Discrete time approximation of BSDEs

B. Bouchard

LPMA University Paris 6 and CREST

SOME MOTIVATIONS

BSDE and finance

- **Hedging of the European option $g(X_1)$ (interest rate r)**

$$\begin{cases} X_t &= X_0 + \int_0^t \mathrm{diag}\,[X_s]\, b(X_s)ds + \int_0^t \mathrm{diag}\,[X_s]\, \sigma(X_s)dW_s\ , \\ Y_t &= g(X_1) - \int_t^1 (Y_s - \phi_s'\mathbf{1})r ds - \int_t^1 \phi_s'\sigma(X_s)dW_s \end{cases}$$

- **Hedging of the European option $g(X_1)$ (interest rate r^b for borrowing, r^l for lending)**

$$\begin{cases} X_t &= X_0 + \int_0^t \mathrm{diag}\,[X_s]\, b(X_s)ds + \int_0^t \mathrm{diag}\,[X_s]\, \sigma(X_s)dW_s\ , \\ Y_t &= g(X_1) - \int_t^1 \left[(Y_s - \phi_s'\mathbf{1})^+ r^l - (Y_s - \phi_s'\mathbf{1})^- r^b\right] ds - \int_t^1 \phi_s'\sigma(X_s)dW_s \end{cases}$$

American option hedging

- **Hedging of the American option $g(X)$ (interest rate r)**

$$\begin{cases} X_t &= X_0 + \int_0^t \text{diag}\,[X_s]\, b(X_s)ds + \int_0^t \text{diag}\,[X_s]\, \sigma(X_s)dW_s \ , \\ Y_t &= g(X_1) - \int_t^1 (Y_s - \phi_s' \mathbf{1}) r ds - \int_t^1 \phi_s' \sigma(X_s)dW_s + K_1 - K_t \\ Y_t &\geq g(X_t) \ , \ t \leq 1 \ \text{ and } \ \int_0^1 (\,Y_s - g(X_s)\,)\text{d}K_s = 0 \ , \end{cases}$$

$\Rightarrow Y_t = \operatorname*{esssup}_{\tau \leq 1} \mathbb{E}\left[e^{-r\tau} g(X_\tau) \mid \mathcal{F}_t\right]$

Game option hedging

- **Hedging of the Game option $g(X)$ (interest rate r)**

$$\begin{cases} X_t &= X_0 + \int_0^t \mathrm{diag}\,[X_s]\, b(X_s) ds + \int_0^t \mathrm{diag}\,[X_s]\, \sigma(X_s) dW_s \;, \\ \\ Y_t &= g(X_1) - \int_t^1 (Y_s - \phi_s' \mathbf{1}) r ds - \int_t^1 \phi_s' \sigma(X_s) dW_s + K_1 - K_t \\ \ell(X_t) &\le Y_t \le h(X_t) \;\;, \quad t \le 1 \\ 0 &= \int_0^1 (\, Y_s - \ell(X_s)\,) \mathrm{d}K_s^+ \;, \\ 0 &= \int_0^1 (\, Y_s - h(X_s)\,) \mathrm{d}K_s^- \;. \end{cases}$$

$\Rightarrow$

$$Y_t = \operatorname*{ess\,inf}_{\theta \le 1} \operatorname*{ess\,sup}_{\tau \le 1} \mathbb{E}\left[e^{-r(\tau \wedge \theta)} \left\{ \left(g(X_\tau) \mathbf{1}_{\tau = T} + \ell(X_\tau) \mathbf{1}_{\tau < T} \right) \mathbf{1}_{\tau \le \theta} + h(X_\theta) \mathbf{1}_{\tau > \theta} \right\} \;\middle|\; \mathcal{F}\right.$$

Default options

- **Free of arbitrage pricing rule:**

$$u_1(t,x) \;:=\; \mathbb{E}_{t,x}\left[g_1(X_T)\mathbf{1}_{\tau>T} + g_0(X_T)\mathbf{1}_{\tau\le T}\right]$$

with default time: $\tau \sim \mathcal{E}(\varrho)$.

- **Decomposition :**

$$u_1(t,x) \;:=\; \mathbb{E}_{t,x}\left[g_1(X_T)e^{-(T-t)\varrho} + \int_t^T u_0(s,X_s)\varrho e^{-\varrho(s-t)}ds\right]$$

$$u_0(t,x) \;:=\; \mathbb{E}_{t,x}\left[g_0(X_T)\right]$$

Default options

$$\frac{\partial}{\partial t}u_0 + \frac{1}{2}x^2\sigma^2 D^2 u_0 = 0 \quad , \quad u_0(T,\cdot) = g_0 = [5-\cdot]^+ \wedge 0.5$$

$$\frac{\partial}{\partial t}u_1 + \frac{1}{2}x^2\sigma^2 D^2 u_1 + \varrho(u_0 - u_1) = 0 \quad , \quad u_1(T,\cdot) = g_1 = [5-\cdot]^+$$

- **BSDE**:

$$\begin{aligned} dX_t &= X_t \sigma dW_t \\ M_t &= \mu((0,t]) \quad [2] \end{aligned}$$

and

$$\begin{aligned} -dY_t &= \left(\varrho \mathbf{1}_{M_t=1} - \lambda\right) U_t dt - U_t \bar{\mu}(dt) - Z_t \cdot dW_t \\ Y_T &= g_1(X_T)\mathbf{1}_{M_T=1} + g_0(X_T)\mathbf{1}_{M_T=0} \end{aligned}$$

BSDE and finance...

- g-expectations

- Optimal investment: Non Lipschitz coefficients !

Semilinear PDEs I: no non-local term

$$\begin{cases} X_t &= X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \;, \\ \\ Y_t &= g(X_1) + \int_t^1 f(X_s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s \end{cases}$$

$\Rightarrow Y_t = u(t, X_t)$ with u solution (in a weak sens) of

$$\begin{aligned} 0 &= u_t + b' Du + \frac{1}{2}\mathrm{Tr}[\sigma\sigma' D^2 u] + f(\cdot, u, Du'\sigma) \\ g &= u(1, \cdot) \;. \end{aligned}$$

$\Rightarrow Z_t = (Du'\sigma)(t, X_t)$

Semilinear PDEs II: with integral term

$$\begin{cases} X_t &= X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_E \beta(X_{s-}, e)\bar{\mu}(de, ds) , \\ Y_t &= g(X_1) + \int_t^1 f(\Theta_s)\, ds - \int_t^1 Z_s dW_s - \int_t^1 \int_E U_s(e)\bar{\mu}(de, ds) \end{cases}$$

with $\Theta := (X, Y, \Gamma, Z)$ and $\Gamma := \int_E \rho(e)U(e)\lambda(de)$.

