First time to exit of a continuous Itô process: general moment estimates and $L^1$-convergence rate for discrete time approximations

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - Ensae-ParisTech
Joint work with S. Geiss and E. Gobet

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Problem formulation

- $X$ approximated by $\tilde{X}$ on a grid $\tilde{\pi}$.
Problem formulation

- $X$ approximated by $\bar{X}$ on a grid $\bar{\pi}$.

- $\theta =$ first exit time of $X$ from $\mathcal{O}$ approximated by $\bar{\theta} =$ first (discrete) exit time of $\bar{X}$ on $\bar{\pi}$. 
Problem formulation

- $X$ approximated by $\tilde{X}$ on a grid $\tilde{\pi}$.

- $\theta = \text{first exit time of } X \text{ from } \mathcal{O} \text{ approximated by } \tilde{\theta} = \text{first (discrete) exit time of } \tilde{X} \text{ on } \tilde{\pi}$.

- Question: $L^p$ control of $\theta - \tilde{\theta}$ and of $X_\theta - \tilde{X}_\tilde{\theta}$.
Example of applications

- Mathematical finance: barrier option pricing

Approximate: \( g(\theta, X_\theta) \) by \( g(\bar{\theta}, \bar{X}_{\bar{\theta}}) \) for \( g \) Lipschitz:

\[
|g(\theta, X_\theta) - g(\bar{\theta}, \bar{X}_{\bar{\theta}})| \leq C (|\theta - \bar{\theta}| + |X_\theta - \bar{X}_{\bar{\theta}}|)
\]
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\]

- BSDEs and semilinear elliptic/parabolic PDEs:

Approximate:

\[
Y_t = g(\theta, X_\theta) + \int_{t \land \theta}^{\theta} f(s, X_s, Y_s, Z_s) ds - \int_{t \land \theta}^{\theta} Z_s dW_s
\]
Plan of the talk

- Known results (in a nutshell)
- Moment estimates on the first time to exit
- Application to exit time and value at the exit time approximation
- References
Known results (in a nutshell)

- **Framework**: $X$ is a Brownian SDE with Lipschitz coefficients and $\tilde{X}$ its Euler scheme on $\tilde{\pi}$.

- **Gobet and Menozzi (10)**: $E[\bar{\theta} \wedge T - \theta \wedge T] = C |\bar{\pi}|^{1/2} + o(|\bar{\pi}|^{1/2})$ for smooth domain and coefficients + uniform ellipticity condition.

- **Bouchard and Menozzi (09)**: $E[|\bar{\theta} \wedge T - \theta \wedge T|] = O(\epsilon (|\bar{\pi}|^{1/2} - \epsilon))$ for piecewise $C^2$ domain, non-characteristic boundary condition.

- **Questions**: Can the $\epsilon$ be removed? What about $E[|\bar{\theta} - \theta|]$? (i.e. unbounded case)
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- **Gobet and Menozzi (10)**:

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\mathbb{E} \left[ \tilde{\theta} \wedge T - \theta \wedge T \right] = C|\tilde{\pi}|^{\frac{1}{2}} + o(|\tilde{\pi}|^{\frac{1}{2}})
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  for smooth domain and coefficients + uniform ellipticity condition.

- **Bouchard and Menozzi (09)**:
  \[ \mathbb{E} \left[ |\tilde{\theta} \land T - \theta \land T| \right] = O_\varepsilon(|\bar{\pi}|^{\frac{1}{2}-\varepsilon}) \]
  for piecewise $C^2$ domain, non-characteristic boundary condition.
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- **Questions**: Can the \( \varepsilon \) be removed? What about \( \mathbb{E} [ |\bar{\theta} - \theta| ] \)? (i.e. unbounded case)
Moment estimates on the first time to exit
General Framework

- **Underlying process**: \((Z_t)_{t \geq 0}\): a continuous and adapted process with values in a metric space \((\mathcal{Z}, d_\mathcal{Z})\).
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- **Domain**: \(\mathcal{O}\) an open set of \(\mathcal{Z}\).
General Framework

