Quenched mass transport of particles towards a target

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Joint work with Boualem Djehiche and Idris Kharroubi

A short introduction to the stochastic target approach

General problem formulation

Soner and Touzi

□ Controlled process : A map : $(t, z, \nu) \in [0, T] \times \mathbb{R}^{d+1} \times \mathcal{U} \mapsto Z^{t,z,\nu}$ a cadlag $\mathbb{F}$-adapted process satisfying $Z_t^{t,z,\nu} = z$.

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☐ Target : G a Borel subset of $\mathbb{R}^{d+1}$.

☐ Problem : Compute

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } Z_T^{t,z,\nu} \in G\}.$$

Geometric Dynamic Principle

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☐ **Theorem :** (Soner and Touzi) Let $\{\theta^\nu,\ \nu \in \mathcal{U}\}$ be a family of stopping times. Then,

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□ In the Markovian diffusion case, Soner and Touzi discovered that it leads to a PDE characterization of the map $(t,z) \mapsto \mathbb{1}_{z \notin V(t)}$.

Focus on the "monotone" case

☐ **Monotonicity assumption :**

(i) $Z^{t,z,\nu} = (X^{t,x,\nu}, Y^{t,z,\nu}) \in \mathbb{R}^d \times \mathbb{R}$, $z = (x, y)$,

(ii) $(x, y) \in G$ implies $(x, y') \in G$ for $y' \geq y$.

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□ **Theorem :** Let $\{\theta^\nu, \ \nu \in \mathcal{U}\}$ be a family of s.t. Then,

(GDP1) If $y > v(t, x)$, then there exists $\nu \in \mathcal{U}$ such that

$$Y^{t,z,\nu}_{\theta^\nu} \geq v(\theta^\nu, X^{t,x,\nu}_{\theta^\nu})$$

(GDP2) If $y < v(t, x)$, then for all $\nu \in \mathcal{U}$

$$\mathbb{P}\left[Y^{t,z,\nu}_{\theta^\nu} > v(\theta^\nu, X^{t,x,\nu}_{\theta^\nu})\right] < 1.$$

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- **PDE** :

$$\sup_{u \in \mathcal{N}^v(t,x,v(t,x))} \left(\mu_Y(t,x,v(t,x),u) - \mathcal{L}_X^u v(t,x)\right) = 0$$

where

$$\mathcal{L}_X^u v := \partial_t v + \mu_X \cdot Dv + \frac{1}{2}\mathrm{Tr}[\sigma_X \sigma_X^\top D^2 v]$$
$$\mathcal{N}^v(t,x,y) := \{u \in U : \sigma_Y(t,x,y,u) = Dv(t,x)\sigma_X(t,x,u)\}$$

when

$$dX = \mu_X(t,X,\nu)dt + \sigma_X(t,X,\nu)dB$$
$$dY = \mu_Y(t,X,Y,\nu)dt + \sigma_Y(t,X,Y,\nu)dB$$

Extensions

Constraint in expectations

B., Elie and Touzi

□ Problem :

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \; \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_T^{t,z,\nu})\right] \geq p\}.$$

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- Reformulation :

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, (\nu,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ s.t. } \ell(Z_T^{t,z,\nu}) \geq M_T^{t,p,\alpha} \ \},$$

where

$$M^{t,p,\alpha} := p + \int_t^{\cdot} \alpha_s dB_s.$$

Expectation maximization under constraint in expectations

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$$\max \left\{ \mathbb{E}\left[L(Z_T^{t,z,\nu})\right],\ \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_T^{t,z,\nu})\right] \geq p \right\}.$$

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where

$$\mathcal{D}:=\{(t,z,p):z\in V(t,p)\}.$$

Quenched mass transport

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- They evolve according to

$$X^\nu_\cdot = \chi + \int_0^\cdot b_s\big(X^\nu_s, \mathbb{P}^B_{X^\nu_s}, \nu_s\big)ds + \int_0^\cdot a_s\big(X^\nu_s, \mathbb{P}^B_{X^\nu_s}, \nu_s\big)dB_s.$$

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- Initial required funds :

$$v(\mu) := \inf\{y \in \mathbb{R} : \exists\, \nu \text{ s.t. } Y^{y,\nu}_T \geq 0 \text{ and } \mathbb{P}^B_{X^\nu_T} \in G\}$$

with $Y^{y,\nu} = y - C^\nu$.

Quenched SDEs

□ We consider the dynamics :

$$X_\cdot^{t,\chi,\nu} = \chi + \int_t^\cdot b_s\big(X_s^{t,\chi,\nu}, \mathbb{P}^B_{X_s^{t,\chi,\nu}}, \nu_s\big)\,ds + \int_0^\cdot a_s\big(X_s^{t,\chi,\nu}, \mathbb{P}^B_{X_s^{t,\chi,\nu}}, \nu_s\big)\,dB_s$$

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☐ Existence and uniqueness are standard, and the solution can be approximated by a particles system.