$\Rightarrow Y_t = u(t, X_t)$ with u solution (in a weak sens) of

$$\begin{aligned} 0 &= \mathcal{L}u + f\left(\cdot, u, Du'\sigma, \mathcal{I}[u]\right) \\ g &= u(1, \cdot)\,, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}u &:= u_t + b'Du + \frac{1}{2}\mathrm{Tr}[\sigma\sigma' D^2 u] + \int_E \{u(\cdot, \cdot + \beta(\cdot, e)) - u - Du'\beta(\cdot, e)\}\,\lambda(de) \\ \mathcal{I}[u] &:= \int_E \{u(\cdot, \cdot + \beta(\cdot, e)) - u\}\,\rho(e)\,\lambda(de)\,. \end{aligned}$$

Semilinear PDEs II: with integral term

$$\left\{ \begin{array}{rcl} X_t &=& X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_E \beta(X_{s-}, e)\bar{\mu}(de, ds) \ , \\ \\ Y_t &=& g(X_1) + \int_t^1 f\left(\Theta_s\right) ds - \int_t^1 Z_s dW_s - \int_t^1 \int_E U_s(e)\bar{\mu}(de, ds) \end{array} \right.$$

with $\Theta := (X, Y, \Gamma, Z)$ and $\Gamma := \int_E \rho(e)U(e)\lambda(de)$.

$\Rightarrow$ $Y_t = u(t, X_t)$ with u solution (in a weak sens) of

$$\begin{array}{rcl} 0 &=& \mathcal{L}u + f\left(\cdot, u, Du'\sigma, \mathcal{I}[u]\right) \\ g &=& u(1, \cdot) \, , \end{array}$$

$\Rightarrow$ $Z_t = (Du'\sigma)(t, X_t)$ and $U_t(e) = u(t, X_{t-} + \beta(X_{t-}, e)) - u(t, X_{t-})$

Semilinear PDEs III: partially coupled systems

Pardoux, Pradeilles and Rao (97), Sow and Pardoux (04).

- **System of κ PDE's ($i = 0, \dots, \kappa - 1$)**

$$\begin{aligned} 0 &= u^i_t + b'_i D u^i + \frac{1}{2}\mathrm{Tr}[\sigma_i \sigma'_i D^2 u^i] + f_i(\cdot, u, (Du^i)'\sigma_i) \\ g_i &= u^i(1, \cdot)\ . \end{aligned}$$

- Define for $i = 0, \dots, \kappa - 1$

$$\tilde{f}(i, x, y, \gamma, z) \ = \ f_i\left(x, (\dots, y + \gamma^{\kappa-2}, y + \gamma^{\kappa-1}, \underbrace{y}_{i}, y + \gamma^1, y + \gamma^2, \dots), z\right)$$

- Set $E = \{1, \dots, \kappa - 1\}$, $\lambda(de) = \lambda \sum_{k=1}^{\kappa-1} \delta_k(e)$ and

$$M_t \ = \ \int_0^t \int_E e\mu(de, ds) \ \ [\kappa]$$

Semilinear PDEs III: partially coupled systems

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$$\begin{aligned} 0 &= u^i_t + b'_i D u^i + \frac{1}{2}\mathrm{Tr}[\sigma_i \sigma'_i D^2 u^i] + f_i(\cdot, u, (Du^i)'\sigma_i) \\ g_i &= u^i(1, \cdot)\ . \end{aligned}$$

$\Rightarrow u^{M_t}(t, X_t) = Y_t$ where

$$\begin{aligned} dX_t &= b_{M_t}(X_t)dt + \sigma_{M_t}(X_t)dW_t \\ -dY_t &= \tilde{f}(M_t, X_t, Y_t, U_t, Z_t)dt - \lambda \sum_{k=1}^{\kappa-1} U(k)_t dt - Z_t dW_t - \int_E U_t(e)\bar{\mu}(de, dt) \\ Y_1 &= g_{M_1}(X_1) \end{aligned}$$

Semilinear PDEs IV: free boundary problems

$$
\begin{cases}
X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s , \\
Y_t = g(X_1) + \int_t^1 f(X_s, Y_s, Z_s)\, ds - \int_t^1 Z_s dW_s + K_1 - K_t \\
Y_t \geq h(X_t) , \quad t \leq 1 \quad \text{and} \quad \int_0^1 (Y_s - h(X_s))\mathrm{d}K_s = 0 ,
\end{cases}
$$

$\Rightarrow Y_t = u(t, X_t)$ where u solves

$$
\begin{aligned}
0 &= \min\left\{-\mathcal{L}u - f(\cdot, u, Du'\sigma) \, , \; u - h\right\} \\
g &= u(1, \cdot)
\end{aligned}
$$

DISCRETE TIME APPROXIMATION

PDE and Tree methods: Some references

Solve the PDE and they use MC simulation

- Ma, Protter and Yong (1994). Solving forward-backward stochastic differential equations explicitly - a four step scheme, *PTRF*.

- Douglas, Ma and Protter (1996). Numerical Methods for Forward-Backward Stochastic Differential Equations, *AAP*.

PDE and Tree methods: Some references

Approximate the Brownian motion by a walk

- Coquet, Mackevičius and Mémin (1998). Stability in D of martingales and backward equations under discretization of filtration, *SPA*.

- Briand, Delyon and Mémin (2001). Donsker-type theorem for BSDE's, *ECP*.

- Ma, Protter, San Martin and Torres (2002). Numerical Method for Backward Stochastic Differential Equations, *AAP*.

- Antonelli and Kohatsu-Higa (2000). Filtration stability of backward SDE's, *SPA*.

Aim

$\Rightarrow$ Use a pure probabilistic approach to obtain the required regularity under weaker conditions.

Discrete approximation approach: Some references

- Zhang J. (01). *Some fine properties of backward stochastic differential equations*. PhD thesis.

- Zhang J. (04). A numerical scheme for BSDEs. *AAP* 14.

- Bally V. and G. Pagès (03). Error analysis of the quantization algorithm for obstacle problems. *SPA* 106.

- Bouchard B. and N. Touzi (04). Discrete-Time Approximation and Monte-Carlo Simulation of Backward Stochastic Differential Equations. *SPA*, 111.

- Bouchard B. and R. Elie (05). Discrete time approximation of decoupled Forward-Backward SDE with jumps. Preprint.

- Ma J. and J. Zhang (05). Representations and regularities for solutions to BSDEs with reflections. *SPA* 115.

- Bouchard B. and J.-F. Chassagneux (06). Discrete time approximation for continuously and discretely reflected BSDEs. Preprint.

