Almost sure hedging with price impact

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Joint works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich), C. Zhou (NUS) and Y. Zou (ex Paris-Dauphine)

Motivation

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- BS and local (stochastic) vol models :
  - Are useful because they provide a clear hedging rule
  - Disregard frictions because do not work at high frequency
  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless. We are not working at the level of the order book.

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  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless. We are not working at the level of the order book.
- However :
  - Do not take price impact and illiquidity into account
  - Problematic when large positions (possibly shared) or illiquid underlying (may run after the delta)
- Question : Can we built a model which
  - Takes price impact and illiquidity into account
  - Leads to a clear hedging and pricing rule
  - Does not have embedded hidden transaction costs (otherwise the super-hedging price would be degenerate)

Some references

□ Many works on hedging with illiquidity or impact : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98, Cetin, Jarrow and Protter 04, Bank and Baum 04, Liu and Yong 05, Cetin, Soner and Touzi 09, Millot and Abergel 11, Frey and Polte 11, Almgren and Li 13, Guéant and Pu 13,...

□ Illiquidity + impact + perfect hedging : Loeper 14/16 (verification arguments).

□ Past and ongoing related works by D. Becherer and T. Bilarev.

Impact rule and continuous time trading dynamics

Impact rule

□ Basic rule (only permanent for the moment) : an order of δ units moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}), \quad \text{[permanent impact]}$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2} \quad \text{[liquidity cost]}.$$

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□ We just model the curve around $\delta = 0$. This should be understood for a "small" order δ. Would obtain the same with

$$X_{t-} \longrightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

and costs

$$\int_0^\delta (X_{t-} + F(X_{t-}, \iota))d\iota$$

if $\partial_\delta F(x,0) = f(x)$, $\partial^2_{\delta x} F(x,0) = f'(x)$ and $F(x,0) = \partial^2_{\delta\delta} F(x,0) = 0$.

Trading signal and discrete trading dynamics

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□ We assume that the stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades (can add a drift - or resilience effect, see Becherer and Bilarev 18).

□ The corresponding dynamics are

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n} \mathbf{1}_{\{t_i^n \le t < t_{i+1}^n\}} + Y_T \mathbf{1}_{\{t=T\}} \ , \ \delta_{t_i^n}^n = Y_{t_i^n}^n - Y_{t_{i-1}^n}^n$$

$$X^n = X_0 + \int_0^{\cdot} \sigma(X_s^n) dW_s + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n),$$

$$V^n = V_0 + \int_0^{\cdot} Y_{s-}^n dX_s^n + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta_{t_i^n}^n)^2 f(X_{t_i^n -}^n),$$

where

$$V^n := \text{ cash part } + Y^n X^n = \text{``portfolio value''}.$$

□ Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$
\begin{aligned}
Y &= Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s \\
X &= X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \underbrace{\int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} a_s (\sigma f')(X_s) ds}_{(Y_{t_i^n}^n - Y_{t_{i-1}^n}^n) f(X_{t_i^n -}^n)} \\
V &= V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \underbrace{\int_0^{\cdot} a_s^2 f(X_s) ds}_{(Y_{t_i^n}^n - Y_{t_{i-1}^n}^n)^2 f(X_{t_i^n -}^n)} ,
\end{aligned}
$$

at a speed $\sqrt{n}$.

Hedging problem(s)

1. Uncovered options.
2. Covered options.
3. Covered options in a generalized model.

The case of uncovered options

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☐ Super-hedging price :

$$\mathrm{v} = \inf\{\text{initial cash} \;:\; \exists(a,b) \text{ s.t. } V_T - Y_T X_T \geq g_0(X_T) \;\text{ and }\; Y_T = g_1(X_T)\}.$$

(Recall that $V = \text{cash} + YX$)

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(Recall that $V = \text{cash} + YX$)

□ Issue : needs to jump to a certain initial or final delta !

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta\nu(d\delta, ds)$$

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□ Jumps δ_i at time τ_i is passed on $[\tau_i, \tau_i + \varepsilon]$ at a rate δ_i/ε.

