Generalized stochastic target problems - Application in optimal book liquidation

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - Ensae

Kyoto 2010

Joint work with Minh Ngoc Dang, CEREMADE and Chevreux C.A.
Motivation

VWAP guaranteed contracts pricing:

Find the minimal $y$ s.t., for some $L \in \mathcal{L}$ with $L_0 = 0$, $X^{L,1}_T = K$, $X^{L,1} \in \Lambda$ and

$$
\mathbb{E} \left[ \ell \left( y + \left[ Y^L(T)/K - \gamma X^{L,2}(T)/\Theta(T) \right] K \right) \right] \geq p,
$$

for $p \in \mathbb{R}$ and $\ell: \mathbb{R} \mapsto \mathbb{R}$ non-decreasing.
Motivation

VWAP guaranteed contracts pricing:

Find the minimal $y$ s.t., for some $L \in \mathcal{L}$ with $L_0 = 0$,

$$X_T^{L,1} = K, X_T^{L,1} \in \Lambda \text{ and }$$

$$\mathbb{E} \left[ \ell \left( y + \frac{Y^L(T)}{K} - \gamma X^{L,2}(T)/\Theta(T) \right) K \right] \geq p,$$

for $p \in \mathbb{R}$ and $\ell : \mathbb{R} \mapsto \mathbb{R}$ non-decreasing.

Leads to a stochastic target problem under expected loss.
Motivation

Following B., Touzi, Elie (2009), we provide a direct PDE representation of the associated pricing function.
Motivation

Following B., Touzi, Elie (2009), we provide a direct PDE representation of the associated pricing function.

Singular quasi-linear operator.
Motivation

Following B., Touzi, Elie (2009), we provide a direct PDE representation of the associated pricing function.

Singular quasi-linear operator.

Novelty

- controls in the form of bounded variation process,
- state constraint.
Motivation

Following B., Touzi, Elie (2009), we provide a direct PDE representation of the associated pricing function.

Singular quasi-linear operator.

Novelty

- controls in the form of bounded variation process,
- state constraint.

Under “good conditions” on the model: comparison holds.
Motivation

Following B., Touzi, Elie (2009), we provide a direct PDE representation of the associated pricing function.

Singular quasi-linear operator.

Novelty

- controls in the form of bounded variation process,
- state constraint.

Under “good conditions” on the model: comparison holds.

Start with a general model (which suits also well to models with proportional transaction costs).
General model

Set of controls : $U \times L$ with

- $U$ : prog. meas. process in $L^2([0, T] \times \Omega)$ with values in $U \subset \mathbb{R}^d$, 

Problem :

$(y, x) \in O(s) \text{ and } y' \geq y \Rightarrow (y', x) \in O(s)$. 

$V(t) := \{ (x, y) : \exists \phi \in U \times L \text{ s.t. } Z\phi(t, x, y(s)) \in O(s) \forall t \leq s \leq T \}$.
General model

Set of controls : $U \times L$ with

- $U$ : prog. meas. process in $L^2([0, T] \times \Omega)$ with values in $U \subset \mathbb{R}^d$,

- $L$ : continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $E[L_T^2] < \infty$. 

Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$:

- $dX_{\phi} = \mu_X(X_{\phi}, \nu) \, dt + \beta_X(X_{\phi}) \, dL + \sigma_X(X_{\phi}, \nu) \, dW$,

- $dY_{\phi} = \mu_Y(Z_{\phi}, \nu) \, dt + \beta_Y(Z_{\phi}) \, dL + \sigma_Y(Z_{\phi}, \nu) \, dW$.

Problem:

- $(y, x) \in O(s)$ and $y' \geq y \Rightarrow (y', x) \in O(s)$.

$V(t) := \{ (x, y) : \exists \phi \in U \times L \text{ s.t. } Z_{\phi}(t, x, y(s)) \in O(s) \ \forall \ t \leq s \leq T \}$. 
General model

Set of controls : $\mathcal{U} \times \mathcal{L}$ with

- $\mathcal{U}$ : prog. meas. process in $L^2([0, T] \times \Omega)$ with values in $U \subset \mathbb{R}^d$,
- $\mathcal{L}$ : continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E}[L_T^2] < \infty$.

Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$ :

\[
dX^\phi = \mu_X(X^\phi, \nu)dr + \beta_X(X^\phi)dL + \sigma_X(X^\phi, \nu)dW \\
dY^\phi = \mu_Y(Z^\phi, \nu)dr + \beta_Y(Z^\phi)^T dL + \sigma_Y(Z^\phi, \nu)^T dW .
\]
General model

Set of controls : $\mathcal{U} \times \mathcal{L}$ with

- $\mathcal{U}$ : prog. meas. process in $L^2([0, T] \times \Omega)$ with values in $U \subset \mathbb{R}^d$,
- $\mathcal{L}$ : continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E}[L_T^2] < \infty$.

Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

\[
\begin{align*}
    dX^\phi &= \mu_X(X^\phi, \nu)dr + \beta_X(X^\phi)dL + \sigma_X(X^\phi, \nu)dW \\
    dY^\phi &= \mu_Y(Z^\phi, \nu)dr + \beta_Y(Z^\phi)^\top dL + \sigma_Y(Z^\phi, \nu)^\top dW.
\end{align*}
\]

Problem :

\[
V(t) := \left\{(x, y) : \exists \phi \in \mathcal{U} \times \mathcal{L} \text{ s.t. } Z_{t,x,y}^\phi(s) \in \mathcal{O}(s) \forall t \leq s \leq T\right\}.
\]
General model

Set of controls : $\mathcal{U} \times \mathcal{L}$ with

- $\mathcal{U}$ : prog. meas. process in $L^2([0, T] \times \Omega)$ with values in $U \subset \mathbb{R}^d$,
- $\mathcal{L}$ : continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E} \left[ L_T^2 \right] < \infty$.

Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$ :

\[
\begin{align*}
    dX^\phi &= \mu_X(X^\phi, \nu)dr + \beta_X(X^\phi)dL + \sigma_X(X^\phi, \nu)dW \\
    dY^\phi &= \mu_Y(Z^\phi, \nu)dr + \beta_Y(Z^\phi)^\top dL + \sigma_Y(Z^\phi, \nu)^\top dW .
\end{align*}
\]

Problem : $(y, x) \in \mathcal{O}(s)$ and $y' \geq y \Rightarrow (y', x) \in \mathcal{O}(s)$.

$V(t) := \left\{ (x, y) : \exists \phi \in \mathcal{U} \times \mathcal{L} \text{ s.t. } Z^\phi_{t,x,y}(s) \in \mathcal{O}(s) \forall t \leq s \leq T \right\}$. 
General model

Set of controls: $\mathcal{U} \times \mathcal{L}$ with

- $\mathcal{U}$: prog. meas. process in $L^2([0, T] \times \Omega)$ with values in $U \subset \mathbb{R}^d$,
- $\mathcal{L}$: continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E} [L_T^2] < \infty$.

Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

\[
\begin{align*}
    dX^\phi &= \mu_X(X^\phi, \nu)dr + \beta_X(X^\phi)dL + \sigma_X(X^\phi, \nu)dW \\
    dY^\phi &= \mu_Y(Z^\phi, \nu)dr + \beta_Y(Z^\phi)^\top dL + \sigma_Y(Z^\phi, \nu)^\top dW .
\end{align*}
\]

Problem: $(y, x) \in \mathcal{O}(s)$ and $y' \geq y \Rightarrow (y', x) \in \mathcal{O}(s)$.

$v(t, x) := \inf \left\{ y : \exists \phi \in \mathcal{U} \times \mathcal{L} \text{ s.t. } Z^\phi_{t,x,y}(s) \in \mathcal{O}(s) \forall t \leq s \leq T \right\}$. 
Relation between both problems

VWAP problem of the form

\[ v(t, x; p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ and } \mathbb{E} \left[ G_{t,x,y}^\phi \right] \geq p \right\}. \]

with \( G_{t,x,y}^\phi := G(Z_{t,x,y}^\phi(T)). \)
Relation between both problems

VWAP problem of the form

\[ \nu(t, x; p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^{\phi} \in \mathcal{O} \text{ and } \mathbb{E} \left[ G_{t,x,y}^{\phi} \right] \geq p \right\} . \]

with \( G_{t,x,y}^{\phi} := G(Z_{t,x,y}^{\phi}(T)) \).

Assume \( G_{t,x,y}^{\phi} \in L^2 \). Then, \( \exists \alpha \in L_{\mathcal{P}}^2 \) such that

\[ G_{t,x,y}^{\phi} = \bar{p} + \int_t^T \alpha_s dW_s =: P_{t,\bar{p}}^{\alpha}(T) \text{ with } \bar{p} := \mathbb{E} \left[ G_{t,x,y}^{\phi} \right] . \]
Relation between both problems

VWAP problem of the form

\[ v(t, x; p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ and } \mathbb{E}\left[ G_{t,x,y}^\phi \right] \geq p \right\} . \]

with \( G_{t,x,y}^\phi := G(Z_{t,x,y}^\phi(T)) \).

Assume \( G_{t,x,y}^\phi \in L^2 \). Then, \( \exists \alpha \in L^2_P \) such that

\[ G_{t,x,y}^\phi = \bar{p} + \int_t^T \alpha_s dW_s =: P_{t,\bar{p}}^\alpha(T) \text{ with } \bar{p} := \mathbb{E}\left[ G_{t,x,y}^\phi \right] . \]

Hence, \( (Z_{t,x,y}^\phi, P_{t,\bar{p}}^\alpha) \in \mathcal{O} \times \mathbb{R} \) and \( G(Z_{t,x,y}^\phi(T)) \geq P_{t,\bar{p}}^\alpha(T) \).
Relation between both problems

VWAP problem of the form

\[
\nu(t, x; p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z^\phi_{t, x, y} \in \mathcal{O} \text{ and } \mathbb{E} \left[ G^\phi_{t, x, y} \right] \geq p \right\}.
\]

with \( G^\phi_{t, x, y} := G(Z^\phi_{t, x, y}(T)) \).

Assume \( G^\phi_{t, x, y} \in L^2 \). Then, \( \exists \alpha \in L^2_P \) such that

\[
G^\phi_{t, x, y} = \bar{p} + \int_t^T \alpha_s dW_s =: P^\alpha_{t, \bar{p}}(T) \text{ with } \bar{p} := \mathbb{E} \left[ G^\phi_{t, x, y} \right].
\]

Hence, \((Z^\phi_{t, x, y}, P^\alpha_{t, p}) \in \mathcal{O} \times \mathbb{R} \) and \( G(Z^\phi_{t, x, y}(T)) \geq P^\alpha_{t, p}(T) \).

Conversely, the above implies \( Z^\phi_{t, x, y} \in \mathcal{O} \) and \( \mathbb{E} \left[ G^\phi_{t, x, y} \right] \geq p \).
Relation between both problems

VWAP problem of the form

\[ v(t, x; p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t, x, y}^{\phi} \in \mathcal{O} \text{ and } \mathbb{E} \left[ G_{t, x, y}^{\phi} \right] \geq p \right\} . \]

with \( G_{t, x, y}^{\phi} := G(\phi_{t, x, y}(T)). \)

Assume \( G_{t, x, y}^{\phi} \in L^2. \) Then, \( \exists \alpha \in L_{\mathcal{P}}^2 \text{ such that} \)

\[ G_{t, x, y}^{\phi} = \bar{p} + \int_t^T \alpha_s dW_s =: P_{t, \bar{p}}^\alpha(T) \text{ with } \bar{p} := \mathbb{E} \left[ G_{t, x, y}^{\phi} \right]. \]

