No marginal arbitrage of the second kind for high production regimes

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - Ensae

SPA 2010

Joint work with Adrien Nguyen Huu, CEREMADE and FiME-EDF

Motivation

- Introduce production capacities in portfolio management

Motivation

- Introduce production capacities in portfolio management
- Study closure properties of the set of hedgeable claims

Motivation

- Introduce production capacities in portfolio management
- Study closure properties of the set of hedgeable claims
- Deduce a dual formulation for hedgeable claims

Motivation

- Introduce production capacities in portfolio management
- Study closure properties of the set of hedgeable claims
- Deduce a dual formulation for hedgeable claims
- Deduce existence result in optimal portfolio management

Previous works and novelty

- Kabanov and Kijima (2006) : continuous time model but retrictions (complete Brownian market) .

Previous works and novelty

- Kabanov and Kijima (2006) : continuous time model but retrictions (complete Brownian market) .
- Bouchard and Pham (2005) : discrete time, general with possible frictions.

Previous works and novelty

- Kabanov and Kijima (2006) : continuous time model but retrictions (complete Brownian market) .
- Bouchard and Pham (2005) : discrete time, general with possible frictions.

In these models the production level depends only on the inventory in production capacities.

Previous works and novelty

- Kabanov and Kijima (2006) : continuous time model but retrictions (complete Brownian market) .
- Bouchard and Pham (2005) : discrete time, general with possible frictions.

In these models the production level depends only on the inventory in production capacities.

Novelties :

- Allow the production level to be "fully" controlled at each time period.

Previous works and novelty

- Kabanov and Kijima (2006) : continuous time model but retrictions (complete Brownian market) .
- Bouchard and Pham (2005) : discrete time, general with possible frictions.

In these models the production level depends only on the inventory in production capacities.

Novelties :

- Allow the production level to be “fully” controlled at each time period.
- Allow for arbitrage du to production (but not marginally for high production regimes).

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\ldots,T}$.

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\dots,T}$.
- Bid-ask matrix : $(\pi_t^{ij},\ i,j \le d)_{t=0,\dots,T}$

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\ldots,T}$.
- Bid-ask matrix : $(\pi_t^{ij},\ i,j \leq d)_{t=0,\ldots,T}$
  - $\pi_t^{ij} \in L^0(\mathcal{F}_t) =$ number of units of asset i needed to obtain one unit of asset j.

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\ldots,T}$.
- Bid-ask matrix : $(\pi_t^{ij},\ i,j \leq d)_{t=0,\ldots,T}$
  - $\pi_t^{ij} \in L^0(\mathcal{F}_t) =$ number of units of asset i needed to obtain one unit of asset j.
  - $\pi_t^{ij}\pi_t^{jk} \geq \pi_t^{ik} > 0$, $\pi_t^{ii} = 1$

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\dots,T}$.
- Bid-ask matrix : $(\pi_t^{ij},\ i,j \leq d)_{t=0,\dots,T}$
  - $\pi_t^{ij} \in L^0(\mathcal{F}_t)$ = number of units of asset i needed to obtain one unit of asset j.
  - $\pi_t^{ij}\pi_t^{jk} \geq \pi_t^{ik} > 0$, $\pi_t^{ii} = 1$
- Financial position : $V \in L^0(\mathbb{R}^d)$ with V^i = number of units of asset i in the portfolio.

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\ldots,T}$.
- Bid-ask matrix : $(\pi_t^{ij},\ i,j \leq d)_{t=0,\ldots,T}$
  - $\pi_t^{ij} \in L^0(\mathcal{F}_t)$ = number of units of asset i needed to obtain one unit of asset j.
  - $\pi_t^{ij}\pi_t^{jk} \geq \pi_t^{ik} > 0$, $\pi_t^{ii} = 1$
- Financial position : $V \in L^0(\mathbb{R}^d)$ with V^i = number of units of asset i in the portfolio.
- Solvency cone process : $K := (K_t)_{t \leq T}$ with

$$K_t(\omega) := \{x \in \mathbb{R}^d : \exists\ a^{ij} \geq 0 \text{ s.t. } x^i + \sum_{j \neq i} a^{ji} - a^{ij}\pi_t^{ij}(\omega) \geq 0\ \forall\ i\}$$

a^{ji} = number of units of i obtained against units of j.

