigma dW^1_r \\ &- \int_t^s X^{1,L}_{t,x}(r)\beta_- dL^1_r + \int_t^s X^{1,L}_{t,x}(r)\beta_+ dL^2_r \end{aligned}$$

with $\beta_-, \beta_+ \geq 0$. In this case, a buying order drives the price up, while a selling order pushes the price down. Note that constraints on the liquidation value of the portfolio $(X^{2,L}, Y^L)$ could also be incorporated by playing with the definition of O. For instance,

$$\begin{aligned} O(t) = &\left\{(x,y) \in (0,\infty) \times \mathbb{R}^2 \ : \ \Lambda(y,x) \geq -c\right\} \mathbf{1}_{t<T} \\ &+ \left\{(x,y) \in (0,\infty) \times \mathbb{R}^2 \ : \ \Lambda(y,x) \geq g(x)\right\} \mathbf{1}_{t=T} \ , t \leq T \ , \end{aligned}$$

means that the liquidation value of the portfolio should never be less than $-c$.

2.3 Dynamic programming

We now come back to the abstract problem (2).

In order to provide a PDE characterization of the value function v, we shall appeal to the *geometric dynamic programming principle* introduced in [22] and [23] in the case $O(t) = \mathbb{R}^{d+1}$ for $t < T$, and extended in [9] in the general case.

It expresses the fact that $z \in V(t)$ if and only if one is able to find a control ϕ such that $Z^{\phi}_{t,z} \in O(\cdot) \cap V(\cdot)$ on $[t,T]$, i.e. $Z^{\phi}_{t,z}(s)$ lies in the domain $O(s)$,

which is our constraint, and $Z^{\phi}_{t,z}(s)$ is such that, starting from this point at time s, one can find a control on $[s,T]$ such that the state process remains in the domains $O(\cdot)$ on $[s,T]$, i.e. $Z^{\phi}_{t,z}(s) \in V(s)$ by definition of V.

This heuristic reasoning can be made rigorous under the following right-continuity assumption:

Standing Assumption 3. [Right-continuity of the target] For all sequence $(t_n, z_n)_n$ of $[0,T] \times \mathbb{R}^{d+1}$ such that $(t_n, z_n) \to (t,z)$, we have

$$t_n \geq t_{n+1} \text{ and } z_n \in O(t_n) \; \forall \; n \geq 1 \Longrightarrow z \in O(t) \; .$$

This also requires an extra technical condition that seems to be missing in [6][2]:

Standing Assumption 4: Fix $(t,z) \in [0,T] \times \mathbb{R}^{d+1}$. Assume that there exists a $\mathbb{P}-$a.s. square integrable progressively measurable process ν with values in U and a continuous and non-decreasing $\mathbb{R}^d$-valued adapted process L such that $Z^{\nu,L}_{t,z}(s) \in O(s)$ for all $s \in [t,T]$ $\mathbb{P}-$a.s. Then, $\mathcal{A}_{t,z} \neq \emptyset$.

Remark 3 Note that the Standing Assumption 4 concerns the integrability condition on (ν, L). As stated, (ν, L) may not belong to $\mathcal{A}$ because the square integrability condition is not satisfied a-priori. However, the Standing Assumption 4 asserts that it should be a-posteriori, after possibly modifying the controls. This may not be the case for the original formulation, but will often be so for an equivalent formulation (i.e. leading to the same set-valued map V). We can then apply the Geometric Dynamic Programming Principle stated below to the equivalent formulation and then deduce it for the original one.

This applies to Example 1 if U is bounded (in which case the assumption is satisfied for the original problem), or if, for instance, g is bounded and 0 belongs to U (in this last case, we can reduce to the situation where Y^{ν} lies in between fixed bounds, and the Itô isometry implies automatically the square integrability of ν). The same holds in Example 2 under an appropriate change of measure, whenever g has polynomial growth, see e.g. the arguments in Lemma 3.3 in [17]. Cases like in Example 3 in which an additional controlled martingale has to be added will be discussed in the context of our main example in Section 4. See Remark 6 below.

In the statement below, we denote by $\mathcal{T}_{[t,T]}$ the set of stopping times with values in $[t,T]$, for $t \leq T$, and use the notation

$$O \bigoplus^{\tau,\theta} V := O(\tau) \; 1_{\tau \leq \theta} + V(\theta) \; 1_{\tau > \theta} \;\; \text{ for } \theta, \tau \in \mathcal{T}_{[0,T]} \; .$$

Theorem 1 (Geometric Dynamic Programming Principle) *For all $t \leq T$ and all family $\{\theta^{\phi}, \; \phi \in \mathcal{A}\} \subset \mathcal{T}_{[t,T]}$:*

$$V(t) = \left\{ z \in \mathbb{R}^{d+1} \; : \; \exists \, \phi \in \mathcal{A} \;\; s.t. \; \{ Z^{\phi}_{t,z}(\theta^{\phi} \wedge \tau) \in O \bigoplus^{\tau,\theta^{\phi}} V \; , \; \forall \, \tau \in \mathcal{T}_{[t,T]} \} \right\} \; .$$

[2] We would like to thank the referee for pointing out to us this technical issue.

Proof See the Appendix. □

Under our Standing Assumption 1, recall Remark 1, Theorem 1 translates in terms of the value function v as follows:

Corollary 1 *Fix* $(t,x,y) \in [0,T] \times \mathbb{R}^{d+1}$ *and a family* $\{\theta^\phi,\ \phi \in \mathcal{A}\} \subset \mathcal{T}_{[t,T]}$.
(GDP1): *If* $y > v(t,x)$ *then there exists* $\phi \in \mathcal{A}$ *such that*

$$Y^\phi_{t,x,y}(\theta^\phi) \geq v(\theta^\phi, X^\phi_{t,x}(\theta^\phi)) \text{ and } Z^\phi_{t,x,y}(s \wedge \theta^\phi) \in O(s \wedge \theta^\phi)\ \forall\, s \in [t,T]\ \mathbb{P}-a.s.$$

(GDP2): *If* $y < v(t,x)$, *then* $\forall\, \phi \in \mathcal{A}$

$$\mathbb{P}\left[Y^\phi_{t,x,y}(\theta^\phi) > v(\theta^\phi, X^\phi_{t,x}(\theta^\phi)) \text{ and } Z^\phi_{t,x,y}(s \wedge \theta^\phi) \in O(s \wedge \theta^\phi)\ \forall\, s \in [t,T]\right] < 1\ .$$

Proof If $y > v(t,x)$ then $(x,y) \in V(t)$, by Remark 1. Applying Theorem 1, this implies that there exists $\phi \in \mathcal{A}$ such that

$$Z^\phi_{t,x,y}(\theta^\phi \wedge \tau) \in O \bigoplus^{\tau,\theta^\phi} V \quad \forall\, \tau \in \mathcal{T}_{[t,T]}.$$

Recalling that $(x',y') \in V(t')$ implies that $y' \geq v(t',x')$, Remark 1, the above combined with Standing Assumption 3 implies (GDP1).

If $y < v(t,x)$ then $(x,y) \notin V(t)$, by Remark 1. In particular, Theorem 1 implies that we can not find $\phi \in \mathcal{A}$ such that

$$Z^\phi_{t,x,y}(\theta^\phi \wedge \tau) \in O \bigoplus^{\tau,\theta^\phi} V \quad \forall\, \tau \in \mathcal{T}_{[t,T]}.$$

This implies that we can not find $\phi \in \mathcal{A}$ such that

$$Z^\phi_{t,x,y}(\theta^\phi \wedge s) \in O \bigoplus^{s,\theta^\phi} V\ \forall\, s \in [t,T]\ \mathbb{P}-\text{a.s.}$$

Again we conclude by using Remark 1. □

3 PDE characterization in the abstract model

Our main result is a direct PDE characterization of the risk constraint based pricing function v.

3.1 Heuristic derivation

Before to state our main result rigorously, let us first explain heuristically how it can be deduced from Corollary 1.

3.1.1 Interior of the domain

In the case where $O(t) = \mathbb{R}^{d+1}$ for all $t < T$ and $\beta_X = \beta_Y = 0$, it is shown in [6] and [23] that v should (in a suitable sense) solve on $[0,T) \times \mathbb{R}^d$

$$\sup \left\{ F^u \left(\cdot, v, \partial_t v, Dv, D^2 v \right) ,\ u \in \mathcal{N}_0(\cdot, v, Dv) \right\} = 0 \tag{5}$$

where, for $\Theta = (x, y, r, p, Q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, $u \in U$, and $\varepsilon \geq 0$,

$$\begin{aligned} F^u(\Theta) &:= \mu_Y(x,y,u) - r - p^\top \mu_X(x,u) - \frac{1}{2}\mathrm{Tr}\left[\sigma_X \sigma_X^\top(x,u) Q\right] \ , \\ \mathcal{N}_\varepsilon(\Theta) &:= \{u \in U \ : \ |N^u(x,y,p)| \leq \varepsilon\} \\ &\qquad \text{with } N^u(x,y,p) := \sigma_Y(x,y,u) - \sigma_X(x,u)^\top p \ . \end{aligned} \tag{6}$$

The reasoning behind the above result is the following. If $y = v(t,x)$, if the infimum in the definition of v is achieved, and if v is smooth, then Theorem 1 implies that there exists $\nu \in \mathcal{U}$ such that, with $\phi = (\nu, 0)$, $dY^\phi_{t,x,y}(t) \geq dv(t, X^\phi_{t,x}(t))$. Formally, this implies that ν_t should thus be such that the volatility terms equal $\sigma_Y(X^\phi_{t,x}(t), Y^\phi_{t,x,y}(t), \nu_t) = \sigma_X(X^\phi_{t,x}(t), \nu_t)^\top Dv(t, X^\phi_{t,x}(t))$ and the drift terms satisfy $\mu_Y(X^\phi_{t,x}(t), Y^\phi_{t,x,y}(t), \nu_t) \geq \mathcal{L}^{\nu_t}_X v(t, X^\phi_{t,x}(t))$ where, for a smooth function φ and $u \in U$,

$$\mathcal{L}^u_X \varphi := \partial_t \varphi + D\varphi^\top \mu_X(\cdot, u) + \frac{1}{2}\mathrm{Tr}\left[\sigma_X \sigma_X^\top(\cdot, u) D^2 \varphi\right] \ .$$

Since $(X^\phi_{t,x}(t), Y^\phi_{t,x,y}(t)) = (x,y) = (x, v(t,x))$, this imposes that the left-hand side of (5) is non-negative. On the other hand, the "optimality" of v should lead to equality in (5).

In our situation where $\beta_X, \beta_Y \neq 0$, one can also use the bounded variation process L in the dynamics (1) to insure that $dY^\phi_{t,x,y}(t) \geq dv(t, X^\phi_{t,x}(t))$. It suffices to find a direction $\ell \in \Delta_+ := [0,\infty)^d \cap B_1(0)$ such that $G^\ell(x, y, Dv(t,x)) > 0$, where

$$G^\ell(x,y,p) := \left(\beta_Y(x,y)^\top - p^\top \beta_X(x)\right) \ell \ ,$$

and to "push in this direction". This corresponds to reflecting the process $(s, X(s), Y(s))_s$ on the boundary of the set $\{(t', x', y') \ : \ y' \geq v(t', x')\}$. Assuming v smooth enough, it is possible only if such a ℓ exists.

It thus follows that v should satisfy either (5) or $G^\ell(x, v(t,x), Dv(t,x)) > 0$ for some $\ell \in \Delta_+$, i.e., at least,

$$H_0(x, v(t,x), \partial_t v(t,x), Dv(t,x), D^2 v(t,x)) \geq 0 \tag{7}$$

where, for $\Theta = (x, y, r, p, Q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, and $\varepsilon \geq 0$,

$$H_\varepsilon(\Theta) := \max\left\{F_\varepsilon(\Theta) \ , \ G(\Theta)\right\}$$

with

$$\begin{aligned} F_\varepsilon(\Theta) &:= \sup\left\{F^u(\Theta),\ u \in \mathcal{N}_\varepsilon(\Theta)\right\} \\ G(\Theta) &:= \max\left\{\left(\beta_Y(x,y)^\top - p^\top \beta_X(x)\right)\ell,\ \ell \in \Delta_+\right\} \ . \end{aligned}$$

3.1.2 Space boundary

We also have to take care of the state constraint $(X, Y) \in O$. To this purpose, we shall assume that the set

$$D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \ : \ (x, y) \in O(t)\} \ . \tag{8}$$

is smooth enough:

Standing Assumption 5: There exists a locally $C^{1,2}$ function δ on $[0, T) \times \mathbb{R}^{d+1}$ such that $\delta > 0$ in $\mathrm{int}(D)$, $\delta = 0$ on $\partial_0 D := \partial D \cap ([0, T) \times \mathbb{R}^{d+1})$, and $\delta < 0$ on $([0, T) \times \mathbb{R}^{d+1}) \setminus D$.

For (t, x) such that $(t, x, y) = (t, x, v(t, x)) \in \partial_0 D$, we can then follow the same reasoning as above, taking into account the fact that now, the control $\phi = (\nu, L)$ should be such that, at the same time, $d\delta(t, X^{\phi}_{t,x}(t), Y^{\phi}_{t,x,y}(t)) \geq 0$ and $dY^{\phi}_{t,x,y}(t) \geq dv(t, X^{\phi}_{t,x}(t))$. As above, this can be achieved either through the drift parts, once the Brownian parts are cancelled, or through the bounded variation process, in the case where a suitable inward direction is available. This leads to

$$H^{\mathrm{in}}_0(x, v(t, x), \partial_t v(t, x), Dv(t, x), D^2 v(t, x)) \geq 0 \tag{9}$$

where, for $\Theta = (t, x, y, r, p, Q) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, $u \in U$, and $\varepsilon \geq 0$,

$$H^{\mathrm{in}}_{\varepsilon}(\Theta) := \max\left\{F^{\mathrm{in}}_{\varepsilon}(\Theta) \ , \ G^{\mathrm{in}}(\Theta)\right\}$$

with

$$\begin{aligned} F^{\mathrm{in}}_{\varepsilon}(\Theta) &:= \sup_{u \in \mathcal{N}^{\mathrm{in}}_{\varepsilon}(\Theta)} \min\{F^u(\Theta) \ , \ \mathcal{L}^u_Z \delta(t, x, y)\} \\ G^{\mathrm{in}}(\Theta) &:= \max_{\ell \in \Delta_+} \min\left\{\left(\beta_Y(x, y)^\top - p^\top \beta_X(t, x)\right)\ell \ , \ D\delta(t, x, y)^\top \beta_Z(x, y)\ell\right\} \ . \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}^{\mathrm{in}}_{\varepsilon}(\Theta) &:= \left\{u \in \mathcal{N}_{\varepsilon}(\Theta) : |D\delta^\top \sigma_Z(x, y, u)| \leq \varepsilon\right\} \ , \\ \mathcal{L}^u_Z \varphi &:= \partial_t \varphi + D\varphi^\top \mu_Z(\cdot, u) + \frac{1}{2}\mathrm{Tr}\left[\sigma_Z \sigma_Z^\top(\cdot, u) D^2 \varphi\right] \\ \mu_Z &:= (\mu_X^\top, \mu_Y)^\top \ , \ \sigma_Z := [\sigma_X^\top \ \ \sigma_Y]^\top \ , \ \beta_Z := [\beta_X^\top \ \ \beta_Y]^\top \end{aligned}$$

for φ smooth.

3.1.3 Terminal condition

In order to fully characterize the value function v, it remains to define appropriate boundary conditions.

We first note that $(x, v(T-, x)) \in O(T)$ can be expressed as

$$v(T-, x) \geq w(x) := \inf \{y \in \mathbb{R} \ : \ (x, y) \in O(T)\} \ .$$

It follows that $v(T-, \cdot)$ should formally satisfy $v(T-, \cdot) \geq w$.