Hence, \( (Z_{t, x, y}^{\phi}, P_{t, \bar{p}}^\alpha) \in \mathcal{O} \times \mathbb{R} \text{ and } G(Z_{t, x, y}^{\phi}(T)) \geq P_{t, \bar{p}}^\alpha(T). \)

Prop. : \( v(t, x; p) = \inf \left\{ y : \exists (\phi, \alpha) \text{ s.t. } (Z_{t, x, y}^{\phi}, P_{t, \bar{p}}^\alpha) \in \bar{\mathcal{O}} \right\} , \)

with \( \bar{\mathcal{O}} := \mathcal{O} \times \mathbb{R} \mathbf{1}_{[0, T]} + \{(x, y, p) \in \mathcal{O} \times \mathbb{R} : G(x, y) \geq p\} \mathbf{1}_{\{T\}}. \)
Relation between both problems

VWAP problem of the form

\[ v(t, x; p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t, x, y}^\phi \in \mathcal{O} \text{ and } \mathbb{E} \left[ G_{t, x, y}^\phi \right] \geq p \right\} \]

with \( G_{t, x, y}^\phi := G(Z_{t, x, y}^\phi(T)) \).

Assume \( G_{t, x, y}^\phi \in L^2 \) for all \((t, x, y, \nu)\). Then, \( \exists \alpha \in L^2_P \) such that

\[ G_{t, x, y}^\phi = \bar{p} + \int_t^T \alpha_s \, dW_s =: P_{t, \bar{p}}^\alpha(T) \text{ with } \bar{p} := \mathbb{E} \left[ G_{t, x, y}^\phi \right]. \]
Key tool: Geometric dynamic programming

- First introduced by Soner and Touzi for super-hedging under Gamma constraints
Key tool: Geometric dynamic programming

- First introduced by Soner and Touzi for super-hedging under Gamma constraints
- Extended to American type constraints: obstacle version of B. and Vu.

Assumption: \[ \text{For all sequence } (t_n, z_n) \text{ of } [0, T] \times \mathbb{R}^d_+ \text{ such that } (t_n, z_n) \to (t, z), \text{ we have } t_n \geq t_{n+1} \text{ and } z_n \in O(t_n) \forall n \geq 1 \Rightarrow z \in O(t). \]
Key tool : Geometric dynamic programming

- First introduced by Soner and Touzi for super-hedging under Gamma constraints
- Extended to American type constraints : obstacle version of B. and Vu.

Assumption : [Right-continuity of the target] For all sequence $(t_n, z_n)_n$ of $[0, T] \times \mathbb{R}^{d+1}$ such that $(t_n, z_n) \to (t, z)$, we have

$$t_n \geq t_{n+1} \text{ and } z_n \in \mathcal{O}(t_n) \forall n \geq 1 \implies z \in \mathcal{O}(t).$$
Key tool: Geometric dynamic programming

- First introduced by Soner and Touzi for super-hedging under Gamma constraints
- Extended to American type contraints: obstacle version of B. and Vu.

Theorem:

\[ V(t) = \left\{ z : \exists \phi \text{ s.t. } Z_{t,z}^\phi(\theta \land \tau) \in O \bigoplus^{\tau,\theta} V \text{ for all } \theta, \tau \in \mathcal{T}_{[t,T]} \right\} \]

where

\[ O \bigoplus^{\tau,\theta} V := O(\tau) 1_{\tau \leq \theta} + V(\theta) 1_{\tau > \theta}. \]
Key tool: Geometric dynamic programming

- First introduced by Soner and Touzi for super-hedging under Gamma constraints
- Extended to American type constraints: obstacle version of B. and Vu.

Theorem: For all \( \phi \in \mathcal{U} \times \mathcal{L} \) and \( \theta \in \mathcal{T}_{[t, T]} \):

**GDP1:**

\[
Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta))
\]

**GDP2:**

\[
y < v(t, x) \Rightarrow \mathbb{P}\left[Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \text{ and } Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, \theta]\right] < 1
\]
Formal derivation of the PDE

Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x)$, $\exists \phi$ such that $Z_{t,z}^\phi \in \mathcal{O}$ on $[t, T]$. 
Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x)$, $\exists \phi$ such that $Z_{t,z}^\phi \in \mathcal{O}$ on $[t, T]$.

Then $Y_{t,z}^\phi(t+) \geq v(t+, X_{t,x}(t+))$ and
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x) \), \( \exists \phi \) such that \( Z^{\phi}_{t,z} \in \mathcal{O} \) on \([t, T]\).

Then \( Y^{\phi}_{t,z}(t+) \geq v(t+, X_{t,x}(t+)) \) and

\[
\mu_Y(z, \nu_t)dt + \sigma_Y(z, \nu_t)dW_t + \beta_Y(z)dL_t
\geq \mathcal{L}^{\nu_t}_X v(t, x)dt + Dv(t, x)\sigma_X(x, \nu_t)dW_t + Dv(t, x)\beta_X(x)dL_t
\]
Formal derivation of the PDE

Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x)$, $\exists \phi$ such that $Z_{t,z}^\phi \in \mathcal{O}$ on $[t, T]$.

Then $Y_{t,z}^\phi(t+) \geq v(t+, X_{t,x}(t+))$ and

$$
\mu_Y(z, \nu_t)dt + \sigma_Y(z, \nu_t)dW_t + 0 \\
\geq \mathcal{L}_X^{\nu_t}v(t, x)dt + Dv(t, x)\sigma_X(x, \nu_t)dW_t + 0
$$
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x) \), \( \exists \phi \) such that \( Z_{t,z}^\phi \in \mathcal{O} \) on \([t, T]\).

