Generalized stochastic target problems - Application in optimal book liquidation

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Joint work with Minh Ngoc Dang, CEREMADE and Chevreux C.A.

Motivation

VWAP guaranteed contracts pricing :

$$\text{Find the minimal } y \text{ s.t., for some } L \in \mathcal{L} \text{ with } L_0 = 0 \, ,$$
$$X_T^{L,1} = K, X^{L,1} \in \Lambda \text{ and}$$
$$\mathbb{E}\left[\ell\left(y + \left[Y^L(T)/K - \gamma X^{L,2}(T)/\Theta(T)\right] K\right)\right] \geq p \, ,$$

for $p \in \mathbb{R}$ and $\ell : \mathbb{R} \mapsto \mathbb{R}$ non-decreasing.

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Leads to a stochastic target problem under expected loss.

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Start with a general model (which suits also well to models with proportional transaction costs).

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Set of controls : $\mathcal{U} \times \mathcal{L}$ with

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Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

$$dX^\phi = \mu_X(X^\phi, \nu)dr + \beta_X(X^\phi)dL + \sigma_X(X^\phi, \nu)dW$$
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Problem :

$$V(t) := \left\{(x,y) : \exists \phi \in \mathcal{U} \times \mathcal{L} \text{ s.t. } Z^\phi_{t,x,y}(s) \in \mathcal{O}(s) \ \forall \ t \le s \le T\right\} \ .$$

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Relation between both problems

VWAP problem of the form

$$v(t, x; p) \quad := \quad \inf \left\{ y : \exists \phi \text{ s.t. } Z^{\phi}_{t,x,y} \in \mathcal{O} \text{ and } \mathbb{E}\left[G^{\phi}_{t,x,y} \right] \geq p \right\} \; .$$

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with $G^{\phi}_{t,x,y} := G(Z^{\phi}_{t,x,y}(T))$.

Assume $G^{\phi}_{t,x,y} \in L^2$. Then, $\exists\ \alpha \in L^2_{\mathcal{P}}$ such that

$$G^{\phi}_{t,x,y} = \bar{p} + \int_t^T \alpha_s dW_s =: P^{\alpha}_{t,\bar{p}}(T) \ \text{ with } \bar{p} := \mathbb{E}\left[G^{\phi}_{t,x,y}\right].$$

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Hence, $(Z^{\phi}_{t,x,y}, P^{\alpha}_{t,p}) \in \mathcal{O} \times \mathbb{R}$ and $G(Z^{\phi}_{t,x,y}(T)) \geq P^{\alpha}_{t,p}(T)$.

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Conversely, the above implies $Z^{\phi}_{t,x,y} \in \mathcal{O}$ and $\mathbb{E}\left[G^{\phi}_{t,x,y}\right] \geq p$.

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Hence, $(Z^{\phi}_{t,x,y}, P^{\alpha}_{t,p}) \in \mathcal{O} \times \mathbb{R}$ and $G(Z^{\phi}_{t,x,y}(T)) \geq P^{\alpha}_{t,p}(T)$.

Prop. : $v(t,x;p) = \inf\left\{y : \exists(\phi,\alpha) \text{ s.t. } (Z^{\phi}_{t,x,y}, P^{\alpha}_{t,p}) \in \bar{\mathcal{O}}\right\}$,
with $\bar{\mathcal{O}} := \mathcal{O} \times \mathbb{R}\mathbf{1}_{[0,T)} + \{(x,y,p) \in \mathcal{O} \times \mathbb{R} : G(x,y) \geq p\}\mathbf{1}_{\{T\}}$.

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with $G^{\phi}_{t,x,y} := G(Z^{\phi}_{t,x,y}(T))$.

Assume $G^{\phi}_{t,x,y} \in L^2$ for all (t,x,y,ν). Then, $\exists\ \alpha \in L^2_{\mathcal{P}}$ such that

$$G^{\phi}_{t,x,y} = \bar{p} + \int_t^T \alpha_s dW_s =: P^{\alpha}_{t,\bar{p}}(T) \;\text{ with }\; \bar{p} := \mathbb{E}\left[G^{\phi}_{t,x,y}\right].$$

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Assumption : [Right-continuity of the target] For all sequence $(t_n, z_n)_n$ of $[0, T] \times \mathbb{R}^{d+1}$ such that $(t_n, z_n) \to (t, z)$, we have

$$t_n \geq t_{n+1} \text{ and } z_n \in \mathcal{O}(t_n) \; \forall \; n \geq 1 \quad \Longrightarrow \quad z \in \mathcal{O}(t) \; .$$