On the other hand, the fact that v satisfies (7) imposes a constraint on v and its gradient through $\mathcal{N}$ and G: $\mathcal{N}_0(x, v(t, x), Dv(t, x)) = 0$ or $G(t, x, v(t, x), Dv(t, x)) \geq 0$. As usual it should propagate up to the boundary. In order to take care of this constraint, we follow [6] and introduce the set valued map

$$\mathbf{N}(x, y, p) := \left\{r \in \mathbb{R}^d : \ r = N^u(x, y, p) \text{ for some } u \in U\right\} \ , \tag{10}$$

together with the signed distance function from its complement set $\mathbf{N}^c$ to the origin:

$$R \ := \ \mathbf{dist}\,(0, \mathbf{N}^c) - \mathbf{dist}\,(0, \mathbf{N}) \ . \tag{11}$$

Then,

$$0 \in \operatorname{int}(\mathbf{N}(x, y, p)) \text{ iff } R(x, y, p) > 0 \ . \tag{12}$$

With these notations, the terminal condition formally reads:

$$\min \{v(T-, x) - w(x) \ , \ M(x, v(T, x), Dv(T, x))\} = 0 \ , \tag{13}$$

where

$$\begin{aligned} &M(x, v(T, x), Dv(T, x)) \\ &:= \max\{R(x, v(T-, x), Dv(T-, x)) \ , \ G(x, v(T-, x), Dv(T-, x))\} \ . \end{aligned}$$

However, the above expression does not incorporate the part of the state constraint that may be imposed on (t, x). In order to take care of this, we shall make the following assumption.

Standing Assumption 6: The function δ admits a locally $C^{1,2}$ extension on $[0, T] \times \mathbb{R}^{d+1}$.

Under the above additional condition, we shall show that $v(T-, \cdot)$ indeed satisfies the constrained boundary condition

$$\min \left\{v(T-, x) - w(x) \ , \ M^{\text{in}}(T, x, v(T-, x), Dv(T-, x))\right\} = 0 \ , \tag{14}$$

when

$$(T, x, v(T-, x)) \in \partial D_T := \{T\} \times \left(\bigcap_{0<r<T} \bigcup_{0<\varepsilon\leq r} \partial O(T - \varepsilon) \right) \ , \tag{15}$$

where

$$M^{\text{in}} := \max\{R^{\text{in}}\ ,\ G^{\text{in}}\}$$

with R^{in} defined as R with $\mathbf{N}^{\text{in}}$ in place of $\mathbf{N}$ and

$$\begin{aligned}&\mathbf{N}^{\text{in}}(t,x,y,p)\\ &:= \{r \in \mathbb{R}^d : r = N^u(x,y,p) \text{ and } D\delta(t,x,y)^\top \sigma_Z(x,y,u) = 0 \text{ for some } u \in U\}\ .\end{aligned}$$

For later use, we set

$$\text{int}(D)_T := (\{T\} \times O(T)) \setminus \partial D_T\ . \tag{16}$$

3.2 Main results

As in [6], the operators F and F^{in} are in general neither upper-semicontinuous nor lower-semicontinuous and need to be relaxed, i.e. we have to consider their semi-relaxed upper- and lower-semicontinuous envelopes.

$$\begin{aligned}F^*(\Theta) &:= \limsup_{\varepsilon \searrow 0, \Theta' \to \Theta} F_\varepsilon(\Theta'), F_*(\Theta) := \liminf_{\varepsilon \searrow 0, \Theta' \to \Theta} F_\varepsilon(\Theta')\\ H^* &:= \max\{F^*, G\}, H_* := \max\{F_*, G\}\end{aligned}$$

and we define similarly $F^{\text{in}*}$, F^{in}_*, $H^{\text{in}*}$, H^{in}_* from F^{in}, H^{in} as well as R^*, R_*, $R^{\text{in}*}$, R^{in}_*, $M^{\text{in}*}$, M^{in}_*, M^*, M_*. For ease of notations, we shall simply write $H_*\varphi$ for $H_*(\cdot, \varphi, \partial_t\varphi, D\varphi, D^2\varphi)$, and use similar notations for all the above defined operators. We shall also write w_* and w^* for the lower- and upper-semicontinuous envelopes of w.

Remark 4 (i) It follows from the convention $\sup \emptyset = -\infty$ that $F_\varepsilon(\Theta) = -\infty$ whenever $\mathcal{N}_\varepsilon(x,y,q) = \emptyset$.

(ii) Since F_ε is non-decreasing in $\varepsilon \geq 0$, we have $H_*(\Theta) = \liminf_{\Theta' \to \Theta} H_0(\Theta')$. In particular, $F_*(\Theta) > -\infty$ implies that there exists a neighborhood of Θ on which $\mathcal{N}_0 \neq \emptyset$.

(iii) The same reasoning holds for F^{in}.

Since the value function v may not be continuous, we also introduce the corresponding semicontinuous envelopes:

$$v_*(t,x) := \liminf_{\substack{(t',x') \to (t,x)\\ (t',x') \in D_Y}} v(t',x')\ ,\ v^*(t,x) := \limsup_{\substack{(t',x') \to (t,x)\\ (t',x') \in D_Y}} v(t',x')\ ,\ (t,x) \in \bar{D}_Y\ , \tag{17}$$

where D_Y is defined as in Standing Assumption 2 and $\bar{D}_Y$ denotes its closure.

Before to state our main results, we need to introduce the following continuity assumption, compare with Assumption 2.1 in [6], which will be used to prove the sub-solution property.

Assumption 1 *Let (t_0, z_0, q_0) be an element of $D \times \mathbb{R}^d$.*

(i) *If $\mathcal{N}_0 \neq \emptyset$ on a neighborhood B of (z_0, q_0), then for every $\varepsilon > 0$ and $u_0 \in \mathcal{N}_0(z_0, q_0)$ there exists a locally Lipschitz map $\hat{\nu}$ defined on a neighborhood of B' of (z_0, q_0) such that $|\hat{\nu}(z_0, q_0) - u_0| \leq \varepsilon$ and $\hat{\nu} \in \mathcal{N}_0$ on B'.*

(ii) *If $\mathcal{N}_0^{\text{in}} \neq \emptyset$ on a neighborhood B of (t_0, z_0, q_0), then for every $\varepsilon > 0$ and $u_0 \in \mathcal{N}_0^{\text{in}}(t_0, z_0, q_0)$ there exists a locally Lipschitz map $\hat{\nu}$ defined on a neighborhood of B' of (t_0, z_0, q_0) such that $|\hat{\nu}(t_0, z_0, q_0) - u_0| \leq \varepsilon$ and $\hat{\nu} \in \mathcal{N}_0^{\text{in}}$ on B'.*

Under the above assumption, we shall show that v is a discontinuous viscosity solution of (7)-(9)-(13)-(14) in the following sense.

Theorem 2 *v_* is a viscosity super-solution on $\bar{D}_Y$ of*

$$\begin{cases} H^*\varphi \geq 0 & \text{on } \bar{D}_Y \cap ([0,T) \times \mathbb{R}^d) \\ \min\left\{(\varphi - w_*)\mathbf{1}_{\{F^*\varphi<\infty, G\varphi<0\}}\ ,\ M^*\varphi\right\} \geq 0 & \text{on } \bar{D}_Y \cap (\{T\} \times \mathbb{R}^d) \end{cases} \tag{18}$$

If Assumption 1 holds, then v^* is a viscosity sub-solution on $\bar{D}_Y \cap ([0,T] \times \mathbb{R}^d)$ of

$$\begin{cases} \begin{array}{c} H_*\varphi \leq 0 \ \text{ if } \ (\cdot,\varphi) \in \text{int}(D) \\ H_*^{\text{in}}\varphi \leq 0 \ \text{ if } \ (\cdot,\varphi) \in \partial D \end{array} & \text{on } \bar{D}_Y \cap ([0,T) \times \mathbb{R}^d) \\ \begin{array}{c} \min\{\varphi - w^*\ ,\ M_*\varphi\} \leq 0 \ \text{ if } \ (\cdot,\varphi) \in \text{int}(D)_T \\ \min\{\varphi - w^*\ ,\ M_*^{\text{in}}\varphi\} \leq 0 \ \text{ if } \ (\cdot,\varphi) \in \partial D_T \end{array} & \text{on } \bar{D}_Y \cap (\{T\} \times \mathbb{R}^d) \end{cases} \tag{19}$$

Note that, as usual, the state constraints appear only on the sub-solution property, see e.g. [19] and [20]. The proof of this result is reported in Sect. 5.

4 Application in optimal book liquidation

In this section, we study an application of our general model to the pricing of a book liquidation contract under a VWAP (Volume Weighted Average Price) constraint.

For sake of simplicity, we shall restrict to the case where W is a one-dimensional Brownian motion although $d \neq 2$, which amounts to set part of the coefficients equal to 0. We shall also consider time-dependent coefficients, which corresponds to adding a component interpreted as time in the process X and can always be done by suitably choosing the drift parameter.

Moreover, the dynamics will be only controlled by a real valued non-decreasing process L. We shall therefore only write X^L and Y^L, and now consider $\mathcal{L}$ as the set of continuous real-valued non-decreasing adapted processes L satisfying $\mathbb{E}\left[L_T^2\right] < \infty$. Still, similar arguments as those used in Example 3 will lead to the introduction of an additional control in $\mathcal{U}$, see Proposition 1 below.

The reader interested by optimal liquidation problems can consult the papers [1], [2], [3], [5], [7] and [18], and the references therein. All of them deal with optimal liquidation strategies, but none of them is concerned by the pricing of related derivatives.

4.1 Description of the model

The optimal book liquidation problem is the following. A financial agent asks a broker to sell on the market a total of $K > 0$ stocks on a time interval $[0, T]$. The broker takes the engagement that he will guarantee to his client a mean selling price which corresponds to (at least) $\gamma \in (0, 1)$ times the mean price of the market, i.e. the observed selling prices weighted by the volume of the corresponding transactions initiated by all the traders that are acting on the market on $[0, T]$. Such contracts are referred to as *VWAP guaranteed*. The financial agent pays to the broker a premium y at time 0.

The cumulated number of stocks sold by the broker on the market since time 0 is described by a continuous real-valued non-decreasing adapted process L. Given $L \in \mathcal{L}$, the dynamic of the broker's portfolio Y^L is given by

$$dY^L(t) = X^{L,1}(t)dL_t \ , \ Y^L(0) = 0$$

where $X^{L,1}$ represents the stock's selling price dynamics and is assumed to solve

$$\begin{aligned} dX^{L,1}(t) = {} & X^{L,1}(t)\mu(t, X^{L,1}(t))dt + X^{L,1}(t)\sigma(t, X^{L,1}(t))dW_t \\ & - X^{L,1}(t)\beta(t, X^{L,1}(t))dL_t \end{aligned}$$

where $\mu, \sigma, \beta : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying

$$\begin{gathered} x \in [0, \infty) \mapsto x\,(\mu(t, x), \sigma(t, x), \beta(t, x)) \text{ is uniformly Lipschitz,} \\ \text{uniformly in } t \in [0, T], \\ \text{and } \beta : [0, T] \times \mathbb{R} \to \mathbb{R}_+ \text{ is } C^2 \text{ in space, uniformly in time.} \end{gathered} \tag{20}$$

Note that we allow the trading strategy of the broker to have an impact on the price dynamics if $\beta \neq 0$.

For sake of simplicity, we model the intensity of all the transactions on the market (more precisely, of aggressive orders initiated by sellers) by a deterministic non-negative continuous process ϑ, so that

$$\Theta(t) := \int_0^t \vartheta(s)ds$$

denotes the cumulated number of stocks sold on the market since time 0. Then, the cumulated amount associated to selling orders in the market, denoted by $X^{L,2}$, has the dynamics

$$dX^{L,2}(t) = X^{L,1}(t)\vartheta(t)dt \ , \ X^{L,2}(0) = 0 \ .$$

To sum up, the liquidation gain of the broker augmented by the premium y paid at time 0 is then given by $y + Y^L(T)$. On the other hand, he has to deliver to his client at least γ times the mean selling price of the market $(X^{L,2}(T)/\Theta(T))$ multiplied by the number of stocks that have to be sold K.

In order to accept the contract, a highly risk adverse broker should then ask for an initial premium y such that

$$y + \left(Y^L(T)/K - \gamma X^{L,2}(T)/\Theta(T)\right) K \geq 0 \text{ for some } L \in \mathcal{L} \text{ s.t. } L_T - L_0 = K,$$

i.e. which a.s. compensates the loss made if the mean selling price of the market is not matched.

In practice, it is clear that the above problem does not make sense and needs to be relaxed. We shall therefore consider problems of the form

$$\begin{gathered}\text{Find the minimal } y \text{ s.t., for some } L \in \mathcal{L} \text{ with } L_0 = 0 \ , \\ L_T = K \text{ and } \mathbb{E}\left[\ell\left(y + \left[Y^L(T)/K - \gamma X^{L,2}(T)/\Theta(T)\right] K\right)\right] \geq p \ ,\end{gathered}$$

for $p \in \mathbb{R}$ and $\ell : \mathbb{R} \to \mathbb{R}$ non-decreasing.

Moreover, practitioners typically impose bounds on the cumulated number of sold stocks $X^{L,3} := L - L_0$. We shall therefore restrict to strategies $L \in \mathcal{L}$ such that

$$X^{L,3}(s) \in [\underline{\Lambda}(s), \overline{\Lambda}(s)] \quad \text{for all } s \leq T \ ,$$

where $\underline{\Lambda}$ and $\overline{\Lambda}$ are assumed here to be C^1 deterministic functions such that

$$\underline{\Lambda} < \bar{\Lambda} \text{ on } [0, T) \ .$$

Remark 5 Up to an obvious change of variables, the initial premium y can be incorporated in the initial condition $Y(0)$ of Y. Similarly, the constant $\gamma K/\Theta(T)$ can be simply written $\gamma > 0$ up to a change of variable. It follows that the above problem could be alternatively written as

$$\begin{gathered}\text{Find the minimal } Y^L(0) \text{ s.t., for some } L \in \mathcal{L} \text{ with } L_0 = 0 \ , \\ X_T^{L,3} = K, \ X^{L,3}(s) \in [\underline{\Lambda}(s), \overline{\Lambda}(s)] \quad \text{for all } s \leq T \ , \\ \text{and } \mathbb{E}\left[\ell\left(Y^L(T) - \gamma X^{L,2}(T)\right)\right] \geq p \ ,\end{gathered}$$

with $\gamma > 0$.

4.2 Value function and problem reduction

In order to define the associated value function, we now extend the above dynamics to arbitrary initial conditions. Given $L \in \mathcal{L}$, we set $X^L := (X^{L,1}, X^{L,2}, X^{L,3})$. We write $Z^L_{t,x,y} = (X^L_{t,x}, Y^L_{t,x,y})$ the corresponding processes satisfying the initial condition $Z^L_{t,x,y}(t) = (x, y)$.

In the following, we restrict to initial conditions $y \geq 0$ and $x = (x^1, x^2, x^3) \in (0, \infty) \times [0, \infty)^2$ to be consistent with the fact that the above quantities should be non-negative and that the process $X^{L,1}$ takes positive values if $X^{L,1}(0) > 0$. In order to simplify our analysis, we make the following assumption:

$$\underline{\Lambda}(T) = \overline{\Lambda}(T) = K \ \text{ and } D\underline{\Lambda}, D\overline{\Lambda} \in (0, M] \text{ on } [0, T] \text{ for some } M > 0. \quad (21)$$

The first condition allows us to impose the constraint $X^{3,L}(T) = K$ via the simpler one $X^{3,L} \in [\underline{\Lambda}, \overline{\Lambda}]$, while the assumption on the right-hand side will be used in the proof of Proposition 5 below in order to provide boundary conditions which will turn easier to handle.

In view of Remark 5 and the left-hand side of (21), the value function associated to the above stochastic target problem can then be written as

$$v(t,x,p) := \inf\{y \geq 0 \ : \ \exists L \in \mathcal{L} \ \text{ s.t. } X^{3,L}_{t,x} \in [\underline{\Lambda}, \overline{\Lambda}] \text{ and } \mathbb{E}\left[\Psi(Z^L_{t,x,y}(T))\right] \geq p\} \ ,$$

with $\Psi(x,y) = \ell(y - \gamma x^2)$ and $\gamma > 0$.

