Almost sure hedging with price impact

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Joint works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich), C. Zhou (NUS) and Y. Zou (ex Paris-Dauphine)
Motivation
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- BS and local (stochastic) vol models:
  - Are useful because they provide a clear hedging rule
  - Disregard frictions because they do not work at high frequency
  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless. We are not working at the level of the order book.

- However:
  - Do not take price impact and illiquidity into account
  - Problematic when large positions (possibly shared) or illiquid underlying (may run after the delta)

Question: Can we build a model which:
  - Takes price impact and illiquidity into account
  - Leads to a clear hedging and pricing rule
  - Does not have embedded hidden transaction costs (otherwise the super-hedging price would be degenerate)
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Some references

- Many works on hedging with illiquidity or impact: Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98, Cetin, Jarrow and Protter 04, Bank and Baum 04, Liu and Yong 05, Cetin, Soner and Touzi 09, Millot and Abergel 11, Frey and Polte 11, Almgren and Li 13, Guéant and Pu 13,...

- Illiquidity + impact + perfect hedging: Loeper 14/16 (verification arguments).

- Past and ongoing related works by D. Becherer and T. Bilarev.
Impact rule and continuous time trading dynamics
Impact rule

Basic rule (only permanent for the moment): an order of $\delta$ units moves the price by

$$X_{t-} \rightarrow X_t = X_{t-} + \delta f(X_{t-}), \quad \text{[permanent impact]}$$

and costs

$$\delta X_{t-} + \frac{1}{2} \delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2} \quad \text{[liquidity cost]}.$$
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- We just model the curve around $\delta = 0$. This should be understood for a “small” order $\delta$. Would obtain the same with

$$X_{t-} \rightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

and costs

$$\int_0^\delta (X_{t-} + F(X_{t-}, \iota))d\iota$$

if $\partial_\delta F(x, 0) = f(x), \quad \partial^2_{\delta x} F(x, 0) = f'(x)$ and $F(x, 0) = \partial^2_{\delta\delta} F(x, 0) = 0.$
Trading signal and discrete trading dynamics

A trading signal is an Itô process controlled by \((a, b)\):

\[
Y = Y_0 + \int_0^t b_s ds + \int_0^t a_s dW_s.
\]
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- Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit: continuous time is an approximation, it should be consistent with discrete (hedging) limits.

Trade at times \(t_n^i = iT/n\) (for simplicity) the quantities:

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\delta_n t_n^i = Y_{t_n^i} - Y_{t_n^{i-1}}.
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- A trading signal is an Itô process controlled by \((a, b)\):

\[ Y = Y_0 + \int_0^t b_s \, ds + \int_0^t a_s \, dW_s. \]

- Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit: continuous time is an approximation, it should be consistent with discrete (hedging) limits.

- Trade at times \(t^n_i = iT/n\) (for simplicity) the quantities \(\delta^n_{t^n_i} = Y_{t^n_i} - Y_{t^n_{i-1}}\).

- We assume that the stock price evolves according to

\[ X = X_{t^n_i} + \int_{t^n_i}^t \sigma(X_s) \, dW_s \]

between two trades (can add a drift - or resilience effect, see Becherer and Bilarev 18).
The corresponding dynamics are

\[ Y^n_t := \sum_{i=0}^{n-1} Y^n_{t^i} \mathbf{1}_{\{t^i \leq t < t^{i+1}\}} + Y^n_T \mathbf{1}_{\{t=T\}} , \quad \delta^n_{t^i} = Y^n_{t^i} - Y^n_{t^{i-1}} \]

\[ X^n = X_0 + \int_0^\cdot \sigma(X^n_s) dW_s + \sum_{i=1}^n \mathbf{1}_{[t^i, T]} \delta^n_{t^i} f(X^n_{t^i-}) , \]

\[ V^n = V_0 + \int_0^\cdot Y^n_{s-} dX^n_s + \sum_{i=1}^n \mathbf{1}_{[t^i, T]} \frac{1}{2} (\delta^n_{t^i})^2 f(X^n_{t^i-}) , \]

where

\[ V^n := \text{cash part} + Y^n X^n = \text{“portfolio value”}. \]
Passing to the limit $n \to \infty$, it converges in $\mathbb{S}_2$ to

$$
\begin{align*}
Y &= Y_0 + \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s \\
X &= X_0 + \int_0^\cdot \sigma(X_s) dW_s + \int_0^\cdot f(X_s) dY_s + \int_0^\cdot a_s(\sigma f')(X_s) ds \\
V &= V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s) ds,
\end{align*}
$$

\((Y_{t_i}^n - Y_{t_{i-1}}^n) f(X_{t_i}^n)\)

at a speed $\sqrt{n}$. 

