Almost sure hedging with price impact

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Joint works with G. Loeper (Monash Univ.) and Y. Zou (Paris-Dauphine)

Motivation

Aim of this work

- Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).
- Not high frequency (no bid-ask spread), but still impact on prices. To be considered as a liquidity model.
- Permanent/resilient impact.

Option pricing with illiquidity or impact in the literature (part of)

□ Equilibrium dynamics (modified price dynamics) : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98.

□ Liquidity curve (but no impact) : Cetin, Jarrow and Protter 04, Cetin, Soner and Touzi 09.

□ Illiquidity + impact : Loeper 14 (verification arguments).

□ Related works : Liu and Yong 05, Almgren and Li 13, Millot and Abergel 11, Guéant and Pu 13,...

Impact rule and continuous time trading dynamics

Impact rule

□ Basic rule (only permanent for the moment) : an order of δ units moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2}.$$

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☐ We just model the curve around $\delta = 0$. This should be understood for a "small" order δ. Would obtain the same with

$$X_{t-} \longrightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

and costs

$$\int_0^\delta (X_{t-} + F(X_{t-}, \iota))d\iota$$

if $\partial_\delta F(x,0) = f(x)$, $\partial^2_{\delta x} F(x,0) = f'(x)$ and

$$F(x,0) = \partial^2_{\delta\delta} F(x,0) = 0.$$

Trading signal and discrete trading dynamics

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□ We assume that the stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades (can add a drift or be multivariate).

- The corresponding dynamics are

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n} \mathbf{1}_{\{t_i^n \le t < t_{i+1}^n\}} + Y_T \mathbf{1}_{\{t=T\}} \ , \ \delta_{t_i^n}^n = Y_{t_i^n}^n - Y_{t_{i-1}^n}^n$$

$$X^n = X_0 + \int_0^{\cdot} \sigma(X_s^n) dW_s + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n),$$

$$V^n = V_0 + \int_0^{\cdot} Y_{s-}^n dX_s^n + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta_{t_i^n}^n)^2 f(X_{t_i^n -}^n),$$

where

$$V^n = \text{ cash part } + Y^n X^n = \text{"portfolio value"}.$$

□ Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$
\begin{aligned}
Y &= Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s \\
X &= X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \underbrace{\int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} a_s(\sigma f')(X_s) ds}_{(Y^n_{t^n_i} - Y^n_{t^n_{i-1}}) f(X^n_{t^n_i -})} \\
V &= V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \underbrace{\int_0^{\cdot} a_s^2 f(X_s) ds}_{(Y^n_{t^n_i} - Y^n_{t^n_{i-1}})^2 f(X^n_{t^n_i -})} ,
\end{aligned}
$$

at a speed $\sqrt{n}$.

Adding a resilience effect

- Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^\cdot \sigma(X_s^n) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n) - \int_0^\cdot \rho R_s^n ds.$$

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- The continuous time dynamics becomes

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$R = R_0 + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds.$$

Hedging problem : two situations

Classical case (uncovered options)

□ Has an initial impact when build the initial position in stocks and a final impact when liquidate it at the end.

▷ B., Loeper, and Zou. Almost-sure hedging with permanent price impact. To appear in *Finance and Stochastics*, 2015.

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Results :
- A quasilinear pde (the volatility term depends on the gradient).
- Perfect hedging by verification when coefficients are smooth enough.
- A modified delta-hedging rule : hedge in delta but at a modified asset price.

Covered options

□ The premium and payoff are paid in cash and stocks with a number of stocks decided by the trader. Avoids any initial and final market impact.

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Easy : find the pde (simple delta hedging, cf. Loeper).
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Results :
- A fully non-linear pde (the volatility term depends on the Hessian).
- Perfect hedging by verification when coefficients are smooth enough.
- Delta-hedging as usual.
- Almost optimal hedging rules constructed in any case.

Covered vs Uncovered

Not a simple approximation !