Problem formulation

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☐ The reachability set at t is defined as :

$$\mathcal{V}(t) \quad = \quad \Big\{\mu \in \mathcal{P}_2 : \; \exists(\chi,\nu) \in \mathrm{X}_t^2 \times \mathcal{U} \text{ s.t. } \mathbb{P}^B_{\chi} = \mu \; \text{ and } \; \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\},$$

where $\mathrm{X}_t^2 = L^2(\mathcal{F}_t)$ and $\mathcal{U}$ is the set of U-valued adapted processes.

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□ Indepence w.r.t. the representent : $\mu \in \mathcal{V}(t) \Leftrightarrow \forall\ \chi \in \mathrm{X}_t^2$ s.t. $\mathbb{P}_\chi^B = \mu$, $\exists\ \nu \in \mathcal{U}$ s.t. $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$.

Dynamic programming

□ GDP : Fix $t \in [0, T]$ and θ a st. time with values in $[t, T]$. Then,

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□ In our exemple :

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(GDP1) If $\mathrm{v}(t,\mu) = 0$, then there exists $(\chi,\nu) \in \mathrm{X}_t^2 \times \mathcal{U}$ such that

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$\Rightarrow$ Appeal to the notion of viscosity solutions.

Derivatives on the lift space

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☐ For a function $w : \mathcal{P}_2 \to \mathbb{R}$, we define its lift as the function W from L^2 to $\mathbb{R}$ such that

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□ We say that w is C^1 if W admits a continuous Fréchet derivative. In this case, there exists a measurable map $\partial_\mu w(\mathbb{P}_X) : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

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□ One can then define $\partial^2_\mu w(\mu)(x, x')$ and $\partial_x \partial_\mu w(\mu)(x)$. If they are continuous and "bounded", we say that $w \in C^2_b$.

Itô's formula

☐ Set $\bar{w}(x_1, \dots, x_N) = w(\mu_X)$ with $\mu_X := N^{-1} \sum_{i=1}^N \delta_{x_i}$. Then,

$$\begin{aligned}\bar{w}(x+h) &= W(X+H) = W(X) + \langle DW(X), H\rangle + o(|H|) \\ &= \bar{w}(x) + \langle \partial_\mu w(\mu_X)(X), H\rangle + o(|H|) \\ &= \bar{w}(x) + \frac{1}{N}\sum_{i=1}^N \partial_\mu w(\mu_X)(x_i) h_i + o(|h|).\end{aligned}$$

Hence,

$$\partial_{x_i} \bar{w}(x) = \frac{1}{N} \partial_\mu w(\mu_X)(x_i).$$

□ Consider iid copies (with respect to ξ)

$$X^\ell = \chi^\ell + \int_t^\cdot b_s^\ell ds + \int_t^\cdot a_s^\ell dB_s,$$

and let $\bar{\mu}^N$ be the empirical measure. Then,

$$\begin{aligned} w(s, \bar{\mu}_s^N) =& w(t, \bar{\mu}_t^N) + \int_t^s \partial_t w(r, \bar{\mu}_r^N) dr + \frac{1}{N} \sum_{\ell=1}^N \int_t^s \partial_\mu w(r, \bar{\mu}_r^N)(X_r^\ell) b_r^\ell dr \\ &+ \frac{1}{N} \sum_{\ell=1}^N \int_t^s \partial_\mu w(r, \bar{\mu}_r^N)(X_r^\ell) a_r^\ell dB_r \\ &+ \frac{1}{2N} \sum_{\ell=1}^N \int_t^s \mathrm{Tr}\left[\partial_x \partial_\mu w(r, \bar{\mu}_r^N)(X_r^\ell) a_r^\ell (a_r^\ell)^\top\right] dr \\ &+ \frac{1}{2N^2} \sum_{\ell,n=1}^N \int_t^s \mathrm{Tr}\left[\partial_\mu^2 w(r, \bar{\mu}_r^N)(X_r^\ell, X_r^n) a_r^\ell (a_r^n)^\top\right] dr. \end{aligned}$$