Hedging problem(s)

1. Uncovered options.

2. Covered options.

3. Covered options in a generalized model.
The case of uncovered options


- Premium paid in cash and one delivers exactly the amount of cash and stocks prescribed by the payoff.
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- Super-hedging price :

\[ v = \inf \{ \text{initial cash} : \exists (a, b) \text{ s.t. } V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T) \}. \]

(Recall that \( V = \text{cash} + Y X \))
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  (Recall that \( V = \text{cash} + YX \))

- Issue: needs to jump to a certain initial or final delta!
Adding jumps and splitting of large orders

We now consider a trading signal of the form

\[ Y = Y_0 - \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s + \int_0^\cdot \delta \nu(d\delta, ds) \]
We now consider a trading signal of the form

\[ Y = Y_0 + \int_0^\cdot b_s \, ds + \int_0^\cdot a_s \, dW_s + \int_0^\cdot \delta \nu(\,d\delta, ds) \]

\[ \text{Jumps } \delta_i \text{ at time } \tau_i \text{ is passed on } [\tau_i, \tau_i + \varepsilon] \text{ at a rate } \delta_i / \varepsilon. \]
The limit dynamics when $\varepsilon \to 0$ is

$$
X = X_0^- + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY^c_s + \int_0^\cdot a_s\sigma f'(X_s)ds
$$

$$
+ \int_0^\cdot \int_0^\cdot \Delta x(X^-_s, \delta)\nu(d\delta, ds)
$$

$$
V = V_0^- + \int_0^\cdot Y_s dX^c_s + \frac{1}{2} \int_0^\cdot a^2_s f(X_s)ds
$$

$$
+ \int_0^\cdot \int_0^\cdot (Y_s^-\Delta x(X^-_s, \delta) + \mathcal{I}(X^-_s, \delta))\nu(d\delta, ds).
$$

in which

$$
\Delta x(x, \delta) + x = x(x, \delta) := x + \int_0^\delta f(x(x, s))ds
$$

and $\mathcal{I}(x, \delta) := \int_0^\delta sf(x(x, s))ds$. 

Dynamic programming

- Intuition (starting from $Y_0 = 0$):

  \[ v \geq v(0, x, 0) \]
  
  "if and only if"

  \[ V_\theta \geq v(\theta, X_\theta, Y_\theta) \text{ for some } (a, b, \nu) \]
Dynamic programming

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- Can not use it directly: because the control $b$ appears (only) linearly in the dynamics, this leads to a singular equation (actually leaving on a submanifold).
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- Use the fact that: $v(t, x) := v(t, x, 0) = v(t, x(x, y), y) - I(x, y)$. Because round trips are possible at zero cost!
Modified geometric dynamic programming:

\[ v \geq v(0, x) \]

"if and only if"

\[ V_\theta \geq v(\theta, x(X_\theta, -Y_\theta)) + J(x(X_\theta, -Y_\theta), Y_\theta) \text{ for some } (a, b, \nu) \]
Modified geometric dynamic programming:

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"if and only if"

\[ V_{\theta} \geq v(\theta, x(x_{\theta}, -y_{\theta})) + I(x(x_{\theta}, -y_{\theta}), y_{\theta}) \] for some \((a, b, \nu)\)

Can then apply standard stochastic target technics.
Pricing equation

A quasi-linear pde

\[ 0 = -\partial_t v - \mu(\cdot, \hat{y}) \partial_x [v + J] - \frac{1}{2} \sigma(\cdot, \hat{y})^2 \partial_{xx} [v + J] \]

where

\[ \mu(\cdot, y) := \frac{1}{2} [\partial_{xx} x \sigma^2](x(\cdot, y), -y) \quad \text{and} \quad \sigma(\cdot, y) := (\sigma \partial_x x)(x(\cdot, y), -y), \]

and

\[ \hat{y}(t, x) := x^{-1}(x, x + f(x) \partial_x v(t, x)). \]
Pricing equation

- A quasi-linear pde

\[ 0 = -\partial_t v - \hat{\mu}(\cdot, \hat{y}) \partial_x [v + \mathcal{I}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx}^2 [v + \mathcal{I}] \]

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\[ \hat{y}(t, x) := x^{-1}(x, x + f(x) \partial_x v(t, x)). \]

- Terminal condition

\[ G(x) := \inf \{ yx(x, y) + g_0(x(x, y)) - \mathcal{I}(x, y) : y = g_1(x(x, y)) \}. \]
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\[ 0 = -\partial_t v - \hat{\mu}(\cdot, \hat{y}) \partial_x [v + \mathcal{I}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx}^2 [v + \mathcal{I}] \]

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- Perfect hedging: Smooth solution under additional conditions, leading to perfect hedging by following \( Y = \hat{y}(\cdot, X). \)
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□ Perfect hedging: Smooth solution under additional conditions, leading to perfect hedging by following \( Y = \hat{y}(\cdot, X) \).