OUTLINE

1. Discretization

a. BSDEs

b. BSDEs with jumps

c. BSDEs with jumps and partially non-Lipschitz driver

d. Reflected BSDEs

2. Numerical implementation and simulation

a. Regression based algorithm

b. Malliavin calculus approach

c. Quantization method

BSDEs

$$\begin{cases} X_t &= X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ , \\ \\ Y_t &= g(X_1) + \int_t^1 f(X_s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s \end{cases}$$

Construction

Step 1: Step-constant driver with $\pi := \{t_i := i/n,\ i \le n\}$

$$\bar{Y}^{\pi}_{t_i} \;=\; \bar{Y}^{\pi}_{t_{i+1}} + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) - \int_{t_i}^{t_{i+1}} Z^{\pi}_t dW_t\,,$$

where

$$\int_{t_i}^{t_{i+1}} Z^{\pi}_t dW_t = \bar{Y}^{\pi}_{t_{i+1}} - \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right]$$

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where

$$\int_{t_i}^{t_{i+1}} Z^\pi_t dW_t = \bar{Y}^\pi_{t_{i+1}} - \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right]$$

Step 2: Best $L^2(\Omega \times [t_i, t_{i+1}])$ approximation of Z^π $\mathcal{F}_{t_i}$-meas. process

$$\bar{Z}^\pi_t \;:=\; n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z^\pi_s ds \mid \mathcal{F}_{t_i}\right] = n\mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right]$$

Construction

Step 1: Step-constant driver with $\pi := \{t_i := i/n,\ i \le n\}$

$$\bar{Y}^\pi_{t_i} \;=\; \bar{Y}^\pi_{t_{i+1}} + \frac{1}{n} f\left(X^\pi_{t_i}, \bar{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}\right) - \int_{t_i}^{t_{i+1}} Z^\pi_t dW_t\,,$$

where

$$\int_{t_i}^{t_{i+1}} Z^\pi_t dW_t = \bar{Y}^\pi_{t_{i+1}} - \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right]$$

Step 2: Best $L^2(\Omega \times [t_i, t_{i+1}])$ approximation of Z^π $\mathcal{F}_{t_i}$-meas. process

$$\bar{Z}^\pi_t \;:=\; n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z^\pi_s ds \mid \mathcal{F}_{t_i}\right] = n\mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right]$$

Step 3: Discrete scheme: $\bar{Y}^\pi_{t_i} = \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^\pi_{t_i}, \bar{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}\right)$.

Construction (an other interpretation)

Step 1: Euler scheme type approximation

$$\bar{Y}^{\pi}_{t_i} \sim \bar{Y}^{\pi}_{t_{i+1}} + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) - \bar{Z}^{\pi}_{t_i}(W_{t_{i+1}} - W_{t_i}),$$

Construction (an other interpretation)

Step 1: Euler scheme type approximation

$$\bar{Y}^{\pi}_{t_i} \;\sim\; \bar{Y}^{\pi}_{t_{i+1}} + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) - \bar{Z}^{\pi}_{t_i}(W_{t_{i+1}} - W_{t_i}),$$

Step 2: By taking expectation

$$\bar{Y}^{\pi}_{t_i} \;=\; \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right)$$

Construction (an other interpretation)

Step 1: Euler scheme type approximation

$$\bar{Y}^{\pi}_{t_i} \sim \bar{Y}^{\pi}_{t_{i+1}} + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) - \bar{Z}^{\pi}_{t_i}(W_{t_{i+1}} - W_{t_i}),$$

Step 2: By taking expectation

$$\bar{Y}^{\pi}_{t_i} = \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right)$$

Step 3: By multiplying by $(W_{t_{i+1}} - W_{t_i})$ and taking expectation

$$\bar{Z}^{\pi}_{t_i} = n\mathbb{E}\left[\bar{Y}^{\pi}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right]$$

Discrete scheme

$$
\begin{aligned}
\bar{Y}_1^{\pi} &= g(X_1^{\pi}) \\
\bar{Z}_{t_i}^{\pi} &:= n\mathbb{E}\left[\bar{Y}_{t_{i+1}}^{\pi}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\
\bar{Y}_{t_i}^{\pi} &= \mathbb{E}\left[\bar{Y}_{t_{i+1}}^{\pi} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X_{t_i}^{\pi}, \bar{Y}_{t_i}^{\pi}, \bar{Z}_{t_i}^{\pi}\right) .
\end{aligned}
$$

Implicit vs Explicit

- Implicit

$$\bar{Y}^{\pi}_{t_i} \;=\; \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) .$$

Implicit vs Explicit

- Implicit

$$\bar{Y}^{\pi}_{t_i} \;=\; \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) \,.$$

- Explicit

$$\bar{Y}^{\pi}_{t_i} \;=\; \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} \mathbb{E}\left[f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_{i+1}}, \bar{Z}^{\pi}_{t_i}\right) \mid \mathcal{F}_{t_i}\right] .$$

Implicit vs Explicit

- Implicit

$$\bar{Y}^{\pi}_{t_i} \;=\; \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}\right) .$$

- Explicit

$$\bar{Y}^{\pi}_{t_i} \;=\; \mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n}\mathbb{E}\left[f\left(X^{\pi}_{t_i}, \bar{Y}^{\pi}_{t_{i+1}}, \bar{Z}^{\pi}_{t_i}\right) \mid \mathcal{F}_{t_i}\right] .$$

- It actually leads to the same approximation error: choose one or the other depending on f.

First bound

Set

$$\mathrm{Err}^2 := \max_i \mathbb{E}\left[\sup_{t_i \le t \le t_{i+1}} |\bar{Y}^\pi_{t_i} - Y_t|^2\right] + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}^\pi_t|^2\right] dt$$

Theorem:

$$\mathrm{Err}^2 \le C\left(|\pi| + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2\right] dt\right)$$

where

$$\bar{Z}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i}\right]$$

$\Longrightarrow$ Regularity of Z ?

Regularity of Z (simplification: f depends only on X)

- Malliavin calculus approach

Proposition: (Y_t, Z_t) admits a Malliavin derivative and (D_sY, D_sZ) solves

$$D_sY_t \;=\; \nabla g(X_1)D_sX_1 + \int_t^1 \nabla f(X_r)D_sX_r dr - \int_t^1 D_sZ_r dW_r$$

- Since

$$Y_t = Y_0 - \int_0^t f(X_r)dr + \int_0^t Z_r dW_r$$

we have

$$\begin{aligned} Z_t = D_tY_t \;&=\; \mathbb{E}\left[\nabla g(X_1)D_sX_1 + \int_t^1 \nabla_x f(X_r)D_sX_r dr \mid \mathcal{F}_t\right] \\ &=\; \mathbb{E}\left[\nabla g(X_1)\nabla X_1 + \int_t^1 \nabla_x f(X_r)\nabla X_r dr \mid \mathcal{F}_t\right](\nabla X_t)^{-1}\sigma(X_t) \end{aligned}$$

Regularity of Z (simplification: f depends only on X)