Quasi-linear vs fully non-linear pde
Modified delta vs standard delta hedging rule.

The case of covered options

See http ://www.ceremade.dauphine.fr/∼bouchard/pdf/BLZ_slides.pdf

for the case of uncovered options.

Hedging and pricing - informal derivation

- Resilience does not play any role, we take $\rho \equiv 0$ so that

$$dY = adW + bdt$$
$$dX = \sigma(X)dW + f(X)dY + a(\sigma f')(X)dt$$
$$dV = YdX + \frac{1}{2}a_s^2 f(X)dt.$$

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with $\sigma^a = \sigma + fa$

Hedging and pricing - informal derivation

- Resilience does not play any role, we take $\rho \equiv 0$ so that

$$dY = adW + bdt = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt$$

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with $\sigma^a = \sigma + fa$ and $\gamma^a = a/(\sigma + fa)$.

□ Super-hedging price :

$$\mathrm{v}(t,x) := \inf\{v = c + yx \ : \ (c,y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}(t,x,v,y) \neq \emptyset\},$$

where $\mathcal{G}(t,x,v,y)$ is the set of (a,b) s.t. $\phi := (y,a,b)$ satisfies

$$V_T^{t,x,v,\phi} \geq g(X_T^{t,x,\phi}).$$

Let us assume that we use the delta-hedging rule :

$$V = \mathrm{v}(\cdot, X) \quad , \quad Y = \partial_x \mathrm{v}(\cdot, X).$$

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Then, equating the *dt* terms implies

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By definition of γ^a and a little bit of algebra :

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} \mathrm{v})} \partial^2_{xx} \mathrm{v} \right] (\cdot, X) = 0.$$

The pricing pde should be

$$-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v} = 0 \quad \text{on } [0, T) \times \mathbb{R},$$
$$\mathrm{v}(T-, \cdot) = g \quad \text{on } \mathbb{R}.$$

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Singular pde :
- Can find smooth solutions s.t. $1 > f\partial^2_{xx}\mathrm{v}$, cf. Loeper 15 (under conditions).
- In general, needs to take care of $1 \neq f\partial^2_{xx}\mathrm{v}$
- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f\partial^2_{xx}\mathrm{v} > 1$ does not make sense.

Hedging with a gamma contraint

Recall

$$dY = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt \ \text{ and } \ dX = \sigma^a(X)dW + \mu_X^{a,b}(X)dt.$$

- We now define v with respect to the gamma constraint

$$\gamma^a(X) \leq \bar{\gamma}(X)$$

with

$$f\bar{\gamma} < 1 - \varepsilon, \ \ \varepsilon > 0.$$

Pricing pde :

$$\min \left\{ -\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})} \partial^2_{xx}\mathrm{v} \; , \; \bar{\gamma} - \partial^2_{xx}\mathrm{v} \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R}.$$

Propagation of the gamma contraint at the boundary :

$$\mathrm{v}(T-, \cdot) = \hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

$$\min \left\{ \varphi - g \; , \; \bar{\gamma} - \partial^2_{xx}\varphi \right\} = 0.$$

See Soner and Touzi 00, and Cheridito, Soner and Touzi 05.

Super-solution property

Use a weak formulation approach and results on small time behavior of double stochastic integrals, see Soner and Touzi 00 and Cheridito, Soner and Touzi 05.

It is based on the Geometric DPP (Soner and Touzi) :
if

$$V_0 > \mathrm{v}(0, X_0)$$

then we can find (a, b, Y_0) such that

$$V_\theta \geq \mathrm{v}(\theta, X_\theta)$$

for any stopping time θ with values in $[0, T]$.

Sub-solution property

□ Main difficulty : can not establish the reverse Geometric DPP, i.e.