□ For \( f \equiv 0 \): recovers the usual delta hedging \( Y = \partial_x v(\cdot, X) \).
The case of covered options


- The trader receives at inception a chosen (by the trader) quantity of cash and stocks, and delivers at maturity a quantity of cash and stocks (chosen by the trader). The initial number of stocks equates the required delta to start the hedging, the quantity of stocks delivered at maturity equates the delta at maturity.
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- Super-hedging price:

\[ v(t, x) := \inf \{ v = c + yx : c, y, (a, b) \text{ s.t. } V_T \geq g(X_T) \}. \]

(Recall that \( V = \text{cash} + YX \))
Hedging and pricing - informal derivation

Let us assume that we use the delta-hedging rule:

\[ V = v(\cdot, X), \quad Y = \partial_x v(\cdot, X). \]
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Then, equating the \( dt \) terms implies

\[ \frac{1}{2} a^2 f(X) = \partial_t v(\cdot, X) + \frac{1}{2} (\sigma a)^2 \partial_{xx}^2 v(\cdot, X), \]
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and applying Itô’s Lemma to \( Y - \partial_x v(\cdot, X) \) leads to

\[ \gamma^a := \frac{a}{\sigma + fa} = \partial_{xx} v(\cdot, X) \in \mathbb{R} \setminus \{1/f\} \]
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By definition of \( \gamma^a \) and a little bit of algebra:

\[ \left[ -\partial_t v - \frac{1}{2} \frac{\sigma^2}{1 - f \partial_{xx}^2 v} \partial_{xx}^2 v \right] (\cdot, X) = 0. \]
The pricing pde should be

\[-\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 v)} \partial_{xx}^2 v = 0 \quad \text{on } [0, T) \times \mathbb{R},\]

\[v(T-, \cdot) = g \quad \text{on } \mathbb{R}.\]
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**Singular pde :**
- Can find smooth solutions s.t. 1 > f \partial_{xx}^2 v, cf. below.
- In general, needs to take care of 1 \neq f \partial_{xx}^2 v
- One possibility: add a gamma constraint \( \partial_{xx}^2 v \leq \tilde{\gamma} \) with \( f \tilde{\gamma} < 1 \).
- A constraint of the form \( f \partial_{xx}^2 v > 1 \) does not make sense.
By a change of variable, we write the dynamics in the form:

\[ dY = \gamma^a(X)dX + \mu^{a,b}(X)dt \quad \text{and} \quad dX = \sigma^a(X)dW + \mu^{a,b}_X(X)dt. \]

We now define \( v \) with respect to the gamma constraint

\[ \gamma^a(X) \leq \bar{\gamma}(X) \]

with

\[ f\bar{\gamma} < 1 - \varepsilon, \quad \varepsilon > 0. \]
Pricing pde :

\[
\min \left\{ -\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} v)} \partial_{xx}^2 v , \ \tilde{\gamma} - \partial_{xx}^2 v \right\} = 0 \ \text{on} \ [0, T) \times \mathbb{R}.
\]

Propagation of the gamma contraint at the boundary :

\[
v(T-, \cdot) = \hat{g} \ \text{on} \ \mathbb{R}
\]

with \( \hat{g} \) the smallest (viscosity) super-solution of

\[
\min \left\{ \varphi - g , \ \tilde{\gamma} - \partial_{xx}^2 \varphi \right\} = 0.
\]

See Soner and Touzi 00, and Cheridito, Soner and Touzi 05.
Super-solution property

Use a weak formulation approach and results on small time behavior of double stochastic integrals, see Soner and Touzi 00 and Cheridito, Soner and Touzi 05.

It is based on the Geometric DPP (Soner and Touzi) : if

$$ V_0 > v(0, X_0) $$

then we can find \((a, b, Y_0)\) such that

$$ V_\theta \geq v(\theta, X_\theta) $$

for any stopping time \(\theta\) with values in \([0, T]\).
Sub-solution property

Main difficulty: cannot establish the reverse Geometric DPP, i.e.