Model description - The financial market

- Probability space : $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t\}_{t=0,\dots,T}$.
- Bid-ask matrix : $(\pi_t^{ij},\ i,j \leq d)_{t=0,\dots,T}$
  - $\pi_t^{ij} \in L^0(\mathcal{F}_t)$ = number of units of asset i needed to obtain one unit of asset j.
  - $\pi_t^{ij}\pi_t^{jk} \geq \pi_t^{ik} > 0$, $\pi_t^{ii} = 1$
- Financial position : $V \in L^0(\mathbb{R}^d)$ with V^i = number of units of asset i in the portfolio.
- Solvency cone process : $K := (K_t)_{t \leq T}$ with

$$K_t(\omega) := \{x \in \mathbb{R}^d : \exists\ a^{ij} \geq 0 \text{ s.t. } x^i + \sum_{j \neq i} a^{ji} - a^{ij}\pi_t^{ij}(\omega) \geq 0\ \forall\ i\}$$

a^{ji} = number of units of i obtained against units of j.

- Set of self-financed exchanges at time t

$$-K_t(\omega) := \{x \in \mathbb{R}^d : \exists\ a^{ij} \geq 0 \text{ s.t. } x^i \leq \sum_{j \neq i} a^{ji} - a^{ij}\pi_t^{ij}(\omega)\ \forall\ i\}$$

Model description - The production

- Family of random maps $(R_t)_{t \leq T}$

Model description - The production

- Family of random maps $(R_t)_{t \leq T}$
- $R_{t+1} : \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \mapsto R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$

Model description - The production

- Family of random maps $(R_t)_{t \leq T}$
- $R_{t+1} : \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \mapsto R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$
    - $\beta^i =$ number of units of asset i consumed and send into the production system at time t

Model description - The production

- Family of random maps $(R_t)_{t \leq T}$
- $R_{t+1} : \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \mapsto R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$
  - β^i = number of units of asset i consumed and send into the production system at time t
  - $R^j_{t+1}(\beta)$ = number of units of asset j obtained at time $t+1$ from the production.

Model description - The production

- Family of random maps $(R_t)_{t\leq T}$
- $R_{t+1} : \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \mapsto R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$
  - $\beta^i =$ number of units of asset i consumed and send into the production system at time t
  - $R^j_{t+1}(\beta) =$ number of units of asset j obtained at time $t+1$ from the production.

Example :

- Asset 1=cash, asset 2 =future on electricity (for a given maturity), asset 3= coal.

Model description - The production

- Family of random maps $(R_t)_{t \leq T}$
- $R_{t+1} : \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \mapsto R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$
  - β^i = number of units of asset i consumed and send into the production system at time t
  - $R^j_{t+1}(\beta)$ = number of units of asset j obtained at time $t+1$ from the production.

Example :

- Asset 1=cash, asset 2 =future on electricity (for a given maturity), asset 3= coal.
- $R_{t+1}(\beta)$ depends only on β^3

Model description - The production

- Family of random maps $(R_t)_{t \leq T}$
- $R_{t+1} : \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \mapsto R_{t+1}(\beta) \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$
  - β^i = number of units of asset i consumed and send into the production system at time t
  - $R^j_{t+1}(\beta)$ = number of units of asset j obtained at time $t+1$ from the production.