- **Underlying process**: \((Z_t)_{t \geq 0}\): a continuous and adapted process with values in a metric space \((Z, d_Z)\).
- **Domain**: \(\mathcal{O}\) an open set of \(Z\).
- **Monitoring times**: \(\pi \subset \mathbb{R}_+\) with two cases

\[
\pi = \mathbb{R}_+ \\
\text{or} \\
0 < \inf_{[0, T] \setminus \pi} (\phi^+ - \phi) \leq \sup_{\mathbb{R}_+} (\phi^+ - \phi) =: |\pi| \leq 1 \forall T \geq T_0,
\]

where

\[
\phi_t := \max\{s \in \pi : s \leq t\} \text{ and } \phi_t^+ := \min\{s \in \pi : s \geq t\}.
\]
Assumptions

Assumption (Z) (Regularity of Z). \( \exists \) loc. bounded \( \kappa : \mathbb{R}_+ \times (0, \infty) \mapsto \mathbb{R}_+ \) s.t.

\[
P_\tau \left[ \sup_{\tau \leq t \leq \tau + T} \| dZ (Z_t, Z_{\phi t \vee \tau}) \| \geq \rho \right] \leq \kappa (T, \rho) |\pi|
\]

\( \forall \tau \in \mathcal{T}, T \geq 0, \) and \( \rho > 0. \)
Assumptions

Assumption (P) (Distance process $\delta(Z)$). \(\exists L\)-Lipschitz \(\delta : Z \mapsto \mathbb{R}\) s.t. \(\delta > 0\) on \(\mathcal{O}\), \(\delta = 0\) on \(\partial \mathcal{O}\) and \(\delta < 0\) on \(\bar{\mathcal{O}}^c\).
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\( P = \delta(Z) \) admits the Itô process decomposition

\[
P_t = P_0 + \int_0^t b_s ds + \int_0^t a_s^\top dW_s
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where

1) \( (P, b, a) \) is a predictable process with values in \( [-L, L]^{d+2} \),
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where

i) \( (P, b, a) \) is a predictable process with values in \([-L, L]^{d+2} \),

ii) \( |a^\top a| \geq L^{-2} \ dt \times d\mathbb{P}\)-a.e. on \( \{|P| \vee d_\mathcal{Z}(Z, Z_\phi) \leq r\} \) for a given \( r \in (0, L^{-3}/4) \).
Definition : For $\tau \in \mathcal{T}$ and $p \in \mathbb{N}^*$ :

\[
\theta(\tau) := \inf \{ t \geq \tau : P_t \leq 0 \}, \quad \theta^\pi(\tau) := \inf \{ t \geq \tau : t \in \pi, P_t \leq 0 \}
\]

\[
\Phi^p(\tau) := \mathbb{E}_\tau \left[ \left( \theta(\tau) - \tau \right)^p \right]^{\frac{1}{p}}, \quad \Phi^{p,\pi}(\tau) := \mathbb{E}_\tau \left[ \left( \theta^\pi(\tau) - \tau \right)^p \right]^{\frac{1}{p}}.
\]
\section*{Assumptions}

\begin{itemize}
  \item \textbf{Definition :} For $\tau \in \mathcal{T}$ and $p \in \mathbb{N}^*$ :

    \[ \theta(\tau) := \inf \{ t \geq \tau : P_t \leq 0 \}, ~ \theta^\pi(\tau) := \inf \{ t \geq \tau : t \in \pi, ~ P_t \leq 0 \} \]

    \[ \Phi^p(\tau) := \mathbb{E}_\tau [ (\theta(\tau) - \tau)^p ]^{\frac{1}{p}}, ~ \Phi^{p,\pi}(\tau) := \mathbb{E}_\tau [ (\theta^\pi(\tau) - \tau)^p ]^{\frac{1}{p}} \].

  \item \textbf{Assumption (L) (uniform bound on expectation of exit time).} $\Phi^{1,\pi}(\tau) + \Phi^1(\tau) \leq L$ for all $\tau \in \mathcal{T}$.
\end{itemize}
Main results

- **Thm 1 (continuous monitoring)**: Fix $0 < \tilde{r} < r$. Then, $\exists c = c(r, r - \tilde{r}, L, d) > 0$ s.t.

  $$\Phi^1(\tau) \leq c(|P_\tau| + |\pi|^{1/2})$$

  $\forall \tau \in \mathcal{T}$ s.t. $Z_\tau \in \bar{O} \cap N_{\tilde{r}}$. If $\tau \in \mathcal{T}^\pi$, it holds if $Z_\tau \in \bar{O} \cap N_r$, and $c$ does not depend on $r - \tilde{r}$.
Main results