- $Z_t = (V_t - \alpha_t)(\nabla X_t)^{-1}\sigma(X_t)$ where

$$V_t = \mathbb{E}\left[\nabla g(X_1)\nabla X_1 + \int_0^1 \nabla_x f(X_r)\nabla X_r dr \mid \mathcal{F}_t\right]$$

$$\alpha_t = \int_0^t \nabla_x f(X_r)\nabla X_r dr$$

- We have for some $\in L^2(\Omega \times [0,1])$

$$V_t = V_0 + \int_0^t \tilde{\sigma}_s dW_s$$

thus

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|V_t - V_{t_i}|^2\right] dt \leq C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \mathbb{E}\left[|\tilde{\sigma}_s|^2\right] ds dt$$

$$\leq C\, |\pi| \int_0^1 \mathbb{E}\left[|\tilde{\sigma}_s|^2\right] ds\ .$$

Discrete time approximation error

Theorem: Assume that all the coefficients are Lipschitz continuous, then

$$\mathsf{Err}^2 \leq C \left(|\pi| + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2 \right] dt \right) \leq C\,|\pi|$$

and

$$\int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2 \right] dt \;\; \leq \;\; \sum_{i \leq n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_t - Z_{t_i}|^2 \right] dt \leq C|\pi|$$

where

$$\bar{Z}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i} \right]$$

Discrete time approximation error

Theorem: Assume that all the coefficients are Lipschitz continuous, then

$$\text{Err}^2 \leq C \left(|\pi| + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2 \right] dt \right) \leq C\, |\pi|$$

and

$$\int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2 \right] dt \;\; \leq \;\; \sum_{i \leq n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_t - Z_{t_i}|^2 \right] dt \leq C|\pi|$$

where

$$\bar{Z}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i} \right]$$

Discrete time approximation error

• Gobet and Labart (2006). Error expansion for the discretization of Backward Stochastic Differential Equations.

⇒ Under smoothness conditions and uniform ellipticity:

$$\bar{Y}^{\pi}_{t_i} - Y_{t_i} \;=\; Du(t_i, X_{t_i})'(X^{\pi}_{t_i} - X_{t_i}) + O_i(|\pi|) + O(|X^{\pi}_{t_i} - X_{t_i}|^2)$$

and weak convergence of $\sqrt{n}(Y^{\pi} - Y)$ where Y^{π} is a continuous-time extension of $\bar{Y}^{\pi}$.

• Similar result for $\bar{Z}^{\pi}$.

Discrete-time scheme II:

BSDEs with jumps

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_E \beta(X_{s-}, e)\bar{\mu}(de, ds) \ , \\ Y_t = g(X_1) + \int_t^1 f\left(\Theta_s\right) ds - \int_t^1 Z_s dW_s - \int_t^1 \int_E U_s(e)\bar{\mu}(de, ds) \end{cases}$$

with $\Theta := (X, Y, \Gamma, Z)$ with $\Gamma := \int_E \rho(e)U(e)\lambda(de)$.

Construction

Step 1: Step-constant driver with $\pi := \{t_i = i/n,\ i \le n\}$

$$\bar{Y}^\pi_{t_i} \;=\; \bar{Y}^\pi_{t_{i+1}} + \frac{1}{n} f\left(X^\pi_{t_i}, \bar{Y}^\pi_{t_i}, \bar{\Gamma}^\pi_{t_i}, \bar{Z}^\pi_{t_i}\right) - \int_{t_i}^{t_{i+1}} Z^\pi_t dW_t - \int_{t_i}^{t_{i+1}} \int_E U^\pi_t \bar{\mu}(de, dt)\,,$$

where

$$\int_{t_i}^{t_{i+1}} Z^\pi_t dW_t + \int_{t_i}^{t_{i+1}} \int_E U^\pi_t \bar{\mu}(de, dt) = \bar{Y}^\pi_{t_{i+1}} - \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right]$$

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Step 2: Best $L^2(\Omega\times[t_i, t_{i+1}])$ approximation of Z^{π} and $\Gamma^{\pi} = \int_E U^{\pi}(e)\rho(e)\lambda(de)$ by $\mathcal{F}_{t_i}$-meas. process

$$\bar{Z}^{\pi}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z^{\pi}_s ds \mid \mathcal{F}_{t_i}\right] = n\mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right]$$

$$\bar{\Gamma}^{\pi}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Gamma^{\pi}_s ds \mid \mathcal{F}_{t_i}\right] = n\mathbb{E}\left[\bar{Y}^{\pi}_{t_{i+1}} \int_E \rho(e)\bar{\mu}(de, (t_i, t_{i+1}]) \mid \mathcal{F}_{t_i}\right]$$

Construction

Step 1: Step-constant driver with $\pi := \{t_i = i/n,\ i \le n\}$

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Step 3: Discrete scheme: $\bar{Y}^\pi_{t_i} = \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X^\pi_{t_i}, \bar{Y}^\pi_{t_i}, \bar{\Gamma}^\pi_{t_i}, \bar{Z}^\pi_{t_i}\right)$.

First bound

Set

$$\mathrm{Err}^2 := \max_i \mathbb{E}\left[\sup_{t_i \le t \le t_{i+1}} |\bar{Y}^{\pi}_{t_i} - Y_t|^2\right] + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}^{\pi}_t|^2 + |\Gamma_t - \bar{\Gamma}^{\pi}_t|^2\right] dt$$

Theorem:

$$\mathrm{Err}^2 \;\le\; C\left(|\pi| + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2\right] dt + \int_0^1 \mathbb{E}\left[|\Gamma_t - \bar{\Gamma}_t|^2\right] dt\right)$$

where

$$\bar{Z}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i}\right] \quad \text{and} \quad \bar{\Gamma}_t := n\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \Gamma_s ds \mid \mathcal{F}_{t_i}\right]$$

$\Longrightarrow$ Regularity of Z and Γ

Discrete time approximation error

- **Assumption :** For each $e \in E$, the map $x \in \mathbb{R}^d \mapsto \beta(x,e)$ admits a Jacobian matrix $\nabla\beta(x,e)$ such that the function

$$(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto a(x,\xi;e) := \xi'(\nabla\beta(x,e) + I_d)\xi$$

satisfies one of the following condition uniformly in $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$

$$a(x,\xi;e) \;\geq\; |\xi|^2 K^{-1} \quad \text{or} \quad a(x,\xi;e) \;\leq\; -|\xi|^2 K^{-1}\,.$$

Proposition: Under the above condition

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_t - Z_{t_i}|^2\right] dt \;\leq\; C\,|\pi|$$

Theorem: Under the above condition

$$\text{Err}^2 \;\leq\; C\,|\pi|\,.$$

- The same results hold if σ, b, $\beta(\cdot,e)$, f and g are C_b^1 functions with K-Lipschitz continuous derivatives, uniformly in $e \in E$.