Example :

- Asset 1=cash, asset 2 =future on electricity (for a given maturity), asset 3= coal.
- $R_{t+1}(\beta)$ depends only on β^3
- $R^i_{t+1}(\beta) = 0$ for $i \neq 1$

Model description - Wealth process

- Strategies

$$(\xi, \beta) \in \mathcal{A}_0 := L^0((-K) \times \mathbb{R}^d_+, \mathbb{F}),$$

i.e. s.t. $(\xi_t, \beta_t) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$ for all $0 \leq t \leq T$

Model description - Wealth process

- Strategies

$$(\xi, \beta) \in \mathcal{A}_0 := L^0((-K) \times \mathbb{R}^d_+, \mathbb{F}),$$

i.e. s.t. $(\xi_t, \beta_t) \in L^0((-K_t) \times \mathbb{R}^d_+, \mathcal{F}_t)$ for all $0 \leq t \leq T$

- Set of portfolio holdings that are attainable at time T by trading from time t with a zero initial holding

$$A^R_t(T) := \left\{ \sum_{s=t}^{T} \xi_s - \beta_s + R_s(\beta_{s-1})1_{s \geq t+1} \, , \; (\xi, \beta) \in \mathcal{A}_0 \right\} \, .$$

No-arbitrage of second kind conditions - Without production

There exists various notions in models with fictions. We focus on one :
$NA2 : (\zeta + A_t^0(T)) \cap L^0(K_T, \mathcal{F}) \neq \{0\} \Rightarrow \zeta \in L^0(K_t, \mathcal{F})$, $\forall\ \zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $t \leq T$.

No-arbitrage of second kind conditions - Without production

There exists various notions in models with fictions. We focus on one :
$NA2 : (\zeta + A_t^0(T)) \cap L^0(K_T, \mathcal{F}) \neq \{0\} \Rightarrow \zeta \in L^0(K_t, \mathcal{F})$,
$\forall\ \zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $t \leq T$.

We assume that the efficient friction assumption holds :

$$EF\ \ :\ \ \pi_t^{ij}\pi_t^{ji} > 1\ \ \forall\ i \neq j,\ \ t \leq T.$$

No-arbitrage of second kind conditions - Without production

There exists various notions in models with fictions. We focus on one :
$NA2 : (\zeta + A_t^0(T)) \cap L^0(K_T, \mathcal{F}) \neq \{0\} \Rightarrow \zeta \in L^0(K_t, \mathcal{F})$,
$\forall\ \zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $t \leq T$.

We assume that the efficient friction assumption holds :

$$EF \ :\ \pi_t^{ij}\pi_t^{ji} > 1 \ \forall\, i \neq j,\ t \leq T.$$

Equivalently under EF :
$NA2 : \zeta \in L^0(K_{t+1}, \mathcal{F}) \Rightarrow \zeta \in L^0(K_t, \mathcal{F}), \forall\ \zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t),\ t < T$.

See Kabanov, Stricker, Rasonyi, Schachermayer for other NA conditions.

No-arbitrage conditions - Strictly consistent price systems

- $\mathcal{M}_t^T$: set of martingales Z on $[t, T]$ s.t. $Z_s \in \mathrm{int} K_s^*$ for $t \leq s \leq T$
  - $K_s^*(\omega) = \left\{ z \in \mathbb{R}^d : 0 \leq z^j \leq z^i \pi_s^{ij}(\omega),\ i, j \leq d \right\}$.

No-arbitrage conditions - Strictly consistent price systems

- $\mathcal{M}_t^T$: set of martingales Z on $[t, T]$ s.t. $Z_s \in \mathrm{int} K_s^*$ for $t \leq s \leq T$
  - $K_s^*(\omega) = \left\{ z \in \mathbb{R}^d : 0 \leq z^j \leq z^i \pi_s^{ij}(\omega), \ i, j \leq d \right\}$.
  - $Z_s \in \mathrm{int} K_s^* : Z_s^j / Z_s^i < \pi_s^{ij}$ (strictly consistent price system - fictitious price).

No-arbitrage conditions - Strictly consistent price systems

- $\mathcal{M}_t^T$: set of martingales Z on $[t, T]$ s.t. $Z_s \in \mathrm{int} K_s^*$ for $t \leq s \leq T$
  - $K_s^*(\omega) = \left\{ z \in \mathbb{R}^d : 0 \leq z^j \leq z^i \pi_s^{ij}(\omega), \; i, j \leq d \right\}$.
  - $Z_s \in \mathrm{int} K_s^* : Z_s^j / Z_s^i < \pi_s^{ij}$ (strictly consistent price system - fictitious price).
  - If there exists fictitious prices that are martingales, then the no-arbitrage condition holds.