In order to convert the above problem into a stochastic target problem in the form of the one studied in the previous sections, we use the key argument of [6] as explained in Example 3. In the following we set $\mathcal{A} := \mathcal{U} \times \mathcal{L}$, where $\mathcal{U}$ denotes the set of all progressively measurable process ν in $L^2([0,T] \times \Omega)$ taking values in $\mathbb{R}$.

Proposition 1 *Assume that ℓ has polynomial growth. Then, for all $(t,x,p) \in [0,T] \times (0,\infty) \times [0,\infty)^2 \times \mathbb{R}$,*

$$v(t,x,p) := \inf\{y \geq 0 \ : \ \mathcal{A}_{t,x,y,p} \neq \emptyset\} \ ,$$

where $\mathcal{A}_{t,x,y,p}$ denotes the set of processes $(\nu, L) \in \mathcal{A}$ such that $(Z^L_{t,x,y}, P^\nu_{t,p}) \in O$ on $[t,T]$ with

$$\begin{aligned} O := & \left\{(x,y,p) \in (0,\infty) \times [0,\infty)^2 \times \mathbb{R}^2 \ : \ x^3 \in [\underline{\Lambda}, \overline{\Lambda}]\right\} \mathbf{1}_{[0,T)} \\ & + \left\{(x,y,p) \in (0,\infty) \times [0,\infty)^2 \times \mathbb{R}^2 \ : \ x^3 = K \ \textit{and} \ \ell(y - \gamma x^2) \geq p\right\} \mathbf{1}_{\{T\}} \ , \end{aligned}$$

and

$$P^\nu_{t,p} := p + \int_t^{\cdot} \nu_s dW_s \ .$$

Proof If ℓ has polynomial growth, then it is clear that $\ell(Y^L_{t,x,y}(T) - \gamma X^{2,L}_{t,x}(T)) \in L^2$ for all $L \in \mathcal{L}$ such that $X^{3,L}_{t,x}(T) \leq K$. It then suffices to reproduce the arguments used in Example 3 or in the proof of Proposition 3.1 in [6]. □

In the following, we set

$$D_Y := \{(t,x,p) \in [0,T) \times (0,\infty) \times [0,\infty)^2 \times \mathbb{R} \ : \ x^3 \in [\underline{\Lambda}(t), \overline{\Lambda}(t)]\} \ ,$$

which is the natural domain on which our problem is stated. It satisfies the Standing Assumption 2 under the additional condition (23) below, see Proposition 3 below. In the following, v_* and v^* are defined as in (17) for D_Y as above.

We conclude this section by showing that one can use an equivalent formulation for which the Standing Assumption 4 is satisfied.

Remark 6 In this remark, we assume that ℓ has polynomial growth. Fix $(\nu, L) \in \mathcal{A}_{t,x,y}$ and first note that $(Z^L_{t,x,y}, P^\nu_{t,p}) \in O$ implies $L_T - L_t = K - x^3$, so that the control L has to be $\mathbb{P}-$a.s. bounded by K. Second, the polynomial growth condition on ℓ, the uniform bound on L and the fact $P^\nu_{t,p}$ is a martingale imply that $P^\nu_{t,p} \leq \mathbb{E}\left[\Psi(Z^L_{t,x,y}(T))|\mathcal{F}_\cdot\right] \leq \varphi(Z^L_{t,x,y})$ on $[t,T]$ with $\varphi(\bar{z}) := C(1+|\bar{z}|^C)$ for some $C > 0$. It follows that

$$\mathcal{A}_{t,x,y,p} = \bar{\mathcal{A}}_{t,x,y,p} \ ,$$

where $\bar{\mathcal{A}}_{t,x,y,p}$ denotes the set of processes $(\nu, L) \in \mathcal{A}$ such that $(Z^L_{t,x,y}, P^\nu_{t,p}) \in \bar{O}$ on $[t,T]$ with

$$\bar{O} := O \cap \left\{(x,y,p) \in (0,\infty) \times [0,\infty)^2 \times \mathbb{R}^2 \ : \ \varphi(x,y) \geq p\right\} .$$

The stochastic target problem defined with $\bar{O}$ now satisfies Standing Assumption 4. Indeed, if L is an adapted continuous and non-decreasing process and ν is a $\mathbb{P}-$a.s. square integrable progressively measurable process such that $(Z^L_{t,x,y}, P^\nu_{t,p}) \in \bar{O}$, then L is bounded and the local martingale $P^\nu_{t,p}$ is bounded from above by the process $\varphi(Z^L_{t,x,y})$ that is easily seen to be uniformly integrable. It is therefore a submartingale. Then, $\Psi(Z^L_{t,x,y}(T)) \geq P^\nu_{t,p}(T)$ implies that $\mathbb{E}\left[\Psi(Z^L_{t,x,y}(T))\right] \geq p$. Since $\Psi(Z^L_{t,x,y}(T)) \in L^2$, the martingale representation theorem implies that $(Z^L_{t,x,y}, P^{\bar{\nu}}_{t,p}) \in \bar{O}$ for some predictable and square integrable process $\bar{\nu}$. The new control $(\bar{\nu}, L)$ belongs to $\bar{\mathcal{A}}_{t,x,y,p}$.

Since the problems defined with O and $\bar{O}$ are equivalent, the validity of the Geometric Dynamic Programming Principle for the latter implies its validity for the former.

4.3 Additional assumptions and a-priori estimates

In the context of the above problem, the sets $(\mathcal{N}_\varepsilon)_\varepsilon$ reads

$$\mathcal{N}_\varepsilon\varphi = \left\{u \in \mathbb{R} \ : \ |uD_p\varphi + x^1\sigma D_{x^1}\varphi| \leq \varepsilon\right\}$$

and

$$F_*\varphi = F^*\varphi = F_0\varphi \text{ if } D_p\varphi \neq 0 \ , \tag{22}$$

where

$$F_0\varphi := -\mathcal{L}_X\varphi - \frac{(x^1\sigma)^2}{2}\left(|D_{x^1}\varphi/D_p\varphi|^2 D^2_p\varphi - 2(D_{x^1}\varphi/D_p\varphi)D^2_{(x^1,p)}\varphi\right)$$

with

$$\mathcal{L}_X\varphi := \partial_t\varphi + x^1\mu D_{x^1}\varphi + x^1\vartheta D_{x^2}\varphi + \frac{1}{2}(x^1\sigma)^2 D^2_{x^1}\varphi \ .$$

In order to provide a PDE characterization in terms of the continuous operator F_0 rather than in terms of F^* and F_*, we need to ensure that $D_p\varphi \neq 0$ for any test function for v_* or v^*. Moreover, the proof of a comparison principle

will require a control of the ratio $D_{x^1}\varphi/D_p\varphi$ which appears in $F_0\varphi$. In order to control the last term, we shall assume from now on that:

$$\ell \text{ admits right- and left-derivatives,}$$
$$\text{there exists } \epsilon > 0 \text{ s.t. } \epsilon \le D^-\ell\ ,\ D^+\ell \le \epsilon^{-1}\ , \tag{23}$$
$$\text{and } \lim_{r\to\infty} D^+\ell(r) = \lim_{r\to\infty} D^-\ell(r) =: D\ell(\infty)\ , \tag{24}$$

where D^+ and D^- denote the right- and left-derivatives respectively.

The above conditions indeed induce the following controls on v, in which we use the notation $e_1 := (1,0,0)$.

Proposition 2 *For all $(t,x,p) \in \bar{D}_Y$ and $h \in (-(x^1 \wedge 1), 1)$*

$$v(t,x,p) \ge \max\{v(t,x,p-\epsilon^{-1}|h|) + |h|\ ,\ v(t,x+he_1,p-C(x)|h|)\}\ ,$$

where

$$C : [0,\infty)^3 \to \mathbb{R}_+ \text{ is a continuous map.} \tag{25}$$

Proof **a.** We start with the first inequality $v(t,x,p) \ge v(t,x,p-\epsilon^{-1}|h|) + |h|$. Fix $y > v(t,x,p)$. Then, there exists $\phi = (\nu, L) \in \mathcal{A}$ such that $(Z^L_{t,x,y}, P^\nu_{t,p}) \in V$ on $[t,T]$. Since, by (23), $\ell(r-|h|) \ge \ell(r) - \epsilon^{-1}|h|$, we have

$$\mathbb{E}\left[\ell\left(Y^L_{t,x,y}(T) - |h| - \gamma X^{L,2}_{t,x}(T)\right)\right] \ge p - \epsilon^{-1}|h|\ .$$

Since $Y^L_{t,x,y} - |h| = Y^L_{t,x,y-|h|}$ and X^L does not depend on the initial value of Y^L, this implies the required result by arbitrariness of $y > v(t,x,p)$ and the definition of the value function v.

b. Before to prove the second inequality, let us observe that standard computations based on Burkholder-Davis-Gundy's inequality, Gronwall's Lemma and the Lipschitz continuity assumption on our coefficients imply that there exists a continuous map $C : [0,\infty)^3 \to \mathbb{R}_+$ such that, for all $h \in [-1,1]$ and $L \in \mathcal{L}$ such that $|L|_T \le K$,

$$\mathbb{E}\left[\left|\Psi\left(Z^L_{t,x+he_1,y}(T)\right) - \Psi\left(Z^L_{t,x,y}(T)\right)\right|\right] \le C(x)|h|\ .$$

c. We now turn to the second inequality $v(t,x,p) \ge v(t,x+he_1,p-C(x)|h|)$. Fix $y > v(t,x,p)$ and consider $\phi = (\nu, L) \in \mathcal{A}$ such that $(Z^L_{t,x,y}, P^\nu_{t,p}) \in V$ on $[t,T]$. It follows from b. above that

$$\mathbb{E}\left[\Psi\big(Z^L_{t,x+he_1,y}(T)\big)\right] \ge \mathbb{E}\left[\Psi\big(Z^L_{t,x,y}(T)\big)\right] - C(x)|h| \ge p - C(x)|h|\ .$$

As above, the required result then follows from the arbitrariness $y > v(t,x,p)$. □

The immediate consequence of the above estimates is a control on $D_{x^1}\varphi/D_p\varphi$ for test functions of v^* or v_*.

Corollary 2 *The function v_* is a viscosity super-solution of*

$$\min\{D_p\varphi-\epsilon,(D_{x^1}\varphi-C(x)D_p\varphi)\mathbf{1}_{x^1>0},-D_{x^1}\varphi+C(x)D_p\varphi\}=0 \text{ on } \bar{D}_Y \tag{26}$$

and v^* is a viscosity sub-solution of

$$\max\{-D_p\varphi+\epsilon,(D_{x^1}\varphi-C(x)D_p\varphi)\mathbf{1}_{x^1>0},-D_{x^1}\varphi+C(x)D_p\varphi\}=0 \text{ on } \bar{D}_Y\ . \tag{27}$$

We now provide additional estimates that will be used later on to establish a comparison principle on the PDE associated to v. We first show that the conditions (21)-(23) allows us to deduce a classical growth condition on v.

Proposition 3 *There exists $\eta>0$ such that*

$$0\le v(t,x,p)\le \epsilon^{-1}|p-\ell(0)|+\gamma\eta(1+|x|) \text{ for all } (t,x,p)\in\bar{D}_Y\ . \tag{28}$$

Proof Define $L:=\max\{x^3,\underline{\Lambda}\}$, which belongs to $\mathcal{L}$ by (21). Then, it follows from (21) again that $X^{L,3}_{t,x}(T)=K$ and $X^{L,3}_{t,x}\in[\underline{\Lambda},\overline{\Lambda}]$ on $[t,T]$. Moreover, standard estimates imply

$$\mathbb{E}\left[|X^L_{t,x}(T)|^2\right]^{\frac{1}{2}}\le\eta(1+|x|)$$

for some $\eta>0$ which does not depend on (t,x). In particular, for $y>0$, the above inequality combined with our assumption (23) leads to

$$\begin{aligned}\mathbb{E}\left[\ell\left(Y^L_{t,x,y}(T)-\gamma X^{L,2}_{t,x}(T)\right)\right]&\ge\mathbb{E}\left[\ell\left(y-\gamma X^{L,2}_{t,x}(T)\right)\right]\\&\ge\epsilon\left(y-\gamma\eta(1+|x|)\right)+\ell(0)\ .\end{aligned}$$

By choosing y equal to the right-hand side of (28), we obtain the inequality $\mathbb{E}\left[\ell\left(Y^L_{t,x,y}(T)-\gamma X^{L,2}_{t,x}(T)\right)\right]\ge p$. The required result follows from the definition of v. □

We finally provide suitable boundary conditions for v.

Proposition 4 *Fix $(t,x,p)\in\bar{D}_Y$. Then,*

$$v_*(t,0,x^2,x^3,p)=v^*(t,0,x^2,x^3,p)=\Psi^{-1}(0,x^2,x^3,p)\ , \tag{29}$$

where

$$\Psi^{-1}(x,p):=\inf\{y\ge0\ :\ \Psi(x,y)\ge p\}\ .$$

Moreover, for all sequence $(t_n,x_n,p_n)_n\subset D_Y$ such that $(t_n,x_n)\to(t,x)\in[0,T]\times[0,\infty)^3$,

$$\lim_{n\to\infty}v_*(t_n,x_n,p_n)=\lim_{n\to\infty}v^*(t_n,x_n,p_n)=0 \text{ if } p_n\to-\infty\ , \tag{30}$$

$$\lim_{n\to\infty}v_*(t_n,x_n,p_n)/p_n=\lim_{n\to\infty}v^*(t_n,x_n,p_n)/p_n=1/D\ell(\infty) \text{ if } p_n\to\infty\ . \tag{31}$$

Proof **a.** We start with the first assertion. Let $(t_n, x_n, p_n)_n$ be a sequence in D_Y that converges to $(t, 0, x^2, x^3, p)$ and fix $y > \Psi^{-1}(x,\, p)$ with $x = (0, x^2, x^3)$. Then, the Lipschitz continuity of the coefficients implies that $X^0_{t_n,x_n}(T) \to (0, x^2, x^3)$ $\mathbb{P}-$ a.s. and in L^q for any $q \geq 2$. Since Ψ is Lipschitz continuous, it follows that $\lim_{n\to\infty} \mathbb{E}\left[\Psi(Z^0_{t_n,x_n,y}(T))\right] = \Psi(x, y) > p$. Hence, $\mathbb{E}\left[\Psi(Z^0_{t_n,x_n,y})\right] \geq p_n$ for n large enough, and therefore $v(t_n, x_n, p_n) \leq y$. The arbitrariness of y thus implies that $\limsup_{n\to\infty} v(t_n, x_n, p_n) \leq \Psi^{-1}(x, p)$. We next deduce from the Lipschitz continuity of the coefficients again that, for any $L^n \in \mathcal{L}$ such that $L^n_T \leq K$ and $(y_n)_{n\geq}$ such that $\mathbb{E}\left[\Psi(Z^{L^n}_{t_n,x_n,y_n}(T))\right] \geq p_n$, we have $Z^{L^n}_{t_n,x_n,y_n}(T) \to (0, x^2, x^3, y)$ $\mathbb{P}-$ a.s. and in L^q for any $q \geq 2$, whenever $y_n \to y$. Hence, $\lim_{n\to\infty} \mathbb{E}\left[\Psi(Z^{L^n}_{t_n,x_n,y}(T))\right] = \Psi(x, y) \geq p$ so that $y \geq \Psi^{-1}(x, p)$. Taking $y_n := v(t_n, x_n, p_n) + 1/n$ with $(t_n, x_n, p_n)_n$ such that $v(t_n, x_n, p_n) \to v_*(t, 0, x^2, x^3, p)$ then shows that $v_*(t, 0, x^2, x^3, p) \geq \Psi^{-1}(x, p)$.

