Stochastic Targets and Optimal Control with Controlled Loss

B. Bouchard, R. Elie, C. Imbert, N. Touzi

Ceremade, Univ. Paris-Dauphine, and, Crest, Ensae

Berlin, June 2009

Outline

Stochastic target problems

Outline

Outline

Outline

Stochastic target problems

Stochastic target with controlled loss

Optimal control under target constraints

Problem Formulation

Dynamics :

$Z^{\nu}_{t,x,y} := (X^{\nu}_{t,x}, Y^{\nu}_{t,x,y}) \in \mathbb{R}^d \times \mathbb{R}$ solution of

$$X^{\nu}(s) = x + \int_t^s \mu_X(X^{\nu}(r), \nu_r)dr + \int_t^s \sigma_X(X^{\nu}(r), \nu_r)dW_r$$

$$Y^{\nu}(s) = y + \int_t^s \mu_Y(Z^{\nu}(r), \nu_r)dr + \int_t^s \sigma_Y(Z^{\nu}(r), \nu_r)dW_r \,.$$

Problem Formulation

Dynamics :

$Z^{\nu}_{t,x,y} := (X^{\nu}_{t,x}, Y^{\nu}_{t,x,y}) \in \mathbb{R}^d \times \mathbb{R}$ solution of

$$X^{\nu}(s) = x + \int_t^s \mu_X(X^{\nu}(r), \nu_r)dr + \int_t^s \sigma_X(X^{\nu}(r), \nu_r)dW_r$$
$$Y^{\nu}(s) = y + \int_t^s \mu_Y(Z^{\nu}(r), \nu_r)dr + \int_t^s \sigma_Y(Z^{\nu}(r), \nu_r)dW_r \,.$$

Controls

$\nu \in \mathcal{U}$, square integrable, prog. meas., valued in $U \subset \mathbb{R}^d$ (may be unbounded).

Problem Formulation

Target

$G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \ : \ g(x, y) \geq 0\}$, with $g \nearrow y$.

Problem Formulation

Target

$G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \ : \ g(x, y) \geq 0\}$, with $g \nearrow y$.

Viability set

$D := \{(t, z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,x,y}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\}$.

Problem Formulation

Target

$G := \{(x,y) \in \mathbb{R}^d \times \mathbb{R} \ : \ g(x,y) \geq 0\}$, with $g \nearrow y$.

Viability set

$D := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,x,y}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\}.$

Value function

$w(t,x) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D\}.$

Example : super-hedging in finance

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Example : super-hedging in finance

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ

Example : super-hedging in finance

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ

Super-hedging price

$Y^\nu(T) \geq \psi(X^\nu(T) \Leftrightarrow g(Z^\nu(T)) \geq 0$ with $g(x, y) := y - \psi(x)$.

Example : super-hedging in finance

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ

Super-hedging price

$Y^\nu(T) \geq \psi(X^\nu(T) \Leftrightarrow g(Z^\nu(T)) \geq 0$ with $g(x,y) := y - \psi(x)$.
$w(t,x) := \inf\{y \in \mathbb{R} \ : \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^\nu_{t,x,y}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\}$.

Geometric dynamic programming principle

Recall

$D := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P}-\text{a.s.}\}.$

$w(t,x) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D\}$

Geometric dynamic programming principle

Recall

$D := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P}-\text{a.s.}\}.$

$w(t,x) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D\}$

Theorem (Soner and Touzi)

For any stopping time $\theta \in [t,T]\ \mathbb{P}-a.s.$

$$D = \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } (\theta, Z^{\nu}_{t,x,y}(\theta)) \in D \ \mathbb{P}-a.s.\} \ .$$

Geometric dynamic programming principle

Recall

$D := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\}.$

$w(t,x) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D\}$

Theorem (Soner and Touzi)

For any stopping time $\theta \in [t, T]$ $\mathbb{P} - a.s.$

$$D = \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } (\theta, Z^{\nu}_{t,x,y}(\theta)) \in D \ \mathbb{P} - a.s.\} \ .$$

Corollary

For any stopping time $\theta \in [t, T]$ $\mathbb{P} - \text{a.s.}$

$y > w(t,x) \Rightarrow \exists\, \nu \text{ s.t. } Y^{\nu}_{t,x,y}(\theta) \geq w(\theta, X^{\nu}_{t,x}(\theta)) \ \mathbb{P} - \text{a.s.}$