Then \( Y_{t,z}^\phi(t+) \geq v(t+, X_{t,x}(t+)) \) and

\[
\mu_Y(z, \nu_t)dt + \sigma_Y(z, \nu_t)dW_t + 0 \geq \mathcal{L}_X^\nu v(t, x)dt + Dv(t, x)\sigma_X(x, \nu_t)dW_t + 0
\]

Ok if

\[
\sup_{u \in Nv} (\mu_Y(\cdot, v, u) - \mathcal{L}_X^u v) \geq 0
\]

with \( Nv := \{ u \in U : \sigma_Y(\cdot, v, u) = Dv\sigma_X(\cdot, u) \} \).
Formal derivation of the PDE

Assume that \( \nu \) is smooth and the inf is achieved.

For \( y = \nu(t, x) \), \( \exists \phi \) such that \( Z_{t,z}^\phi \in \mathcal{O} \) on \([t, T]\).

Then \( Y_{t,z}^\phi(t+) \geq \nu(t+, X_{t,x}(t+)) \) and

\[
\mu_Y(z, \nu_t) dt + \sigma_Y(z, \nu_t) dW_t + \beta_Y(z) dL_t \\
\geq \mathcal{L}_X^{\nu_t} \nu(t, x) dt + D\nu(t, x) \sigma_X(x, \nu_t) dW_t + D\nu(t, x) \beta_X(x) dL_t
\]
Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x)$, $\exists \phi$ such that $Z_{t,z}^\phi \in \mathcal{O}$ on $[t, T]$.

Then $Y_{t,z}^\phi(t+) \geq v(t+, X_{t,x}(t+))$ and

$$
\mu_Y(z, \nu_t)dt + \sigma_Y(z, \nu_t)dW_t + [\beta_Y(z)^\top - Dv(t, x)\beta_X(x)]dL_t \\
\geq \mathcal{L}^\nu_t v(t, x)dt + Dv(t, x)\sigma_X(x, \nu_t)dW_t +
$$
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x) \), \( \exists \phi \) such that \( Z_{t,z}^\phi \in \mathcal{O} \) on \([t, T]\).

Then \( Y_{t,z}^\phi(t+) \geq v(t+, X_{t,x}(t+)) \) and

\[
\mu_Y(z, \nu_t)dt + \sigma_Y(z, \nu_t)dW_t + [\beta_Y(z)^\top - Dv(t, x)\beta_X(x)]dL_t \\
\geq \mathcal{L}_X^\nu_t v(t, x)dt + Dv(t, x)\sigma_X(x, \nu_t)dW_t +
\]

Ok if

\[
\max\{[\beta_Y(z)^\top - Dv(t, x)\beta_X(x)]\ell, \ \ell \in \Delta_+\} > 0
\]

with \( \Delta_+ := \mathbb{R}^d_+ \cap \partial B_1(0) \).
Formal derivation of the PDE

Set

$$F_\varepsilon v := \sup \{ \mu_Y(\cdot, v, u) - L_X^u v, \; u \in N_\varepsilon v \}$$

$$G v := \max \left\{ [\beta_Y(z)^\top - Dv(t, x)\beta_X(x)]\ell, \; \ell \in \Delta_+ \right\}$$

with

$$N_\varepsilon v := \{ u \in U : |\sigma_Y(\cdot, v, u) - Dv\sigma_X(\cdot, u)| \leq \varepsilon \}$$

$$\Delta_+ := \mathbb{R}_+^d \cap \partial B_1(0).$$
Formal derivation of the PDE

Set

\[ F_\varepsilon v := \sup \{ \mu_Y (\cdot, v, u) - \mathcal{L}^u_X v, \ u \in N_\varepsilon v \} \]

\[ Gv := \max \left\{ [\beta_Y(z)^\top - Dv(t, x)\beta_X(x)]\ell, \ \ell \in \Delta_+ \right\} \]

PDE characterization in the interior of the domain

\[ \max \{ F_0 v, \ Gv \} = 0 \text{ on } (t, x, v(t, x)) \in \text{int}(D) \]

where \( D := \{(t, x, y) : (x, y) \in O(t)\} \).
Formal derivation of the PDE

Set

\[ F_\varepsilon v := \sup \{ \mu_Y(\cdot, v, u) - \mathcal{L}_X^u v, \ u \in N_\varepsilon v \} \]

\[ Gv := \max \left\{ \left[ \beta_Y(z)^\top - Dv(t, x)\beta_X(x) \right] \ell, \ \ell \in \Delta_+ \right\} \]

PDE characterization in the interior of the domain

\[ \max \{ F_0 v, \ Gv \} = 0 \text{ on } (t, x, v(t, x)) \in \text{int}(D) \]

where \( D := \{(t, x, y) : (x, y) \in O(t)\} \).

Need to be relaxed in \( \varepsilon, t, x, v, Dv, D^2v \) to ensure proper definitions.
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is
\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

Assumption : \(D \in C^{1,2}\) (or intersection of \(C^2\) domains).
PDE on the space boundary \((x, y) \in \partial\mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^2\) domains).

Take \(\delta \in C^2\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.\]

Assumption : \(D \in C^{1,2}\) (or intersection of \(C^2\) domains).

Take \(\delta \in C^2\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d \delta(Z_{t,z}^\phi(t)) \geq 0\) if \((t, z) \in \partial D\).
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

Assumption : \(D \in C^{1,2}\) (or intersection of \(C^2\) domains).

Take \(\delta \in C^2\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(Z_{t,z}^\phi(t)) \geq 0\) if \((t, z) \in \partial D\).

As above it implies : either

\[ \mathcal{L}_Z^u \delta(t, x, y) \geq 0 \text{ and } D\delta(t, x, y)\sigma_Z(x, y, u) = 0 \]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^2\) domains).

Take \(\delta \in C^2\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(Z_{t,z}^\phi(t)) \geq 0\) if \((t, z) \in \partial D\).