□ The limit dynamics when $\varepsilon \to 0$ is

$$\begin{aligned} X &= X_{0-} + \int_0^{\cdot} \sigma(X_s)dW_s + \int_0^{\cdot} f(X_s)dY^c_s + \int_0^{\cdot} a_s \sigma f'(X_s)ds \\ &\quad + \int_0^{\cdot}\int \Delta\mathrm{x}(X_{s-},\delta)\nu(d\delta,ds) \\ V &= V_{0-} + \int_0^{\cdot} Y_s dX^c_s + \frac{1}{2}\int_0^{\cdot} a^2_s f(X_s)ds \\ &\quad + \int_0^{\cdot}\int \left(Y_{s-}\Delta\mathrm{x}(X_{s-},\delta) + \mathfrak{I}(X_{s-},\delta)\right)\nu(d\delta,ds). \end{aligned}$$

in which

$$\Delta\mathrm{x}(x,\delta) + x = \mathrm{x}(x,\delta) := x + \int_0^{\delta} f(\mathrm{x}(x,s))ds$$

$$\text{and}\quad \mathfrak{I}(x,\delta) := \int_0^{\delta} s f(\mathrm{x}(x,s))ds.$$

Dynamic programming

- Intuition (starting from $Y_0 = 0$) :

$$v \geq \mathrm{v}(0, x, 0)$$

"if and only if"

$$V_\theta \geq \mathrm{v}(\theta, X_\theta, Y_\theta) \text{ for some } (a, b, \nu)$$

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☐ Use the fact that : $\mathrm{v}(t, x) := \mathrm{v}(t, x, 0) = \mathrm{v}(t, \mathrm{x}(x, y), y) - \mathfrak{I}(x, y)$. Because round trips are possible at zero cost !

□ Modified geometric dynamic programming :

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- Can then apply standard stochastic target technics.

Pricing equation

- A quasi-linear pde

$$0 = -\partial_t \mathrm{v} - \hat{\mu}(\cdot, \hat{y})\partial_x[\mathrm{v} + \mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2 \partial^2_{xx}[\mathrm{v} + \mathfrak{I}]$$

where

$$\hat{\mu}(\cdot, y) := \frac{1}{2}[\partial^2_{xx}\mathrm{x}\sigma^2](\mathrm{x}(\cdot, y), -y) \text{ and } \hat{\sigma}(\cdot, y) := (\sigma\partial_x\mathrm{x})(\mathrm{x}(\cdot, y), -y),$$

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- Terminal condition

$$G(x) := \inf\left\{y\mathrm{x}(x, y) + g_0(\mathrm{x}(x, y)) - \mathfrak{I}(x, y) : \; y = g_1(\mathrm{x}(x, y))\right\}.$$

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☐ Perfect hedging : Smooth solution under additional conditions, leading to perfect hedging by following $Y = \hat{y}(\cdot, X)$.

☐ For $f \equiv 0$: recovers the usual delta hedging $Y = \partial_x \mathrm{v}(\cdot, X)$.

The case of covered options

B., G. Loeper, and Y. Zou. Hedging of covered options with linear market impact and gamma constraint. *SIAM Journal on Control and Optimization*, 55(5), 3319-3348, 2017.

□ The trader receives at inception a chosen (by the trader) quantity of cash and stocks, and delivers at maturity a quantity of cash and stocks (chosen by the trader). The initial number of stocks equates the required delta to start the hedging, the quantity of stocks delivered at maturity equates the delta at maturity.