No-arbitrage conditions - Strictly consistent price systems

- $\mathcal{M}_t^T$: set of martingales Z on $[t, T]$ s.t. $Z_s \in \mathrm{int} K_s^*$ for $t \leq s \leq T$
  - $K_s^*(\omega) = \left\{ z \in \mathbb{R}^d : 0 \leq z^j \leq z^i \pi_s^{ij}(\omega),\ i,j \leq d \right\}$.
  - $Z_s \in \mathrm{int} K_s^*$: $Z_s^j / Z_s^i < \pi_s^{ij}$ (strictly consistent price system - fictitious price).
  - If there exists fictitious prices that are martingales, then the no-arbitrage condition holds.
- **Theorem :** Under EF : $NA2 \Leftrightarrow PCE^0$: $\exists\, Z \in \mathcal{M}_t^T$ s.t. $Z_t = X$, $\forall\, t \leq T$ and $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$.

No-arbitrage conditions - Strictly consistent price systems

- $\mathcal{M}_t^T$: set of martingales Z on $[t, T]$ s.t. $Z_s \in \mathrm{int} K_s^*$ for $t \leq s \leq T$
  - $K_s^*(\omega) = \left\{ z \in \mathbb{R}^d : 0 \leq z^j \leq z^i \pi_s^{ij}(\omega),\ i,j \leq d \right\}$.
  - $Z_s \in \mathrm{int} K_s^*$: $Z_s^j / Z_s^i < \pi_s^{ij}$ (strictly consistent price system - fictitious price).
  - If there exists fictitious prices that are martingales, then the no-arbitrage condition holds.
- **Theorem :** Under *EF* : *NA*2 $\Leftrightarrow$ *PCE*^{0} : $\exists\, Z \in \mathcal{M}_t^T$ s.t. $Z_t = X$, $\forall\, t \leq T$ and $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$.

- Plays the same role as martingale measures :

$$G \in \zeta + A_t^0(T) \text{ "iff" } \mathbb{E}\left[Z_T(G - \zeta) | \mathcal{F}_t\right] \leq 0 \ \forall Z \in \mathcal{M}_t^T.$$

No-arbitrage conditions - In the model with production

We work under a version of the $NA2$ condition.

No-arbitrage conditions - In the model with production

We work under a version of the $NA2$ condition.

NA of the second kind for L ($NA2^L$) : $\forall\ (\zeta,\beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$

(i) $\zeta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \zeta \in K_t$,

No-arbitrage conditions - In the model with production

We work under a version of the $NA2$ condition.

NA of the second kind for L ($NA2^L$) : $\forall\ (\zeta, \beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$

(i) $\zeta \quad -\beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \zeta \in K_t$,

No-arbitrage conditions - In the model with production

We work under a version of the $NA2$ condition.

NA of the second kind for L ($NA2^L$) : $\forall\ (\zeta, \beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$

(i) $\zeta \quad -\beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \zeta \in K_t$,

(ii) $-\beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \beta = 0$.

No-arbitrage conditions - In the model with production

We work under a version of the $NA2$ condition.

NA of the second kind for L ($NA2^L$) : $\forall\,(\zeta,\beta)\in L^0(\mathbb{R}^d\times\mathbb{R}^d_+,\mathcal{F}_t)$

(i) $\zeta\ \ -\beta+L_{t+1}\beta\in L^0(K_{t+1},\mathcal{F}_{t+1})\Rightarrow\zeta\in K_t$,

(ii) $-\beta+L_{t+1}\beta\in L^0(K_{t+1},\mathcal{F}_{t+1})\Rightarrow\beta=0$.