As above it implies: or

\[
\max\{DD(t, x, y)\beta_z^\top(x, y)\ell, \ell \in \Delta_+\} > 0.
\]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

The GDP and the need for a reflexion on the boundary leads to the definition of

\[
N_{\varepsilon}^\text{in} v := \{ u \in N_{\varepsilon} v : |D\delta(\cdot, v)\sigma Z(\cdot, v, u)| \leq \varepsilon \} \\
F_{\varepsilon}^\text{in} v := \sup_{u \in N_{\varepsilon}^\text{in} v} \min \{ \mu_Y(\cdot, v, u) - L_X^u v, L_Z^u \delta(t, x, y) \} \\
G^\text{in} v := \max_{\ell \in \Delta_+} \min \left\{ \left[ \beta_Y(\cdot, v)^\top - Dv \beta_X \right] \ell, D\delta(\cdot, v) \beta_Z^\top(\cdot, v) \ell \right\}
\]

Then, the PDE on the boundary reads

\[
\max \{ F_{\varepsilon}^\text{in} 0 v, G^\text{in} v \} = 0 \text{ on } (t, x, v(t, x)) \in \partial \mathcal{D}.
\]

Need to be relaxed as above. As usual, the constraint appears only on the subsolution part.
The GDP and the need for a reflexion on the boundary leads to the definition of

\[ N^{\text{in}}_{\varepsilon} v := \{ u \in N_{\varepsilon} v : |D\delta(\cdot, v)\sigma Z(\cdot, v, u)| \leq \varepsilon \} \]

\[ F^{\text{in}}_{\varepsilon} v := \sup_{u \in N^{\text{in}}_{\varepsilon} v} \min \{ \mu_Y(\cdot, v, u) - \mathcal{L}_X^u v, \mathcal{L}_Z^u \delta(t, x, y) \} \]

\[ G^{\text{in}} v := \max_{\ell \in \Delta_+} \min \left\{ [\beta_Y(\cdot, v)^T - Dv \beta_X]\ell, D\delta(\cdot, v)\beta_z^T(\cdot, v)\ell \right\} \]

Then, the PDE on the boundary reads

\[ \max\{ F^{\text{in}}_0 v, G^{\text{in}} v \} = 0 \text{ on } (t, x, v(t, x)) \in \partial D. \]
PDE on the space boundary $(x, y) \in \partial \Omega(t)$

The GDP and the need for a reflexion on the boundary leads to the definition of

$$
N_{\varepsilon}^{in} v := \{ u \in N_{\varepsilon} v : |D \delta(\cdot, v) \sigma Z(\cdot, v, u)| \leq \varepsilon \}
$$

$$
F_{\varepsilon}^{in} v := \sup_{u \in N_{\varepsilon}^{in} v} \min \{ \mu_Y(\cdot, v, u) - \mathcal{L}_X^u v, \mathcal{L}_Z^u \delta(t, x, y) \}
$$

$$
G^{in} v := \max_{\ell \in \Delta_+} \min \left\{ [\beta_Y(\cdot, v)^\top - Dv \beta_X] \ell, D \delta(\cdot, v) \beta_Z^\top (\cdot, v) \ell \right\}
$$

Then, the PDE on the boundary reads

$$
\max\{F_0^{in} v, G^{in} v\} = 0 \text{ on } (t, x, v(t, x)) \in \partial D.
$$

Need to be relaxed as above.
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

The GDP and the need for a reflexion on the boundary leads to the definition of

\[
N_{\varepsilon}^{\text{in}} \nu := \{ u \in N_{\varepsilon} \nu : |D\delta(\cdot, \nu)\sigma_{\mathcal{Z}}(\cdot, \nu, u)| \leq \varepsilon \}
\]

\[
F_{\varepsilon}^{\text{in}} \nu := \sup_{u \in N_{\varepsilon}^{\text{in}} \nu} \min \left\{ \mu_{\mathcal{Y}}(\cdot, \nu, u) - \mathcal{L}_{X}^{u} \nu, \mathcal{L}_{Z}^{u} \delta(t, x, y) \right\}
\]

\[
G_{\varepsilon}^{\text{in}} \nu := \max_{\ell \in \Delta_{+}} \min \left\{ [\beta_{\mathcal{Y}}(\cdot, \nu)^{\top} - D\nu \beta_{X}] \ell, D\delta(\cdot, \nu)\beta_{Z}^{\top}(\cdot, \nu) \ell \right\}
\]

Then, the PDE on the boundary reads

\[
\max\{F_{0}^{\text{in}} \nu, G_{\varepsilon}^{\text{in}} \nu\} = 0 \text{ on } (t, x, \nu(t, x)) \in \partial \mathcal{D}.
\]

Need to be relaxed as above.

As usual, the constraint appears only on the subsolution part.
Terminal condition at $t = T$

Must have

$$v(T-, \cdot) \geq g(x) := \inf\{y : (x, y) \in O(T)\}.$$
Terminal condition at $t = T$

Must have

$$v(T-,\cdot) \geq g(x) := \inf\{y : (x,y) \in O(T)\}.$$ 

Must also have

either $N_0^{(in)}v(T-\cdot) \neq \emptyset$
Terminal condition at $t = T$

Must have

$$v(T-, \cdot) \geq g(x) := \inf \{y : (x, y) \in \mathcal{O}(T)\}.$$ 

Must also have

either $N_{0}^{(\text{in})}v(T- \cdot) \neq \emptyset$ or $G^{(\text{in})}v(T-\cdot) \geq 0$. 

Terminal condition at $t = T$

Must have

$$v(T-, \cdot) \geq g(x) := \inf\{y : (x, y) \in O(T)\}.$$ 

Must also have

either $N_0^{(in)} v(T-\cdot) \neq \emptyset$ or $G^{(in)} v(T-\cdot) \geq 0$.

This writes

$$\min \left\{ v - g, \max \{ R^{(in)}(\cdot, v, Dv), G^{(in)}v \} \right\}(T-, \cdot) = 0.$$
Back to the VWAP guaranteed pricing problem

Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
Back to the VWAP guaranteed pricing problem

Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.