If (a, b, Y_0) are such that

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□ Problem :
- at θ we have a position Y_θ that may not match with the position $\hat{Y}_\theta$ associated to $\mathrm{v}(\theta, X_\theta)$. Can not jump from Y_θ to $\hat{Y}_\theta$...
- can neither go smoothly to it as it will move X because of the impact, and therefore $\hat{Y}$ (sort of fixed point problem), compare with Cheridito, Soner, and Touzi 2005.

The smoothing approach

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Step 1. Using Perron's method + comparison, construct a (bounded) viscosity solution w^{ι} of

$$\min\left\{-\partial_t \varphi - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\varphi)}\partial^2_{xx}\varphi \;,\; \bar{\gamma} - \partial^2_{xx}\varphi\right\} = 0 \quad \text{on } [0, T) \times \mathbb{R},$$

with terminal condition

$$\mathrm{w}^{\iota}(T, \cdot) = \hat{g} + \iota \quad \text{on } \mathbb{R}$$

with $\iota > 0$.

Step 2. Up to replacing w^{ι} by an approximating sequence of quasi-concave functions (by quadratic inf-convolution), we can assume that w^{ι} is quasi-concave

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$$\min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \ , \ \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\right\} \geq 0 \ \text{ a.e.}$$

with $\partial^2_{xx}\mathrm{w}^\iota$ the density of the absolute continuous part of the second order derivative measure

See Jensen 88.

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with $\partial^2_{xx}\mathrm{w}^\iota$ the density of the absolute continuous part of the second order derivative measure, and

$$\mathrm{w}^\iota(T, \cdot) \geq \hat{g} + \iota/2.$$

See Jensen 88.

Step 3. Consider a (non-negative) smooth kernel ψ with support $[-1,0] \times [-1,1]$, take a window size $\delta > 0$, and set

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$$0 \le \min\left\{ -\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \ ,\ \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota \right\} \star \psi_\delta$$

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The pde operator is concave

$$\begin{aligned}
0 &\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta\\
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The pde operator is concave decreasing, and $\partial^2_{xx}\mathrm{w}^\iota_\delta\le\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta$ (by quasi-concavity),

$$\begin{aligned}
0 &\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta\\
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while, for δ small with respect to ι,

$$\mathrm{w}^\iota_\delta(T,\cdot)\ge\hat g.$$

Step 4. We have produced a smooth function satisfying

$$\min\left\{-\partial_t \mathrm{w}_\delta^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial_{xx}^2\mathrm{w}_\delta^\iota)}\partial_{xx}^2\mathrm{w}_\delta^\iota\ ,\ \bar{\gamma}-\partial_{xx}^2\mathrm{w}_\delta^\iota\right\}\geq 0$$

and

$$\mathrm{w}_\delta^\iota(T,\cdot)\geq \hat{g}.$$

Step 4. We have produced a smooth function satisfying

$$\min\left\{-\partial_t \mathrm{w}_\delta^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}_\delta^\iota)}\partial^2_{xx}\mathrm{w}_\delta^\iota\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{w}_\delta^\iota\right\} \geq 0$$

and

$$\mathrm{w}_\delta^\iota(T,\cdot) \geq \hat{g}.$$

Taking

$$V = \mathrm{w}_\delta^\iota(\cdot, X) \quad \text{and} \quad Y = \partial_x \mathrm{w}_\delta^\iota(\cdot, X),$$

we obtain

$$V_T \geq \hat{g}(X_T) \geq g(X_T).$$

Step 4. We have produced a smooth function satisfying

$$\min\left\{-\partial_t \mathrm{w}^\iota_\delta - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota_\delta)}\partial^2_{xx}\mathrm{w}^\iota_\delta\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{w}^\iota_\delta\right\} \geq 0$$

and

$$\mathrm{w}^\iota_\delta(T,\cdot) \geq \hat{g}.$$

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This implies that $\mathrm{v} \leq \mathrm{w}^\iota_\delta \to \mathrm{w}^\iota$, as $\delta \to 0$.