Geometric dynamic programming principle

Recall

$D := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\}.$

$w(t,x) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D\}$

Theorem (Soner and Touzi)

For any stopping time $\theta \in [t, T]$ $\mathbb{P} - a.s.$

$$D = \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } (\theta, Z^{\nu}_{t,x,y}(\theta)) \in D \ \mathbb{P} - a.s.\}\ .$$

Corollary

For any stopping time $\theta \in [t, T]$ $\mathbb{P} - \text{a.s.}$

$y > w(t,x) \Rightarrow \exists\, \nu$ s.t. $Y^{\nu}_{t,x,y}(\theta) \geq w(\theta, X^{\nu}_{t,x}(\theta))$ $\mathbb{P} - \text{a.s.}$

$y < w(t,x) \Rightarrow \nexists\, \nu$ s.t. $Y^{\nu}_{t,x,y}(\theta) > w(\theta, X^{\nu}_{t,x}(\theta))$ $\mathbb{P} - \text{a.s.}$

PDE Derivation

For "$y = w(t,x)$", $\exists\ \nu \in \mathcal{U}$ s.t. $Y^{\nu}_{t,x,y}(t+) \geq w(t+, X^{\nu}_{t,x}(t+))$.

PDE Derivation

For "$y = w(t,x)$", $\exists\ \nu \in \mathcal{U}$ s.t. $Y^{\nu}_{t,x,y}(t+) \geq w(t+, X^{\nu}_{t,x}(t+))$.
Thus,

$$
\begin{aligned}
dY^{\nu}_{t,x,y}(t) &= \mu_Y(x,y,\nu_t)dt + \sigma_Y(x,y,\nu_t)\, dW_t \\
&\geq dw(t, X^{\nu}_{t,x}(t)) \\
&= \mathcal{L}^{\nu_t}_X w(t,x)dt + Dw(t,x)\sigma_X(x,\nu_t)\, dW_t
\end{aligned}
$$

PDE Derivation

For "$y = w(t,x)$", $\exists\ \nu \in \mathcal{U}$ s.t. $Y^{\nu}_{t,x,y}(t+) \geq w(t+, X^{\nu}_{t,x}(t+))$.
Thus,

$$\begin{aligned} dY^{\nu}_{t,x,y}(t) &= \mu_Y(x,y,\nu_t)dt + \sigma_Y(x,y,\nu_t)\,dW_t \\ &\geq dw(t, X^{\nu}_{t,x}(t)) \\ &= \mathcal{L}^{\nu_t}_X w(t,x)dt + Dw(t,x)\sigma_X(x,\nu_t)\,dW_t \end{aligned}$$

This leads to

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}^u_X w(t,x) \geq 0$$

where $\mathcal{N}(t,x,w) := \{u \in U \ : \ \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x,u)\}$

PDE Derivation

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \{u \in U \ : \ \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x,u)\}$,

PDE Derivation

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y\left(x, w(t,x), u\right) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \{u \in U \ : \ \sigma_Y\left(x, w(t,x), u\right) = Dw(t,x)\sigma_X\left(x,u\right)\}$,

then "we can find" $\hat{u}$ s.t. $\hat{u}(t',x') \in N(t',x',w)$ for (t',x').

PDE Derivation

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \{u \in U \ : \ \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x,u)\}$,

then “we can find” $\hat{u}$ s.t. $\hat{u}(t',x') \in N(t',x',w)$ for (t',x').

For $\nu := \hat{\nu}(\cdot, X^{\nu}_{t,x}(\cdot))$ and $y := w(t,x) - \varepsilon$, we have

$$dY^{\nu}_{t,x,y} \quad > \quad dw(\cdot, X^{\nu}_{t,x}(\cdot))$$

PDE Derivation

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y\left(x, w(t,x), u\right) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \left\{u \in U \ : \ \sigma_Y\left(x, w(t,x), u\right) = Dw(t,x)\sigma_X\left(x,u\right)\right\}$,

then "we can find" $\hat{u}$ s.t. $\hat{u}(t',x') \in N(t',x',w)$ for (t',x').
For $\nu := \hat{\nu}(\cdot, X^\nu_{t,x}(\cdot))$ and $y := w(t,x) - \varepsilon$, we have

$$dY^\nu_{t,x,y} \quad > \quad dw(\cdot, X^\nu_{t,x}(\cdot))$$

and therefore $Y^\nu_{t,x,y}(\theta) > w(\theta, X^\nu_{t,x}(\theta))$ for θ well chosen.