If \((a, b, Y_0)\) are such that

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Problem:
- At \(\theta\) we have a position \(Y_\theta\) that may not match with the position \(\hat{Y}_\theta\) associated to \(v(\theta, X_\theta)\). Cannot jump from \(Y_\theta\) to \(\hat{Y}_\theta\)…
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- Problem:
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  - can neither go smoothly to it as it will move \(X\) because of the impact, and therefore \(\hat{Y}\) (sort of fixed point problem), compare with Cheridito, Soner, and Touzi 05.
The smoothing approach

In place, we use a smoothing/verification approach initiated by B. and Nutz 13 (inspired from Jensen’s and Krylov’s ideas).
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Conclusion: $v$ is the (unique) viscosity solution.
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3. By PDE comparison $v \geq w$
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1. Using the concavity of the PDE, create a sequence $w^\delta_\iota$ of smooth super-solutions that converges to a viscosity solution $w$.

2. By verification $w^\delta_\iota \geq v$.

3. By PDE comparison $v \geq w \leftarrow_{\delta, \iota \to 0} w^\delta_\iota \geq v$.

Conclusion: $v$ is the (unique) viscosity solution.
Adding a resilience effect

Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^\cdot \sigma(X^n_s) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^n 1_{[t^n_i, T]} \delta^n_{t^n_i} f(X^n_{t^n_i^-}) - \int_0^\cdot \rho R^n_s ds.$$
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Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^\cdot \sigma(X^n_s) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^n 1_{[t^n_i, T]} \delta^n_{t^n_i} f(X^n_{t^n_i}) - \int_0^\cdot \rho R^n_s ds.$$

The continuous time dynamics becomes

$$X = X_0 + \int_0^\cdot \sigma(X_s) dW_s + \int_0^\cdot f(X_s) dY_s + \int_0^\cdot (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$R = R_0 + \int_0^\cdot f(X_s) dY_s + \int_0^\cdot (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$V = V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2} \int_0^\cdot \dot{a}_s^2 f(X_s) ds.$$
Extension: abstract impact model


A general impact function:

\[ X = x + \int_t^\cdot \mu(s, X_s, \gamma_s, b_s)ds + \int_t^\cdot \sigma(s, X_s, \gamma_s)dW_s \]

\[ Y = y + \int_t^\cdot b_s ds + \int_t^\cdot \gamma_s dX_s \]

\[ V = v + \int_t^\cdot F(s, X_s, \gamma_s)ds + \int_t^\cdot Y_s dX_s \]

This allows to model: permanent impact, immediate partial relaxation of the impact, modified liquidity cost, and can easily add resilience.
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This allows to model: permanent impact, immediate partial relaxation of the impact, modified liquidity cost, and can easily add resilience.

Relaxation of the gamma constraint. Can be as close as one wants to the singularity:

\[
\min \{-\partial_t v - \bar{F}(\cdot, \partial_{xx}^2 v) \, , \, \bar{\gamma} - \partial_{xx}^2 v\} = 0 \text{ on } [0, T) \times \mathbb{R},
\]

where

\[
\bar{F}(t, x, z) := \frac{1}{2} \sigma(t, x, z)^2 z - F(t, x, z)
\]

and

\[
\{ \bar{F} < \infty \} = \{ F < \infty \} = \{(t, x, z) : z < \bar{\gamma}(t, x)\}. \]
Scheme of proof

- Super-solution is obtained as before.
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- Sub-solution: Lack of concavity $\Rightarrow$ the smoothing procedure does not apply. In place, use of parabolic regularity for fully non-linear equations to provide smooth (approximate) solutions - in place of smoothing. It requires more smoothness of the coefficients than in the previous situation...
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For this, we need a-priori estimates: If $u$ with $\partial_{xx}^2 u < \tilde{\gamma}$ solves the PDE, then $w := \tilde{F}(\cdot, \partial_{xx}^2 u)$ solves

$$\partial_t w + \partial_z \tilde{F}(\cdot, \partial_{xx}^2 u) \partial_{xx}^2 w = \frac{\partial_t \tilde{F}(\cdot, \partial_{xx}^2 u)}{\tilde{F}(\cdot, \partial_{xx}^2 u)} w.$$
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\]

Then,

\[
w(t, x) = \mathbb{E}[w(T, \bar{X}_{T_t}^{x,T})e^{-\int_t^T (\partial_t \bar{F}(\cdot, \partial_{xx}^2 u)/\bar{F}(\cdot, \partial_{xx}^2 u))(s, \bar{X}_s^{t,x})ds}]
\]

where \( \bar{X} = x + \int_t^T (2\partial_z \bar{F}(\cdot, \partial_{xx}^2 u)(s, \bar{X}_s))^{\frac{1}{2}} dW_s. \)
Scheme of proof

- Super-solution is obtained as before.