Price dynamics:

$$dX^{L,1} = X^{L,1} \mu(X^{L,1})dt + X^{L,1} \sigma(X^{L,1})dW_t - X^{L,1} \beta(X^{L,1}(t))dL_t$$
Back to the VWAP guaranteed pricing problem

Controls: \( L \uparrow \) adapted and continuous. \( L_t = \# \) of sold stocks.

Price dynamics:

\[
dX^{L,1} = X^{L,1} \mu(X^{L,1})dt + X^{L,1} \sigma(X^{L,1})dW_t - X^{L,1} \beta(X^{L,1}(t))dL_t
\]

Cumulated liquidation cost: \( dY^L = X^{L,1} dL_t \)
Back to the VWAP guaranteed pricing problem

Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.

Price dynamics:

$$dX^{L,1} = X^{L,1} \mu(X^{L,1})dt + X^{L,1} \sigma(X^{L,1})dW_t - X^{L,1} \beta(X^{L,1}(t))dL_t$$

Cumulated liquidation cost: $dY^L = X^{L,1} dL_t$

Volume weighted market price: $dX^{L,2} = \vartheta dX^{L,1}$. 
Back to the VWAP guaranted pricing problem

Controls : $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.

Price dynamics :

$$dX^{L,1} = X^{L,1} \mu(X^{L,1})dt + X^{L,1} \sigma(X^{L,1})dW_t - X^{L,1} \beta(X^{L,1}(t))dL_t$$

Cumulated liquidation cost : $dY^L = X^{L,1} dL_t$

Volume weighted market price : $dX^{L,2} = \vartheta dX^{L,1}$.

Cumulated $\#$ of sold stocks : $X^{L,3} := L \in [\underline{\Lambda}, \bar{\Lambda}] \rightarrow \{K\}$
Back to the VWAP guaranteed pricing problem

Controls : \( L \uparrow \) adapted and continuous. \( L_t = \# \) of sold stocks.

Price dynamics :
\[
dX^{L,1} = X^{L,1} \mu(X^{L,1}) dt + X^{L,1} \sigma(X^{L,1}) dW_t - X^{L,1} \beta(X^{L,1}(t)) dL_t
\]

Cumulated liquidation cost : \( dY^{L} = X^{L,1} dL_t \)

Volume weighted market price : \( dX^{L,2} = \vartheta dX^{L,1} \).

Cumulated \( \# \) of sold stocks : \( X^{L,3} := L \in [\Lambda, \bar{\Lambda}] \to \{K\} \)

Pricing function (with \( \Psi(x, y) = \ell(y - \gamma Kx^2 / \Theta(T)) \) and \( \gamma > 0 \))
\[
v(t, x, p) := \inf \{ y \geq 0 : \exists L \text{ s.t. } X_{t,x}^{L,3} \in [\Lambda, \bar{\Lambda}] , \mathbb{E} \left[ \Psi(Z_{t,x}^{L}(Y(T))) \right] \geq p \},
\]
Back to the VWAP guaranteed pricing problem

Controls : $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.

Price dynamics :

$$dX^{L,1} = X^{L,1} \mu(X^{L,1}) dt + X^{L,1} \sigma(X^{L,1}) dW_t - X^{L,1} \beta(X^{L,1}(t)) dL_t$$

Cumulated liquidation cost : $dY^{L} = X^{L,1} dL_t$

Volume weighted market price : $dX^{L,2} = \vartheta dX^{L,1}$.

Cumulated $\#$ of sold stocks : $X^{L,3} := L \in [\underline{\Lambda}, \overline{\Lambda}] \to \{K\}$

Pricing function (with $\Psi(x, y) = \ell(y - \gamma x^2)$ and $\gamma > 0$)

$$v(t, x, p) := \inf \{y \geq 0 : \exists L \text{ s.t. } X^{L,3}_{t, x} \in [\underline{\Lambda}, \overline{\Lambda}] \text{, } \mathbb{E} \left[ \Psi(Z^{L}_{t, x, y}(T)) \right] \geq p \},$$
Representation as a stochastic target problem

\[ \nu(t, x, p) := \inf \{ y \geq 0 : \exists L \text{ s.t. } X_{t,x}^{L,3} \in [\Lambda, \Lambda], \mathbb{E} [\Psi(Z_{t,x,y}(T))] \geq p \} \]

\[ = \inf \{ y \geq 0 : \exists (L, \nu) \text{ s.t. } X_{t,x}^{L,3} \in [\Lambda, \Lambda], \Psi(Z_{t,x,y}(T)) \geq P_{t,p}^\nu(T) \} \]

with \( \nu \in L^2_P \) and \( P_{t,p}^\nu := p + \int_t^T \nu_s dW_s \).
PDE characterization

**Proposition** Under “good assumptions”, $v_*$ is a viscosity supersolution on $[0, T)$ of

$$\max \left\{ F_0 \varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} = 0 \quad \text{if} \quad \Lambda \leq x^3 \leq \bar{\Lambda}$$

and $v_*$ is a subsolution on $[0, T)$ of

$$\min \left\{ \varphi, \max \left\{ F_0 \varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} \right\} = 0 \quad \text{if} \quad \Lambda < x^3 < \bar{\Lambda}$$

$$\min \left\{ \varphi, x^1 + \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} = 0 \quad \text{if} \quad \Lambda = x^3$$

$$\min \left\{ \varphi, F_0 \varphi \right\} = 0 \quad \text{if} \quad x^3 = \bar{\Lambda},$$

where

$$F_0 \varphi := -L_\varphi x \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right).$$