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□ Super-hedging price :

$$\mathrm{v}(t,x) := \inf\{v = c + yx \ : \ c, y, (a,b) \text{ s.t. } V_T \geq g(X_T)\}.$$

(Recall that $V = \text{cash} + YX$)

Hedging and pricing - informal derivation

Let us assume that we use the delta-hedging rule :

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and applying Itô's Lemma to $Y - \partial_x \mathrm{v}(\cdot, X)$ leads to

$$\gamma^a := \frac{a}{\sigma + fa} = \partial^2_{xx} \mathrm{v}(\cdot, X) \in \mathbb{R} \setminus \{1/f\}$$

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By definition of γ^a and a little bit of algebra :

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})} \partial^2_{xx} \mathrm{v} \right] (\cdot, X) = 0.$$

The pricing pde should be

$$-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v} = 0 \quad \text{on } [0,T) \times \mathbb{R},$$

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Singular pde :
- Can find smooth solutions s.t. $1 > f\partial^2_{xx}\mathrm{v}$, cf. below.
- In general, needs to take care of $1 \neq f\partial^2_{xx}\mathrm{v}$
- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \le \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f\partial^2_{xx}\mathrm{v} > 1$ does not make sense.

Hedging with a gamma contraint

□ By a change of variable, we write the dynamics in the form :

$$dY = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt \;\text{ and }\; dX = \sigma^a(X)dW + \mu_X^{a,b}(X)dt.$$

□ We now define v with respect to the gamma constraint

$$\gamma^a(X) \leq \bar{\gamma}(X)$$

with

$$f\bar{\gamma} < 1 - \varepsilon, \;\; \varepsilon > 0.$$

Pricing pde :

$$\min\left\{-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v}\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{v}\right\}=0 \quad \text{on } [0,T)\times\mathbb{R}.$$

Propagation of the gamma contraint at the boundary :

$$\mathrm{v}(T-,\cdot)=\hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

$$\min\left\{\varphi - g\ ,\ \bar{\gamma}-\partial^2_{xx}\varphi\right\}=0.$$

See Soner and Touzi 00, and Cheridito, Soner and Touzi 05.

Super-solution property

Use a weak formulation approach and results on small time behavior of double stochastic integrals, see Soner and Touzi 00 and Cheridito, Soner and Touzi 05.

It is based on the Geometric DPP (Soner and Touzi) :
if

$$V_0 > \mathrm{v}(0, X_0)$$

then we can find (a, b, Y_0) such that

$$V_\theta \geq \mathrm{v}(\theta, X_\theta)$$

for any stopping time θ with values in $[0, T]$.

Sub-solution property

□ Main difficulty : can not establish the reverse Geometric DPP, i.e.

If (a, b, Y_0) are such that

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- at θ we have a position Y_θ that may not match with the position $\hat{Y}_\theta$ associated to $\mathrm{v}(\theta, X_\theta)$. Can not jump from Y_θ to $\hat{Y}_\theta$...
- can neither go smoothly to it as it will move X because of the impact, and therefore $\hat{Y}$ (sort of fixed point problem), compare with Cheridito, Soner, and Touzi 05.

The smoothing approach

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The smoothing approach

In place, we use a smoothing/verification approach initiated by B. and Nutz 13 (inspired from Jensen's and Krylov's ideas).

1. Using the concavity of the PDE, create a sequence w_δ^ι of smooth super-solutions that converges to a viscosity solution w.

2. By verification $\mathrm{w}_\delta^\iota \geq \mathrm{v}$.

3. By PDE comparison $\mathrm{v} \geq \mathrm{w} \underbrace{\longleftarrow}_{\delta,\iota\to 0} \mathrm{w}_\delta^\iota \geq \mathrm{v}$.

Conclusion : v is the (unique) viscosity solution.

Adding a resilience effect

- Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^\cdot \sigma(X_s^n) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n) - \int_0^\cdot \rho R_s^n ds.$$

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- The continuous time dynamics becomes

$$X = X_0 + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (a_s(\sigma f')(X_s) - \rho R_s)ds$$

$$R = R_0 + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (a_s(\sigma f')(X_s) - \rho R_s)ds$$

$$V = V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2}\int_0^\cdot a_s^2 f(X_s)ds.$$

Extension : abstract impact model

B., G. Loeper, M. Soner and C. Zhou. Second order stochastic target problems with generalized market impact.