PDE Derivation

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \{u \in U \ : \ \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x,u)\}$,

then "we can find" $\hat{u}$ s.t. $\hat{u}(t',x') \in N(t',x',w)$ for (t',x').
For $\nu := \hat{\nu}(\cdot, X^{\nu}_{t,x}(\cdot))$ and $y := w(t,x) - \varepsilon$, we have

$$dY^{\nu}_{t,x,y} \quad > \quad dw(\cdot, X^{\nu}_{t,x}(\cdot))$$

and therefore $Y^{\nu}_{t,x,y}(\theta) > w(\theta, X^{\nu}_{t,x}(\theta))$ for θ well chosen.
This leads to

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) \leq 0 \, .$$

PDE Derivation

Theorem (Soner and Touzi ; B., Elie and Imbert)

w is a viscosity solution (in the discontinuous sense) of

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) = 0 \quad (t,x) \in [0,T) \times \mathbb{R}^d$$

where

$$\mathcal{N}(t,x,w) := \{u \in U \ : \ \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x,u)\} .$$

A powerful tool

Robust approach

A powerful tool

Robust approach

- Dynamic programming principle is robust.

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- New proofs under minimal conditions (B., R. Elie and N. Touzi).

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- New proofs under minimal conditions (B., R. Elie and N. Touzi).
- Can be extended to jumps (B.) or American type constraints (B. and V. T. Nam)

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- New proofs under minimal conditions (B., R. Elie and N. Touzi).
- Can be extended to jumps (B.) or American type constraints (B. and V. T. Nam)

Dual formulation ?

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- New proofs under minimal conditions (B., R. Elie and N. Touzi).
- Can be extended to jumps (B.) or American type constraints (B. and V. T. Nam)

Dual formulation ?

- Works only for financial models.

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- New proofs under minimal conditions (B., R. Elie and N. Touzi).
- Can be extended to jumps (B.) or American type constraints (B. and V. T. Nam)

Dual formulation ?

- Works only for financial models.
- Event not always (large investors, gamma contraints,...)

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- New proofs under minimal conditions (B., R. Elie and N. Touzi).
- Can be extended to jumps (B.) or American type constraints (B. and V. T. Nam)

Dual formulation ?

- Works only for financial models.
- Event not always (large investors, gamma contraints,...)
- What about insurance, power plant management,...

Outline

Stochastic target problems

Stochastic target with controlled loss

Optimal control under target constraints

Problem Formulation

Viability set

$$D(p) := \{(t, z) \;:\; \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p\},\; p \in \mathbb{R}.$$

Problem Formulation

Viability set

$$D(p) := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p\},\ p \in \mathbb{R}.$$

Value function

$$w(t,x;p) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D(p)\}.$$

Example

Interpretation

Stocks/Factors : X^{ν}. Wealth : Y^{ν}. Portfolio strategy : $\nu \in \mathcal{U}$.

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ ;

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x,y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Super-hedging price for $p = 1$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Super-hedging price for $p = 1$

$$
\begin{aligned}
& \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T))\right] \geq 1\} \\
= \; & \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } Y^\nu_{t,x,y}(T) \geq \psi(X^\nu_{t,x}(T)) \; \mathbb{P}-\text{a.s.}\}
\end{aligned}
$$

Example

Interpretation

Stocks/Factors : X^{ν}. Wealth : Y^{ν}. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ;

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Quantile-hedging costs for $0 < p < 1$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ ; Define : $g(x,y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Quantile-hedging costs for $0 < p < 1$

$$
\begin{aligned}
& \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T))\right] \geq p\} \\
= \quad & \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^\nu_{t,x,y}(T) \geq \psi(X^\nu_{t,x}(T))\right] \geq p\}
\end{aligned}
$$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ;

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \ell([y - \psi(x)]^-)$, $\ell \nearrow$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \ell([y - \psi(x)]^-)$, $\ell \nearrow$

Loss function price

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x,y) := \ell([y-\psi(x)]^-)$, $\ell \nearrow$

Loss function price

$$\begin{aligned} & \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } -\mathbb{E}\left[g(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T))\right] \geq p\} \\ = \; & \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\ell\left([Y^\nu_{t,x,y}(T) - \psi(X^\nu_{t,x}(T))]^-\right)\right] \leq -p\} \end{aligned}$$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ ;