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Theorem: Under the above condition

$$\text{Err}^2 \ \leq \ C\,|\pi| \, .$$

- The same results hold with $h^{1-\varepsilon}$ without condition.

Discrete time approximation error: extensions

- b, σ, β, and f can depend on t if 1/2-Hölder in t

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- Consider the system

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\begin{aligned}
M_t &= M_0 + \int_0^t b_M(M_r)dr + \int_0^t \int_E \beta_M(M_{r-}, e)\bar{\mu}(de, dr) \\
X_t &= X_0 + \int_0^t b(M_r, X_r)dr + \int_0^t \sigma(M_r, X_r)dW_r + \int_0^t \int_E \beta(M_{r-}, X_{r-}, e)\bar{\mu}(de, dr) \\
Y_t &= g(M_1, X_1) + \int_t^1 f\left(M_r, \Theta_r\right) dr - \int_t^1 Z_r dW_r - \int_t^1 \int_E U_r(e)\bar{\mu}(de, dr)
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$\Rightarrow$ M independent of X and its Malliavin derivative equals zero.

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\end{aligned}
$$

$\Rightarrow$ M independent of X and its Malliavin derivative equals zero.

$\Rightarrow$ Inversibility condition on $\nabla\beta$ or smoothness of $b(m,\cdot), \sigma(m,\cdot), \beta(m,\cdot)$, $f(m,\cdot)$ and $g(m,\cdot)$ (for each m) sufficient.

Discrete-time scheme III

systems of semilinear PDEs

$$
\begin{aligned}
M_t &= \int_0^t \int_E e\mu(de, ds) \;\; [\kappa] \\
dX_t &= b(M_t, X_t)dt + \sigma(M_t, X_t)dW_t \\
-dY_t &= f(M_t, X_t, Y_t, Z_t, \Gamma_t)dt - Z_t dW_t - \int_E U_t(e)\bar{\mu}(de, dt) \\
Y_1 &= g(M_1, X_1)
\end{aligned}
$$

Where f is not Lipschitz in M !

Construction

- $M = \int_0^{\cdot} e\mu(de, dt)$ $[\kappa]$ can be simulated perfectly.

- By adding the jump times of M in the Euler scheme X^π of X one has

$$\mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |X_t - X^\pi_{t_i}|^2\right] \le C\,|t_{i+1} - t_i|$$

even if b, σ and β are not Lipschitz in m.

Construction

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even if b, σ and β are not Lipschitz in m.

- The approximation scheme for Y is defined as follows:

$$\begin{aligned}
\bar{Z}^\pi_t &:= n\,\mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}\Delta W_{i+1} \mid \mathcal{F}_{t_i}\right] \\
\bar{\Gamma}^\pi_t &:= n\,\mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}\int_E \rho(e)\bar{\mu}(de,(t_i,t_{i+1}]) \mid \mathcal{F}_{t_i}\right] \\
\bar{Y}^\pi_t &:= \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \mathbb{E}\left[\int_{t_i}^{t_{i+1}} f\left(M_s, X^\pi_{t_i}, \bar{Y}^\pi_{t_{i+1}}, \bar{Z}^\pi_{t_i}, \bar{\Gamma}^\pi_{t_i}\right) ds \mid \mathcal{F}_{t_i}\right]
\end{aligned}$$

for $t \in [t_i, t_{i+1})$, with the terminal condition $\bar{Y}^\pi_1 = g(M_1, X^\pi_1)$.

Discrete time approximation error

- We assume that, for each m, $b(m,\cdot), \sigma(m,\cdot), \beta(m,\cdot), f(m,\cdot), g(m,\cdot)$ satisfy the previous Lipschitz continuity assumption.

Discrete time approximation error

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Proposition: Let $(F_t)_{t\le T}$ and $(\bar{F}_t)_{t\le T}$ be defined, for $t \in [t_i, t_{i+1}]$, by

$$F_t := f(M_t, X_{t_i}, Y_{t_i}, \bar{Z}_{t_i}, \bar{\Gamma}_{t_i}), \quad \bar{F}_t := \mathbb{E}\left[n \int_{t_i}^{t_{i+1}} f\left(M_s, X_{t_i}, Y_{t_i}, \bar{Z}_{t_i}, \bar{\Gamma}_{t_i}\right) ds \mid \mathcal{F}_{t_i} \right]$$

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Then,

$$\text{Err}^2 \le C\left(|\pi| + \int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}_t|^2 + |\Gamma_t - \bar{\Gamma}_t|^2 + |F_t - \bar{F}_t|^2\right] dt\right).$$

Discrete time approximation error

- We assume that, for each m, $b(m,\cdot),\sigma(m,\cdot),\beta(m,\cdot),f(m,\cdot),g(m,\cdot)$ satisfy the previous Lipschitz continuity assumption.

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$$F_t \;:=\; f(M_t,X_{t_i},Y_{t_i},\bar Z_{t_i},\bar\Gamma_{t_i})\,,\quad \bar F_t \;:=\; \mathbb{E}\left[n\int_{t_i}^{t_{i+1}} f\left(M_s,X_{t_i},Y_{t_i},\bar Z_{t_i},\bar\Gamma_{t_i}\right)ds \mid \mathcal{F}_{t_i}\right]$$

Then,

$$\mathrm{Err}^2 \;\le\; C\left(|\pi| + \int_0^1 \mathbb{E}\left[|Z_t-\bar Z_t|^2 + |\Gamma_t-\bar\Gamma_t|^2 + |F_t-\bar F_t|^2\right]dt\right)\;.$$

Proposition: Under the above conditions

$$\mathrm{Err}^2 \;\le\; C\left(|\pi| + \int_0^1 \mathbb{E}\left[|F_t-\bar F_t|^2\right]dt\right) \longrightarrow 0\;.$$

Discrete-time scheme IV

Reflected BSDEs and free boundary problems

$$\begin{cases} X_t &= X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ , \\ \\ Y_t &= g(X_1) + \int_t^1 f\,(X_s, Y_s, Z_s)\,ds - \int_t^1 Z_s dW_s + K_1 - K_t \\ Y_t &\geq h(X_t) \ , \ \ t \leq 1 \ \text{ and } \ \int_0^1 (\,Y_s - h(X_s)\,)\mathrm{d}K_s = 0 \ , \end{cases}$$

Previous works

- f independent of Z: Bally, Pages and Printemps (02,...), B. and Touzi (04).

- f depends on Z: Bally (with randomization of the time horizon), Ma and Zhang.