- **Thm 1 (continuous monitoring)**: Fix \( 0 < \tilde{r} < r \). Then, there exists \( c = c(r, r - \tilde{r}, L, d) > 0 \) such that

\[
\Phi^{1}(\tau) \leq c(|P_{\tau}| + |\pi|^{\frac{1}{2}})
\]

for all \( \tau \in \mathcal{T} \) such that \( Z_\tau \in \tilde{O} \cap N_{\tilde{r}} \). If \( \tau \in \mathcal{T}^{\pi} \), it holds if \( Z_\tau \in \tilde{O} \cap N_{r} \), and \( c \) does not depend on \( r - \tilde{r} \).

- **Thm 2 (discrete monitoring)**: Take \( \pi \neq \mathbb{R}_{+} \) and \( |\pi| \leq \varepsilon \) (given explicitly), fix \( 0 < \tilde{r} < r \). Then, there exists \( c = c(r, r - \tilde{r}, L, d) > 0 \) such that

\[
\Phi^{1,\pi}(\tau) \leq c \left(|P_{\tau}| + |\pi|^{\frac{1}{2}}\right)
\]

for all \( \tau \in \mathcal{T} \) such that \( Z_\tau \in \tilde{O} \cap N_{\tilde{r}} \). If \( \tau \in \mathcal{T}^{\pi} \), it holds if \( Z_\tau \in \tilde{O} \cap N_{r} \), and \( c \) does not depend on \( r - \tilde{r} \).
Scheme of proof

Freidlin type inequalities on exit times moments:

\[(\Phi^p(\tau))^p \leq c_p \Phi^1(\tau) \wedge L(p) \quad \text{and} \quad (\Phi^{p,\pi}(\tau))^p \leq c_p \Phi^{1,\pi}(\tau) \wedge L(p)\]

where \(c_p := p!L^{p-1} =: pL^{(p-1)}\).
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where \( c_p := p!L^{p-1} =: pL^{(p-1)} \).

Indeed

\[ \frac{(\Phi^{p+1,\pi}(\tau))^{p+1}}{p + 1} = \int_\tau^\infty \mathbb{E}_\tau \left[ (\theta^{\pi}(\tau) - t)^p 1_{\theta^{\pi}(\tau) > t} \right] dt \]

\[ = \int_\tau^\infty \mathbb{E}_\tau \left[ \mathbb{E}_{t \vee \tau} \left[ (\theta^{\pi}(t \vee \tau) - t \vee \tau)^p \right] 1_{\theta^{\pi}(\tau) > t} \right] dt \]

\[ \leq \int_\tau^\infty L^p(\tau) \mathbb{E}_\tau \left[ 1_{\theta^{\pi}(\tau) > t} \right] dt \leq L^p(\tau) \Phi^{1,\pi}(\tau). \]
Scheme of proof

An a-priori control in terms of the probability of strictly sub-harmonic paths: \( \exists \ c > 0 \ s.t. \)

\[
\Phi^1(\tau) \leq c \mathbb{P}_\tau[A^\tau], \text{ for all } \tau \in \mathcal{T}.
\]

where

\[
(A^\tau)^c := \{2Pb + a^\top a \geq L^{-2}/2 \text{ on } [\tau, \theta(\tau)]\}.
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Scheme of proof

▷ An a-priori control in terms of the probability of strictly sub-harmonic paths: \( \exists \ c > 0 \text{ s.t.} \)

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\Phi^1(\tau) \leq c \mathbb{P}_\tau [A^\tau], \text{ for all } \tau \in \mathcal{T}.
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(A^\tau)^c := \{2Pb + a^\top a \geq L^{-2}/2 \text{ on } [\tau, \theta(\tau)]\}.
\]

Indeed, on \((A^\tau)^c\),

\[
\frac{\theta(\tau) - \tau}{2L^2} \leq \int_\tau^{\theta(\tau)} (2P_s b_s + a_s^\top a_s) ds
\]

\[
= |P_{\theta(\tau)}|^2 - |P_\tau|^2 - \int_\tau^{\theta(\tau)} 2P_s a_s^\top dW_s
\]

\[
\leq - \int_\tau^{\theta(\tau)} 2P_s a_s^\top dW_s.
\]
Scheme of proof

Conclude with a control on \( \mathbb{P}_\tau [A^\tau] : \forall \iota > 0 \exists \eta(\iota) > 0 \) s.t.

