First time to exit of a continuous Itô process: general moment estimates and L^1-convergence rate for discrete time approximations

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- X approximated by $\bar{X}$ on a grid $\bar{\pi}$.

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□ Question : L^p control of $\theta - \bar{\theta}$ and of $X_\theta - \bar{X}_{\bar{\theta}}$.

Example of applications

□ Mathematical finance : barrier option pricing

$\hookrightarrow$ Approximate : $g(\theta, X_\theta)$ by $g(\bar{\theta}, \bar{X}_{\bar{\theta}})$ for g Lipschitz :

$$|g(\theta, X_\theta) - g(\bar{\theta}, \bar{X}_{\bar{\theta}})| \leq C \left(|\theta - \bar{\theta}| + |X_\theta - \bar{X}_{\bar{\theta}}|\right)$$

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☐ BSDEs and semilinear elliptic/parabolic PDEs :

↪ Approximate :

$$Y_t = g(\theta, X_\theta) + \int_{t\wedge\theta}^{\theta} f(s, X_s, Y_s, Z_s)ds - \int_{t\wedge\theta}^{\theta} Z_s dW_s$$

Plan of the talk

- Known results (in a nutshell)
- Moment estimates on the first time to exit
- Application to exit time and value at the exit time approximation
- References

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$$\mathbb{E}\left[\bar{\theta} \wedge T - \theta \wedge T\right] = C|\bar{\pi}|^{\frac{1}{2}} + o(|\bar{\pi}|^{\frac{1}{2}})$$

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$$\mathbb{E}\left[|\bar{\theta} \wedge T - \theta \wedge T|\right] = O_{\varepsilon}(|\bar{\pi}|^{\frac{1}{2}-\varepsilon})$$

for piecewise C^2 domain, non-characteristic boundary condition.

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☐ Questions : Can the ε be removed ? What about $\mathbb{E}\left[|\bar{\theta} - \theta|\right]$? (i.e. unbounded case)

Moment estimates on the first time to exit

General Framework

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- Monitoring times : $\pi \subset \mathbb{R}_+$ with two cases

$$\pi = \mathbb{R}_+$$

or

$$0 < \inf_{[0,T]\setminus\pi} (\phi^+ - \phi) \leq \sup_{\mathbb{R}_+}(\phi^+ - \phi) =: |\pi| \leq 1 \; \forall \; T \geq T_0,$$

where

$$\phi_t := \max\{s \in \pi \;\; : \;\; s \leq t\} \text{ and } \phi_t^+ := \min\{s \in \pi \;\; : \;\; s \geq t\}.$$

Assumptions

▷ **Assumption (Z) (Regularity of Z).** $\exists$ loc. bounded κ : $\mathbb{R}_+ \times (0, \infty) \mapsto \mathbb{R}_+$ s.t.

$$\mathbb{P}_\tau \left[\sup_{\tau \le t \le \tau + T} d_{\mathcal{Z}} \left(Z_t, Z_{\phi_t \vee \tau} \right) \ge \rho \right] \le \kappa(T, \rho) |\pi|$$

$\forall\ \tau \in \mathcal{T},\ T \ge 0$, and $\rho > 0$.

Assumptions

▷ **Assumption (P) (Distance process $\delta(Z)$).** $\exists$ L-Lipschitz $\delta : \mathcal{Z} \mapsto \mathbb{R}$ s.t. $\delta > 0$ on $\mathcal{O}$, $\delta = 0$ on $\partial\mathcal{O}$ and $\delta < 0$ on $\bar{\mathcal{O}}^c$.

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$$P_t = P_0 + \int_0^t b_s ds + \int_0^t a_s^\top dW_s$$

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where

i) (P, b, a) is a predictable process with values in $[-L, L]^{d+2}$,

ii) $|a^\top a| \geq L^{-2}$ $dt \times d\mathbb{P}$-a.e. on $\{|P| \vee d_{\mathcal{Z}}(Z, Z_\phi) \leq r\}$ for a given $r \in (0, L^{-3}/4)$.