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := U(y - \psi(x))$, U concave $\nearrow$

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ ; Define : $g(x, y) := U(y - \psi(x))$, U concave $\nearrow$

Indifference price

Example

Interpretation

Stocks/Factors : X^ν. Wealth : Y^ν. Portfolio strategy : $\nu \in \mathcal{U}$.
Option payoff : ψ ; Define : $g(x,y) := U(y-\psi(x))$, U concave $\nearrow$

Indifference price

$$\begin{aligned} & \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } -\mathbb{E}\left[g(X^\nu_{t,x}(T), Y^\nu_{t,x,y_0+y}(T))\right] \geq p\} \\ = \; & \inf\{y : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[U\left(Y^\nu_{t,x,y_0+y}(T) - \psi(X^\nu_{t,x}(T))\right)\right] \geq p\} \end{aligned}$$

Dynamic Programming

Recall

$D(p) := \{(t,z)\ :\ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p\ \mathbb{P}-\text{a.s.}\}.$

$w(t,x;p) := \inf\{y \in \mathbb{R}\ :\ (t,x,y) \in D(p)\}$

Dynamic Programming

Recall

$D(p) := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p\ \mathbb{P}-\text{a.s.}\}.$

$w(t,x;p) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D(p)\}$

Geometric Dynamic Programming ?

For a stopping time $\theta \in [t,T]\ \mathbb{P}-\text{a.s.}$

$$D(p) \neq \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } (\theta, Z^{\nu}_{t,z}(\theta)) \in D(p)\ \mathbb{P}-\text{a.s.}\}\ .$$

Dynamic Programming

Recall

$D(p) := \{(t,z)\ :\ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p\ \mathbb{P}-\text{a.s.}\}.$

$w(t,x;p) := \inf\{y \in \mathbb{R}\ :\ (t,x,y) \in D(p)\}$

Geometric Dynamic Programming

For a stopping time $\theta \in [t,T]\ \mathbb{P}-\text{a.s.}$

$$D(p) = \{(t,z)\ :\ \exists\, \nu \in \mathcal{U} \text{ s.t. } (\theta, Z^{\nu}_{t,z}(\theta)) \in D(P)\ \mathbb{P}-\text{a.s.}\}\ .$$

with "$P := \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\ |\ \mathcal{F}_{\theta}\right]$"

Dynamic Programming

Recall

$D(p) := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p \ \mathbb{P}-\text{a.s.}\}.$

$w(t,x;p) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D(p)\}$

Geometric Dynamic Programming

For a stopping time $\theta \in [t,T]\ \mathbb{P}-\text{a.s.}$

$$D(p) = \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } (\theta, Z^{\nu}_{t,z}(\theta)) \in D(P)\ \mathbb{P}-\text{a.s.}\}\ .$$

with "$P := \mathbb{E}\left[g(Z^{\nu}_{t,z}(T)) \mid \mathcal{F}_\theta\right]$" $= P^{\alpha}_{t,p}(\theta) := p + \int_t^\theta \alpha_s dW_s$

Dynamic Programming

Recall

$D(p) := \{(t,z) \ : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p \ \mathbb{P} - \text{a.s.}\}.$

$w(t,x;p) := \inf\{y \in \mathbb{R} \ : \ (t,x,y) \in D(p)\}$

Theorem (B., Elie and Touzi)

$(t,z) \in D(p)$ *iff there exists* (ν,α) *s.t., for any stopping time* $\theta \in [t,T]\, \mathbb{P} - a.s.$,

$$(\theta, Z^{\nu}_{t,z}(\theta)) \in D(P^{\alpha}_{t,p}(\theta))\ \mathbb{P} - a.s.$$

Dynamic Programming

Recall

$D(p) := \{(t,z)\ :\ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p\ \mathbb{P}-\text{a.s.}\}.$

$w(t,x;p) := \inf\{y \in \mathbb{R}\ :\ (t,x,y) \in D(p)\}$

Theorem (B., Elie and Touzi)

$(t,z) \in D(p)$ *iff there exists* (ν,α) *s.t., for any stopping time* $\theta \in [t,T]\ \mathbb{P}-$ *a.s.,*

$$(\theta, Z^{\nu}_{t,z}(\theta)) \in D(P^{\alpha}_{t,p}(\theta))\ \mathbb{P}-a.s.$$

Back to a.s. stochastic target problems

Apply the previous approach to the new controlled process $(Z^{\nu}_{t,z}, P^{\alpha}_{t,p})$ and controls (ν,α).