Moreover, $v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p)$. 
Proposition Under “good assumptions”, $v_*$ is a viscosity supersolution on $[0, T)$ of

$$\max \left\{ F_0 \varphi , x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} = 0 \text{ if } \Lambda \leq x^3 \leq \overline{\Lambda}$$

and $v_*$ is a subsolution on $[0, T)$ of

$$\min \left\{ \varphi , \max \left\{ F_0 \varphi , x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} \right\} = 0 \text{ if } \Lambda < x^3 < \overline{\Lambda}$$

$$\min \left\{ \varphi , x^1 + \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} = 0 \text{ if } \Lambda = x^3$$

$$\min \left\{ \varphi , F_0 \varphi \right\} = 0 \text{ if } x^3 = \overline{\Lambda},$$

where

$$F_0 \varphi := -L_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1,p)}^2 \varphi \right).$$

Moreover, $v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p).$
The “good assumptions”

On $\Lambda$, $\bar{\Lambda}$:

\[ \Lambda, \bar{\Lambda} \in C^1, \; \Lambda < \bar{\Lambda} \text{ on } [0, T), \; D\Lambda, D\bar{\Lambda} \in (0, M] \]
The “good assumptions”

On $\Lambda, \bar{\Lambda}$:

$\Lambda, \bar{\Lambda} \in C^1$, $\Lambda < \bar{\Lambda}$ on $[0, T)$, $D\Lambda, D\bar{\Lambda} \in (0, M]$

On the loss function $\ell$:

$\exists \epsilon > 0$ s.t. $\epsilon \leq D^-\ell$, $D^+\ell \leq \epsilon^{-1}$,

and $\lim_{r \to \infty} D^+\ell(r) = \lim_{r \to \infty} D^-\ell(r)$. 

Proposition $v^*$ is a viscosity supersolution of

$\min \{D^p\phi - \epsilon, (D^x_1\phi - C D^p\phi)_1 > 0, -D^x_1\phi + C D^p\phi\} = 0$ (∗)

and $v^*$ is a viscosity subsolution of

$\max \{-D^p\phi + \epsilon, (D^x_1\phi - C D^p\phi)_1 > 0, -D^x_1\phi + C D^p\phi\} = 0$ (∗∗)

where $C$ is continuous and depends only on $x$.

Provides a control on the ratio $D^x_1\phi / D^p\phi$. 
The “good assumptions”

On $\Lambda, \bar{\Lambda}$:

\[ \Lambda, \bar{\Lambda} \in C^1, \Lambda < \bar{\Lambda} \text{ on } [0, T), \quad D\Lambda, D\bar{\Lambda} \in (0, M] \]

On the loss function $\ell$:

\[ \exists \epsilon > 0 \text{ s.t. } \epsilon \leq D^-\ell, \quad D^+\ell \leq \epsilon^{-1}, \]

and

\[ \lim_{r \to \infty} D^+\ell(r) = \lim_{r \to \infty} D^-\ell(r). \]

Proposition $v_*$ is a viscosity supersolution of

\[ \min \left\{ D_p\varphi - \epsilon, \ (D_{x^1}\varphi - CD_p\varphi)1_{x^1 > 0}, \ -D_{x^1}\varphi + CD_p\varphi \right\} = 0 \quad (*) \]

and $v^*$ is a viscosity subsolution of

\[ \max \left\{ -D_p\varphi + \epsilon, \ (D_{x^1}\varphi - CD_p\varphi)1_{x^1 > 0}, \ -D_{x^1}\varphi + CD_p\varphi \right\} = 0. \quad (***) \]

where $C$ is continuous and depends only on $x$. 
The “good assumptions”

On $\Lambda, \bar{\Lambda}$:

$$\Lambda, \bar{\Lambda} \in C^1, \Lambda < \bar{\Lambda} \text{ on } [0, T), \ D\Lambda, D\bar{\Lambda} \in (0, M]$$

On the loss function $\ell$:

$$\exists \epsilon > 0 \text{ s.t. } \epsilon \leq D^-\ell, \ D^+\ell \leq \epsilon^{-1},$$

and

$$\lim_{r \to \infty} D^+\ell(r) = \lim_{r \to \infty} D^-\ell(r).$$

Proposition $v_*$ is a viscosity supersolution of

$$\min \{D_p\varphi - \epsilon, (D_{x^1}\varphi - CD_p\varphi)1_{x^1 > 0}, -D_{x^1}\varphi + CD_p\varphi\} = 0 \ (\ast)$$

and $v^*$ is a viscosity subsolution of

$$\max \{-D_p\varphi + \epsilon, (D_{x^1}\varphi - CD_p\varphi)1_{x^1 > 0}, -D_{x^1}\varphi + CD_p\varphi\} = 0. \ (\ast\ast)$$

where $C$ is continuous and depends only on $x$.

Provides a control on the ratio $D_{x^1}\varphi/D_p\varphi$. 
More controls on $v$

It also implies that $\exists \eta > 0$ s.t.

$$0 \leq v(t, x, p) \leq \epsilon - 1 \left| p - \ell(0) \right| + \gamma \eta \left( 1 + \left| x \right| \right),$$

and that for $(t_n, x_n, p_n)_n$ s.t.

$$\lim_{n \to \infty} v^*(t_n, x_n, p_n) = \lim_{n \to \infty} v^*(t_n, x_n, p_n) = 0 \text{ if } p_n \to -\infty,$$

$$\lim_{n \to \infty} v^*(t_n, x_n, p_n) = \ell(\infty) \text{ if } p_n \to \infty.$$
More controls on $v$

It also implies that $\exists \eta > 0$ s.t.

$$0 \leq v(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma\eta(1 + |x|),$$
More controls on $\nu$

It also implies that $\exists \eta > 0$ s.t.

$$0 \leq \nu(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma \eta(1 + |x|) ,$$

and that for $(t_n, x_n, p_n)_n$ s.t. $(t_n, x_n) \to (t, x)$:

$$\lim_{n \to \infty} \nu^*(t_n, x_n, p_n) = \lim_{n \to \infty} \nu^*(t_n, x_n, p_n) = 0 \text{ if } p_n \to -\infty ,$$

$$\lim_{n \to \infty} \frac{\nu^*(t_n, x_n, p_n)}{p_n} = \lim_{n \to \infty} \frac{\nu^*(t_n, x_n, p_n)}{p_n} = \frac{1}{D\ell(\infty)} \text{ if } p_n \to \infty .$$
More controls on $\nu$