Assumptions

☐ **Definition :** For $\tau \in \mathcal{T}$ and $p \in \mathbb{N}^*$:

$$\theta(\tau) := \inf\{t \geq \tau \ : \ P_t \leq 0\} \ , \ \theta^\pi(\tau) := \inf\{t \geq \tau \ : \ t \in \pi, \ P_t \leq 0\}$$

$$\Phi^p(\tau) := \mathbb{E}_\tau \left[(\theta(\tau) - \tau)^p\right]^{\frac{1}{p}} \ , \ \Phi^{p,\pi}(\tau) := \mathbb{E}_\tau \left[(\theta^\pi(\tau) - \tau)^p\right]^{\frac{1}{p}} .$$

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▷ **Assumption (L) (uniform bound on expectation of exit time).** $\Phi^{1,\pi}(\tau) + \Phi^1(\tau) \leq L$ *for all* $\tau \in \mathcal{T}$.

Main results

□ **Thm 1 (continuous monitoring) :** Fix $0 < \tilde{r} < r$. Then, $\exists$ $c = c(r, r-\tilde{r}, L, d) > 0$ s.t.

$$\Phi^1(\tau) \le c(|P_\tau| + |\pi|^{\frac{1}{2}})$$

$\forall\ \tau \in \mathcal{T}$ s.t. $Z_\tau \in \bar{\mathcal{O}} \cap N_{\tilde{r}}$. If $\tau \in \mathcal{T}^\pi$, it holds if $Z_\tau \in \bar{\mathcal{O}} \cap N_r$, and c does not depend on $r - \tilde{r}$.

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□ **Thm 2 (discrete monitoring) :** Take $\pi \neq \mathbb{R}_+$ and $|\pi| \leq \varepsilon$ (given explicitly), fix $0 < \tilde{r} < r$. Then, $\exists$ $c = c(r, r - \tilde{r}, L, d) > 0$ s.t.

$$\Phi^{1,\pi}(\tau) \leq c\left(|P_\tau| + |\pi|^{\frac{1}{2}}\right)$$

$\forall\ \tau \in \mathcal{T}$ s.t. $Z_\tau \in \bar{\mathcal{O}} \cap N_{\tilde{r}}$. If $\tau \in \mathcal{T}^\pi$, it holds if $Z_\tau \in \bar{\mathcal{O}} \cap N_r$, and c does not depend on $r - \tilde{r}$.

Scheme of proof

▷ **Freidlin type inequalities on exit times moments :**

$$(\Phi^p(\tau))^p \leq c_p \Phi^1(\tau) \wedge L^{(p)} \quad \text{and} \quad (\Phi^{p,\pi}(\tau))^p \leq c_p \Phi^{1,\pi}(\tau) \wedge L^{(p)}$$

where $c_p := p! L^{p-1} =: pL^{(p-1)}$.

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where $c_p := p! L^{p-1} =: p L^{(p-1)}$.

Indeed

$$\begin{aligned}
\frac{(\Phi^{p+1,\pi}(\tau))^{p+1}}{p+1} &= \int_\tau^\infty \mathbb{E}_\tau \left[(\theta^\pi(\tau) - t)^p \mathbf{1}_{\theta^\pi(\tau)>t} \right] dt \\
&= \int_\tau^\infty \mathbb{E}_\tau \left[\mathbb{E}_{t\vee\tau}[(\theta^\pi(t\vee\tau) - t\vee\tau)^p] \mathbf{1}_{\theta^\pi(\tau)>t} \right] dt \\
&\leq \int_\tau^\infty L^{(p)} \mathbb{E}_\tau \left[\mathbf{1}_{\theta^\pi(\tau)>t} \right] dt \leq L^{(p)} \Phi^{1,\pi}(\tau).
\end{aligned}$$

Scheme of proof

$\triangleright$ **An a-priori control in terms of the probability of strictly sub-harmonic paths :** $\exists\, c > 0$ s.t.