b. We now turn to the second assertion. It follows from the following easy observation. Fix $(t, x) \in [0, T) \times (0, \infty) \times [0, \infty)^2$ such that $x^3 \in [\underline{\Lambda}(t), \overline{\Lambda}(t)]$. Then, for L defined by $L = \underline{\Lambda} + x^3 - \underline{\Lambda}(t)$, one obtains $X^{L,3}_{t,x} \in [\underline{\Lambda}, \overline{\Lambda}]$ on $[t, T]$, recall (21), and $\underline{p}(t, x) := \mathbb{E}\left[\Psi\left(Z^L_{t,x,0}(T)\right)\right] > -\infty$. It follows that $v(t, x, p) = 0$ for $p \leq \underline{p}(t, x)$, where the function $\underline{p}$ is clearly locally bounded.

c. We finally prove the last assertion. Since $p_n \to \infty$ and, for any strategy $L^n \in \mathcal{L}$ such that $L^n_T \leq K$, $Z^{L^n}_{t_n,x_n,0}(T)$ is uniformly bounded in any L^q, $q \geq 2$, one has $y_n \to \infty$ whenever $\mathbb{E}\left[\Psi(Z^{L^n}_{t_n,x_n,y_n}(T))\right] \geq p_n$ for all n. Using (24), one deduces that, for all $\varepsilon > 0$, $\exists\, r_\varepsilon \in \mathbb{R}$ such that

$$\Psi(Z^{L^n}_{t_n,x_n,y_n}(T)) \leq \ell(r_\varepsilon) + (D\ell(\infty) + \varepsilon)\left(y_n + Y^{L^n}_{t_n,x_n,0}(T) - \gamma X^{2,L^n}_{t_n,x_n}(T) - r_\varepsilon\right)$$

for n large $\mathbb{P}-$ a.s. It follows that

$$1 \leq \limsup_{n\to\infty} \mathbb{E}\left[\Psi(Z^{L^n}_{t_n,x_n,y_n}(T))\right]/p_n \leq (D\ell(\infty) + \varepsilon) \limsup_{n\to\infty} y_n/p_n\ .$$

This implies that $\liminf_{n\to\infty} v_*(t_n, x_n, p_n)/p_n \geq 1/\left(D\ell(\infty) + \varepsilon\right)$. Choosing ε arbitrarily small leads to the required result. On the other hand, for $y_n := p_n(D\ell(\infty) - \varepsilon)^{-1}$ with $\varepsilon \in (0, D\ell(\infty))$, we have, by similar arguments,

$$\Psi(Z^0_{t_n,x_n,y_n}(T))/p_n \to D\ell(\infty)(D\ell(\infty) - \varepsilon)^{-1} > 1$$

so that

$$\liminf_{n\to\infty} \mathbb{E}\left[\Psi(Z^0_{t_n,x_n,y_n}(T))\right]/p_n > 1$$

and therefore $v(t_n, x_n, p_n) \leq p_n(D\ell(\infty) - \varepsilon)^{-1}$ for n large enough. This implies that $\limsup_{n\to\infty} v^*(t_n, x_n, p_n)/p_n \leq (D\ell(\infty) - \varepsilon)^{-1}$, which yields the required result by arbitrariness of $\varepsilon > 0$. □

4.4 PDE characterization

We can now provide the main results of this section. We first report the PDE characterization of v.

Proposition 5 *The functions v_* is a viscosity supersolution on D_Y of*

$$\max\left\{F_0\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0\ . \tag{32}$$

The function v^* is a sub-solution on D_Y of

$$\begin{array}{rl} \min\left\{\varphi\ ,\ \max\left\{F_0\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\}\right\} = 0 & \textit{if}\ \ \underline{\Lambda} < x^3 < \overline{\Lambda} \\ \min\left\{\varphi\ ,\ x^1 + \beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0 & \textit{if}\ \ \underline{\Lambda} = x^3 \\ \min\left\{\varphi\ ,\ F_0\varphi\right\} = 0 & \textit{if}\ \ x^3 = \overline{\Lambda}\ . \end{array} \tag{33}$$

Moreover,

$$v_*(T,x,p) = v^*(T,x,p) = \Psi^{-1}(x,p)\ \forall\ (x,p) \in [0,\infty)^2 \times \{K\} \times \mathbb{R}\ . \tag{34}$$

Proof **a.** We first discuss the PDE characterization. In view of Theorem 2, we already know that v_* is a supersolution on D_Y of

$$\max\left\{F^*\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0$$

and that v^* is a sub-solution on $D_Y \cap \{v^* > 0\}$ of

$$\begin{array}{c} \max\left\{F_*\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0 \text{ if } \underline{\Lambda} < x^3 < \overline{\Lambda} \\ \max\left\{\min\{F_*\varphi\ ,\ -D\underline{\Lambda}\}\ ,\ \min\{x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\ ,\ 1\}\right\} = 0 \text{ if } \underline{\Lambda} = x^3 \\ \max\left\{\min\{F_*\varphi\ ,\ D\overline{\Lambda}\}\ ,\ \min\{x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\ ,\ -1\}\right\} = 0 \text{ if } x^3 = \overline{\Lambda}\ . \end{array}$$

Note that the boundary $x^2 = 0$ does not play any role here since the process $X^{2,L}$ is non-decreasing.

Since, by Proposition 2, v is strictly increasing in its p variable, any test function φ such that (t_0, x_0, p_0) achieves a local maximum (resp. minimum) of $v^* - \varphi$ (resp. $v_* - \varphi$) must then satisfy $D_p\varphi(t_0, x_0, p_0) > 0$. It follows that F^* and F_* can be replaced by F_0, see (22).

In order to simplify the sub-solution property for $t < T$, it then suffices to use the fact that $D\underline{\Lambda} > 0$ and $D\overline{\Lambda} > 0$ by assumption (21).

b. It remains to prove the boundary condition at T.

b.1. We first discuss the supersolution property at T. Let $(t_n, x_n, p_n)_{n\geq 1}$ be a sequence in D_Y, with $t_n < T$ for all n, such that $(t_n, x_n, p_n) \to (T, x_0, p_0) \in \bar{D}_Y$ and $v(t_n, x_n, p_n) \to v_*(T, x_0, p_0)$. Set $y_n := v(t_n, x_n, p_n) + 1/n$ and let L^n be such that $\mathbb{E}\left[\Psi(Z^n(T))\right] \geq p_n$ where $Z^n = (X^n, Y^n) := (X^{L^n}_{t_n,x_n}, Y^{L^n}_{t_n,x_n,y_n})$. Since $X^{n,2}(T) = K$, we have $L^n(T) - L^n(t_n) = K - x_n^3$. Since L^n is non-decreasing, this shows that $\sup_{t_n\leq t\leq T} L^n(t) - L^n(t_n) \to 0$ in L^∞. This implies that $Z^n(T) \to z_0 := (x_0, y_0)$ in any L^q, $q \geq 2$. It then follows from the dominated convergence theorem that

$$\Psi(z_0) - p_0 = \lim_{n\to\infty} \mathbb{E}\left[\Psi(Z^n(T))\right] - p_n \geq 0\ .$$

This shows that $v_*(T,x_0,p_0) \geq \Psi^{-1}(x_0,p_0)$.
b.2. We finally prove the sub-solution property. Let $(T,x_0,p_0) \in \bar{D}_Y$ and φ be a smooth function such that

$$\begin{aligned}&(T,x_0,p_0) \text{ achieves a strict local maximum of } v^* - \varphi \text{ such that}\\ &\quad (v^*-\varphi)(T,x_0,p_0) = 0 \ \text{ and } \varphi(T,x_0,p_0) > \Psi^{-1}(x_0,p_0)\ . \end{aligned} \tag{35}$$

Let $(t_n,x_n,p_n)_{n\geq 1}$ be a sequence in D_Y, with $t_n < T$ for all n, which converges to (T,x_0,p_0) and such that $v(t_n,x_n,p_n) \to v^*(T,x_0,p_0)$. Set $y_n := v(t_n,x_n,p_n) - 1/n$. Since $\underline{\Lambda}$ and $\overline{\Lambda}$ are C^1, there exists L^n such that $X^{L^n,2}_{t_n,x_n}$ is reflected on the boundary of $[\underline{\Lambda},\overline{\Lambda}]$. It takes the form

$$dL^n_t = \alpha^n_t dt \tag{36}$$

where α^n is a predictable process satisfying, recall (21),

$$\sup_{t\in[t_n,T]} |\alpha^n_t| \leq \sup_{n\geq 1}\sup_{t\in[t_n,T]} \big(D\underline{\Lambda}(t) \vee D\overline{\Lambda}(t)\big) \leq M\ . \tag{37}$$

Since φ is smooth, we can also define the control

$$\nu^n := -\sigma(\cdot, X^{L^n,1}_{t_n,x_n})(D_{x^1}\varphi/D_p\varphi)(\cdot, X^{L^n,1}_{t_n,x_n}),$$

recall from the above discussion that we must have $D_p\varphi > 0$ on a neighborhood of (T,x_0,p_0). For ease of notations, we write $Z^n = (X^n,Y^n) := (X^{L^n}_{t_n,x_n}, Y^{L^n}_{t_n,x_n,y_n})$ and $P^n := P^{\nu^n}_{t_n,p_n}$.

Let $\tilde{\varphi}$ be defined by $\tilde{\varphi}(t,x,p) := \varphi(t,x,p) + \sqrt{T-t+\iota} - \sqrt{\iota}$ for some $\iota > 0$. It follows from the identity $v(T,\cdot) = \Psi^{-1}$ and (35) that

$$\max_{(\{T\}\times\bar{B}_\varepsilon(x_0,p_0))\cup([T-\varepsilon,T]\times\partial B_\varepsilon(x_0,p_0))} (v^* - \tilde{\varphi}) := -\zeta < 0\ ,$$

for $\varepsilon > 0$ small enough. Moreover, for $\varepsilon,\iota > 0$ small enough, one has

$$\inf_{|\alpha|\leq M} \big(x^1\alpha + F_0\tilde{\varphi} + \beta(x^1)\alpha D_{x^1}\tilde{\varphi} - \alpha D_{x^3}\tilde{\varphi}\big) \geq 0 \text{ on } [T-\varepsilon,T]\times\partial B_\varepsilon(x_0,p_0)\ , \tag{38}$$

since $\partial_t\tilde{\varphi} \to -\infty$ as $t\to T$ and $\iota \to 0$. We next define the stopping times

$$\begin{aligned}\theta^o_n &:= \inf\{s\geq t_n \ :\ (s,X^n(s),P^n(s)) \notin [T-\varepsilon,T]\times B_\varepsilon(x_0,p_0)\}\,,\\ \theta_n &:= \inf\{s\geq t_n \ :\ |Y^n(s) - \varphi(s,X^n(s),P^n(s))| \geq \varepsilon\} \wedge \theta^o_n\ .\end{aligned}$$

Using (36)-(37)-(38), the same arguments as in the proof of Proposition 9 below show that

$$Y^n(\theta_n) - v\left(\theta_n, X^n(\theta_n), P^n(\theta_n)\right) > 0$$

for n large enough. Recalling that $y_n = v(t_n,x_n,p_n) - n^{-1} < v(t_n,x_n,p_n)$, this is in contradiction with (GDP2) of Corollary 1. □

4.5 Heuristic description of the optimal strategy

The rigorous statement of a verification result is beyond the scope of this paper. Still we explain here how an optimal strategy could be constructed.

Let us assume that there exists a positive smooth function $\hat{v}$ which is a strong solution of (33) on D_Y and satisfies (34). Set

$$\mathcal{R} := \{(t,x,p) \in D_Y \ : \ F_0\hat{v}(t,x,p) = 0\}.$$

Fix $(0,x_0,p_0) \in D_Y$, $y_0 := \hat{v}(0,x_0,p_0)$. Assume that we can find $(\nu, L) \in \mathcal{A}$ such that

$$\nu = -X^{L,1}\tfrac{\sigma D_{x^1}\hat{v}}{D_p\hat{v}}(\cdot, X^L, P^\nu) \ \ Leb \times d\mathbb{P} \text{ on } [0,T] \tag{39}$$

$$0 = \int_0^T \mathbf{1}_{\{(s,X^L_s,P^\nu_s)\in\mathcal{R}\}} dL_s \tag{40}$$

$$(\cdot, X^L, P^\nu) \in \mathcal{R} \subset D_Y \ \ Leb \times d\mathbb{P} \text{ on } [0,T] \tag{41}$$

where $(X^L, Y^L, P^\nu) := (X^L_{0,x_0}, Y^L_{0,x_0,y_0}, P^\nu_{0,p_0})$. It then follows from Itô's Lemma, the fact that $\hat{v} > 0$ solves (33) and (39)-(40)-(41) that

$$\begin{aligned} Y^L_t - \hat{v}(t, X^L_t, P^\nu_t) &= \int_0^t F_0\hat{v}(s, X^L_s, P^\nu_s)ds + \int_0^T X^{L,1}_s dL_s \\ &\quad + \int_0^t (X^{L,1}_s(\beta D_{x^1}\hat{v})(s, X^L_s, P^\nu_s) - D_{x^3}\hat{v}(s, X^L_s, P^\nu_s))dL_s \\ &\quad - \int_0^t (\nu_s D_p\hat{v}(s, X^L_s, P^\nu_s) + X^{L,1}_s(\sigma D_{x^1}\hat{v})(s, X^L_s, P^\nu_s))dW_s \\ &= 0 \ , \ t \in [0,T] \ . \end{aligned}$$

Since (34) holds for $\hat{v}$, this implies that $\Psi(Y^L_T, X^L_T) \geq P^\nu_T$. By taking expectation, we obtain $\mathbb{E}\left[\Psi(Y^L_T, X^L_T)\right] \geq p$. Since $X^{L,3}_T = K$ by (21)-(41), this shows that L is optimal, whenever $v(0,x_0,p_0) = \hat{v}(0,x_0,p_0)$. The latter can be deduced from a comparison principle, see the next section.

Note that solving (39)-(40)-(41) reduces to prove existence to a multidimensional stochastic differential equation reflected on the boundary of $\mathcal{R}$. This comes from the fact that the control ν can be identified to a Markovian control by (39). Existence of a solution requires a minimum of smoothness on the boundary of the domain, see e.g. [14] and the references therein. The same argument is actually used in the proof of Proposition 9, see the "Second case".

Again, a rigorous statement is beyond the scope of this paper. The approximation of the optimal strategy when the system is only observed at discrete times is also left for further research. Note however that, if $\hat{v}$ is strictly increasing in its p-variable and satisfies $\hat{v}(t,x,\infty) = \infty$ and $\hat{v}(t,x,-\infty) = 0$, compare with (30)-(31) and Corollary 2, then the process P^ν can be recovered from the observation of Y^L and X^L. Indeed, the above argument shows that we should have $Y^L = \hat{v}(\cdot, X^L, P^\nu)$. Hence, it suffices to invert the latter function: $P^\nu = \hat{v}^{-1}(\cdot, X^L, Y^L)$.

A formal "optimal" strategy in discrete time can be inferred from the above discussion. At the observation time t, compute $P_t := \hat{v}^{-1}(t, X^L_{t-}, Y^L_{t-})$. Then, find the minimal $\Delta L_t \geq 0$ such that (assuming it is possible) $X^{L,3}_t \in [\underline{\Lambda}(t), \overline{\Lambda}(t)]$ and $(t, X^L_t, P_t) \in \mathcal{R}$ where $X^L_{t-} := (X^{L,1}_t(1 - \beta(t, X^{L,1}_t)\Delta L_t), X^{L,2}_{t-}, X^{L,3}_{t-} + \Delta L_t)$. Update $Y^L_t := Y^L_{t-} + X^L_{t-}\Delta L_t$.

This requires the computation of the function $\hat{v}$. Note that the comparison results stated in the following section open the door to the study of deterministic numerical schemes. However, the construction of a monotone and consistent scheme is made difficult by the presence of a ratio in the operator F_0. We also leave this point for further research.

4.6 Comparison principle and uniqueness

In order to complete the characterization of Proposition 5, it remains to provide a comparison theorem for (32)-(33).