***Arxiv :1806.08533*, 2018.**

□ A general impact function :

$$X = x + \int_t^{\cdot} \mu(s, X_s, \gamma_s, b_s) ds + \int_t^{\cdot} \sigma(s, X_s, \gamma_s) dW_s$$
$$Y = y + \int_t^{\cdot} b_s ds + \int_t^{\cdot} \gamma_s dX_s$$
$$V = v + \int_t^{\cdot} F(s, X_s, \gamma_s) ds + \int_t^{\cdot} Y_s dX_s$$

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This allows to model : permanent impact, immediate partial relaxation of the impact, modified liquidity cost, and can easily add resilience.

□ Relaxation of the gamma constraint. Can be as close as one wants to the singularity :

$$\min\{-\partial_t \mathrm{v} - \bar{F}(\cdot, \partial^2_{xx}\mathrm{v}) \ , \ \bar{\gamma} - \partial^2_{xx}\mathrm{v}\} = 0 \text{ on } [0, T) \times \mathbb{R},$$

where

$$\bar{F}(t, x, z) := \frac{1}{2}\sigma(t, x, z)^2 z - F(t, x, z)$$

and

$$\{\bar{F} < \infty\} = \{F < \infty\} = \{(t, x, z) : z < \bar{\gamma}(t, x)\}.$$

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□ For this, we need a-priori estimates : If u with $\partial^2_{xx} u < \bar{\gamma}$ solves the PDE, then $w := \bar{F}(\cdot, \partial^2_{xx} u)$ solves

$$\partial_t w + \partial_z \bar{F}(\cdot, \partial^2_{xx} u) \partial^2_{xx} w = \frac{\partial_t \bar{F}(\cdot, \partial^2_{xx} u)}{\bar{F}(\cdot, \partial^2_{xx} u)} w.$$

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Then,

$$w(t,x) = \mathbb{E}[w(T, \bar{X}^{t,x}_T) e^{-\int_t^T (\partial_t \bar{F}(\cdot, \partial^2_{xx} u)/\bar{F}(\cdot, \partial^2_{xx} u))(s, \bar{X}^{t,x}_s) ds}]$$

where $\bar{X} = x + \int_t^{\cdot} (2\partial_z \bar{F}(\cdot, \partial^2_{xx} u)(s, \bar{X}_s))^{\frac{1}{2}} dW_s$.

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where $\bar{X} = x + \int_t^{\cdot} (2\partial_z \bar{F}(\cdot, \partial^2_{xx} u)(s, \bar{X}_s))^{\frac{1}{2}} dW_s$.

Provides a uniform bound if $\partial^2_{xx} u(T, \cdot) \le \bar{\gamma} - \iota$ with $\iota > 0$.

Expansion around 0 impact

☐ Scaling :

$$X = x + \int_t^{\cdot} \mu(s, X_s, \epsilon\gamma_s, b_s)ds + \int_t^{\cdot} \sigma(s, X_s, \epsilon\gamma_s)dW_s$$

$$V = v + \int_t^{\cdot} \epsilon^{-1} F(s, X_s, \epsilon\gamma_s)ds + \int_t^{\cdot} Y_s dX_s$$

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□ Expansion performed around the solution v^0 of $(\partial_z \bar{F}(\cdot, 0) =: \partial_z \bar{F}_0)$

$$\partial_t \mathrm{v}^0 + \partial_z \bar{F}_0 \partial^2_{xx} \mathrm{v}^0 = 0 \text{ on } [0, T) \times \mathbb{R} \text{ and } \mathrm{v}^0(T, \cdot) = \hat{g} \text{ on } \mathbb{R}.$$

□ Proposition :

$$\begin{aligned}
\mathrm{v}^{\epsilon}(0,x) =& \mathrm{v}^{0}(0,x) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T [\partial^2_{zz}\bar{F}_0|\partial^2_{xx}\mathrm{v}^0|^2](s,\tilde{X}^0_s)ds\right] + o(\epsilon) \\
=& \mathrm{v}^{0}(0,x) + \frac{\epsilon}{2}\,\mathbb{E}\left[\partial_x\hat{g}(T,\tilde{X}^0_T)\tilde{Y}_T\right] + o(\epsilon)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{X}^z &= x + \int_t^{\cdot} (2\partial_z\bar{F}(\cdot, z\partial^2_{xx}\mathrm{v}^0))^{\frac{1}{2}}(s,\tilde{X}^z_s)dW_s, \\
\tilde{Y} &= \frac{1}{\sqrt{2}}\int_t^{\cdot} \frac{\partial_x\partial_z\bar{F}_0(s,\tilde{X}^0_s)\tilde{Y}_s + \partial^2_{zz}\bar{F}_0\partial^2_{xx}\mathrm{v}^0(s,\tilde{X}^0_s)}{\sqrt{\partial_z\bar{F}_0(s,\tilde{X}^0_s)}}dW_s.
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□ **Proposition :**