$$\Phi^1(\tau) \leq c\mathbb{P}_\tau\left[\mathcal{A}^\tau\right], \text{ for all } \tau \in \mathcal{T}.$$

where

$$(\mathcal{A}^\tau)^c := \{2Pb + a^\top a \geq L^{-2}/2 \text{ on } [\tau, \theta(\tau)]\}.$$

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Indeed, on $(\mathcal{A}^\tau)^c$,

$$\begin{aligned}
\frac{\theta(\tau)-\tau}{2L^2} &\leq \int_\tau^{\theta(\tau)} (2P_s b_s + a_s^\top a_s) ds \\
&= |P_{\theta(\tau)}|^2 - |P_\tau|^2 - \int_\tau^{\theta(\tau)} 2P_s a_s^\top dW_s \\
&\leq -\int_\tau^{\theta(\tau)} 2P_s a_s^\top dW_s.
\end{aligned}$$

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▷ **Conclude with a control on** $\mathbb{P}_\tau\left[\mathcal{A}^\tau\right]$: $\forall\ \iota > 0\ \exists\ \eta(\iota) > 0$ s.t.

$$\mathbb{P}_\tau[\mathcal{A}^\tau] \leq \eta(\iota)(P_\tau + |\pi|) + \iota\ \Phi^1(\tau)$$

for all $\tau \in \mathcal{T}^\pi$ such that $P_\tau \in [0, r]$. (use the non-characteristic boundary condition to exit with high probability before leaving the neighborhood of the boundary)

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Recall that :

$$\Phi^1(\tau) \leq c\mathbb{P}_\tau[\mathcal{A}^\tau]$$

And use : passing to $\tau \in \mathcal{T}$ to $\tau \in \mathcal{T}^\pi$ costs $|\pi|^{\frac{1}{2}}$.

Scheme of proof

▷ **Extension to $\Phi^{1,\pi}(\tau)$:** Picture on the board...

Application to exit time approximation

Exit time approximation

□ **Problem :** X with θ on $\mathbb{R}_+$ approximated by $\bar{X}$ with $\bar{\theta}$ on $\bar{\pi}$ (discrete grid). Set $P := d_{\mathcal{Z}}(X)$ and $\bar{P} := d_{\mathcal{Z}}(\bar{X})$.

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□ **Thm :** If $\exists\ \rho > 0$ s.t.

$$\mathbb{E}\left[|P_\vartheta - \bar{P}_\vartheta|^2\right] \leq \rho|\bar{\pi}| \quad \forall\ \vartheta \in \mathcal{T} \text{ s.t. } \vartheta \leq \theta.$$

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Then, $\exists\ c = c(r, L, d, \rho) > 0$ and $\varepsilon = \varepsilon(r, L) > 0$ s.t.

$$\mathbb{E}\left[|\theta - \bar{\theta}|\right] \le \mathbb{E}\left[\mathbb{E}_\vartheta\left[|\theta - \bar{\theta}|\right]^2\right]^{\frac{1}{2}} \le c\,|\bar{\pi}|^{\frac{1}{2}}, \quad \text{if } |\bar{\pi}| \le \varepsilon,$$

where $\vartheta := \phi_\theta^+ \wedge \bar{\theta}$.

Exit time approximation

□ **Lemma :** Fix $\vartheta \in \mathcal{T}$. Assume $\exists\ \rho > 0$ and $0 < c_2 < c_1$ s.t.

$$\mathbb{P}[\vartheta \geq T] \leq \rho e^{-c_1 T} \quad \text{and} \quad \sup_{t \in [0,T]} \mathbb{E}\left[d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right)^4\right]^{\frac{1}{2}} \leq \rho T |\bar{\pi}| e^{\frac{1}{2} c_2 T}.$$

Then, $\exists\ c = c(\rho, d, c_1, c_2) > 0$ s.t.

$$\mathbb{E}\left[d_{\mathcal{Z}}\left(X_\vartheta, \bar{X}_\vartheta\right)^2\right] \leq c\ |\bar{\pi}|^{\frac{1}{2}}.$$

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