Note that the term $D_{x^1}\varphi/D_p\varphi$ which appears in the definition of F_0 can be shown to be bounded because viscosity super- and sub-solution of (32)-(33) have to satisfy $D_p\varphi \geq \epsilon$ and $|D_{x^1}\varphi/D_p\varphi| \leq C$ in the viscosity sense. Still obtaining a general comparison theorem for the above PDE in an unbounded domain remains an open question.

In what follows, we shall therefore reduce to a bounded domain by adding the following condition:

$$\exists\, \hat{x}^1 > 0 \text{ s.t. } \mu(\cdot, \hat{x}^1) = \sigma(\cdot, \hat{x}^1) = 0\ . \tag{42}$$

This implies that $\hat{x}^1$ is an absorbing point for $X^{L,1}$. In particular, it remains bounded as well as $X^{L,2}$ which is bounded by $\Theta(T)\hat{x}^1$. In particular, we can then restrict to the bounded domain

$$\hat{D}_Y := D_Y \cap \big([0,T) \times (0, 2\hat{x}^1) \times [0, 2\Theta(T)\hat{x}^1) \times [0,K] \times \mathbb{R}\big)\ .$$

For $x^1 > \hat{x}^1$, the value function v can be easily computed explicitly, since the problem becomes deterministic, and is continuous:

$$v(t,x,p) = \Psi^{-1}\left(x^1, \gamma\left(x^2 + \int_t^T x^1\vartheta(s)ds\right) - x^1(K - x^3), x^3, p\right) \text{ for } x^1 > \hat{x}^1\ . \tag{43}$$

Note that it is not a real limitation for practical applications, since $\hat{x}^1$ can be arbitrary large.

Proposition 6 *Assume that* (42) *holds. Let U (resp. V) be a non-negative lower-semicontinuous supersolution of* (32) *(resp. upper-semicontinuous subsolution of* (33)*) on $\hat{D}_Y$, such that U and V are continuous in x^3. Assume that*

$$U(t,x,p) \geq V(t,x,p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\}, \tag{44}$$

and that there exists $c_+ > 0$ *and* $c_- \in \mathbb{R}$ *such that*

$$\limsup_{(t',x',p')\to(t,x,\infty)} V(t',x',p')/p' \le c_+ \le \liminf_{(t',y',p')\to(t,y,\infty)} U(t',y',p')/p' \ , \quad (45)$$

$$\limsup_{(t',x',p')\to(t,x,-\infty)} V(t',x',p') \le c_- \le \liminf_{(t',y',p')\to(t,y,-\infty)} U(t',y',p') \quad (46)$$

$\forall\ (t,x) \in [0,T) \times [0,\infty)^3$. *If either* U *is a viscosity supersolution of* (26) *on* $\hat{D}_Y$ *which is continuous in* p*, or that* V *is a viscosity sub-solution of* (27) *on* $\hat{D}_Y$ *which is continuous in* p*, then*

$$U \ge V \quad \text{on } \hat{D}_Y .$$

Before to provide the proof of the above result, we state the following immediate corollary which shows that the characterization of v in Proposition 5 is indeed sharp.

Corollary 3 *Assume that* (42) *holds. Then,* v *is continuous on the closure of* $\hat{D}_Y$*. Moreover, it is the unique non-negative viscosity solution of* (32) *in the class of continuous functions that are either super-solutions of* (26) *or sub-solutions of* (27)*, and satisfy the boundary conditions* (29)-(30)-(31)-(34)-(43)*.*

This follows from Proposition 4, Corollary 2, Proposition 5, Proposition 6 and the following continuity result.

Proposition 7 *The function* v *is continuous in its* p *and* x^3 *variables, and, therefore, so are* v_* *and* v^**.*

Proof Since ℓ is non-decreasing, so is v, in the p-variable. It thus suffices to show that $\limsup_{|h|\to 0} v(t,x,p+|h|) \le v(t,x,p)$. To see this fix $y > v(t,x,p)$ and $L \in \mathcal{L}$ such that $Z^L_{t,x,y} \in V$ and $\mathbb{E}\left[\Psi(Z^L_{t,x,y}(T))\right] > p$, which is possible since ℓ is strictly increasing and $y > v(t,x,p)$. It follows that $y \ge v(t,x,p+|h|)$ for h small enough. Sending $|h| \to 0$ and then $y \to v(t,x,p)$ leads to the required result.

We now turn to the continuity with respect to x^3. Fix $h \in \mathbb{R}$ such that $|h| \le \min\{x^3 - \underline{\Lambda}(t), \overline{\Lambda}(t) - x^3\}$. Denote $e_3 := (0,0,1)$, and let $L \in \mathcal{L}$ and $y \ge 0$ be such that $Z^L_{t,x,y} \in V$. Then, $L_T - L_t \le K$ and $\sup_{t\le s\le T} |Z^L_{t,x,0}| \le c$ for some $c > 0$. Set $L^h := \mathbf{1}_{[t,T]}\left((L - L_t + x^3 + h) \wedge \overline{\Lambda}\right) \vee \underline{\Lambda}$. Then, $Z^{L^h}_{t,x+he_3,y} \in V$ and $\sup_{t\le s\le T} |Z^L_{t,x,0} - Z^{L^h}_{t,x+he_3,0}| \le c|h|$ where $c > 0$ does not depend on (t,x,p). This implies that $v(t,x+he_3,p) \le v(t,x,p) + c|h|$. Similarly, $v(t,x,p) \le v(t,x+he_3,p) + c|h|$. □

We conclude with the proof of Proposition 6.

Proof of Proposition 6. We assume that U is a viscosity supersolution of (26) which is continuous in p. The case where V is a viscosity sub-solution of (27) which is continuous in p is treated similarly. As usual, we argue by contradiction and assume that there exists $(t_0,x_0,p_0) \in \hat{D}_Y$ such that

$$(V - U)(t_0,x_0,p_0) =: 4\eta_0 > 0 \ .$$

Given $\kappa > 0$ and $\zeta > 1$, we define $\tilde{U}_\zeta$ and $\tilde{V}$ by $\tilde{U}_\zeta(t,x,p) := e^{\kappa(t+x^3)}U(t,x,p\zeta)$ and $\tilde{V}(t,x,p) := e^{\kappa(t+x^3)}V(t,x,p)$. Since U is continuous in its p-variable, one has

$$\sup_{\hat{D}_Y}(\tilde{V} - \tilde{U}_\zeta) =: 2\eta \geq 2\eta_0 > 0 \ , \tag{47}$$

for $\kappa > 0$ and $\zeta > 1$ small enough.

1. Note that $\eta < \infty$ and that

$$2\eta = \sup_{\hat{D}_Y}(\tilde{V} - \tilde{U}_\zeta) = (\tilde{V} - \tilde{U}_\zeta)(\hat{t}_\zeta, \hat{x}_\zeta, \hat{p}_\zeta) \tag{48}$$

for some $\hat{z}_\zeta := (\hat{t}_\zeta, \hat{x}_\zeta, \hat{p}_\zeta)$ in the closure of $\hat{D}_Y$. Indeed, if $(t_k, x_k, p_k)_{k\geq 1}$ is a maximizing sequence, then $(t_k, x_k)_{k\geq 1}$ is bounded and therefore converges along a subsequence. If $p_k \to -\infty$, we obtain a contradiction by appealing to (46). If $p_k \to \infty$, then $\limsup_{k\to\infty} V(t_k, x_k, p_k)/p_k \leq c_+ < c_+\zeta \leq \zeta \limsup_{k\to\infty} U(t_k, x_k, p_k\zeta)/(p_k\zeta)$, which also leads to a contradiction. The fact that the supremum is achieved then follows from the upper-semicontinuity of $\tilde{V} - \tilde{U}_\zeta$. From now on, we assume that

$$\hat{x}^3_\zeta = \underline{\Lambda}(\hat{t}_\zeta) \tag{49}$$

for all $\zeta > 1$ small enough. The case where $\hat{x}^3_\zeta = \overline{\Lambda}(\hat{t}_\zeta)$ (resp. $\hat{x}^3_\zeta \in (\underline{\Lambda}(\hat{t}_\zeta), \overline{\Lambda}(\hat{t}_\zeta))$) can be treated similarly by replacing $|x^3 - y^3 - \delta|^2$ in the definition of Λ_n below by $|y^3 - x^3 - \delta|^2$ (resp. 0).

2. For $n \geq 1$, we now set

$$\Theta_n : (t,x,y,p,q) \mapsto \tilde{V}(t,x,p) - \tilde{U}_\zeta(t,y,q) - \Lambda_n(t,x,y,p,q)$$

where

$$\begin{aligned}\Lambda_n(t,x,y,p,q) := \frac{n}{2}\left(|p-q|^2 + |x^1 - y^1|^2 + |x^3 - y^3 - \delta|^2\right) \\ + \frac{1}{2}\left(|t - \hat{t}_\zeta|^2 + |x^3 - \hat{x}^3_\zeta|^4\right)\end{aligned}$$

for some $\delta > 0$. Note that, by continuity of V in x^3 and (48),

$$\lim_{\delta\to 0}\inf_{n\geq 1} \Theta_n(\hat{t}_\zeta, \hat{x}_\zeta + \delta e_3, \hat{x}_\zeta, \hat{p}_\zeta, \hat{p}_\zeta) = 2\eta \ ,$$

with $e_3 := (0,0,1)$. It follows that

$$\sup_{\hat{D}^2_Y} \Theta_n \geq \eta \text{ for all } n \geq 1 \ , \tag{50}$$

for $\delta > 0$ small enough, where

$$\hat{D}^2_Y := \{(t,x,y,p,q) \in [0,T]\times[0,\infty)^6 \times \mathbb{R}^2 : (t,x,p),(t,y,q) \in \hat{D}_Y \ , \ x^2 = y^2\} \ .$$

Moreover, the same arguments as in step 1. above show that there exists $\bar{z}_n := (t_n, x_n, y_n, p_n, q_n)$ in the closure of $\hat{D}_Y$ satisfying

$$\Theta_n(\bar{z}_n) = \sup_{\hat{D}_Y^2} \Theta_n \geq \eta \text{ for all } n \geq 1 \ . \tag{51}$$

It then follows from standard arguments, combined with the ones used in step 1. above, see e.g. [10], that

$$\bar{z}_n \to \bar{z}_{\zeta,\delta} := (t_{\zeta,\delta}, x_{\zeta,\delta}, y_{\zeta,\delta}, p_{\zeta,\delta}, q_{\zeta,\delta}) \text{ in the closure of } \hat{D}_Y^2 \text{ as } n \to \infty \tag{52}$$

where

$$\begin{gathered} \lim_{n\to\infty} n|x_n^1 - y_n^1|^2 + n|p_n - q_n|^2 + n|x_n^3 - y_n^3 - \delta|^2 = 0 \\ \lim_{n\to\infty} (\tilde{V} - \tilde{U}_\zeta)(\bar{z}_n) = (\tilde{V} - \tilde{U}_\zeta)(\bar{z}_{\zeta,\delta}) \\ \text{and } \lim_{\delta\to 0}(t_{\zeta,\delta}, x^3_{\zeta,\delta}) = (\hat{t}_\zeta, \hat{x}^3_\zeta) \ , \ \lim_{\delta\to 0}(\tilde{V} - \tilde{U}_\zeta)(\bar{z}_{\zeta,\delta}) = (\tilde{V} - \tilde{U}_\zeta)(\hat{z}_\zeta) = 2\eta > \eta_0 \ . \end{gathered} \tag{53}$$

Note that, combined with (49), this implies that

$$x^1_{\zeta,\delta} = y^1_{\zeta,\delta},\ x^2_{\zeta,\delta} = y^2_{\zeta,\delta},\ p_{\zeta,\delta} = q_{\zeta,\delta} \text{ and } \underline{\Lambda}(t_{\zeta,\delta}) < y^3_{\zeta,\delta} + \delta = x^3_{\zeta,\delta} < \overline{\Lambda}(t_{\zeta,\delta}), \tag{54}$$

for $\delta > 0$ small enough.

3. a. Clearly, we cannot have $\tilde{V}(t_{\zeta,\delta}, x_{\zeta,\delta}, p_{\zeta,\delta}) = 0$ since $\tilde{U}_\zeta \geq 0$ by assumption.

b. We can neither have $t_{\zeta,\delta} = T$ since this would imply $x^3_{\zeta,\delta} = y^3_{\zeta,\delta}$ by definition of $\hat{D}_Y$ and (21), a contradiction to (54).

c. We can also not have $x^1_{\zeta,\delta} \in \{0, 2\hat{x}^1\}$ for all $\delta > 0$ small enough. To see this assume the contrary and note that the fact that U is a supersolution of (26) implies that it is non-decreasing in p. First assume that $q_{\zeta,\delta} \geq 0$ for all $\zeta > 1$ small enough. Since $\zeta > 1$, it then follows from the upper-semicontinuity of $V, -U$ and from (54) that $V(t_{\zeta,\delta}, x_{\zeta,\delta}, p_{\zeta,\delta}) - U(t_{\zeta,\delta}, y_{\zeta,\delta}, \zeta q_{\zeta,\delta}) \leq (V - U)(t_{\zeta,\delta}, x_{\zeta,\delta}, p_{\zeta,\delta}) + O(\delta)$. Recalling (44) and the definition of $(\tilde{V}, \tilde{U}_\zeta)$, we obtain $\tilde{V}(t_{\zeta,\delta}, x_{\zeta,\delta}, p_{\zeta,\delta}) - \tilde{U}_\zeta(t_{\zeta,\delta}, y_{\zeta,\delta}, q_{\zeta,\delta}) \leq O(\delta) < \eta_0$ for δ small enough whenever $x^1_{\zeta,\delta} \in \{0, 2\hat{x}^1\}$, a contradiction to (53).

Now assume that $p_{\zeta,\delta} < 0$ for all $\zeta > 1$ small enough. Then (46), the fact that $(t_{\zeta,\delta}, x_{\zeta,\delta})$ takes values in a compact set and the definition of $\eta \geq \eta_0 > 0$ imply that $|p_{\zeta,\delta}| \leq \xi$ for some $\xi > 0$ which does not depend on ζ or δ. In particular, as $\zeta \to 1$, $(t_{\zeta,\delta}, x_{\zeta,\delta}, y_{\zeta,\delta}, p_{\zeta,\delta}, q_{\zeta,\delta})_{\zeta>1}$ converges to some $(t_{1,\delta}, x_{1,\delta}, y_{1,\delta}, p_{1,\delta}, q_{1,\delta})$ such that $x^1_{1,\delta} \in \{0, 2\hat{x}^1\}$, (54) holds at the limit $\zeta = 1$, and $\limsup_{\zeta\to 1}(\tilde{V}(t_{\zeta,\delta}, x_{\zeta,\delta}, p_{\zeta,\delta}) - \tilde{U}_\zeta(t_{\zeta,\delta}, y_{\zeta,\delta}, q_{\zeta,\delta})) \leq \tilde{V}(t_{1,\delta}, x_{1,\delta}, p_{1,\delta}) - \tilde{U}_1(t_{1,\delta}, x_{1,\delta} - \delta e_3, p_{1,\delta}) \leq O(\delta)$ by upper-semicontinuity of V, $-U$, and (44). This shows that $\tilde{V}(t_{\zeta,\delta}, x_{\zeta,\delta},\ p_{\zeta,\delta}) - \tilde{U}_\zeta(t_{\zeta,\delta}, y_{\zeta,\delta}, q_{\zeta,\delta}) < \eta_0 < 2\eta$ for ζ sufficiently close to 1 and $\delta > 0$ small enough, a contradiction to (53).