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where $\bar{X} = x + \int_t^T (2\partial_z \tilde{F}(\cdot, \partial_{xx}^2 u)(s, \bar{X}_s))^{\frac{1}{2}} dW_s$.

Provides a uniform bound if $\partial_{xx}^2 u(T, \cdot) \leq \tilde{\gamma} - \iota$ with $\iota > 0$. 
Expansion around 0 impact

Scaling:

\[ X = x + \int_t^\cdot \mu(s, X_s, \epsilongamma_s, b_s) ds + \int_t^\cdot \sigma(s, X_s, \epsilongamma_s) dW_s \]

\[ V = v + \int_t^\cdot \epsilon^{-1} F(s, X_s, \epsilongamma_s) ds + \int_t^\cdot Y_s dX_s \]
Expansion around 0 impact

- Scaling:

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- In the initial model, it amongs to considering \( \epsilon f \) in place of \( f \).
Expansion around 0 impact

□ Scaling:

\[ X = x + \int_t^\cdot \mu(s, X_s, \epsilon \gamma_s, b_s)ds + \int_t^\cdot \sigma(s, X_s, \epsilon \gamma_s)dW_s \]

\[ V = v + \int_t^\cdot \epsilon^{-1}F(s, X_s, \epsilon \gamma_s)ds + \int_t^\cdot Y_s dX_s \]

□ In the initial model, it amongs to considering \( \epsilon f \) in place of \( f \).

□ Expansion performed around the solution \( v^0 \) of \( (\partial_z \bar{F}(\cdot, 0) =: \partial_z \bar{F}_0) \)

\[ \partial_t v^0 + \partial_z \bar{F}_0 \partial_{xx}^2 v^0 = 0 \text{ on } [0, T) \times \mathbb{R} \text{ and } v^0(T, \cdot) = \hat{g} \text{ on } \mathbb{R}. \]
Proposition:

\[ v^\varepsilon(0, x) = v^0(0, x) + \frac{\varepsilon}{2} \mathbb{E} \left[ \int_0^T \left| \partial_{zz}^2 F_0 \partial_{xx}^2 v^0 \right|^2(s, \tilde{X}^0_s) ds \right] + o(\varepsilon) \]

\[ = v^0(0, x) + \frac{\varepsilon}{2} \mathbb{E} \left[ \partial_x \hat{g}(T, \tilde{X}^0_T) \tilde{Y}_T \right] + o(\varepsilon) \]

where

\[ \tilde{X}^z = x + \int_t^T \left( 2 \partial_z F(\cdot, z \partial_{xx}^2 v^0) \right)^{\frac{1}{2}}(s, \tilde{X}^z_s) dW_s, \]

\[ \tilde{Y} = \frac{1}{\sqrt{2}} \int_t^T \frac{\partial_x \partial_z F_0(s, \tilde{X}^0_s) \tilde{Y}_s + \partial_{zz}^2 F_0 \partial_{xx}^2 v^0(s, \tilde{X}^0_s)}{\sqrt{\partial_z F_0(s, \tilde{X}_s^0)}} dW_s. \]
\( \square \) Proposition:

\[
v^\epsilon(0, x) = v^0(0, x) + \frac{\epsilon}{2} \mathbb{E} \left[ \int_0^T \left[ \partial_{zz}^2 F_0 \partial_{xx}^2 v^0 \right]^2(s, \tilde{X}^0_s) ds \right] + o(\epsilon)
\]

\[
= v^0(0, x) + \frac{\epsilon}{2} \mathbb{E} \left[ \partial_x \hat{g}(T, \tilde{X}^0_T) \tilde{Y}_T \right] + o(\epsilon)
\]

where

\[
\tilde{X}^z = x + \int_t^T (2 \partial_z F(\cdot, z \partial_{xx}^2 v^0))^{\frac{1}{2}}(s, \tilde{X}^z_s) dW_s,
\]

\[
\tilde{Y} = \frac{1}{\sqrt{2}} \int_t^T \frac{\partial_x \partial_z F_0(s, \tilde{X}_s^0) \tilde{Y}_s + \partial_{zz}^2 F_0 \partial_{xx}^2 v^0(s, \tilde{X}_s^0)}{\sqrt{\partial_z F_0(s, \tilde{X}_s^0)}} dW_s.
\]