PDE Derivation

Theorem (B., Elie and Imbert)

w is a viscosity solution (in the discontinuous sense) of

$$\sup_{(u,\alpha)\in\mathcal{N}(t,x,p,w)} \mu_Y(x, w(t,x,p), u) - \mathcal{L}^{u,\alpha}_{X,P} w(t,x,p) = 0$$

where

$$\mathcal{N}(\cdot, w) := \left\{(u,\alpha) \in U \times \mathbb{R}^d : \sigma_Y(\cdot, w, u) = D_x w \sigma_X(\cdot, u) + \alpha D_p w\right\}.$$

Example : Quantile hedging in B.-S.

Dynamics

$$dX = X\mu dt + X\sigma dW \;,\; dY^{\nu} = \nu dX/X$$

Example : Quantile hedging in B.-S.

Dynamics

$$dX = X\mu dt + X\sigma dW \ , \ dY^{\nu} = \nu dX/X$$

Quantile hedging problem

$$w(t,x,p) = \inf\{y : \exists\, \nu \in \mathcal{U},\ \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq \psi(X_{t,x}(T))\right] \geq p\}$$

Example : Quantile hedging in B.-S.

Dynamics

$$dX = X\mu dt + X\sigma dW \ , \ dY^{\nu} = \nu dX/X$$

Quantile hedging problem

$$w(t,x,p) = \inf\{y : \exists\, \nu \in \mathcal{U},\ \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq \psi(X_{t,x}(T))\right] \geq p\}$$

PDE

$$0 = \sup_{u\sigma s = \sigma x w_x + \alpha w_p} \left(u\mu x - \mu x w_x - \frac{1}{2}\sigma^2 x^2 w_{xx} - \alpha\sigma x w_{xp} - \frac{1}{2}\alpha^2 w_{pp}\right)$$

Example : Quantile hedging in B.-S.

Dynamics

$$dX = X\mu dt + X\sigma dW \ , \ dY^{\nu} = \nu dX/X$$

Quantile hedging problem

$$w(t,x,p) = \inf\{y : \exists\, \nu \in \mathcal{U},\ \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq \psi(X_{t,x}(T))\right] \geq p\}$$

PDE and convexity ($w_{pp} \geq 0$)

$$0 = -w_t - \frac{1}{2}\sigma^2 s^2 w_{xx} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}w_p - \sigma x w_{xp}\right)^2}{w_{pp}}$$

Example : Quantile hedging in B.-S.

Legendre transform

$\tilde{w}(t,x,q) := \sup_{p\in[0,1]} \{pq - w(t,x,p)\}$.

Example : Quantile hedging in B.-S.

Legendre transform

$$\tilde{w}(t,x,q) := \sup_{p\in[0,1]} \{pq - w(t,x,p)\} \ .$$

Boundary condition for $t = T$

$$w(T-,x,1) = \psi(x),\ w(T-,x,0) = 0$$

Example : Quantile hedging in B.-S.

Legendre transform

$\tilde{w}(t,x,q) := \sup_{p\in[0,1]} \{pq - w(t,x,p)\}$.

Boundary condition for $t = T$

$w(T-,x,1) = \psi(x)$, $w(T-,x,0) = 0$

and w concave in $p \Rightarrow w(T-,x,p) = p\psi(x)$

Example : Quantile hedging in B.-S.

Legendre transform

$\tilde{w}(t,x,q) := \sup_{p\in[0,1]} \{pq - w(t,x,p)\}$.

Boundary condition for $t = T$

$w(T-,x,1) = \psi(x)$, $w(T-,x,0) = 0$

and w concave in $p \Rightarrow w(T-,x,p) = p\psi(x)$

$\Rightarrow \tilde{w}(T-,x,q) = [q - \psi(x)]^+$.

Example : Quantile hedging in B.-S.

Legendre transform

$$\tilde{w}(t,x,q) := \sup_{p\in[0,1]} \{pq - w(t,x,p)\} \ .$$

Boundary condition for $t = T$

$w(T-,x,1) = \psi(x)$, $w(T-,x,0) = 0$

and w concave in $p \Rightarrow w(T-,x,p) = p\psi(x)$

$\Rightarrow \tilde{w}(T-,x,q) = [q - \psi(x)]^+$.