$NMA2$: $\exists\,(c,L)\in L^\infty(\mathbb{R}^d\times\mathbb{M}^d,\mathbb{F})$ s.t. $NA2^L$ and

$$c_{t+1}+L_{t+1}\beta-R_{t+1}(\beta)\in L^0(K_{t+1},\mathcal{F}_{t+1})\ \forall\,\beta\in L^0(\mathbb{R}^d_+,\mathcal{F}_t),\ t<T.$$

No-arbitrage conditions - In the model with production

We work under a version of the $NA2$ condition.

NA of the second kind for L ($NA2^L$) : $\forall\,(\zeta,\beta)\in L^0(\mathbb{R}^d\times\mathbb{R}^d_+,\mathcal{F}_t)$

(i) $\zeta\ \ -\beta+L_{t+1}\beta\in L^0(K_{t+1},\mathcal{F}_{t+1})\Rightarrow\zeta\in K_t$,

(ii) $-\beta+L_{t+1}\beta\in L^0(K_{t+1},\mathcal{F}_{t+1})\Rightarrow\beta=0$.

$NMA2$: $\exists\,(c,L)\in L^\infty(\mathbb{R}^d\times\mathbb{M}^d,\mathbb{F})$ s.t. $NA2^L$ and

$c_{t+1}+L_{t+1}\beta-R_{t+1}(\beta)\in L^0(K_{t+1},\mathcal{F}_{t+1})\ \forall\,\beta\in L^0(\mathbb{R}^d_+,\mathcal{F}_t),\ t<T.$

Think at L such that

$$\lim_{\eta\to\infty}R_t(\eta\beta)/\eta=L_t\beta.$$

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?
  - Consume β at s to produce

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?
  - Consume β at s to produce
  - Receive $L_{s+1}\beta$ at $s+1$

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?
  - Consume β at s to produce
  - Receive $L_{s+1}\beta$ at $s+1$

Price at s is zero (need 0 to build it)

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?
  - Consume β at s to produce
  - Receive $L_{s+1}\beta$ at $s+1$

Price at s is zero (need 0 to build it)

- Price of a position : $L_{s+1}\beta - \beta$ in the price system Z ?

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?
  - Consume β at s to produce
  - Receive $L_{s+1}\beta$ at $s+1$

Price at s is zero (need 0 to build it)

- Price of a position : $L_{s+1}\beta - \beta$ in the price system Z ?

$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right]$

Main results - No-arbitrage characterization

- Price of a position $L_{s+1}\beta - \beta$?
  - Consume β at s to produce
  - Receive $L_{s+1}\beta$ at $s+1$

Price at s is zero (need 0 to build it)

- Price of a position : $L_{s+1}\beta - \beta$ in the price system Z ?

$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0$

if Z is striclty more favorable than π.

Main results - No-arbitrage characterization

Definitions :

- $\mathcal{L}_t^T$: set of martingales Z s.t. for $t \leq s < T$

$$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0 \; \forall \; \beta_s \in L^0(\mathbb{R}^d_+ \setminus \{0\}, \mathcal{F}_s).$$

Main results - No-arbitrage characterization

Definitions :

- $\mathcal{L}_t^T$: set of martingales Z s.t. for $t \leq s < T$

$$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0 \; \forall \; \beta_s \in L^0(\mathbb{R}_+^d \setminus \{0\}, \mathcal{F}_s).$$

- PCE^L : $\exists \; Z \in \mathcal{M}_t^T \cap \mathcal{L}_t^T$ s.t. $Z_t = X$, $\forall \; t \leq T$, $X \in L^1(\mathrm{int}K_t^*, \mathcal{F}_t)$.

Main results - No-arbitrage characterization

Definitions :

- $\mathcal{L}_t^T$: set of martingales Z s.t. for $t \leq s < T$

$$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0 \ \forall \ \beta_s \in L^0(\mathbb{R}_+^d \setminus \{0\}, \mathcal{F}_s).$$

- PCE^L : $\exists \ Z \in \mathcal{M}_t^T \cap \mathcal{L}_t^T$ s.t. $Z_t = X$, $\forall \ t \leq T$, $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$.

Theorem : $NA2^L \Leftrightarrow PCE^L$.