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=& \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\,\mathbb{E}\left[\partial_x \hat{g}(T,\tilde{X}^0_T)\tilde{Y}_T\right] + o(\epsilon)
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\tilde{Y} &= \frac{1}{\sqrt{2}}\int_t^{\cdot} \frac{\partial_x\partial_z\bar{F}_0(s,\tilde{X}^0_s)\tilde{Y}_s + \partial^2_{zz}\bar{F}_0\partial^2_{xx}\mathrm{v}^0(s,\tilde{X}^0_s)}{\sqrt{\partial_z\bar{F}_0(s,\tilde{X}^0_s)}}dW_s.
\end{aligned}$$

□ The leading order term allows for super-hedging with $\mathbf{L}^{\infty}$-error controlled by ϵ^2.

Dual formulation

- In the concave case :

$$\mathrm{v}(t,x) = \sup_{\mathfrak{s}} \mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$
$$= \sup_{\mathfrak{s}} \mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$

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$$X^{t,x,\mathfrak{s}} = x + \int_t^{\cdot} \mathfrak{s}_s dW_s.$$

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in which

$$X^{t,x,\mathfrak{s}} = x + \int_t^{\cdot} \mathfrak{s}_s dW_s.$$

□ In the previous model :

$$\bar{F}^*(t,x,s^2) = \frac{1}{2}\frac{(s-\sigma(t,x))^2}{f(x)}, \quad \text{for } s \geq 0.$$

Open problems

No constraint at all on the gamma ?

Dual formulation in a non-Markovian framework ?

Generic completeness ?

Existence/stability of FBSDE with impact ?

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Thank you !

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Details on the smoothing approach

□ Assume $f, \sigma, \bar{\gamma}$ are constant, and $\hat{g}$ bounded and uniformly continuous, for simplicity.

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Step 1. Using Perron's method + comparison, construct a (bounded) viscosity solution w^ι of

$$\min\left\{-\partial_t\varphi - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\varphi)}\partial^2_{xx}\varphi \;,\; \bar{\gamma} - \partial^2_{xx}\varphi\right\} = 0 \quad \text{on } [0, T) \times \mathbb{R},$$

with terminal condition

$$\mathrm{w}^\iota(T, \cdot) = \hat{g} + \iota \quad \text{on } \mathbb{R}$$

with $\iota > 0$.

Step 2. Up to replacing w^{ι} by an approximating sequence of quasi-concave functions (by quadratic inf-convolution), we can assume that w^{ι} is quasi-concave

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$$\min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \; , \; \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\right\} \geq 0 \text{ a.e.}$$

with $\partial^2_{xx}\mathrm{w}^\iota$ the density of the absolute continuous part of the second order derivative measure

See Jensen 88.

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with $\partial^2_{xx}\mathrm{w}^\iota$ the density of the absolute continuous part of the second order derivative measure, and

$$\mathrm{w}^\iota(T,\cdot) \geq \hat{g} + \iota/2.$$

See Jensen 88.