Construction

- Approximation scheme

$$
\begin{aligned}
\bar{Z}^\pi_{t_i} &= n\, \mathbb{E}_i \left[\bar{Y}^\pi_{t_{i+1}} (W_{t_{i+1}} - W_{t_i}) \right] \\
\tilde{Y}^\pi_{t_i} &= \mathbb{E}_i \left[\bar{Y}^\pi_{t_{i+1}} \right] + n^{-1}\, f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\
\bar{Y}^\pi_{t_i} &= \mathcal{R} \left(t_i\ ,\ X^\pi_{t_i}\ ,\ \tilde{Y}^\pi_{t_i} \right)\ ,\ i \le N-1\ ,
\end{aligned}
$$

with the terminal condition

$$
\bar{Y}^\pi_T = g(X^\pi_T)\ .
$$

where for $\Re = \{r_j,\ 0 \le j \le \kappa\} \supset \pi$

$$
\mathcal{R}(t,x,y) := y + [h(x) - y]^+ \mathbf{1}_{\{t \in \Re \setminus \{0,T\}\}}\ ,\quad (t,x,y) \in [0,T] \times \mathbb{R}^{d+1}\ .
$$

Construction

- Approximation scheme

$$
\begin{aligned}
\bar{Z}^{\pi}_{t_i} &= n\,\mathbb{E}_i\left[\bar{Y}^{\pi}_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})\right] \\
\tilde{Y}^{\pi}_{t_i} &= \mathbb{E}_i\left[\bar{Y}^{\pi}_{t_{i+1}}\right] + n^{-1}\, f(X^{\pi}_{t_i}, \tilde{Y}^{\pi}_{t_i}, \bar{Z}^{\pi}_{t_i}) \\
\bar{Y}^{\pi}_{t_i} &= \mathcal{R}\left(t_i\ ,\ X^{\pi}_{t_i}\ ,\ \tilde{Y}^{\pi}_{t_i}\right)\ ,\ i \le N-1\ ,
\end{aligned}
$$

with the terminal condition

$$
\bar{Y}^{\pi}_{T} = g(X^{\pi}_{T})\ .
$$

where for $\Re = \{r_j,\ 0 \le j \le \kappa\} \supset \pi$

$$
\mathcal{R}(t,x,y) := y + [h(x) - y]^{+} \mathbf{1}_{\{t \in \Re \setminus \{0,T\}\}}\ ,\quad (t,x,y) \in [0,T] \times \mathbb{R}^{d+1}\ .
$$

- f independent of Z (Bally and Pages): When $\Re = \pi$, error on Y controled in $|\pi|^{\frac{1}{2}}$ and by $|\pi|$ when h is semi-convex.

Regularity of Z:

- Ma and Zhang approach: Integration by parts $\Rightarrow$ uniform ellipticity condition on σ

$$Z_t = \mathbb{E}\left[g(X_1)N_1^t + \int_t^1 f(\Theta_s)N_s^t ds + \int_t^1 N_s^t dK_s \mid \mathcal{F}_t\right]$$

where

$$N_r^t \ := \ (r-t)^{-1}\int_t^r \sigma(X_s)^{-1}\nabla X_s dW_s \, (\nabla X_t)^{-1}\sigma(X_t)$$

Regularity of Z:

- Ma and Zhang approach: Integration by parts $\Rightarrow$ uniform ellipticity condition on σ

Theorem: (Ma and Zhang) If σ is uniformly elliptic, σ, $b \in C_b^1$ and $h \in C_b^2$, then

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_t - Z_{t_i}|^2\right] dt \ \leq \ C\, |\pi|^{\frac{1}{2}}\ .$$

If $\Re = \pi$, then

$$\mathbb{E}\left[\max_{i \leq n} |Y_{t_i} - \bar{Y}^{\pi}_{t_i}|^2\right] + \max_{i \leq n} \mathbb{E}\left[\sup_{t \in [t_i, t_{i+1}]} |Y_t - \bar{Y}^{\pi}_{t_{i+1}}|^2\right] + \|Z^{\pi} - Z\|^2_{\mathbf{H}^2} \ \leq \ C\, |\pi|^{\frac{1}{2}}\ ,$$

Regularity of Z: Discretely reflected case

- Discretely reflected BSDE (BC06 with simplication $f = f(x)$)

$$\tilde{Y}_t^b = Y_{r_{j+1}}^b + \int_t^{r_{j+1}} f(X_s)\mathrm{d}s - \int_t^{r_{j+1}} Z_s^b \mathrm{d}W_s \ ,$$
$$Y_t^b = \mathcal{R}\left(t \ , \ X_t \ , \ \tilde{Y}_t^b\right) \quad \text{on each } [r_j, r_{j+1}] \ , \ j \le \kappa - 1 \ .$$

with $Y_1^b = g(X_1)$ and $\Theta^b = (X, \tilde{Y}^b, Z^b)$.

Proposition: There is a version of Z^b such that for each $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$:

$$Z_t^b = \mathbb{E}\left[\nabla g(X_1)\nabla X_1 \mathbf{1}_{\{\tau_j=1\}} + \nabla h(X_{\tau_j})\nabla X_{\tau_j}\mathbf{1}_{\{\tau_j<1\}} \mid \mathcal{F}_t\right](\nabla X_t)^{-1}\sigma(X_t)$$
$$+ \ \mathbb{E}\left[\int_t^{\tau_j} \nabla_x f(X_u)\nabla X_u \mathrm{d}u \mid \mathcal{F}_t\right](\nabla X_t)^{-1}\sigma(X_t) \ .$$

where $\tau_j := \inf\{t \in \Re \mid t \ge r_{j+1}, \ h(X_t) > \tilde{Y}_t^b\} \wedge 1$ and $s \in [r_j, r_{j+1})$.

Regularity of Z: Discretely reflected case

• Main idea to conclude: Assume $h = g$ is C^1_L and $f \equiv 0$. For simplicity: $\Lambda \equiv (\nabla X_t)^{-1}\sigma(X_t) \equiv 1$.

$$Z^b_t = V^j_t := \mathbb{E}\left[\nabla h(X_{\tau_j})\nabla X_{\tau_j} \mid \mathcal{F}_t\right]$$

is a martingale, thus, with i_j s.t. $t_{i_j} = r_j$,

$$\begin{aligned}
\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z^b_t - Z^b_{t_i}|^2 dt\right] &= \sum_{j=0}^{\kappa-1} \sum_{k=i_j}^{i_{j+1}-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|V^j_t - V^j_{t_k}|^2\right] dt \\
&\le |\pi| \sum_{j=0}^{\kappa-1} \sum_{k=i_j}^{i_{j+1}-1} \mathbb{E}\left[|V^j_{t_{k+1}}|^2 - |V^j_{t_k}|^2\right] \\
&= |\pi| \left(\mathbb{E}\left[|V^{\kappa-1}_{r_\kappa}|^2 - |V^0_{r_0}|^2\right]\right. \\
&\qquad \left. + \sum_{j=1}^{\kappa-2} \mathbb{E}\left[|V^j_{r_{j+1}}|^2 - |V^{j+1}_{r_{j+1}}|^2\right]\right)
\end{aligned}$$