It also implies that $\exists \eta > 0$ s.t.

$$0 \leq \nu(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma\eta(1 + |x|) ,$$

and that for $(t_n, x_n, p_n)_n$ s.t. $(t_n, x_n) \to (t, x)$:

$$\lim_{n \to \infty} \nu(t_n, x_n, p_n) = \lim_{n \to \infty} \nu^*(t_n, x_n, p_n) = 0 \text{ if } p_n \to -\infty ,$$

$$\lim_{n \to \infty} \frac{\nu(t_n, x_n, p_n)}{p_n} = \lim_{n \to \infty} \frac{\nu^*(t_n, x_n, p_n)}{p_n} = \frac{1}{D\ell(\infty)} \text{ if } p_n \to \infty .$$

A little more: $\nu$ is continuous in $p$ and $x^3$. 
Uniqueness

Want a comparison result in the class of functions with the above limit and growth conditions.
Uniqueness

Want a comparison result in the class of functions with the above limit and growth conditions.

Recall that

$$F_0 \varphi := -\mathcal{L}_X \varphi - \frac{(x_1^1 \sigma)^2}{2} \left( \left| \frac{D_{x_1} \varphi}{D_p \varphi} \right|^2 D_p^2 \varphi - 2 \left( \frac{D_{x_1} \varphi}{D_p \varphi} \right) D_{(x_1,p)}^2 \varphi \right).$$
Uniqueness

Want a comparison result in the class of functions with the above limit and growth conditions.

Recall that

\[ F_0 \phi := -\mathcal{L}_x \phi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \phi / D_p \phi|^2 D^2_p \phi - 2(D_{x^1} \phi / D_p \phi) D^2_{(x^1, p)} \phi \right). \]

We now control \( D_{x^1} \phi / D_p \phi \).
Uniqueness

Want a comparison result in the class of functions with the above limit and growth conditions.

Recall that

$$F_0 \varphi := -\mathcal{L}_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D^{2}_{(x^1, p)} \varphi \right).$$

We now control $D_{x^1} \varphi / D_p \varphi$.

This is not enough... If we need to penalize in $x^1$ (stock price) then the term $|D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi$ will blow up as $n \to \infty$, where $n$ comes from the usual penalisation $n|x^1_1 - x^1_2|^2$ due to the doubling of constants.
Uniqueness

Want a comparison result in the class of functions with the above limit and growth conditions.

Recall that

$$F_0\phi := -\mathcal{L}_X\phi - \frac{(x^1\sigma)^2}{2} \left( |D_{x^1}\phi/D_p\phi|^2 D_p^2\phi - 2(D_{x^1}\phi/D_p\phi)D^2_{(x^1,p)}\phi \right).$$

We now control $D_{x^1}\phi/D_p\phi$.

Assumption:

$$\exists \hat{x}^1 > 0 \text{ s.t. } \mu(\hat{x}^1) \leq 0 = \sigma(\hat{x}^1).$$
Uniqueness

Want a comparison result in the class of functions with the above limit and growth conditions.

Recall that

\[ F_0 \varphi := -\mathcal{L}_X \varphi - \frac{(x_1 \sigma)^2}{2} \left( |D_{x_1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x_1} \varphi / D_p \varphi) D_{(x_1,p)}^2 \varphi \right). \]

We now control \( D_{x_1} \varphi / D_p \varphi \).

**Assumption:**

\[ \exists \hat{x}_1 > 0 \text{ s.t. } \mu(\hat{x}_1) \leq 0 = \sigma(\hat{x}_1). \]

Bound on the stock price...
**Comparison**

**Theorem:** Let $U$ (resp. $V$) be a non-negative super- and subsolutions which are continuous in $x^3$. Assume that

$$U(t, x, p) \geq V(t, x, p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\},$$

and that $\exists c_+ > 0$ and $c_- \in \mathbb{R}$ s.t.

$$\limsup_{(t', x', p') \to (t, x, \infty)} V(t', x', p')/p' \leq c_+ \leq \liminf_{(t', y', p') \to (t, y, \infty)} U(t', y', p')/p',$$

$$\limsup_{(t', x', p') \to (t, x, -\infty)} V(t', x', p') \leq c_- \leq \liminf_{(t', y', p') \to (t, y, -\infty)} U(t', y', p').$$

If either $U$ is a supersolution of (*) which is continuous in $p$, or $V$ is a subsolution of (**) which is continuous in $p$, then

$$U \geq V.$$
Remarks on models with proportional transaction costs

Typical model

\[ X_{t,x}^1(s) = x^1 + \int_t^s X_{t,x}^1(r) \mu dr + \int_t^s X_{t,x}^1(r) \sigma dW_r \]

\[ X_{t,x}^{2,L}(s) = x^2 + \int_t^s \frac{X_{t,x}^{2,L}(r)}{X_{t,x}^1(r)} dX_{t,x}^1(r) - \int_t^s dL_1^r + \int_t^s dL_2^r \]

\[ Y_{t,y}^L(s) = y + \int_t^s (1 - \lambda) dL_1^r - \int_t^s (1 + \lambda) dL_2^r. \]

Our general results allow for pricing derivatives under loss constraints,...
Remarks on optimal management under shortfall constraints

Serves as a building block for problems of the form

$$\sup_{\phi \in \mathcal{A}_{t,z}} \mathbb{E} \left[ U(Z_{t,z}^\phi(T)) \right]$$

with

$$\mathcal{A}_{t,z} := \{ \phi \in \mathcal{A} : Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \}.$$ 

see B., Elie and Imbert (2010).
Some references