\( \square \) The leading order term allows for super-hedging with \( L^\infty \)-error controlled by \( \epsilon^2 \).
Dual formulation

□ In the concave case:

\[ v(t, x) = \sup_{s} \mathbb{E} \left[ \hat{g}(X_T^{t, x, s}) - \int_t^T \bar{F}^*(s, X_s^{t, x, s}, s_s^2) ds \right] \]

\[ = \sup_{s} \mathbb{E} \left[ g(X_T^{t, x, s}) - \int_t^T \bar{F}^*(s, X_s^{t, x, s}, s_s^2) ds \right] \]

in which

\[ X_T^{t, x, s} = x + \int_t^T s_s dW_s. \]
Dual formulation

□ In the concave case:

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v(t, x) = \sup_s \mathbb{E} \left[ \hat{g}(X_T^{t, x, s}) - \int_t^T \bar{F}^*(s, X_s^{t, x, s}, s^2) \, ds \right]
\]

\[
= \sup_s \mathbb{E} \left[ g(X_T^{t, x, s}) - \int_t^T \bar{F}^*(s, X_s^{t, x, s}, s^2) \, ds \right]
\]

in which

\[ X^{t, x, s} = x + \int_t^s s \, dW_s. \]

□ In the previous model:

\[
\bar{F}^*(t, x, s^2) = \frac{1}{2} \frac{(s - \sigma(t, x))^2}{f(x)}, \quad \text{for } s \geq 0.
\]
Open problems

No constraint at all on the gamma?

Dual formulation in a non-Markovian framework?

Generic completeness?

Existence/stability of FBSDE with impact?
Open problems

No constraint at all on the gamma?

Dual formulation in a non-Markovian framework?

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Thank you!
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Details on the smoothing approach
Assume $f, \sigma, \bar{\gamma}$ are constant, and $\hat{g}$ bounded and uniformly continuous, for simplicity.
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Step 1. Using Perron's method + comparison, construct a (bounded) viscosity solution $w^\iota$ of

$$
\min \left\{ -\partial_t \varphi - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} \varphi)^2} \partial_{xx} \varphi, \ \bar{\gamma} - \partial_{xx} \varphi \right\} = 0 \quad \text{on} \ [0, T) \times \mathbb{R},
$$

with terminal condition

$$
w^\iota(T, \cdot) = \hat{g} + \iota \quad \text{on} \ \mathbb{R}
$$

with $\iota > 0$. 
Step 2. Up to replacing $w^t$ by an approximating sequence of quasi-concave functions (by quadratic inf-convolution), we can assume that $w^t$ is quasi-concave.
**Step 2.** Up to replacing $w^t$ by an approximating sequence of quasi-concave functions (by quadratic inf-convolution), we can assume that $w^t$ is quasi-concave and then

$$\min \left\{ -\partial_tw^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}w^t)} \partial_{xx}^2 w^t, \bar{\gamma} - \partial_{xx}^2 w^t \right\} \geq 0 \text{ a.e.}$$

with $\partial_{xx}^2 w^t$ the density of the absolute continuous part of the second order derivative measure.

See Jensen 88.
Step 2. Up to replacing $w^t$ by an approximating sequence of quasi-concave functions (by quadratic inf-convolution), we can assume that $w^t$ is quasi-concave and then

$$\min \left\{ -\partial_t w^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w^t)} \partial_{xx}^2 w^t, \, \bar{\gamma} - \partial_{xx}^2 w^t \right\} \geq 0 \quad \text{a.e.}$$

with $\partial_{xx}^2 w^t$ the density of the absolute continuous part of the second order derivative measure, and

$$w^t(T, \cdot) \geq \hat{g} + \ell/2.$$

See Jensen 88.
Step 3. Consider a (non-negative) smooth kernel $\psi$ with support $[-1, 0] \times [-1, 1]$, take a window size $\delta > 0$, and set

$$\psi_\delta = \delta^{-1} \psi(\delta^{-1} \cdot)$$
Step 3. Consider a (non-negative) smooth kernel $\psi$ with support $[-1, 0] \times [-1, 1]$, take a window size $\delta > 0$, and set