\[
\mathbb{P}_\tau [A^\tau] \leq \eta(\iota)(P_\tau + |\pi|) + \iota \Phi^1(\tau)
\]

for all \( \tau \in T^\pi \) such that \( P_\tau \in [0, r] \). (use the non-characteristic boundary condition to exit with high probability before leaving the neighborhood of the boundary)
Scheme of proof

Conclude with a control on $\mathbb{P}_\tau[A^\tau] : \forall \nu > 0 \exists \eta(\nu) > 0$ s.t.

$$\mathbb{P}_\tau[A^\tau] \leq \eta(\nu)(P_\tau + |\pi|) + \nu \Phi^1(\tau)$$

for all $\tau \in T^\pi$ such that $P_\tau \in [0, r]$. (use the non-characteristic boundary condition to exit with high probability before leaving the neighborhood of the boundary)

Recall that :

$$\Phi^1(\tau) \leq c\mathbb{P}_\tau[A^\tau]$$

And use : passing to $\tau \in T$ to $\tau \in T^\pi$ costs $|\pi|^{1/2}$. 
Scheme of proof

- **Extension to** $\Phi^{1,\pi}(\tau)$: Picture on the board...
Application to exit time approximation
Problem: \( X \) with \( \theta \) on \( \mathbb{R}_+ \) approximated by \( \bar{X} \) with \( \bar{\theta} \) on \( \bar{\pi} \) (discrete grid). Set \( P := d_{\mathcal{Z}}(X) \) and \( \bar{P} := d_{\mathcal{Z}}(\bar{X}) \).
Exit time approximation

Problem: $X$ with $\theta$ on $\mathbb{R}_+$ approximated by $\bar{X}$ with $\bar{\theta}$ on $\bar{\pi}$ (discrete grid). Set $P := d_{\mathbb{Z}}(X)$ and $\bar{P} := d_{\mathbb{Z}}(\bar{X})$.

Thm: If $\exists \rho > 0$ s.t.

$$\mathbb{E} \left[ |P_\vartheta - \bar{P}_\vartheta|^2 \right] \leq \rho |\bar{\pi}| \quad \forall \vartheta \in \mathcal{T} \text{ s.t. } \vartheta \leq \theta.$$
Exit time approximation

Problem: $X$ with $\theta$ on $\mathbb{R}_+$ approximated by $\bar{X}$ with $\bar{\theta}$ on $\bar{\pi}$ (discrete grid). Set $P := d\mathcal{Z}(X)$ and $\bar{P} := d\mathcal{Z}(\bar{X})$.

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$$\mathbb{E} \left[ |P_\vartheta - \bar{P}_\vartheta|^2 \right] \leq \rho |\bar{\pi}| \quad \forall \vartheta \in \mathcal{T} \text{ s.t. } \vartheta \leq \theta.$$  

Then, $\exists \ c = c(r, L, d, \rho) > 0$ and $\varepsilon = \varepsilon(r, L) > 0$ s.t.

$$\mathbb{E} \left[ |\theta - \bar{\theta}| \right] \leq \mathbb{E} \left[ \mathbb{E}_\vartheta \left[ |\theta - \bar{\theta}| \right]^2 \right]^{\frac{1}{2}} \leq c |\bar{\pi}|^{\frac{1}{2}}, \text{ if } |\bar{\pi}| \leq \varepsilon,$$

where $\vartheta := \phi_\theta^+ \wedge \bar{\theta}$.  

**Lemma**: Fix \( \vartheta \in \mathcal{T} \). Assume \( \exists \rho > 0 \) and \( 0 < c_2 < c_1 \) s.t.

\[
\mathbb{P} [ \vartheta \geq T ] \leq \rho e^{-c_1 T} \quad \text{and} \quad \sup_{t \in [0, T]} \mathbb{E} \left[ d_Z (X_t, \bar{X}_t)^4 \right]^{\frac{1}{2}} \leq \rho T | \bar{\pi} | e^{\frac{1}{2} c_2 T}.
\]

Then, \( \exists \ c = c(\rho, d, c_1, c_2) > 0 \) s.t.

\[
\mathbb{E} \left[ d_Z (X_{\vartheta}, \bar{X}_{\vartheta})^2 \right] \leq c | \bar{\pi} |^{\frac{1}{2}}.
\]
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