4. Now observe that, by assumption, $\tilde{U}_\zeta$ and $\tilde{V}$ are super and sub-solutions on $\hat{D}_Y$ of

$$\max\left\{\kappa\varphi + \tilde{F}_0\varphi \ , \ \kappa\varphi + x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} \geq 0 \tag{55}$$

and

$$\max\left\{\kappa\varphi + \tilde{F}_0\varphi\ ,\ \kappa\varphi + x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} \leq 0 \text{ if } \underline{\Lambda} < x^3 < \overline{\Lambda}\ \ \&\ \ \tilde{V} > 0 \tag{56}$$

respectively, with

$$\begin{aligned}\tilde{F}_0\varphi := &-\partial_t\varphi - x^1\mu D_{x^1}\varphi - x^1\vartheta(t)D_{x^2}\varphi \\ &-\frac{(x^1\sigma)^2}{2}\left(D^2_{x^1}\varphi + |D_{x^1}\varphi/D_p\varphi|^2 D^2_p\varphi - 2(D_{x^1}\varphi/D_p\varphi)D^2_{(x^1,p)}\varphi\right)\ ,\end{aligned}$$

and that $\tilde{U}_\zeta$ is a viscosity supersolution of

$$\min\left\{\zeta^{-1}D_p\varphi - \epsilon\ ,\ D_{x^1}\varphi - C(x)\zeta^{-1}D_p\varphi\ ,\ -D_{x^1}\varphi + C(x)\zeta^{-1}D_p\varphi\right\} = 0. \tag{57}$$

Let $\bar{\mathcal{P}}^+\tilde{V}$ and $\bar{\mathcal{P}}^-\tilde{U}_\zeta$ denote the super- and subjets of $\tilde{V}$ and $\tilde{U}_\zeta$, with (t, x^2, x^3) taken as a first order term. It then follows from Ishii's Lemma, see e.g. [10], that we can find 2-dimensional symmetric matrices $(X_n, Y_n) \in \mathbb{S}^2$ such that

$$\begin{aligned}(D_t\Lambda_n, (D_x\Lambda_n, D_p\Lambda_n), X_n)(t_n, x_n, y_n, p_n, q_n) &\in \bar{\mathcal{P}}^+\tilde{V}(t_n, x_n, p_n)\\ (-D_t\Lambda_n, -(D_y\Lambda_n, D_q\Lambda_n), Y_n)(t_n, x_n, y_n, p_n, q_n) &\in \bar{\mathcal{P}}^-\tilde{U}_\zeta(t_n, y_n, q_n),\end{aligned}$$

where the super- and subjets are defined with (t, x^2, x^3) viewed as a first order term, that satisfy

$$\begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{58}$$

where I denotes the 2-dimensional identity matrix.

We now study two cases:

Case 1. We first assume that

$$\kappa\tilde{U}_\zeta(t_n, y_n, q_n) + y^1_n - y^1_n\beta(t_n, y^1_n)D_{y^1}\Lambda_n(z_n) + D_{y^3}\Lambda_n(z_n) \geq 0\ ,$$

along a subsequence. Then, by (51), (52), (54), (56), step 3. and the Lipschitz of $x \mapsto x\beta(t, x)$,

$$\begin{aligned}-\kappa\eta &\geq \kappa(\tilde{U}_\zeta(t_n, y_n, q_n) - \tilde{V}(t_n, x_n, p_n))\\ &\geq -(y^1_n - x^1_n) + y^1_n\beta(t_n, y^1_n)D_{y^1}\Lambda_n(z_n) + x^1_n\beta(t_n, x^1_n)D_{x^1}\Lambda_n(z_n)\\ &\quad -D_{y^3}\Lambda_n(z_n) - D_{x^3}\Lambda_n(z_n)\\ &\geq O(n|x^1_n - y^1_n|^2 + |x^1_n - y^1_n|) + O(|x^3_n - \hat{x}^3_\zeta|)\ .\end{aligned}$$

It then follows from (52)-(53) that

$$-\kappa\eta \geq 0$$

which leads to a contradiction since $\eta > 0$.

Case 2. We now assume that

$$\kappa\tilde{U}_\zeta(t_n, y_n, q_n) + y_n^1 - y_n^1\beta(t_n, y_n^1)D_{y^1}\Lambda_n(z_n) + D_{y^3}\Lambda_n(z_n) < 0 \ ,$$

along a subsequence. Then, using (52), (54), (55), (56), step 3. again, we deduce that

$$\begin{aligned} 0 \leq & \left(\kappa\tilde{U}_\zeta + D_t\Lambda_n + y_n^1\mu(t_n, y_n^1)D_{y^1}\Lambda_n + y_n^1\vartheta(t_n)D_{y^2}\Lambda_n\right)(\bar{z}_n) \\ & -\frac{(y_n^1\sigma(t_n, y_n^1))^2}{2}\left(Y_n^{11} + |A_n|^2 Y_n^{22} - 2A_nY_n^{12}\right) \end{aligned} \tag{59}$$

and

$$\begin{aligned} 0 \geq & \left(\kappa\tilde{V} - D_t\Lambda_n - x_n^1\mu(t_n, x_n^1)D_{x^1}\Lambda_n - x_n^1\vartheta(t_n)D_{x^2}\Lambda_n\right)(\bar{z}_n) \\ & -\frac{(x_n^1\sigma(t_n, x_n^1))^2}{2}\left(X_n^{11} + |A_n|^2 X_n^{22} - 2A_nX_n^{12}\right) \ , \end{aligned} \tag{60}$$

where

$$A_n := \frac{-D_{y^1}\Lambda_n(\bar{z}_n)}{-D_q\Lambda_n(\bar{z}_n)} = \frac{D_{x^1}\Lambda_n(\bar{z}_n)}{D_p\Lambda_n(\bar{z}_n)} \ .$$

Moreover, the continuity of C, recall (25), and the viscosity supersolution properties of $\tilde{U}_\zeta$ in (57), together with (52), imply that

$$|A_n| \leq \zeta^{-1}C(y_{\zeta,\delta}) + 1 \tag{61}$$

for n large. We now use (58), the Lipschitz continuity of the coefficients and (61) to obtain

$$\begin{aligned} \kappa(\tilde{V} - \tilde{U}_\zeta)(\bar{z}_n) \leq & \ n|x_n^1\mu(t_n, x_n^1) - y_n^1\mu(t_n, y_n^1)||x_n^1 - y_n^1| + O(|\hat{z}_\zeta - (t_n, x_n, p_n)|) \\ & + O\left((1 + |A_n|^2)|x_n^1\sigma(t_n, x_n^1) - y_n^1\sigma(t_n, y_n^1)|^2\right) \\ \leq & \ O(n|x_n^1 - y_n^1|^2) + O(|\hat{z}_\zeta - (t_n, x_n, p_n)|) \ . \end{aligned}$$

Recalling (52)-(53) and sending $n \to \infty$ and then $\delta \to 0$ then leads to a contradiction since $\eta > 0$. □

5 Proof of the viscosity property in the abstract model

We now provide the proof of Theorem 2. It is divided in several subsections.

5.1 Viscosity solution property on $[0,T)$

5.1.1 Super-solution property on $[0,T)$

We first consider the case $(t_0,x_0)\in\bar{D}_Y$ with $t_0<T$. The proof follows from almost exactly the same arguments as in [6]. The only difference comes from the part of the control with bounded variations, however it is easily handled. We provide it for completeness.

Proposition 8 *Let $(t_0,x_0)\in\bar{D}_Y$, with $t_0<T$, and let φ be a smooth function such that*

$$\text{(strict)}\min_{[0,T]\times\mathbb{R}^d}(v_*-\varphi)=(v_*-\varphi)(t_0,x_0)=0\ . \tag{62}$$

Then, $H^*\varphi(t_0,x_0)\geq 0$.

Proof We assume to the contrary that

$$H^*\varphi(t_0,x_0)\leq -2\eta \tag{63}$$

for some $\eta>0$, and work towards a contradiction. It follows from (63) and the definition of H^* that we may find $\varepsilon>0$ such that

$$\begin{aligned}\max_{\xi\in\Delta_+}\left[\beta_Y(x,y)^\top-D\varphi(t,x)^\top\beta_X(x)\right]\xi\leq-\eta\\ \mu_Y(x,y,u)-\mathcal{L}^u\varphi(t,x)\ \leq-\eta\ \ \forall\ u\in\mathcal{N}_\varepsilon(x,y,D\varphi(t,x))\\ \forall\ (t,x,y)\in[0,T]\times\mathbb{R}^{d+1}\text{ s.t. }(t,x)\in B_\varepsilon(t_0,x_0)\cap\bar{D}_Y,\ |y-\varphi(t,x)|\leq\varepsilon\ .\end{aligned} \tag{64}$$

For later use, observe that, by (62) and the definition of φ,

$$\zeta:=\min_{\partial_pB_\varepsilon(t_0,x_0)}(v_*-\varphi)>0\ , \tag{65}$$

where $\partial_pB_\varepsilon(t_0,x_0)$ denotes the parabolic boundary of $B_\varepsilon(t_0,x_0)$.

Let $(t_n,x_n)_{n\geq1}$ be a sequence in D_Y which converges to (t_0,x_0) and such that $v(t_n,x_n)\to v_*(t_0,x_0)$. Set $y_n=v(t_n,x_n)+n^{-1}$ and observe that

$$\gamma_n:=y_n-\varphi(t_n,x_n)\to 0\ . \tag{66}$$

For each $n\geq1$, we have $y_n>v(t_n,x_n)$. It thus follows from (GDP1) of Corollary 1, that there exists some $\phi^n=(\nu^n,L^n)\in\mathcal{A}$ such that

$$Y^n(t\wedge\theta_n)\geq v(t\wedge\theta_n,X^n(t\wedge\theta_n))\quad\text{for }t\geq t_n, \tag{67}$$

where

$$Z^n:=(X^n,Y^n):=\left(X^{\nu^n}_{t_n,x_n},Y^{\nu^n}_{t_n,x_n,y_n}\right)\ \text{ and }\theta_n:=\theta^o_n\wedge\theta^1_n\ ,$$

with

$$\begin{aligned}\theta^o_n&:=\inf\left\{s\geq t_n:(s,X^{\phi^n}_{t_n,x_n}(s))\notin B_\varepsilon(t_0,x_0)\right\}\\ \theta^1_n&:=\inf\left\{s\geq t_n:|Y^{\phi^n}_{t_n,x_n,y_n}(s)-\varphi(s,X^{\phi^n}_{t_n,x_n}(s))|\geq\varepsilon\right\}\ .\end{aligned}$$

Let us define

$$A_n := \left\{ s \in [t_n, \theta_n] \ : \ \mu_Y \left(Z^n(s), \nu_s^n \right) - \mathcal{L}^{\nu_s^n} \varphi \left(s, X^n(s) \right) > -\eta \right\} \ , \quad (68)$$

and observe that (64) implies that the process

$$\psi_s^n := N^{\nu^n} \left(Z_s^n, D\varphi(s, X_s^n) \right) \text{ satisfies } |\psi_s^n| > \varepsilon \text{ for } s \in A_n, \quad (69)$$

recall (6). Since $L^n \in \mathcal{L}$ is continuous, so is the path of Z^n. Using (67), the definition of ζ in (65) and the definition of θ_n, thus leads to

$$\begin{aligned} Y^n(t \wedge \theta_n) &\geq \varphi \left(t \wedge \theta_n, X^n(t \wedge \theta_n) \right) + \left(\zeta \mathbf{1}_{\{\theta_n^o = \theta_n\}} + \varepsilon \mathbf{1}_{\{\theta_n^o > \theta_n\}} \right) \mathbf{1}_{\{t = \theta_n\}} \\ &\geq \varphi \left(t \wedge \theta_n, X^n(t \wedge \theta_n) \right) + \left(\zeta \wedge \varepsilon \right) \mathbf{1}_{\{t = \theta_n\}} \ , \ \ t \geq t_n \ . \end{aligned}$$

Since φ is smooth, it then follows from Itô's Lemma, (64), (66) and the definition of ψ^n that

$$\begin{aligned} -\left(\zeta \wedge \varepsilon \right) \mathbf{1}_{\{t < \theta_n\}} &\leq K_t^n + \int_{t_n}^{t \wedge \theta_n} b_s^n \mathbf{1}_{A_n^c}(s) ds \\ &\quad + \int_{t_n}^{t \wedge \theta_n} \left[\beta_Y^\top (Z^n(s)) - D\varphi(s, X^n(s))^\top \beta_X(X^n(s)) \right] dL_s^n \\ &\leq K_t^n \ , \end{aligned} \quad (70)$$

where

$$K_t^n := \gamma_n - \left(\zeta \wedge \varepsilon \right) + \int_{t_n}^{t \wedge \theta_n} b_s^n \mathbf{1}_{A_n}(s) ds + \int_{t_n}^{t \wedge \theta_n} \psi_s^n dW_s \ ,$$

and

$$b_s^n := \left[\mu_Y \left(Z^n(s), \nu_s^n \right) - \mathcal{L}^{\nu_s^n} \varphi \left(s, X^n(s) \right) \right] \ .$$

Let M^n be the exponential local martingale defined by $M_{t_n}^n = 1$ and, for $s \geq t_n$,

$$dM_s^n = -M_s^n \left(b_s^n |\psi_s^n|^{-2} (\psi_s^n)^\top \right) \mathbf{1}_{A_n}(s) dW_s \ ,$$

which is well defined by (69), the Lipschitz continuity of the coefficients and our definition of the set of admissible controls $\mathcal{U}$. By Itô's formula and (70), we see that $M^n K^n$ is a local martingale which is bounded from below by the submartingale $-\left(\zeta \wedge \varepsilon \right) M^n$. Then, $M^n K^n$ is a supermartingale, and it follows from (70) that

$$0 = \mathbb{E} \left[-\left(\zeta \wedge \varepsilon \right) \mathbf{1}_{\{\theta_n < \theta_n\}} \right] \leq \mathbb{E} \left[M_{\theta_n}^n K_{\theta_n}^n \right] \leq \gamma_n - \left(\zeta \wedge \varepsilon \right) < 0 \ ,$$

for n large enough, recall (66), which leads to a contradiction. □

5.1.2 Sub-solution property on $[0,T)$

We first consider the case where $(t_0, x_0, v^*(t_0,x_0)) \in \mathrm{int}(D)$. The first part of the proof is similar to those provided in [6]. The novelty comes from the second part where we play with the part of the control with bounded variations to obtain a contradiction.

Proposition 9 *Let* $(t_0,x_0) \in \bar{D}_Y$*, with* $t_0 < T$*, and* φ *be a smooth function such that*

$$\text{(strict)} \max_{\bar{D}_Y}(v^* - \varphi) = (v^* - \varphi)(t_0,x_0) = 0 \ . \tag{71}$$

Assume that $(t_0, x_0, v^*(t_0,x_0)) \in \mathrm{int}(D)$*. Then,* $H_*\varphi(t_0,x_0) \le 0$*.*

Proof We assume to the contrary that

$$(t_0, x_0, v^*(t_0,x_0)) \in \mathrm{int}(D) \ \text{ and } \ H_*\varphi(t_0,x_0) \ge 2\eta \tag{72}$$

for some $\eta > 0$, and work towards a contradiction. For later use note that (72) implies that, for $\varepsilon > 0$ small enough,

$$\big\{(t,x,y) \in [0,T] \times \mathbb{R}^{d+1} \text{ s.t. } (t,x) \in B_\varepsilon(t_0,x_0),\ |y - \varphi(t,x)| \le \varepsilon\big\} \subset \mathrm{int}(D) \tag{73}$$

Also observe that, by (71) and the definition of φ,

$$-\zeta := \max_{\partial_p B_\varepsilon(t_0,x_0)} (v^* - \varphi) < 0 \ . \tag{74}$$

Moreover, we can find a sequence $(t_n,x_n)_{n\ge1}$ in D_Y which converges to (t_0,x_0) and such that $v(t_n,x_n) \to v^*(t_0,x_0)$. Set $y_n = v(t_n,x_n) - n^{-1}$ and observe that

$$\gamma_n := y_n - \varphi(t_n,x_n) \to 0 \ . \tag{75}$$

We now consider two cases.