PDE

$$-\tilde{w}_t - \frac{1}{2}\sigma^2\tilde{w}_{xx} - (\mu/\sigma)q\sigma x\tilde{w}_{xq} - \frac{1}{2}(\mu/\sigma)^2 q^2\tilde{w}_{qq} = 0$$

Example : Quantile hedging in B.-S.

Legendre transform

$\tilde{w}(t,x,q) := \sup_{p\in[0,1]} \{pq - w(t,x,p)\}$.

Boundary condition for $t = T$

$w(T-,x,1) = \psi(x)$, $w(T-,x,0) = 0$
and w concave in $p \Rightarrow w(T-,x,p) = p\psi(x)$
$\Rightarrow \tilde{w}(T-,x,q) = [q-\psi(x)]^+$.

PDE

$-\tilde{w}_t - \frac{1}{2}\sigma^2\tilde{w}_{xx} - (\mu/\sigma)q\sigma x\tilde{w}_{xq} - \frac{1}{2}(\mu/\sigma)^2q^2\tilde{w}_{qq} = 0$

Explicit solution by Feynman-Kac.

General results

Already done

- Viscosity characterization for stochastic target problems with unbounded controls (with minimal assumptions).
- Derivation of the boundary conditions for stochastic target problems with controlled probability of loss at $p = 0, 1$ and $t = T$.

General results

Already done

- Viscosity characterization for stochastic target problems with unbounded controls (with minimal assumptions).
- Derivation of the boundary conditions for stochastic target problems with controlled probability of loss at $p = 0, 1$ and $t = T$.

Remains to do

- Comparison results ?
- American version ?

Outline

Problem formulation

Stochastic target constraints problem ($\mathbb{P}$ − a.s. sense)

$$V(t,z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P}-\text{a.s.}\right\}$$

Problem formulation

Stochastic target constraints problem ($\mathbb{P}$ – a.s. sense)

$$
\begin{aligned}
V(t,z) &:= \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right] \\
\text{with } \mathcal{U}_{t,z} &:= \left\{\nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\right\}
\end{aligned}
$$

Target constraints in expectation/probability

$$
\begin{aligned}
V(t,z,p) &:= \sup_{\nu \in \mathcal{U}_{t,z,p}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right] \\
\text{with } \mathcal{U}_{t,z,p} &:= \left\{\nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[g(Z^{\nu}_{t,z}(T))\right] \geq p \ \mathbb{P} - \text{a.s.}\right\}
\end{aligned}
$$

Problem formulation

Stochastic target constraints problem ($\mathbb{P}$ – a.s. sense)

$$V(t,z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \ \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P}-\text{a.s.}\right\}$$

Example : Super-hedging constraint/Index tracking

$$\mathcal{U}_{t,x,y} := \left\{\nu \in \mathcal{U} \ \text{ s.t. } Y^{\nu}_{t,x,y}(T) \geq \psi(X^{\nu}_{t,x}(T)) \ \mathbb{P}-\text{a.s.}\right\} \ ,$$

Problem formulation

Stochastic target constraints problem ($\mathbb{P}$ – a.s. sense)

$$V(t,z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\right\} ,$$

Example : Quantile-hedging constraint/Index tracking

$$\mathcal{U}_{t,x,y,p} := \left\{\nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq \psi(X^{\nu}_{t,x}(T))\right] \geq p\right\} ,$$

Problem formulation

Stochastic target constraints problem ($\mathbb{P}$ − a.s. sense)

$$V(t,z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^\nu_{t,z}(T))\right]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \ \text{ s.t. } g(Z^\nu_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\right\} ,$$

Example : Loss constraint

$$\mathcal{U}_{t,x,y,p} := \left\{\nu \in \mathcal{U} \ \text{ s.t. } \mathbb{E}\left[\ell\left(\left[Y^\nu_{t,x,y}(T) - \psi(X^\nu_{t,x}(T))\right]^-\right)\right] \leq -p\right\} ,$$

Problem re-formulation

Stochastic target constraints problem ($\mathbb{P}$ − a.s. sense)

$V(t,z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right]$

with $\mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \; \mathbb{P} - \text{a.s.}\right\}$.

Problem re-formulation

Stochastic target constraints problem ($\mathbb{P}$ − a.s. sense)

$V(t,z) := \sup_{\nu\in\mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right]$
with $\mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\right\}$.