$$
\psi_\delta = \delta^{-1} \psi(\delta^{-1} \cdot) \text{ and } w_\delta^t = w^t \ast \psi_\delta := \int w^t(t', x') \psi_\delta(t' - \cdot, x' - \cdot) dt' dx'.
$$
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$$\psi_\delta = \delta^{-1} \psi(\delta^{-1} \cdot) \quad \text{and} \quad w_\delta^t = w^t \star \psi_\delta := \int w^t(t', x') \psi_\delta(t' - \cdot, x' - \cdot) dt' dx'.$$

$$0 \leq \min \left\{ -\partial_t w^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w^t)} \partial_{xx}^2 w^t, \bar{\gamma} - \partial_{xx}^2 w^t \right\} \star \psi_\delta$$
Step 3. Consider a (non-negative) smooth kernel $\psi$ with support $[-1,0] \times [-1,1]$, take a window size $\delta > 0$, and set

$$\psi_{\delta} = \delta^{-1} \psi(\delta^{-1} \cdot)$$

and

$$w_{\delta}^t = w^t \ast \psi_{\delta} := \int w^t(t', x') \psi_{\delta}(t' - \cdot, x' - \cdot) dt' dx'.$$

The PDE operator is concave

$$0 \leq \min \left\{ -\partial_t w^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w^t)} \partial_{xx}^2 w^t, \bar{\gamma} - \partial_{xx}^2 w^t \right\} \ast \psi_{\delta}$$

$$\leq \min \left\{ -\partial_t w^t \ast \psi_{\delta} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w^t \ast \psi_{\delta})} \partial_{xx}^2 w^t \ast \psi_{\delta}, \bar{\gamma} - \partial_{xx}^2 w^t \ast \psi_{\delta} \right\}$$
Step 3. Consider a (non-negative) smooth kernel $\psi$ with support $[-1, 0] \times [-1, 1]$, take a window size $\delta > 0$, and set

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\psi_\delta = \delta^{-1} \psi(\delta^{-1} \cdot) \quad \text{and} \quad w_\delta^t = w^t \ast \psi_\delta := \int w^t(t', x') \psi_\delta(t' - \cdot, x' - \cdot) dt' dx'.
$$

The pde operator is concave decreasing, and $\partial_{xx}^2 w_\delta^t \leq \partial_{xx}^2 w^t \ast \psi_\delta$ (by quasi-concavity),

$$
0 \leq \min \left\{ -\partial_t w^t - \frac{1}{2} \sigma^2 \frac{\partial_{xx}^2 w^t}{(1 - f \partial_{xx}^2 w^t)} \partial_{xx}^2 w^t, \bar{\gamma} - \partial_{xx}^2 w^t \right\} \ast \psi_\delta
$$

$$
\leq \min \left\{ -\partial_t w^t \ast \psi_\delta - \frac{1}{2} \sigma^2 \frac{\partial_{xx}^2 w^t \ast \psi_\delta}{(1 - f \partial_{xx}^2 w^t \ast \psi_\delta)} \partial_{xx}^2 w^t \ast \psi_\delta, \bar{\gamma} - \partial_{xx}^2 w^t \ast \psi_\delta \right\}
$$

$$
\leq \min \left\{ -\partial_t w_\delta^t - \frac{1}{2} \sigma^2 \frac{\partial_{xx}^2 w_\delta^t}{(1 - f \partial_{xx}^2 w_\delta^t)} \partial_{xx}^2 w_\delta^t, \bar{\gamma} - \partial_{xx}^2 w_\delta^t \right\}
$$
Step 3. Consider a (non-negative) smooth kernel $\psi$ with support $[-1, 0] \times [-1, 1]$, take a window size $\delta > 0$, and set

$$
\psi_\delta = \delta^{-1} \psi(\delta^{-1} \cdot) \quad \text{and} \quad w^t_\delta = w^t \ast \psi_\delta := \int w^t(t', x') \psi_\delta(t' - \cdot, x' - \cdot) dt' dx'.
$$

The pde operator is concave decreasing, and $\partial^2_{xx} w^t_\delta \leq \partial^2_{xx} w^t \ast \psi_\delta$ (by quasi-concavity),

$$
0 \leq \min \left\{ -\partial_t w^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} w^t)} \partial^2_{xx} w^t, \bar{\gamma} - \partial^2_{xx} w^t \right\} \ast \psi_\delta
$$

$$
\leq \min \left\{ -\partial_t w^t \ast \psi_\delta - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} w^t \ast \psi_\delta)} \partial^2_{xx} w^t \ast \psi_\delta, \bar{\gamma} - \partial^2_{xx} w^t \ast \psi_\delta \right\}
$$