Step 3. Consider a (non-negative) smooth kernel ψ with support $[-1, 0] \times [-1, 1]$, take a window size $\delta > 0$, and set

$$\psi_\delta = \delta^{-1}\psi(\delta^{-1}\cdot)$$

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$$0\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta$$

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The pde operator is concave

$$0\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta$$

$$\le \min\left\{-\partial_t\mathrm{w}^\iota\star\psi_\delta-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta)}\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta,\bar{\gamma}-\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta\right\}$$

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The pde operator is concave decreasing, and $\partial^2_{xx}\mathrm{w}^\iota_\delta \le \partial^2_{xx}\mathrm{w}^\iota \star \psi_\delta$ (by quasi-concavity),

$$\begin{aligned}
0 &\le \min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \ , \ \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta \\
&\le \min\left\{-\partial_t \mathrm{w}^\iota\star\psi_\delta - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta)}\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta, \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta\right\} \\
&\le \min\left\{-\partial_t \mathrm{w}^\iota_\delta - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota_\delta)}\partial^2_{xx}\mathrm{w}^\iota_\delta \ , \ \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota_\delta\right\}
\end{aligned}$$

Step 3. Consider a (non-negative) smooth kernel ψ with support $[-1,0]\times[-1,1]$, take a window size $\delta>0$, and set

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$$\begin{aligned}0&\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta\\&\le \min\left\{-\partial_t\mathrm{w}^\iota\star\psi_\delta-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta)}\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta,\bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta\right\}\\&\le \min\left\{-\partial_t\mathrm{w}^\iota_\delta-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota_\delta)}\partial^2_{xx}\mathrm{w}^\iota_\delta\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota_\delta\right\}\end{aligned}$$

while, for δ small with respect to ι,

$$\mathrm{w}^\iota_\delta(T,\cdot)\ge\hat g.$$

Step 4. We have produced a smooth function satisfying

$$\min\left\{-\partial_t \mathrm{w}_\delta^\iota - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial_{xx}^2 \mathrm{w}_\delta^\iota)}\partial_{xx}^2 \mathrm{w}_\delta^\iota \ , \ \bar{\gamma} - \partial_{xx}^2 \mathrm{w}_\delta^\iota\right\} \geq 0$$

and

$$\mathrm{w}_\delta^\iota(T, \cdot) \geq \hat{g}.$$

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$$\mathrm{w}_\delta^\iota(T,\cdot) \geq \hat{g}.$$

Taking

$$V = \mathrm{w}_\delta^\iota(\cdot, X) \quad \text{and} \quad Y = \partial_x \mathrm{w}_\delta^\iota(\cdot, X),$$

we obtain

$$V_T \geq \hat{g}(X_T) \geq g(X_T).$$

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we obtain

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This implies that $\mathrm{v} \leq \mathrm{w}_\delta^\iota \to \mathrm{w}^\iota$, as $\delta \to 0$.

Step 5. Since w^ι is solution of

$$\min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{w}^\iota\right\}=0$$

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$\mathrm{w}^\iota \to \mathrm{w}$ where w is solution of

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with

$$\mathrm{w}(T,\cdot)=\hat{g}.$$

It satisfies $\mathrm{w} \leftarrow \mathrm{w}^\iota \geq \mathrm{v}$.

Step 5. Since w^ι is solution of

$$\min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \;,\; \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\right\} = 0$$

with

$$\mathrm{w}^\iota(T,\cdot) = \hat{g} + \iota,$$

$\mathrm{w}^\iota \to \mathrm{w}$ where w is solution of

$$\min\left\{-\partial_t \mathrm{w} - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w})}\partial^2_{xx}\mathrm{w} \;,\; \bar{\gamma} - \partial^2_{xx}\mathrm{w}\right\} = 0$$

with

$$\mathrm{w}(T,\cdot) = \hat{g}.$$

It satisfies $\mathrm{w} \leftarrow \mathrm{w}^\iota \geq \mathrm{v}$.

Step 6. But v is a super-solution of the same equation : $\mathrm{w} \leq \mathrm{v}$ by comparison, and therefore $\mathrm{w} = \mathrm{v}$ by the above.

To sum up :

$$\underbrace{\mathrm{v}}_{\text{super-solution}} \geq \underbrace{\mathrm{w}}_{\text{solution}} \underbrace{\longleftarrow}_{\delta,\iota\to 0} \underbrace{\mathrm{w}_\delta^\iota}_{\text{super-hedging}} \geq \mathrm{v}$$