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Theorem :

$$V(t) = \left\{ z : \exists\, \phi \text{ s.t. } Z^{\phi}_{t,z}(\theta \wedge \tau) \in \mathcal{O} \bigoplus^{\tau,\theta} V \;\text{ for all }\; \theta, \tau \in \mathcal{T}_{[t,T]} \right\}$$

where

$$\mathcal{O} \bigoplus^{\tau,\theta} V := \mathcal{O}(\tau)\, 1_{\tau \leq \theta} + V(\theta)\, 1_{\tau > \theta} \; .$$

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Theorem : For all $\phi \in \mathcal{U} \times \mathcal{L}$ and $\theta \in \mathcal{T}_{[t,T]}$:
GDP1 :

$$Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y^{\phi}_{t,z}(\theta) \geq v(\theta, X^{\phi}_{t,x}(\theta))$$

GDP2 :

$$y < v(t,x) \Rightarrow \mathbb{P}\left[Y^{\phi}_{t,z}(\theta) \geq v(\theta, X^{\phi}_{t,x}(\theta)) \text{ and } Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t,\theta] \right] < 1$$

Formal derivation of the PDE

Assume that v is smooth and the inf is achieved.

For $y = v(t,x)$, $\exists\ \phi$ such that $Z_{t,z}^{\phi} \in \mathcal{O}$ on $[t, T]$.

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\begin{aligned}
&\mu_Y(z,\nu_t)dt + \sigma_Y(z,\nu_t)dW_t + \beta_Y(z)dL_t \\
&\geq \mathcal{L}^{\nu_t}_X v(t,x)dt + Dv(t,x)\sigma_X(x,\nu_t)dW_t + Dv(t,x)\beta_X(x)dL_t
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Ok if

$$\sup_{u \in Nv} \left(\mu_Y(\cdot, v, u) - \mathcal{L}^u_X v\right) \geq 0$$

with $Nv := \{u \in U : \sigma_Y(\cdot, v, u) = Dv\sigma_X(\cdot, u)\}$.

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Ok if

$$\max\{[\beta_Y(z)^\top - Dv(t,x)\beta_X(x)]\ell,\ \ell \in \Delta_+\} > 0$$

with $\Delta_+ := \mathbb{R}^d_+ \cap \partial B_1(0)$.

Formal derivation of the PDE

Set

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\begin{aligned}
F_\varepsilon v &:= \sup\left\{\mu_Y(\cdot, v, u) - \mathcal{L}_X^u v,\ \ u \in N_\varepsilon v\right\} \\
Gv &:= \max\left\{[\beta_Y(z)^\top - Dv(t,x)\beta_X(x)]\ell,\ \ell \in \Delta_+\right\}
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PDE characterization in the interior of the domain

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\max\{F_0 v\ ,\ Gv\} = 0 \text{ on } (t, x, v(t,x)) \in \mathrm{int}(D)
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Need to be relaxed in $\varepsilon, t, x, v, Dv, D^2 v$ to ensure proper definitions.

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As above it implies : either

$$\mathcal{L}^u_Z \delta(t, x, y) \geq 0 \text{ and } D\delta(t, x, y)\sigma_Z(x, y, u) = 0$$

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As above it implies : or

$$\max\{D\delta(t, x, y)\beta_z^{\top}(x, y)\ell,\ \ell \in \Delta_+\} > 0\ .$$

PDE on the space boundary $(x, y) \in \partial\mathcal{O}(t)$

The GDP and the need for a reflexion on the boundary leads to the definition of

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\begin{aligned}
N_\varepsilon^{\rm in} v &:= \left\{u \in N_\varepsilon v : |D\delta(\cdot, v)\sigma_Z(\cdot, v, u)| \le \varepsilon\right\} \\
F_\varepsilon^{\rm in} v &:= \sup_{u \in N_\varepsilon^{\rm in} v} \min\left\{\mu_Y(\cdot, v, u) - \mathcal{L}_X^u v \;,\; \mathcal{L}_Z^u \delta(t, x, y)\right\} \\
G^{\rm in} v &:= \max_{\ell \in \Delta_+} \min\left\{[\beta_Y(\cdot, v)^\top - Dv\beta_X]\ell \;,\; D\delta(\cdot, v)\beta_z^\top(\cdot, v)\ell\right\}
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Then, the PDE on the boundary reads

$$\max\{F_0^{\text{in}} v \ , \ G^{\text{in}} v\} = 0 \ \text{ on } (t, x, v(t, x)) \in \partial D \ .$$

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N_\varepsilon^{\text{in}} v &:= \{u \in N_\varepsilon v : |D\delta(\cdot, v)\sigma_Z(\cdot, v, u)| \le \varepsilon\} \\
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As usual, the constraint appears only on the subsolution part.