Regularity of Z: Discretely reflected case

where

$$\begin{aligned}\mathbb{E}\left[|V^j_{r_{j+1}}|^2 - |V^{j+1}_{r_{j+1}}|^2\right] &\leq \mathbb{E}\left[\eta_{r_{j+1}}|V^j_{r_{j+1}} - V^{j+1}_{r_{j+1}}|\right] \\ &= \mathbb{E}\left[\eta_{r_{j+1}}|\mathbb{E}\left[\nabla h(X_{\tau_j})\nabla X_{\tau_j} - \nabla h(X_{\tau_{j+1}})\nabla X_{\tau_{j+1}} \mid \mathcal{F}_{r_{j+1}} \\ &\leq \mathbb{E}\left[\hat{\eta}(\tau_{j+1} - \tau_j)^{\frac{1}{2}}\right]\end{aligned}$$

so that

$$\sum_{j=1}^{\kappa-2} \mathbb{E}\left[|V^j_{r_{j+1}}|^2 - |V^{j+1}_{r_{j+1}}|^2\right] \leq \sqrt{\kappa}\,\mathbb{E}\left[\hat{\eta}(\tau_{\kappa-1} - \tau_1)^{\frac{1}{2}}\right]$$

- Similarly if $\nabla h \in C^2$:

$$\mathbb{E}\left[|V^j_{r_{j+1}}|^2 - |V^{j+1}_{r_{j+1}}|^2\right] \leq \mathbb{E}\left[\hat{\eta}(\tau_{j+1} - \tau_j)\right]$$

so that

$$\sum_{j=1}^{\kappa-2} \mathbb{E}\left[|V^j_{r_{j+1}}|^2 - |V^{j+1}_{r_{j+1}}|^2\right] \leq \mathbb{E}\left[\hat{\eta}(\tau_{\kappa-1} - \tau_1)\right]$$

Regularity of Z: Discretely reflected case

Theorem: If $h \in C^1_L$ with L-Lipschitz derivative, then

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z^b_t - Z^b_{t_i}|^2 dt\right] \;\leq\; C\,\sqrt{\kappa}\,|\pi|\,.$$

If $h \in C^2_L$ with L-Lipschitz derivatives + $\sigma \in C^1_b$, then

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z^b_t - Z^b_{t_i}|^2 dt\right] \;\leq\; C\,|\pi|\,.$$

Convergence speed: Discretely reflected case

- **Assumptions:** b, σ, g and f are Lipschitz-continuous.

- Use $\mathbf{H}_i$: $h \in C^i_L$ with L-Lipschitz derivatives of order up to i, with $i = 1$ or 2.

Theorem: Let $\mathbf{H}_1$ hold. Then,

$$\mathbb{E}\left[\max_{i\le n}|\bar{Y}^\pi_{t_i} - Y^b_{t_i}|^2\right] + \max_{i\le n-1} \mathbb{E}\left[\sup_{t\in(t_i,t_{i+1}]}|\bar{Y}^\pi_{t_{i+1}} - Y^b_t|^2\right] \;\le\; C\,\kappa^{\frac{1}{2}}\,|\pi|$$

and

$$\sum_i \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\bar{Z}^\pi_{t_i} - Z^b_t|^2 dt\right] \;\le\; C\,\kappa\,|\pi|$$

Convergence speed: Discretely reflected case

- **Assumptions:** b, σ, g and f are Lipschitz-continuous.

- Use $\mathbf{H}_i$: $h \in C^i_L$ with L-Lipschitz derivatives of order up to i, with $i = 1$ or 2.

Theorem: Let $\mathbf{H}_2$ hold $+ \; \sigma \in C^1_b$. Then,

$$\mathbb{E}\left[\max_{i \le n} |\bar{Y}^{\pi}_{t_i} - Y^b_{t_i}|^2\right] + \max_{i \le n-1} \mathbb{E}\left[\sup_{t \in (t_i, t_{i+1}]} |\bar{Y}^{\pi}_{t_{i+1}} - Y^b_t|^2\right] \;\le\; C \; |\pi|$$

and

$$\sum_i \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\bar{Z}^{\pi}_{t_i} - Z^b_t|^2 dt\right] \;\le\; C \, \kappa \, |\pi|$$

Convergence speed: Discretely reflected case

- **Assumptions:** b, σ, g and f are Lipschitz-continuous.

- Use $\mathbf{H}_i$: $h \in C^i_L$ with L-Lipschitz derivatives of order up to i, with $i = 1$ or 2.

Theorem: Let $\mathbf{H}_1$ hold and $X^\pi = X$ on π. Then,

$$\sum_i \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\bar{Z}^\pi_{t_i} - Z^b_t|^2 dt\right] \;\leq\; C\,\kappa^{\frac{1}{2}}\,|\pi|$$

Let $\mathbf{H}_2$ hold and $X^\pi = X$ on π. Then,

$$\sum_i \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\bar{Z}^\pi_{t_i} - Z^b_t|^2 dt\right] \;\leq\; C\,|\pi|$$

Convergence speed: Continuously reflected case

Theorem: Take $\Re = \pi$. Let $\mathbf{H}_1$ hold. Then,

$$\mathbb{E}\left[\max_{i \le n} |Y^\pi_{t_i} - Y_{t_i}|^2\right] + \max_{i \le n-1} \mathbb{E}\left[\sup_{t \in (t_i, t_{i+1}]} |Y^\pi_{t_{i+1}} - Y_t|^2\right] \le C_L\, \alpha(\pi) ,$$

with $\alpha(\pi) = |\pi|^{\frac{1}{2}}$ under $\mathbf{H}_1$ and $\alpha(\pi) = |\pi|$ under $\mathbf{H}_2$.