State constraint problem formulation

$\mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z}(s) \in D \ \mathbb{P} - \text{a.s. } \forall s \in [t,T]\right\}$, where
$D = \{(t,x) \ : \ \mathcal{U}_{t,z} \neq \emptyset\}$.

Problem re-formulation

Stochastic target constraints problem ($\mathbb{P}$ – a.s. sense)

$V(t,z) := \sup_{\nu\in\mathcal{U}_{t,z}} \mathbb{E}\left[f(Z^{\nu}_{t,z}(T))\right]$
with $\mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \ \text{ s.t. } g(Z^{\nu}_{t,z}(T)) \geq 0 \ \mathbb{P} - \text{a.s.}\right\}$.

State constraint problem formulation

$\mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U} \ \text{ s.t. } Z^{\nu}_{t,z}(s) \in D \ \mathbb{P} - \text{a.s. } \forall s \in [t,T]\right\}$, where
$D = \{(t,x) \ : \ \mathcal{U}_{t,z} \neq \emptyset\}$.

Important point

D is given by "the" viscosity solution of a PDE. Not a-priori. More complex but implies reflexion on the boundary automatically.

PDE formulation

Assumption

The value function w of the target problem is continuous in the domain, with a continuous extension at T

PDE formulation

Assumption

The value function w of the target problem is continuous in the domain, with a continuous extension at T

Decomposition of the domain

$$\begin{aligned} \mathrm{int}_p D &:= \{(t,x,y) \in [0,T) \times \mathbb{R}^{d+1} : y > w(t,x)\} \\ \partial_p D &:= \{(t,x,y) \in [0,T) \times \mathbb{R}^{d+1} : y = w(t,x)\} \\ \partial_T D &:= \{(t,x,y) \in [0,T] \times \mathbb{R}^{d+1} : y \geq w(t,x),\ t = T\} \end{aligned}$$

PDE formulation

On $\mathrm{int}_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

PDE formulation

On $\text{int}_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

- $\forall\ \nu$, $\exists\ \theta > t\ \mathbb{P}-$a.s. s.t. $Y^{\nu}_{t,x,y}(\theta) > w(\theta, X^{\nu}_{t,x}(\theta))\ \mathbb{P}-$a.s.

PDE formulation

On $\text{int}_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

- $\forall\ \nu$, $\exists\ \theta > t\ \mathbb{P}-$a.s. s.t. $Y^{\nu}_{t,x,y}(\theta) > w(\theta, X^{\nu}_{t,x}(\theta))\ \mathbb{P}-$a.s.
- The state constraint does not play any role.

PDE formulation

On $\mathrm{int}_p D := \{(t,x,y) \in [0,T) \times \mathbb{R}^{d+1} : y > w(t,x)\}$

- $\forall\ \nu,\ \exists\ \theta > t\ \mathbb{P}-$ a.s. s.t. $Y^{\nu}_{t,x,y}(\theta) > w(\theta, X^{\nu}_{t,x}(\theta))\ \mathbb{P}-$ a.s.
- The state constraint does not play any role.
- Usual HJB equation

$$\inf_u -\mathcal{L}^u_{X,Y} V(t,x,y) = 0\ .$$

PDE formulation

On $\mathrm{int}_p D := \{(t,x,y) \in [0,T) \times \mathbb{R}^{d+1} : y > w(t,x)\}$

- $\forall\ \nu,\ \exists\ \theta > t\ \mathbb{P}-$ a.s. s.t. $Y^{\nu}_{t,x,y}(\theta) > w(\theta, X^{\nu}_{t,x}(\theta))\ \mathbb{P}-$ a.s.
- The state constraint does not play any role.
- Usual HJB equation

$$\inf_u -\mathcal{L}^u_{X,Y} V(t,x,y) = 0\ .$$

On $\partial_T D := \{(t,x,y) \in [0,T] \times \mathbb{R}^{d+1} : y \geq w(t,x),\ \ t = T\}$

Standard boundary condition $V(T-,x,y) = f(x,y)$.