<u>First case.</u> We first assume that

$$F_*\varphi(t_0,x_0) \ge 2\eta \ . \tag{76}$$

Then it follows from Assumption 1 and Remark 4 that may find $\varepsilon > 0$ such that

$$\begin{gathered} \mu_Y(\cdot,\hat{\nu}) - \mathcal{L}^{\hat{\nu}}\varphi > \eta \\ \forall\ (t,x,y) \in [0,T] \times \mathbb{R}^{d+1} \text{ s.t. } (t,x) \in B_\varepsilon(t_0,x_0),\ |y - \varphi(t,x)| \le \varepsilon \ , \end{gathered} \tag{77}$$

where $\hat{\nu}$ is a locally Lipschitz map satisfying

$$\hat{\nu}(x,y,D\varphi(t,x)) \in \mathcal{N}_0(x,y,D\varphi(t,x)) \text{ if } (t,x) \in B_\varepsilon(t_0,x_0) \text{ and } |y - \varphi(t,x)| \le \varepsilon \ . \tag{78}$$

We now fix n large enough so that $(t_n, x_n) \in B_\varepsilon(t_0, x_0)$ and let $Z^n := (X^n, Y^n)$ denote the solution of (1) associated to the Markovian control $\hat{\phi}^n$ and the initial condition $Z^n(t_n) = (x_n, y_n)$, where

$$\hat{\phi}^n = (\hat{\nu}^n, \hat{L}^n) := (\hat{\nu}(\cdot, X^n, Y^n), 0) \ .$$

We next define the stopping times

$$\begin{aligned} \theta_n^o &:= \inf\{s \geq t_n \ : \ (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\}\,, \\ \theta_n &:= \inf\{s \geq t_n \ : \ |Y^n(s) - \varphi(s, X^n(s))| \geq \varepsilon\} \wedge \theta_n^o \ . \end{aligned}$$

Note that, by definition of $\hat{\phi}^n$ and (77), $Y^n - \varphi(\cdot, X^n)$ is non-decreasing on $[t_n, \theta_n]$, so that

$$Y^n(\theta_n) - \varphi(\theta_n, X^n(\theta_n)) \geq y_n - \varphi(t_n, x_n) = \gamma_n > -(\varepsilon \wedge \zeta)/2 \tag{79}$$

for n large enough, recall (75). Since $\varphi \geq v^* \geq v$, it follows that

$$\begin{aligned} Y^n(\theta_n) - v\left(\theta_n, X^n(\theta_n)\right) &\geq \mathbf{1}_{\{\theta_n < \theta_n^o\}} \left\{Y^n(\theta_n) - \varphi\left(\theta_n, X^n(\theta_n)\right)\right\} \\ &\qquad + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left\{Y^n(\theta_n^o) - v^*\left(\theta_n^o, X^n(\theta_n^o)\right)\right\} \\ &= \varepsilon \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left\{Y^n(\theta_n^o) - v^*\left(\theta_n^o, X^n(\theta_n^o)\right)\right\} \\ &\geq \varepsilon \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left\{Y^n(\theta_n^o) + \zeta - \varphi\left(\theta_n^o, X^n(\theta_n^o)\right)\right\} \\ &\geq \varepsilon \wedge \zeta + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left\{Y^n(\theta_n^o) - \varphi\left(\theta^o, X^n(\theta_n^o)\right)\right\} \ . \end{aligned}$$

In view of (79), this leads to

$$Y^n(\theta_n) - v\left(\theta_n, X^n(\theta_n)\right) \geq (\varepsilon \wedge \zeta)/2$$

for n large enough. Recalling (73) and the fact that $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$, this is clearly in contradiction with (GDP2) of Corollary 1.

<u>Second case.</u> If (76) does not hold, then it follows from (72) that we can find $\hat{\ell} \in \Delta_+$ such that, for $\varepsilon > 0$ small enough,

$$\left[\beta_Y^\top - D\varphi^\top \beta_X\right] \hat{\ell} > \eta \tag{80}$$

$$\forall\ (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0),\ |y - \varphi(t, x)| \leq \varepsilon \ . \tag{81}$$

Set $O_n := \{(t, x, y) \ : \ (t, x) \in B_{2\varepsilon}(t_0, x_0)\ ,\ |y - \varphi(t, x)| < 2\varepsilon\ ,\ y - \varphi(t, x) > -|\gamma_n|\}$. It follows from (80) that we can find $r > 0$ such that

$$\bigcup_{0 \leq \lambda \leq r} B_{\lambda r}((t, x, y) - \lambda \hat{\gamma}(t, x, y)) \subset O_n^c \ \text{ for all } (t, x, y) \in \partial O_n \text{ satisfying (81)}\ ,$$

where $\hat{\gamma}(t, x, y)^\top := (0, \beta_X(x), \beta_Y^\top(x, y))\hat{\ell}$. Given $u \in U$, we thus deduce from Theorem 4.8 of [14] and the assumption made on our coefficients, that there exists an adapted process $Z^n = (X^n, Y^n)$ and a continuous real-valued adapted

non-decreasing process M^n satisfying

$$\begin{aligned}
X^n(s) &= x_n + \int_{t_n}^s \mu_X(X^n(r),u)dr + \int_{t_n}^s \beta_X(X^n(r))\hat{\ell}\, dM_r^n \\
&\quad + \int_{t_n}^s \sigma_X(X^n(r),u)dW_r \\
Y^n(s) &= y_n + \int_{t_n}^s \mu_Y(Z^n(r),u)dr + \int_{t_n}^s \beta_Y(Z^n(r))^\top \hat{\ell}\, dM_r^n \\
&\quad + \int_{t_n}^s \sigma_Y(Z^n(r),u)^\top dW_r \ , \\
Y^n(s\wedge\theta_n) &\geq \varphi(s\wedge\theta_n, X^n(s\wedge\theta_n)) - 2|\gamma_n| \quad \text{for all } s\geq t_n \ ,
\end{aligned} \tag{82}$$

where

$$\begin{aligned}
\theta_n^o &:= \inf\{s\geq t_n \ : \ (s,X^n(s))\notin B_\varepsilon(t_0,x_0)\}\,, \\
\theta_n &:= \inf\{s\geq t_n \ : \ |Y^n(s)-\varphi(s,X^n(s))|\geq\varepsilon\}\wedge\theta_n^o \ .
\end{aligned}$$

Observe that Z^n coincides with the solution of (1) for the control $(u,\hat{\ell}M^n)$. In view of (75) and (82), we have $Y^n(\theta_n)-\varphi(\theta_n,X^n(\theta_n))\geq -2|\gamma_n|>-\varepsilon$ for n large enough. Following the arguments after (79) above then leads to the required contradiction. □

We now turn to the case where $(t_0,x_0,v^*(t_0,x_0))\in\partial D$.

Proposition 10 *Let $(t_0,x_0)\in\bar{D}_Y$, with $t_0<T$, and φ be a smooth function such that*

$$(\text{strict})\max_{\bar{D}_Y}(v^*-\varphi) = (v^*-\varphi)(t_0,x_0) = 0 \ . \tag{83}$$

Assume that $(t_0,x_0,v^*(t_0,x_0))\in\partial D$. Then, $H_*^{\text{in}}\varphi(t_0,x_0)\leq 0$.

Proof The fact that

$$(t_0,x_0,v^*(t_0,x_0))\in\partial D \ \text{ and } \ H_*^{\text{in}}\varphi(t_0,x_0)\geq 2\eta \tag{84}$$

for some $\eta>0$, leads to a contradiction to (GDP2) of Corollary 1 follows exactly from the same arguments as in the proof of Proposition 9. We therefore only sketch the case where $F_*^{\text{in}}\varphi(t_0,x_0)\geq 2\eta$. In this case, it follows from the definition of F_*^{in} and (ii) of Assumption 1, see also Remark 4, that we may find $\varepsilon>0$ such that

$$\begin{aligned}
&\min\{\mu_Y(\cdot,\hat{\nu})-\mathcal{L}^{\hat{\nu}}\varphi \ , \ \mathcal{L}_Z^{\hat{\nu}}\delta\}>\eta \\
&\forall\ (t,x,y)\in[0,T]\times\mathbb{R}^{d+1} \text{ s.t. } (t,x)\in B_\varepsilon(t_0,x_0),\ |y-\varphi(t,x)|\leq\varepsilon \ ,
\end{aligned} \tag{85}$$

where $\hat{\nu}$ is a locally Lipschitz map satisfying

$$\hat{\nu}(t,x,y)\in\mathcal{N}_0^{\text{in}}(t,x,y,D\varphi(t,x)) \text{ for } (t,x)\in B_\varepsilon(t_0,x_0) \text{ and } |y-\varphi(t,x)|\leq\varepsilon \ . \tag{86}$$

Let $(t_n, x_n)_{n\geq 1}$ be a sequence in D_Y which converges to (t_0, x_0) and such that $v(t_n, x_n) \to v^*(t_0, x_0)$. Set $y_n = v(t_n, x_n) - n^{-1}$ and observe that

$$\gamma_n := y_n - \varphi(t_n, x_n) \to 0 \ . \tag{87}$$

Let $Z^n := (X^n, Y^n)$ denote the solution of (1) associated to the Markovian control $(\hat{\nu}^n, 0)$ and the initial condition $Z^n(t_n) = (x_n, y_n)$, where

$$\hat{\nu}^n = \hat{\nu}(\cdot, X^n, Y^n) \ .$$

We next define the stopping times

$$\begin{aligned} \theta_n^o &:= \inf\{s \geq t_n \ : \ (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\} \ , \\ \theta_n &:= \inf\{s \geq t_n \ : \ |Y^n(s) - \varphi(s, X^n(s))| \geq \varepsilon\} \wedge \theta_n^o \ . \end{aligned}$$

The same arguments as in the proof of Proposition 9 show that

$$Y^n(\theta_n) - v\left(\theta_n, X^n(\theta_n)\right) > 0$$

for n large enough. Moreover, it follows from (85), (86) and Itô's Lemma that

$$\delta(\cdot, X^n, Y^n) \geq 0 \quad \text{on } [t_n, \theta_n] \ .$$

Recalling that $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$, this is in contradiction with (GDP2) of Corollary 1. □

5.2 Viscosity solution property at T

In this part we follow standard arguments which consist in propagating the boundary condition backward on some small time $[T - \varepsilon, T]$ so as to be in position to repeat the arguments used to derive the viscosity solution property on $[0, T)$. The arguments being standard, see e.g. [6] or [23], we only sketch them.

We begin with the super-solution property.

Proposition 11 *Fix $(T, x_0) \in \bar{D}_Y$, and let φ be a smooth function such that*

$$\text{(strict)} \min_{[0,T]\times\mathbb{R}^d} (v_* - \varphi) = (v_* - \varphi)(T, x_0) = 0 \ . \tag{88}$$

Then, $M^*\varphi(T, x_0) \geq 0$. If moreover $F^*\varphi(T, x_0) < \infty$ and $G\varphi(T, x_0) < 0$, then $\varphi(T, x_0) - w_*(x_0) \geq 0$.

Proof The fact that $M^*\varphi(T, x_0) \geq 0$ is deduced from Proposition 8 and the upper-semicontinuity of M^* by standard arguments, see e.g. the proof of Lemma 5.2 in [22]. We now prove the second assertion. Assume that

$$F^*\varphi(T, x_0) < \infty \ , \ G\varphi(T, x_0) < 0 \text{ and } \varphi(T, x_0) = v_*(T, x_0) < w_*(x_0) \ ,$$

and let us work towards a contradiction. Since $v(T, .) = w$ by the definition of the problem, there is a constant $\eta > 0$ such that $\varphi - v(T, \cdot) \leq$

$\varphi - w_* \leq -\eta$ on $B_\varepsilon(x_0)$ for some $\varepsilon > 0$. Since x_0 is a strict minimizer, $2\zeta := \min_{x\in\partial B_\varepsilon(x_0)} (v_*(T,x) - \varphi(T,x)) > 0$ and it follows that there exists $r > 0$ such that $v(t,x) - \varphi(t,x) \geq \zeta > 0$ for all $(t,x) \in [T-r,T] \times \partial B_\varepsilon(x_0)$. Hence,

$$v(t,x)-\varphi(t,x) \geq \zeta\wedge\eta > 0 \ \text{ for } (t,x) \in ([T-r,T]\times\partial B_\varepsilon(x_0))\cup(\{T\}\times B_\varepsilon(x_0)) \ . \tag{89}$$

Since $F^*\varphi(T,x_0) < \infty$ and $G\varphi(T,x_0) < 0$, we can assume, after possibly changing $\varepsilon > 0$, that

$$\begin{aligned} G\varphi(t,x) \leq 0 \text{ and } \mu_Y(x,y,u) - \mathcal{L}^u_X\varphi(t,x) \leq C \ \text{ for all } \ u \in \mathcal{N}_\varepsilon(x,y,D\varphi(t,x)) \\ \text{and } (t,x,y) \in [0,T]\times\mathbb{R}^{d+1} \ \text{ s.t. } \ (t,x) \in B_\varepsilon(T,x_0) \text{ and } |y-\varphi(x)| \leq \varepsilon \end{aligned}$$

for some constant $C > 0$. Let $\tilde\varphi(t,x) := \varphi(x) - \sqrt{T-t+\varepsilon} + \sqrt{\varepsilon}$. Then, for sufficiently small $\varepsilon > 0$, we have

$$v - \tilde\varphi \geq \tfrac{1}{2}(\zeta\wedge\eta) > 0 \ \text{ on } ([T-\varepsilon,T]\times\partial B_\varepsilon(x_0))\cup(\{T\}\times B_\varepsilon(x_0))$$

and

$$\begin{aligned} G\tilde\varphi(t,x) \leq 0 \text{ and } \ \mu_Y(x,y,u) - \mathcal{L}^u_X\tilde\varphi(t,x) \leq 0 \ \text{ for all } \ u \in \mathcal{N}_\varepsilon(x,y,D\tilde\varphi(t,x)) \\ (t,x,y) \in [T-\varepsilon,T]\times\mathbb{R}^{d+1} \ \text{ s.t. } \ x \in B_\varepsilon(x_0) \text{ and } |y-\tilde\varphi(t,x)| \leq \varepsilon. \end{aligned}$$

By following the arguments in the proof of Proposition 8, the latter inequalities lead to a contradiction of (GDP1) of Corollary 1. □

We now turn to the sub-solution property. As in the previous section, we first consider the case where $(t_0,x_0,v^*(t_0,x_0)) \in \mathrm{int}(D)_T$.

Proposition 12 *Let $(t_0,x_0) \in \bar D_Y$, with $t_0 = T$, and φ be a smooth function such that*

$$(\text{strict}) \max_{\bar D_Y}(v^* - \varphi) = (v^* - \varphi)(t_0,x_0) = 0 \ . \tag{90}$$

Assume that $(T,x_0,v^*(T,x_0)) \in \mathrm{int}(D)_T$. Then,

$$\min\{\varphi(T,x_0) - w^*(x_0) \ , \ M_*\varphi(T,x_0)\} \leq 0.$$

Proof Assume to the contrary that

$$\min\{\varphi(T,x_0) - w^*(x_0) \ , \ M_*\varphi(T,x_0)\} =: 2\eta > 0 \ .$$

Let $\tilde\varphi$ be defined by $\tilde\varphi(t,x) := \varphi(t,x) + \sqrt{T-t+\varepsilon} - \sqrt{\varepsilon}$ for $\varepsilon > 0$ small. Clearly, (T,x_0) achieves a strict maximum of $v^* - \tilde\varphi$, and it follows from the identity $v(T,\cdot) = w(x_0)$, the fact that $\varphi(T,x_0) - w^*(x_0) > 0$ and (90) that

$$\max_{(\{T\}\times\bar B_\varepsilon(x_0))\cup([T-\varepsilon,T]\times\partial B_\varepsilon(x_0))} (v^* - \tilde\varphi) := -\zeta < 0 \ .$$

Also observe that, the fact that $M_*\varphi(T,x_0) > 0$, means that

$$\max\{R_*\tilde\varphi(T,x_0) \ , \ G\tilde\varphi(T,x_0)\} > 0 \ .$$

Since $\partial_t\tilde{\varphi} \to -\infty$ as $t \to T$ and $\varepsilon \to 0$, we can find $\varepsilon > 0$ small enough such that

$$\max\{F_*\tilde{\varphi}(T,x_0)\ ,\ G\tilde{\varphi}(T,x_0)\} > 0\ .$$

Moreover, the assumption $(T,x_0,\tilde{\varphi}(T,x_0)) = (T,x_0,v^*(T,x_0)) \in \mathrm{int}(D)_T$, recall (15)-(16), implies that, for $\varepsilon > 0$ small enough,

$$\big\{(t,x,y) \in [0,T) \times \mathbb{R}^{d+1} \text{ s.t. } (t,x) \in B_\varepsilon(T,x_0),\ |y - \tilde{\varphi}(t,x)| \le \varepsilon\big\} \subset \mathrm{int}(D)\ .$$

Following line by line the arguments of Proposition 9 then leads to a contradiction to (GDP2) of Corollary 1. □

We finally consider the case $(T,x_0,v^*(T,x_0)) \in \partial D_T$.