Moreover, under $\mathbf{H}1$,

$$\sum_i \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\bar{Z}^\pi_{t_i} - Z_t|^2 dt\right] \le C_L\, |\pi|^{\frac{1}{2}} ,$$

If $\mathbf{H}_2 + \sigma \in C^1_b$ holds and $X^\pi = X$ on π, then

$$\sum_i \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\bar{Z}^\pi_{t_i} - Z_t|^2 dt\right] \le C_L\, |\pi| .$$

Numerical methods

a. Regression based algorithm

b. Malliavin calculus approach

c. Quantization method

How to compute the conditional expectations ?

$$
\begin{aligned}
\bar{Y}_1^\pi &= g(X_1^\pi) \\
\bar{Z}_{t_i}^\pi &:= n\mathbb{E}\left[\bar{Y}_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\
\bar{Y}_{t_i}^\pi &= \mathbb{E}\left[\bar{Y}_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(X_{t_i}^\pi, \bar{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi\right) .
\end{aligned}
$$

Regression based algorithms

- Simulate N path $(X^{\pi,j})_{j \leq N}$ of X^{π}

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Regression based algorithms

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- Compute the non parametric regression: $(\bar{Y}_{t_{i+1}}^{\pi,j})_{j \le N}$ and $(\bar{Y}_{t_{i+1}}^{\pi,j}(W_{t_{i+1}}^j - W_{t_i}^j))_{j \le N}$ on $(X_{t_i}^{\pi,j})_{j \le N}$ and set

$$\hat{\mathbb{E}}\left[\bar{Y}_{t_{i+1}}^\pi \mid X_{t_i}^\pi = X_{t_i}^{\pi,j}\right] = \sum_{k \le \kappa} \hat{\alpha}_k p_k(X_{t_i}^{\pi,j})$$

$$\hat{\mathbb{E}}\left[\bar{Y}_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid X_{t_i}^\pi = X_{t_i}^{\pi,j}\right] = \sum_{k \le \kappa} \hat{\beta}_k p_k(X_{t_i}^{\pi,j})$$

- Gives

$$\bar{Z}_{t_i}^{\pi,j} = n \sum_{k \le \kappa} \hat{\beta}_k p_k(X_{t_i}^{\pi,j})$$

$$\bar{Y}_{t_i}^{\pi,j} = \sum_{k \le \kappa} \hat{\alpha}_k p_k(X_{t_i}^{\pi,j}) + \frac{1}{n} f\left(X_{t_i}^{\pi,j}, \bar{Y}_{t_i}^{\pi,j}, \bar{Z}_{t_i}^{\pi,j}\right).$$

Regression based algorithms

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Malliavin calculus approach

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Malliavin calculus approach

- Simulate N path $(X^{\pi,j})_{j\leq N}$ of X^π
- Initialize: $\bar{Y}_1^{\pi,j} = g(X_1^{\pi,j})$ for $j \leq N$
- Estimate the conditional expectation using the representation

$$\mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mid X^\pi_{t_i} = X^{\pi,j}_{t_i}\right] = \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}} \mathbf{1}_{\{X^\pi_{t_i} \geq X^{\pi,j}_{t_i}\}} \Phi_i\right] / \mathbb{E}\left[\mathbf{1}_{\{X^\pi_{t_i} \geq X^{\pi,j}_{t_i}\}} \Phi_i\right]$$

$$\sim \left(\sum_l \bar{Y}^{\pi,l}_{t_{i+1}} \mathbf{1}_{\{X^{\pi,l}_{t_i} \geq X^{\pi,j}_{t_i}\}} \Phi^l_i\right) / \left(\sum_l \mathbf{1}_{\{X^{\pi,l}_{t_i} \geq X^{\pi,j}_{t_i}\}} \Phi^l_i\right)$$

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$$\sim \quad \left(\sum_l \bar{Y}^{\pi,l}_{t_{i+1}} \mathbf{1}_{\{X^{\pi,l}_{t_i} \ge X^{\pi,j}_{t_i}\}} \Phi_i^l\right) / \left(\sum_l \mathbf{1}_{\{X^{\pi,l}_{t_i} \ge X^{\pi,j}_{t_i}\}} \Phi_i^l\right)$$

- Set for $j \le N$

$$\bar{Z}^{\pi,j}_{t_i} := n\hat{\mathbb{E}}\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid X^\pi_{t_i} = X^{\pi,j}_{t_i}\right]$$

$$\bar{Y}^{\pi,j}_{t_i} = \hat{\mathbb{E}}\left[\bar{Y}^\pi_{t_{i+1}} \mid X^\pi_{t_i} = X^{\pi,j}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi,j}_{t_i}, \bar{Y}^{\pi,j}_{t_i}, {}^{\pi,j}_{t_i}\right).$$

With a good algorithm, the complexity IS NOT in N^2 but $N\ln(N)^d$!

Malliavin calculus approach

- Simulate N path $(X^{\pi,j})_{j\leq N}$ of X^π

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$$\sim \quad \left(\sum_l \bar{Y}^{\pi,l}_{t_{i+1}} \mathbf{1}_{\{X^{\pi,l}_{t_i} \geq X^{\pi,j}_{t_i}\}} \Phi^l_i\right) / \left(\sum_l \mathbf{1}_{\{X^{\pi,l}_{t_i} \geq X^{\pi,j}_{t_i}\}} \Phi^l_i\right)$$

- Set for $j \leq N$

$$\begin{aligned} \bar{Z}^{\pi,j}_{t_i} &:= n\hat{\mathbb{E}}\left[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid X^\pi_{t_i} = X^{\pi,j}_{t_i}\right] \\ \bar{Y}^{\pi,j}_{t_i} &= \hat{\mathbb{E}}\left[\bar{Y}^\pi_{t_{i+1}} \mid X^\pi_{t_i} = X^{\pi,j}_{t_i}\right] + \frac{1}{n} f\left(X^{\pi,j}_{t_i}, \bar{Y}^{\pi,j}_{t_i}, {}^{\pi,j}_{t_i}\right). \end{aligned}$$

$\Rightarrow$ With a good algorithm, the complexity IS NOT in N^2 but $N\ln(N)^d$!

Malliavin calculus approach

- Fournié, Lasry, Lebuchoux, and Lions (2001). Applications of Malliavin calculus to Monte Carlo methods in finance II, *Finance and Stochastics*.

- Bouchard, Ekeland and Touzi (2004). On the Malliavin approach to Monte Carlo approximation of conditional expectations, *Finance and Stochastics*.

- Bouchard and Touzi (2004). Discrete-Time Approximation and Monte-Carlo Simulation of Backward Stochastic Differential Equations. *Stochastic Processes and their Applications*.

- Lions and Regnier (2001). Calcul du prix et des sensibilités d'une option américaine par une méthode de Monte Carlo, preprint.

Quantization method

• Replace X^π by a process $\hat{X}^\pi$ taking values in a sequence of grids $(\Gamma_{t_i})_{t_i}$ such that

1. $\hat{X}^\pi_{t_i}$ is the projection of $X^\pi_{t_i}$ on Γ_{t_i}

2. Γ_{t_i} minimize the L^2 distance between $\hat{X}^\pi_{t_i}$ and $X^\pi_{t_i}$ among all the grids having the same size.

• Write the algorithm on this process after having computed the transition probabilities of $\hat{X}^\pi$ by MC methods: finite dimensional space, everything becomes explicit.

⇒ Bally and Pages (2002). A quantization algorithm for solving discrete time multidimensional optimal stopping problems. *Bernoulli*.

⇒ and many others... see the web page of Pages.