PDE formulation

On $\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

PDE formulation

On $\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY^{\nu}_{t,x,y}(t) \geq dw(t, X^{\nu}_{t,x}(t))$

PDE formulation

On $\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY^{\nu}_{t,x,y}(t) \geq dw(t, X^{\nu}_{t,x}(t))$
- This implies
  - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$

PDE formulation

On $\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY^{\nu}_{t,x,y}(t) \geq dw(t, X^{\nu}_{t,x}(t))$
- This implies
  - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$
  - $\mu_Y(x, y, u) - \mathcal{L}^u_X w(t, x) \geq 0$

PDE formulation

On $\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY^{\nu}_{t,x,y}(t) \geq dw(t, X^{\nu}_{t,x}(t))$
- This implies
  - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$
  - $\mu_Y(x, y, u) - \mathcal{L}^u_X w(t, x) \geq 0$
  - Defines a set $U(t, x, y, w)$.

PDE formulation

On $\partial_p D := \{(t,x,y) \in [0,T) \times \mathbb{R}^{d+1} : y = w(t,x)\}$

- Must choose ν s.t. $dY^{\nu}_{t,x,y}(t) \geq dw(t, X^{\nu}_{t,x}(t))$
- This implies
  - $\sigma_Y(x,y,u) = Dw(t,x)\sigma_X(x,u)$
  - $\mu_Y(x,y,u) - \mathcal{L}^u_X w(t,x) \geq 0$
  - Defines a set $U(t,x,y,w)$.
- Constrained HJB equation

$$\inf_{u \in U(t,x,y,w)} -\mathcal{L}^u_{X,Y} V(t,x,y) = 0 \ .$$

PDE formulation

Precise formulation on

$\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

PDE formulation

Precise formulation on
$\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- V_* is a super-solution of

$$\inf_{\varphi \in \mathcal{T}^*(t,x)} \inf_{u \in U(t,x,y,\varphi)} -\mathcal{L}^u_{X,Y} V_*(t, x, y) \geq 0 .$$

$$\mathcal{T}^*(t, x) := \{\varphi \in C^{1,2} \text{ s.t. } 0 = \max(w - \varphi) = (w - \varphi)(t, x)\}.$$

PDE formulation

Precise formulation on
$\partial_p D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- V_* is a super-solution of

$$\inf_{\varphi \in \mathcal{T}^*(t,x)} \inf_{u \in U(t,x,y,\varphi)} -\mathcal{L}^u_{X,Y} V_*(t, x, y) \geq 0 \; .$$

$$\mathcal{T}^*(t, x) := \{\varphi \in C^{1,2} \text{ s.t. } 0 = \max(w - \varphi) = (w - \varphi)(t, x)\}.$$

- V^* is a sub-solution of

$$\sup_{\varphi \in \mathcal{T}_*(t,x)} \inf_{u \in U(t,x,y,\varphi)} -\mathcal{L}^u_{X,Y} V^*(t, x, y) \leq 0 \; .$$

$$\mathcal{T}_*(t, x) := \{\varphi \in C^{1,2} \text{ s.t. } 0 = \min(w - \varphi) = (w - \varphi)(t, x)\}.$$

Results

Already done

Results

Already done

- PDE characterization (discontinuous viscosity solutions and relaxation of the operators).

Results

Already done

- PDE characterization (discontinuous viscosity solutions and relaxation of the operators).
- Constrained subsolution property on the boundary $\partial_p D$ under realistic assumptions.

Results

Already done

- PDE characterization (discontinuous viscosity solutions and relaxation of the operators).
- Constrained subsolution property on the boundary $\partial_p D$ under realistic assumptions.
- Change of variables on the boundary $\partial_p D$ when $w \in C^{1,2}$. Allows to rewrite the boundary condition as a Dirichlet condition $V(t,x,y) = \mathcal{V}(t,x)$ at $y = w(t,x)$ where $\mathcal{V} = V(\cdot, w(\cdot))$ solves a suitable PDE.

Results

To be done

Results

To be done

- Boundary conditions when the constraint is in expectation/probability.

Results

To be done

- Boundary conditions when the constraint is in expectation/probability.
- Comparison/Numerical schemes.

Possible extensions

Could be extended to

Possible extensions

Could be extended to

- Jump diffusion processes (in progress by L. Moreau).

Possible extensions

Could be extended to

- Jump diffusion processes (in progress by L. Moreau).
- American type constraints (Dynamic programming by B. and V. T. Nam)

Possible extensions

Could be extended to

- Jump diffusion processes (in progress by L. Moreau).
- American type constraints (Dynamic programming by B. and V. T. Nam)
- Multiple constraints (no real problem).