Proposition 13 *Let $(t_0,x_0) \in \bar{D}_Y$, with $t_0 = T$, and φ be a smooth function such that*

$$(\text{strict}) \max_{\bar{D}_Y}(v^* - \varphi) = (v^* - \varphi)(t_0,x_0) = 0\ . \tag{91}$$

Assume that $(T,x_0,v^*(T,x_0)) \in \partial D_T$. Then,

$$\min\big\{\varphi(T,x_0) - w^*(x_0)\ ,\ M_*^{\mathrm{in}}\varphi(T,x_0)\big\} \le 0.$$

Proof The result follows from an obvious combination of the arguments used in the proofs of Proposition 12 and Proposition 10 above. □

Appendix: Proof of Theorem 1

For the reader's convenience, we provide here the main elements of the proof of Theorem 1. It follows line by line the arguments in [9], which in turn are based on [23]. One difference is that we formulate our result in terms of a family of stopping times $\{\theta^\phi,\ \phi \in \mathcal{A}\} \subset \mathcal{T}_{[t,T]}$ and not in terms of a fixed one θ. This does not change anything, except at the level of notations. Another one is that their assumption **A1** (pasting property) is not "formally" satisfied by elements of $\mathcal{L}$. This is however easily handled.

Let $t \in [0,T]$ be fixed. We must show that $V(t) = \bar{V}(t)$ where

$$\bar{V}(t) := \{z \in \mathbb{R}^{d+1} : \exists \phi \in \mathcal{A} \text{ s.t } \{Z^\phi_{t,z}(\tau \wedge \theta^\phi) \in O \overset{\tau,\theta^\phi}{\bigoplus} V \text{ for all } \tau \in \mathcal{T}_{[t,T]}\}\}\ .$$

In the following, we work on the d-dimensional canonical space endowed with the Wiener measure $\mathbb{P}$. The filtration is given by the augmentation of the rough filtration generated by W. The canonical path is denoted by $\omega = (\omega_t)_{t\in[0,T]}$. We use the notations $\omega^s := (\omega_{t\wedge s})_{t\in[0,T]}$ and $\delta_s\omega := (\omega_{t\vee s} - \omega_s)_{t\in[0,T]}$, $s \in [0,T]$.

We split the proof in several Lemmas.

Lemma 1 $V(t) \subset \bar{V}(t)$.

Proof. Fix $z \in V(t)$ and $\phi \in \mathcal{A}$ such that $Z^{\phi}_{t,z}(s) \in O(s)$ for all $s \in [t,T]$ $\mathbb{P}-$ a.s. By the flow property, we have $Z^{\phi}_{\theta^{\phi},\xi}(s \vee \theta^{\phi}) \in O(s \vee \theta^{\phi})$ for all $s \in [t,T]$ $\mathbb{P}-$ a.s., where $\xi := Z^{\phi}_{t,z}(\theta^{\phi})$. Hence,

$$1 = \mathbb{P}\left[Z^{\phi}_{\theta^{\phi},\xi}(s \vee \theta^{\phi}) \in O(s \vee \theta^{\phi}) \mid \mathcal{F}_{\theta^{\phi}}\right] \text{ for all } s \in [t,T] \ \mathbb{P}-\text{a.s.}$$

It follows that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$1 = \mathbb{P}\left[Z^{\tilde{\phi}_{\omega}}_{\theta^{\phi}(\omega),\xi(\omega)}(s \vee \theta^{\phi}(\omega)) \in O(s \vee \theta^{\phi}(\omega))\right] \text{ for all } s \in [t,T] \ ,$$

where $\tilde{\phi}_{\omega}$ is the map $\tilde{\omega} \in \Omega \mapsto \phi(\omega^{\theta(\omega)}, \delta_{\theta(\omega)}\tilde{\omega})$, with ϕ viewed as map on Ω. Using the right-continuity of $Z^{\tilde{\phi}_{\omega}}_{\theta^{\phi}(\omega),\xi(\omega)}$ and Standing Assumption 3, we then deduce that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $Z^{\tilde{\phi}_{\omega}}_{\theta^{\phi}(\omega),\xi(\omega)}(s \vee \theta^{\phi}(\omega))(\tilde{\omega}) \in O(s \vee \theta^{\phi}(\omega))$ for all $s \in [\theta^{\phi}(\omega), T]$ for $\mathbb{P}$-a.e. $\tilde{\omega} \in \Omega$. This shows that $\xi = Z^{\phi}_{t,z}(\theta^{\phi}) \in V(\theta^{\phi})$ $\mathbb{P}-$ a.s. Since we already know that $Z^{\phi}_{t,z}(\tau) \in O(\tau)$ for all $\tau \in \mathcal{T}_{[t,T]}$, this shows that $z \in \bar{V}(t)$. □

It remains to prove the opposite inclusion. In the following we equip $\mathcal{A} = \mathcal{U} \times \mathcal{L}$ with the distance

$$d((\nu, L), (\nu', L')) := \mathbb{E}\left[\int_0^T |\nu_s - \nu'_s|^2 ds + \sup_{s \le T} |L_s - L'_s|^2\right]^{\frac{1}{2}}$$

and denote by $\mathcal{B}_{\mathcal{A}}$ the induced Borel tribe.

Lemma 2 $\bar{V}(t) \subset V(t)$.

Proof. We now fix $z \in \bar{V}(t)$ and $\phi = (\nu, L) \in \mathcal{A}$ such that, for all $\tau \in \mathcal{T}_{[t,T]}$,

$$Z^{\phi}_{t,z}(\tau \wedge \theta^{\phi}) \in O \bigoplus^{\tau,\theta^{\phi}} V \ . \tag{92}$$

1. We first work on the event set $\{\theta^{\phi} < \tau\}$. On this set, we have $Z^{\phi}_{t,z}(\theta^{\phi}) \in V(\theta^{\phi})$ and therefore

$$(\theta^{\phi}, Z^{\phi}_{t,z}(\theta^{\phi})) \in \mathcal{D} := \{(t,z) \in [0,T] \times \mathbb{R}^{d+1} : z \in V(t)\}.$$

Let $\mathcal{B}_{\mathcal{D}}$ denote the collection of Borel subsets of $\mathcal{D}$. Applying Lemma 3 below to the measure induced by $(\theta^{\phi}, Z^{\phi}_{t,z}(\theta^{\phi}))$ on $[0,T] \times \mathbb{R}^{d+1}$, we can construct a measurable map $\bar{\phi} : (\mathcal{D}, \mathcal{B}_{\mathcal{D}}) \to (\mathcal{A}, \mathcal{B}_{\mathcal{A}})$ such that

$$Z^{\bar{\phi}(\theta^{\phi}, Z^{\phi}_{t,z}(\theta^{\phi}))}_{\theta^{\phi}, Z^{\phi}_{t,z}(\theta^{\phi})}(\vartheta) \in O(\vartheta) \text{ for all } \ \vartheta \in \mathcal{T}_{[\theta^{\phi},T]} \ .$$

Because $\mathcal{A}$ is a separable metric space, we can then find a progressively measurable process $\phi_1 = (\nu^1, L^1)$ such that $\phi_1 = \bar{\phi}(\theta^\phi, Z^\phi_{t,z}(\theta^\phi))$ on $[\theta^\phi, T]$ $Leb\times\mathbb{P}$-a.e., see Lemma 2.1 in [23]. We then define

$$\hat{\phi} := \phi 1_{[0,\theta^\phi)} + (\nu^1, L^1 - L^1_{\theta^\phi} + L_{\theta^\phi}) 1_{[\theta^\phi, T]} \ .$$

Since the dynamics of $Z^{\phi_1}_{\theta^\phi, Z^\phi_{t,z}(\theta^\phi)}$ only depends on the increments of L^1 on $[\theta^\phi, T]$, the above defined control satisfies

$$Z^{\hat{\phi}}_{\theta^\phi, Z^\phi_{t,z}(\theta^\phi)}(\tau) \in O(\tau) \text{ on } \{\theta^\phi < \tau\} \ .$$

2. Let $\hat{\phi}$ be defined as above and note that, by (92), we also have

$$Z^{\hat{\phi}}_{t,z}(\tau) = Z^\phi_{t,z}(\tau) \in O \bigoplus^{\tau,\theta^\phi} V = O(\tau) \ \text{ on } \{\tau \le \theta^\phi\} \ .$$

3. Combining the two above steps and recalling our Standing Assumption 4, this shows that $z \in V(t)$. □

It remains to prove the following result which was used in the previous proof.

Lemma 3 *For any probability measure μ on $[0,T]\times\mathbb{R}^{d+1}$, there exists a Borel measurable function $\bar{\phi} : (\mathcal{D}, \mathcal{B}_\mathcal{D}) \to (\mathcal{A}, \mathcal{B}_\mathcal{A})$ such that*

$$\bar{\phi}(t,z) \in \mathcal{A}_{t,z} \ \ \textit{for } \mu - \textit{a.e.} \ \ (t,z) \in \mathcal{D} \ .$$

Proof. Set $B := \{(t,z,\phi) \in [0,T] \times \mathbb{R}^{d+1} \times \mathcal{A} : \phi \in \mathcal{A}_{t,z}\}$. It follows from our Lipschitz continuity assumptions that the map $(t,z,\phi) \in [0,T] \times \mathbb{R}^{d+1} \times \mathcal{A} \to Z^\phi_{t,z}(r) \in L^2(\Omega)$ is continuous, and therefore Borel measurable, for any $r \le T$. Then, for any bounded continuous function f, the map $\psi^r_f : (t,z,\phi) \in [0,T] \times \mathbb{R}^{d+1} \times \mathcal{A} \to \mathbb{E}\left[f(Z^\phi_{t,z}(r))\right]$ is Borel measurable. Since $O(r)$ is a Borel set, the map $1_{O(r)}$ is the limit of a sequence of bounded continuous functions $(f^n)_{n\ge1}$. Therefore, $\psi^r_{1_{O(r)}} = \lim_{n\to\infty} \psi^r_{f^n}$ is a Borel function. This implies that

$$\begin{aligned} B^r &:= \{(t,z,\phi) \in [0,T] \times \mathbb{R}^{d+1} \times \mathcal{A} : \psi^r_{1_O}(t,z,\phi) \ge 1\} \\ &= \{(t,z,\phi) \in [0,T] \times \mathbb{R}^{d+1} \times \mathcal{A} : Z^\phi_{t,z}(r) \in O(r)\} \end{aligned}$$

is a Borel set. Appealing to the Standing Assumption 3 and the right-continuity of $Z^\phi_{t,z}$, we then deduce that $B = \bigcap_{r\le T,\ r\in\mathbb{Q}} B^r$. This shows that B is a Borel set and therefore an analytic subset of $[0,T] \times \mathbb{R}^{d+1} \times \mathcal{A}$, see [4].

Applying the Jankov-von Neumann Theorem (see [4] Proposition 7.49), we then deduce that there exists an analytically measurable function $\bar{\phi} : \mathcal{D} \to \mathcal{A}$ such that

$$\bar{\phi}(t,z) \in \mathcal{A}_{t,z} \ \text{ for all } (t,z) \in \mathcal{D} \ .$$

Since an analytically measurable map is also universally measurable, the required result follows from Lemma 7.27 in [4]. □

References

1. R. Almgren. Optimal trading in a dynamic market, Technical Report 2, (2009)
2. R. Almgren B. Harts. A dynamic algorithm for smart order routing, Technical report, StreamBase, (2008)
3. R.F. Almgren N. Chriss. Optimal execution of portfolio transactions, Journal of Risk, 3, 5–39 (2000)
4. D.P. Bertsekas S.E. Shreve. Stochastic Optimal Control: The Discrete Time Case. Mathematics in Science and Engineering. Academic Press, San Diego (1978)
5. D. Bertsimas, A. W. Lo P. Hummel. Optimal control of execution costs for portfolios, Computing in Science and Engg., 1, 40–53 (1999)
6. B. Bouchard R. Elie N. Touzi, Stochastic target problems with controlled loss. SIAM Journal on Control and Optimization, 48(5), 3123–3150 (2009)
7. B. Bouchard, N.M. Dang C.A. Lehalle. Optimal control of trading algorithms: a general impulse control approach, SIAM Journal on Financial Mathematics, 2, 404–438 (2011)
8. B. Bouchard N. Touzi, Explicit Solution of the Multivariate Super-Replication Problem under Transaction Costs, Annals of Applied Probability, 10, 685–708 (2000)
9. B. Bouchard T. N. Vu, The American version of the geometric dynamic programming principle: Application to the pricing of american options under constraints, Applied Mathematics and Optimization, 61(2), 235–265 (2010)
10. M. Crandall H. Ishii P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, American Mathematical Society, 27, 1–67 (1992)
11. P. Cheridito M. Soner N. Touzi, The multi-dimensional super-replication problem under Gamma constraints, Annales de l'Institut Henri Poincaré, Série C: Analyse Non-Linéaire 22, 633–666 (2005)
12. J. Cvitanić H. Pham N. Touzi, Super-replication in stochastic volatility models with portfolio constraints, Journal of Applied Probability, 36, 523–545 (1999)
13. J. Cvitanić H. Pham N. Touzi, A closed-form solution to the problem of super-replication under transaction costs, Finance and Stochastics, 3, 35–54 (1999)
14. P. Dupuis and H. Ishii. SDEs with oblique reflection on nonsmooth domains, The Annals of Probability, 21(1):554–580 (1993)
15. H. Föllmer P. Leukert, Quantile Hedging, Finance and Stochastics, 3(3), 251–273 (1999)
16. H. Föllmer P. Leukert, Efficient hedging : cost versus shortfall risk, Finance and Stochastics, 4, 117–146 (2000)
17. Y.M. Kabanov, Hedging and liquidation under transaction costs in currency markets, Finance and Stochastics, 3(2), 237–248 (1999).
18. G. Pagès, S. Laruelle C.-A. Lehalle. Optimal split of orders across liquidity pools: a stochastic algorithm, approach, Technical report, (2009).
19. H. M. Soner, Optimal control with state-space constraint. I, SIAM Journal on Control and Optimization, 24(3):552–561 (1986)
20. H. M. Soner, Optimal control with state-space constraint. II, SIAM Journal on Control and Optimization, 24(6): 1110–1122 (1986)
21. H.M. Soner N. Touzi, Super-replication under Gamma constraint, Journal on Control and Optimization, 39, 73–96 (2000)
22. H. M. Soner N. Touzi, Stochastic target problems, dynamic programming and viscosity solutions, SIAM Journal on Control and Optimization, 41:404–424 (2002)
23. H. M. Soner N. Touzi, Dynamic programming for stochastic target problems and geometric flows, Journal of the European Mathematical Society, 4:201–236 (2002)
24. H.M. Soner N. Touzi, The problem of super-replication under constraints, to appear in Paris-Princeton Lectures in Mathematical Finance, Lecture Notes in Mathematics, Springer-Verlag
25. N. Touzi, Direct characterization of the value of super-replication under stochastic volatility and portfolio constraints, Stochastic Processes and their Applications, 88, 305-328 (2000)