Main results - No-arbitrage characterization

Definitions :

- $\mathcal{L}_t^T$: set of martingales Z s.t. for $t \leq s < T$

$$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0 \;\forall\; \beta_s \in L^0(\mathbb{R}^d_+ \setminus \{0\}, \mathcal{F}_s).$$

- PCE^L : $\exists\; Z \in \mathcal{M}_t^T \cap \mathcal{L}_t^T$ s.t. $Z_t = X$, $\forall\; t \leq T$, $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$.

Theorem : $NA2^L \Leftrightarrow PCE^L$.

Remark : Argument splitted on the different time intervals $[t, t+1]$ and not globally on $[0, T]$ like for the other no-arbitrage conditions.

Main results - No-arbitrage characterization

Definitions :

- $\mathcal{L}_t^T$: set of martingales Z s.t. for $t \leq s < T$

  $$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0 \;\forall\; \beta_s \in L^0(\mathbb{R}_+^d \setminus \{0\}, \mathcal{F}_s).$$

- PCE^L : $\exists\; Z \in \mathcal{M}_t^T \cap \mathcal{L}_t^T$ s.t. $Z_t = X$, $\forall\; t \leq T$, $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$.

Theorem : $NA2^L \Leftrightarrow PCE^L$.

Remark : Argument splitted on the different time intervals $[t, t+1]$ and not globally on $[0, T]$ like for the other no-arbitrage conditions. No need to prove a closure property first : construct the Z ω by ω.

Main results - No-arbitrage characterization

Definitions :

- $\mathcal{L}_t^T$: set of martingales Z s.t. for $t \leq s < T$

$$\mathbb{E}\left[Z'_{s+1}(L_{s+1}\beta_s - \beta_s)|\mathcal{F}_s\right] < 0 \;\forall\; \beta_s \in L^0(\mathbb{R}_+^d \setminus \{0\}, \mathcal{F}_s).$$

- PCE^L : $\exists\; Z \in \mathcal{M}_t^T \cap \mathcal{L}_t^T$ s.t. $Z_t = X$, $\forall\; t \leq T$, $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$.

Theorem : $NA2^L \Leftrightarrow PCE^L$.

Remark : Argument splitted on the different time intervals $[t, t+1]$ and not globally on $[0, T]$ like for the other no-arbitrage conditions. No need to prove a closure property first : construct the Z ω by ω. Allows for dynamic programming type arguments also general pasting is not possible (as for density processes of martingale measures).

Main results - Closure property

Definition : $A \subset L^0(\mathbb{R}^d, \mathcal{F})$ is Fatou-closed if for any sequence $(g^n)_{n\geq 1} \subset A$ which converges $\mathbb{P}-$ a.s. to some $g \in L^0(\mathbb{R}^d, \mathcal{F})$ and such that, for some $\kappa \in \mathbb{R}^d$, $g^n + \kappa \in K_T$ for all $n \geq 1$, then $g \in A$.

Main results - Closure property

Definition : $A \subset L^0(\mathbb{R}^d, \mathcal{F})$ is Fatou-closed if for any sequence $(g^n)_{n\geq 1} \subset A$ which converges $\mathbb{P}-$ a.s. to some $g \in L^0(\mathbb{R}^d, \mathcal{F})$ and such that, for some $\kappa \in \mathbb{R}^d$, $g^n + \kappa \in K_T$ for all $n \geq 1$, then $g \in A$.

Theorem : $A_0^L(T)$ is Fatou-closed under $NA2^L$.

Main results - Closure property

Definition : $A \subset L^0(\mathbb{R}^d, \mathcal{F})$ is Fatou-closed if for any sequence $(g^n)_{n \geq 1} \subset A$ which converges $\mathbb{P}-$a.s. to some $g \in L^0(\mathbb{R}^d, \mathcal{F})$ and such that, for some $\kappa \in \mathbb{R}^d$, $g^n + \kappa \in K_T$ for all $n \geq 1$, then $g \in A$.