Step 5. Since w^ι is solution of

$$\min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \ , \ \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\right\} = 0$$

with

$$\mathrm{w}^\iota(T, \cdot) = \hat{g} + \iota,$$

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It satisfies $\mathrm{w} \leftarrow \mathrm{w}^\iota \geq \mathrm{v}$.

Step 6. But v is a super-solution of the same equation : $\mathrm{w} \leq \mathrm{v}$ by comparison, and therefore $\mathrm{w} = \mathrm{v}$ by the above.

To sum up :

$$\underbrace{\mathrm{v}}_{\text{super-solution}} \geq \underbrace{\mathrm{w}}_{\text{solution}} \underbrace{\longleftarrow}_{\delta,\iota\to 0} \underbrace{\mathrm{w}_{\delta}^{\iota}}_{\text{super-hedging}} \geq \mathrm{v}$$

Remark : almost optimal hedging rule

□ w_δ^ι allows one to hedge by a usual delta-hedging strategy and

$$\mathrm{v} \leftarrow \epsilon_\delta^\iota + \mathrm{v} \geq \mathrm{w}_\delta^\iota \geq \mathrm{v}$$

$\Rightarrow$ can be as close as one wants to the super-hedging price, for small δ, ι.

General case

□ Non-constant coefficients

▷ start with a solution of the pde with shaken coefficients in the sens of Krylov :

$$\min_{x'\in B_\varepsilon(x)} \min\left\{-\partial_t\varphi - \frac{1}{2}\frac{\sigma^2(x')}{(1-f(x')\partial^2_{xx}\varphi)}\partial^2_{xx}\varphi\ ,\ \bar\gamma(x') - \partial^2_{xx}\varphi\right\}(t,x) = 0.$$

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or equivalently (formally)

$$\min_{x' \in B_\varepsilon(x)} \min \left\{ -\partial_t \varphi - \frac{1}{2} \frac{\sigma^2(x)}{(1 - f(x)\partial^2_{xx}\varphi)} \partial^2_{xx}\varphi \; , \; \bar{\gamma}(x) - \partial^2_{xx}\varphi \right\} (t', x') = 0,$$

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so that we can freeze the coefficients at their value at the center of the ball before integrating on this ball.

□ $\hat{g}$ uniformly continuous with linear growth

▷ use a space dependent window for the kernel (to handle the linear growth)

▷ further approximate $\hat{g}$ from above by functions with affine behavior outside of a compact set (to keep uniform convergence when using a symmetric kernel).

Numerical example

- Constant impact and constraint.
- Bachelier model : $dX_t = 0.2\, dW_t$.
- Butterfly option : $g(x) = (x+1)^+ - 2x^+ + (x-1)^+,\ T = 2$.

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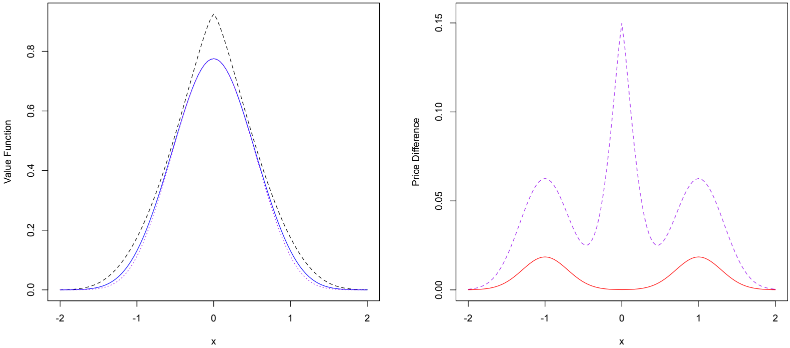

Figure : **Left : Dashed line : $f = 0.5$, $\bar{\gamma} = 1.75$; solid line : $f = 0$, $\bar{\gamma} = 1.75$; dotted line : $f = 0$, $\bar{\gamma} = +\infty$.**

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