□ Take $\mathbb{E}_B$ and let $N \to \infty$:

$$
\begin{aligned}
w(s, \mathbb{P}^B_{X_s}) &= w(t, \mathbb{P}^B_{\chi}) \\
&\quad + \int_t^s \mathbb{E}_B \left[\partial_t w(r, \mathbb{P}^B_{X_r}) + \partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) b_r \right] dr \\
&\quad + \frac{1}{2} \int_t^s \mathbb{E}_B \left[\mathrm{Tr} \left(\partial_x \partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) a_r a_r^\top \right) \right] dr \\
&\quad + \frac{1}{2} \int_t^s \mathbb{E}_B \left[\tilde{\mathbb{E}}_B \left[\mathrm{Tr} \left(\partial_\mu^2 w(r, \mathbb{P}^B_{X_r})(X_r, \tilde{X}_r) a_r \tilde{a}_r^\top \right) \right] \right] dr \\
&\quad + \int_t^s \mathbb{E}_B \left[\partial_\mu w(r, \mathbb{P}^B_{X_r})(X_r) a_r(X_r, \mathbb{P}^B_{X_r}, \nu_r)) \right] dB_r
\end{aligned}
$$

where "tilde" stands for independent copy (given B).

HJB formulation

□ The value function $\mathrm{v} : \mu \mapsto 1 - \mathbb{1}_{\mathcal{V}}$ is a viscosity solution (in the discontinuous viscosity solution sens) of

$$-\partial_t \mathrm{v}(t,\mu) + H\big(t,\mu,\partial_\mu \mathrm{v}(t,\mu),\partial_\mu\partial_x \mathrm{v}(t,\mu),\partial^2_\mu \mathrm{v}(t,\mu)\big) = 0 \ ,$$

in which

$$H\big(t,\mu,\partial_\mu \mathrm{v}(t,\mu),\partial_\mu\partial_x \mathrm{v}(t,\mu),\partial^2_\mu \mathrm{v}(t,\mu)\big) := \sup_{u\in N(t,\mu,\partial_\mu \mathrm{v}(t,\mu))} \left(-L^u_t[\mathrm{v}](\mu)\right)$$

with

$$N(t,\mu,\partial_\mu \mathrm{v}(t,\mu)) := \left\{ u \in L^0(\mathbb{R}^d;\mathrm{U}) : \int \partial_\mu \mathrm{v}(t,\mu)(x) a_t(x,\mu,u(x))\mu(dx) = 0 \right\}$$

and

$$L^u_t[\mathrm{v}](\mu) :=$$
$$\int\int \Big\{ b_t(x,\mu,u(x))^\top \partial_\mu \mathrm{v}(t,\mu)(x) + \frac{1}{2}\mathrm{Tr}\left[\partial_x\partial_\mu \mathrm{v}(t,\mu)(x)(a_t a_t^\top)(x,\mu,u(x))\right]$$
$$+\frac{1}{2}\mathrm{Tr}\left[\partial^2_\mu \mathrm{v}(t,\mu)(x,\tilde{x}) a_t(x,\mu,u(x)) a_t^\top(\tilde{x},\mu,u(\tilde{x}))\right] \Big\} \mu(dx)\mu(d\tilde{x}).$$

Back to the example

- The function

$$\mathrm{v}(0,\mu) := \inf\{y \in \mathbb{R} : \exists\, \nu \text{ s.t. } Y_T^{y,\nu} \geq 0 \text{ and } \mathbb{P}^B_{X_T^\nu} \in G\},$$

with $Y^{y,\nu} = y - \int_0^\cdot c(\nu_s)ds$, is a viscosity solution of

$$-\partial_t \mathrm{v}(t,\mu) + H\big(t,\mu,\partial_\mu \mathrm{v}(t,\mu),\partial_\mu\partial_x \mathrm{v}(t,\mu),\partial^2_\mu \mathrm{v}(t,\mu)\big) = 0\ ,$$

in which

$$\begin{aligned} &H\big(t,\mu,\partial_\mu \mathrm{v}(t,\mu),\partial_\mu\partial_x \mathrm{v}(t,\mu),\partial^2_\mu \mathrm{v}(t,\mu)\big) \\ &:= \sup_{u \in N(t,\mu,\partial_\mu \mathrm{v}(t,\mu))} \Big(-c(u) - L_t^{X,u}[\mathrm{v}](\mu)\Big) \end{aligned}$$

with

$$N(t,\mu,\partial_\mu \mathrm{v}(t,\mu)) := \left\{u \in L^0(\mathbb{R}^d;\mathrm{U}) : \int \partial_\mu \mathrm{v}(t,\mu)(x) a_t(x,\mu,u(x))\mu(dx) = 0\right\}$$

The case of a global control

□ If ν is required to be adapted to the filtration of the Brownian motion, we have the same formulation but with

$$N(t, \mu, \partial_\mu \mathrm{v}(t, \mu)) := \left\{ u \in \mathrm{U} : \int \partial_\mu \mathrm{v}(t, \mu)(x) a_t(x, \mu, u) \mu(dx) = 0 \right\}.$$

Thank you !

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