Theorem : $A_0^L(T)$ is Fatou-closed under $NA2^L$. The same holds for $A_0^R(T)$ under $NMA2$ and USC.
where

$$USC: \limsup_{\beta \in \mathbb{R}^d_+, \beta \to \beta^0} R_t(\beta) - R_t(\beta^0) \in -K_t \text{ for all } \beta^0 \in \mathbb{R}^d_+,$$

where the limsup is taken component by component.

Applications - Super-hedging

Proposition : Assume that $NMA2$ holds and that $A_0^R(T)$. Let $V \in L^0(\mathbb{R}^d, \mathcal{F})$ be such that $V + \kappa \in L^0(K_T, \mathcal{F})$ for some $\kappa \in \mathbb{R}^d$. Then, the following are equivalent :

(i) $V \in A_0^R(T)$,

(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T$.

Applications - Super-hedging

Proposition : Assume that $NMA2$ holds and that $A_0^R(T)$. Let $V \in L^0(\mathbb{R}^d, \mathcal{F})$ be such that $V + \kappa \in L^0(K_T, \mathcal{F})$ for some $\kappa \in \mathbb{R}^d$.
Then, the following are equivalent :
(i) $V \in A_0^R(T)$,
(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T$.
If moreover

$$\lim_{\eta \to \infty} R_t(\eta\beta)/\eta = L_t\beta \quad \text{for all } \beta \in \mathbb{R}_+^d,\ t \leq T\ ,$$

then the following are equivalent :
(i) $V \in A_0^R(T)$,
(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T \cap \mathcal{L}_0^T$.

Applications - Super-hedging

Proposition : Assume that $NMA2$ holds and that $A_0^R(T)$. Let $V \in L^0(\mathbb{R}^d, \mathcal{F})$ be such that $V + \kappa \in L^0(K_T, \mathcal{F})$ for some $\kappa \in \mathbb{R}^d$.
Then, the following are equivalent :
(i) $V \in A_0^R(T)$,
(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T$.
If moreover

$$\lim_{\eta \to \infty} R_t(\eta\beta)/\eta = L_t\beta \quad \text{for all } \beta \in \mathbb{R}_+^d,\ t \leq T\ ,$$

then the following are equivalent :
(i) $V \in A_0^R(T)$,
(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ for all $Z \in \mathcal{M}_0^T \cap \mathcal{L}_0^T$.

If $R = L$ then $\alpha^R = 0$.

Applications - Optimal management

Setting

- U : $\mathbb{P}-$ a.s.-upper semi-continuous and concave random map from $\mathbb{R}^d$ to $[-\infty, 1]$

Applications - Optimal management

Setting

- $U : \mathbb{P}-$ a.s.-upper semi-continuous and concave random map from $\mathbb{R}^d$ to $[-\infty, 1]$
- $U(V) = -\infty$ on $\{V \notin K_T\}$

Applications - Optimal management

Setting

- $U : \mathbb{P}-$ a.s.-upper semi-continuous and concave random map from $\mathbb{R}^d$ to $[-\infty, 1]$
- $U(V) = -\infty$ on $\{V \notin K_T\}$
- $\mathcal{U}(x_0) := \left\{V \in A_0^R(T) \ : \ \mathbb{E}\left[|U(x_0 + V)|\right] < \infty\right\} \neq \emptyset.$

Applications - Optimal management

Setting

- $U : \mathbb{P}-$ a.s.-upper semi-continuous and concave random map from $\mathbb{R}^d$ to $[-\infty, 1]$
- $U(V) = -\infty$ on $\{V \notin K_T\}$
- $\mathcal{U}(x_0) := \left\{V \in A_0^R(T) \ : \ \mathbb{E}\left[|U(x_0+V)|\right] < \infty\right\} \neq \emptyset.$

Corollary : If $NMA2$, USC hold and $A_0^R(T)$ is convex, then $\exists$ $V(x_0) \in A_0^R(T)$ such that

$$\mathbb{E}\left[U(x_0 + V(x_0))\right] = \sup_{V \in \mathcal{U}(x_0)} \mathbb{E}\left[U(x_0+V)\right] .$$