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This writes

$$\min\left\{v - g \ , \ \max\{R^{(\mathrm{in})}(\cdot, v, Dv) \ , \ G^{(\mathrm{in})} v\}\right\}(T-,\cdot) = 0 \ .$$

Back to the VWAP guaranted pricing problem

Controls : $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.

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Representation as a stochastic target problem

$$v(t,x,p)$$
$$:=$$
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$$=$$
$$\inf\{y \geq 0 : \exists (L,\nu) \ \text{ s.t. } X^{L,3}_{t,x} \in [\underline{\Lambda}, \overline{\Lambda}] \ , \ \Psi(Z^L_{t,x,y}(T)) \geq P^{\nu}_{t,p}(T)\}$$

with $\nu \in L^2_{\mathcal{P}}$ and $P^{\nu}_{t,p} := p + \int_t^{\cdot} \nu_s dW_s$.

PDE characterization

Proposition Under “good assumptions”, v_* is a viscosity supersolution on $[0, T)$ of

$$\max\left\{F_0\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0 \ \text{ if } \underline{\Lambda} \le x^3 \le \overline{\Lambda}$$

and v^* is a subsolution on $[0, T)$ of

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On $\underline{\Lambda}, \overline{\Lambda}$:

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Provides a control on the ratio $D_{x^1}\varphi/D_p\varphi$.

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A little more : v is continuous in p and x^3.

Uniqueness

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This is not enough... If we need to penalize in x^1 (stock price) then the term $|D_{x^1}\varphi/D_p\varphi|^2 D_p^2\varphi$ will blow up as $n\to\infty$, where n comes from the usual penalisation $n|x_1^1 - x_2^1|^2$ du to the doubling of constants.

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Bound on the stock price...

Comparison

Theorem : Let U (resp. V) be a non-negative super- and subsolutions which are continuous in x^3. Assume that

$$U(t,x,p) \geq V(t,x,p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\},$$

and that $\exists\ c_+ > 0$ and $c_- \in \mathbb{R}$ s.t.

$$\limsup_{(t',x',p')\to(t,x,\infty)} V(t',x',p')/p' \leq c_+ \leq \liminf_{(t',y',p')\to(t,y,\infty)} U(t',y',p')/p' \ ,$$
$$\limsup_{(t',x',p')\to(t,x,-\infty)} V(t',x',p') \leq c_- \leq \liminf_{(t',y',p')\to(t,y,-\infty)} U(t',y',p') \ .$$

If either U is a supersolution of (*) which is continuous in p, or V is a subsolution of (**) which is continuous in p, then

$$U \geq V \ .$$

Remarks on models with proportional transaction costs

Typical model

$$
\begin{aligned}
X^1_{t,x}(s) &= x^1 + \int_t^s X^1_{t,x}(r)\mu dr + \int_t^s X^1_{t,x}(r)\sigma dW_r \\
X^{2,L}_{t,x}(s) &= x^2 + \int_t^s \frac{X^{2,L}_{t,x}(r)}{X^1_{t,x}(r)} dX^1_{t,x}(r) - \int_t^s dL^1_r + \int_t^s dL^2_r \\
Y^L_{t,y}(s) &= y + \int_t^s (1-\lambda) dL^1_r - \int_t^s (1+\lambda) dL^2_r \ .
\end{aligned}
$$

Our general results allow for pricing derivatives under loss constraints,...

Remarks on optimal management under shortfall constraints

Serves as a building block for problems of the form

$$\sup_{\phi \in \mathcal{A}_{t,z}} \mathbb{E}\left[U(Z^{\phi}_{t,z}(T))\right]$$

with

$$\mathcal{A}_{t,z} := \{\phi \in \mathcal{A} : Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t, T]\} \ .$$

see B., Elie and Imbert (2010).

Some references

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