$$
\leq \min \left\{ -\partial_t w^t_\delta - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} w^t_\delta)} \partial^2_{xx} w^t_\delta, \bar{\gamma} - \partial^2_{xx} w^t_\delta \right\}
$$

while, for $\delta$ small with respect to $\iota$,

$$
w^t_\delta(T, \cdot) \geq \hat{g}.
$$
Step 4. We have produced a smooth function satisfying

\[
\min \left\{ -\partial_t w_\delta^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} w_\delta^t)} \partial_{xx}^2 w_\delta^t, \bar{\gamma} - \partial_{xx}^2 w_\delta^t \right\} \geq 0
\]

and

\[
w_\delta^t(T, \cdot) \geq \hat{g}.
\]
Step 4. We have produced a smooth function satisfying

$$\min \left\{ -\partial_t w^t_\delta - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w^t_\delta)} \partial_{xx}^2 w^t_\delta, \bar{\gamma} - \partial_{xx}^2 w^t_\delta \right\} \geq 0$$

and

$$w^t_\delta(T, \cdot) \geq \hat{g}.$$  

Taking

$$V = w^t_\delta(\cdot, X) \quad \text{and} \quad Y = \partial_x w^t_\delta(\cdot, X),$$

we obtain

$$V_T \geq \hat{g}(X_T) \geq g(X_T).$$
Step 4. We have produced a smooth function satisfying

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and

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Taking

$$V = w^t_\delta(\cdot, X) \quad \text{and} \quad Y = \partial_x w^t_\delta(\cdot, X),$$

we obtain

$$V_T \geq \hat{g}(X_T) \geq g(X_T).$$

This implies that $v \leq w^t_\delta \to w^t$, as $\delta \to 0$. 
Step 5. Since $w^\iota$ is solution of

$$\min \left\{ -\partial_t w^\iota - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} w^\iota)} \partial_{xx}^2 w^\iota, \ \bar{\gamma} - \partial_{xx}^2 w^\iota \right\} = 0$$

with

$$w^\iota(T, \cdot) = \hat{g} + \iota,$$
Step 5. Since $w^\iota$ is solution of

$$\min \left\{ -\partial_t w^\iota - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} w^\iota)} \partial^2_{xx} w^\iota, \; \tilde{\gamma} - \partial^2_{xx} w^\iota \right\} = 0$$

with

$$w^\iota(T, \cdot) = \hat{g} + \iota,$$

$w^\iota \to w$ where $w$ is solution of

$$\min \left\{ -\partial_t w - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} w)} \partial^2_{xx} w, \; \tilde{\gamma} - \partial^2_{xx} w \right\} = 0$$

with

$$w(T, \cdot) = \hat{g}.$$
Step 5. Since $w^\iota$ is solution of

$$\min \left\{ -\partial_t w^\iota - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w^\iota)} \partial_{xx}^2 w^\iota, \, \tilde{\gamma} - \partial_{xx}^2 w^\iota \right\} = 0$$

with

$$w^\iota(T, \cdot) = \hat{g} + \iota,$$

$w^\iota \to w$ where $w$ is solution of

$$\min \left\{ -\partial_t w - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 w)} \partial_{xx}^2 w, \, \tilde{\gamma} - \partial_{xx}^2 w \right\} = 0$$

with

$$w(T, \cdot) = \hat{g}.$$

It satisfies $w \leftarrow w^\iota \geq v$. 

Step 6. But $v$ is a super-solution of the same equation: $w \leq v$ by comparison, and therefore $w = v$ by the above.
Step 5. Since $w^t$ is solution of
\[
\min \left\{ -\partial_t w^t - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} w^t)} \partial_{xx}^2 w^t, \bar{\gamma} - \partial_{xx}^2 w^t \right\} = 0
\]
with
\[
w^t(T, \cdot) = \hat{g} + \iota,
\]
$w^t \rightarrow w$ where $w$ is solution of
\[
\min \left\{ -\partial_t w - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} w)} \partial_{xx}^2 w, \bar{\gamma} - \partial_{xx}^2 w \right\} = 0
\]
with
\[
w(T, \cdot) = \hat{g}.
\]
It satisfies $w \leftarrow w^t \geq v$.

Step 6. But $v$ is a super-solution of the same equation: $w \leq v$ by comparison, and therefore $w = v$ by the above.
To sum up:

\[
\begin{align*}
\{v\} & \supseteq \{w\} \\
\text{super-solution} & \rightarrow \text{solution} & \delta, \iota \rightarrow 0 & \text{super-hedging} \\
\{w^\iota_\delta\} & \supseteq v
\end{align*}
\]