Almost sure hedging under permanent price impact

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - Ensae-ParisTech

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Joint work with G. Loeper (BNP-Paribas) and Y. Zou (Paris-Dauphine)
Motivation
Aim of this work

- Aim:
  - Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).
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□ Aim :

• Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).

• Not high frequency (no bid-ask spread), but still impact on prices. To be considered as a liquidity model.

• Here, only permanent impact.
Option pricing with illiquidity or impact in the literature (part of)

- Equilibrium dynamics (modified price dynamics) : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98.

- Liquidity curve (but no impact) : Cetin, Jarrow and Protter 04, Cetin, Soner and Touzi 09.

- Illiquidity + impact : Loeper 14 (verification arguments).

- Related works : Liu and Yong 05, Almgren and Li 2013, Millot and Abergel 2011, Guéant and Pu 2013,...
Impact rule and continuous time trading dynamics
Impact rule

- Basic rule: A small order \( \delta \) moves the price by

\[
X_{t-} \rightarrow X_t = X_{t-} + \delta f(X_{t-}),
\]

and costs

\[
\delta X_{t-} + \frac{1}{2} \delta^2 f(X_{t-}) = \delta \left( \frac{1}{2} X_{t-} + \frac{1}{2} X_t \right).
\]
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Would obtain the same with

$$X_{t-} \rightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

if $F(x, 0) = 0$ and $\partial_\delta F(x, \delta) = f(x) + o(\delta)$. 
Trading signal and discrete trading dynamics

- A trading signal is an Itô process of the form

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Trading signal and discrete trading dynamics

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- Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.
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- Trade at times \( t^n_i = iT/n \) the quantity \( \delta^n_{t^n_i} = Y_{t^n_i} - Y_{t^n_{i-1}} \).
A trading signal is an Itô process of the form

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Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.

Trade at times \( t_i^n = iT/n \) the quantity \( \delta_{t_i^n} = Y_{t_i^n} - Y_{t_i^n}^{-1} \).

We assume that the stock price evolves according to

\[ X = X_{t_i^n} + \int_{t_i^n}^\cdot \sigma(X_s) \, dW_s \]

between two trades (can add a drift or be multivariate without extra complications).
Trading signal and discrete trading dynamics

- Passing to the limit \( n \to \infty \), it converges in \( S_2 \) to

\[
Y = Y_0 + \int_0^t b_s ds + \int_0^t a_s dW_s
\]
\[
X = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t f(X_s) dY_s + \int_0^t a_s \sigma(X_s) f'(X_s) ds
\]
\[
V = V_0 + \int_0^t Y_s dX_s + \frac{1}{2} \int_0^t a_s^2 f(X_s) ds,
\]

at a speed \( \sqrt{n} \), where

\[
V = \text{cash part} + YS = \text{“portfolio value”}.
\]
How to define the super-hedging problem?
Super-hedging problem

- Fix a claim \( g = (g_0, g_1) \) with
  - \( g_0 \) = cash part
  - \( g_1 \) = \# of stocks to deliver.
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- $g_0 =$ cash part
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Super-hedging price = minimal initial cash so that
\[ V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T). \]
(Recall that $V = \text{cash} + YX$)
\[ \hat{w}(0, X_0, Y_0) \] is the min over \( V_0 \) such that super-hedge for some \((a, b)\), starting from \( Y_0 \).
\( \hat{w}(0, X_0, Y_0) \) is the min over \( V_0 \) such that super-hedge for some \( (a, b) \), starting from \( Y_0 \).

Problems:
\( \hat{\nu}(0, X_0, Y_0) \) is the min over \( V_0 \) such that super-hedge for some \((a, b)\), starting from \( Y_0 \).

Problems:
- certainly needs an initial jump of \( Y \) at 0 to have \( Y_{0+} = \partial_x \hat{\nu}(0, X_{0+}, Y_{0+}) \)
\( \hat{w}(0, X_0, Y_0) \) is the min over \( V_0 \) such that super-hedge for some \((a, b)\), starting from \( Y_0 \).

**Problems:**

- certainly needs an initial jump of \( Y \) at 0 to have “\( Y_{0+} = \partial_x \hat{w}(0, X_{0+}, Y_{0+}) \)”
- from the pde point of view, will be on a curve \( Y = \partial_x \hat{w}(\cdot, X, Y) \)!
Another difficulty

- Expending the dynamics leads to

\[ Y = Y_0 + \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s \]

\[ X = X_0 + \int_0^\cdot (\sigma + a_s f)(X_s) dW_s + \int_0^\cdot (a_s \sigma f' + b_s f)(X_s) ds \]

\[ V = V_0 + \int_0^\cdot Y_s (\sigma + a_s f)(X_s) dW_s + \int_0^\cdot Y_s (a_s \sigma f' + b_s f)(X_s) ds \]

\[ + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s) ds, \]
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- \( b \) appears linearly and is not constrained a-priori \( \Rightarrow \) singular control problem!
Formally, if $\nu = \hat{w}(t, x, y)$, we can find $(a, b)$ such that

$$0 = d(V^\nu - \hat{w}(\cdot, X., Y.))$$

but not better (i.e. with $>$).
Formally, if \( \nu = \hat{\nu}(t, x, y) \), we can find \((a, b)\) such that

\[
0 = d(V^\nu - \hat{\nu}(\cdot, X, Y)) = b[Yf(X) - (f \partial_x \hat{\nu} + \partial_y \hat{\nu})(\cdot, X, Y)]dt + \cdots
\]

but not better (i.e. with \(>\)).
Formally, if $v = \hat{w}(t, x, y)$, we can find $(a, b)$ such that

$$0 = d(V^v - \hat{w}(\cdot, X, Y)) = b[Yf(X) - (f \partial_x \hat{w} + \partial_y \hat{w})(\cdot, X, Y)]dt + \cdots$$

but not better (i.e. with $>$). In particular, we should have

$$yf(x) = f(x)\partial_x \hat{w}(t, x, y) + \partial_y \hat{w}(t, x, y).$$
Learning from the above definition

- The fact that (formally)
  \[ yf(x) = f(x)\partial_x \hat{w}(t, x, y) + \partial_y \hat{w}(t, x, y) \]
  implies
  \[ \hat{w}(t, x, y) - \mathcal{I}(x(x, -y), y) = \hat{w}(t, x(x, -y), 0) =: w(t, x(x, -y)) \]
in which
  \[ x(x, \delta) = x + \int_0^\delta f(x(x, s))ds \quad \text{and} \quad \mathcal{I}(x, \delta) := \int_0^\delta sf(x(x, s))ds. \]
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Interpretation:
- \( x(x, \delta) \): impact of a jump \( \delta \) on \( Y \) by using the splitting rule,

Split \( \delta \) in \( \delta/n \) then
\[ x + \frac{\delta}{n}f(x) \simeq x(x, \frac{\delta}{n}) \sim x(x(x, \frac{\delta}{n}), \frac{\delta}{n})) = x(x, \frac{2\delta}{n}) \sim \ldots \simeq x(x, \delta) \]
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- The fact that (formally)

\[ yf(x) = f(x)\partial_x \hat{w}(t, x, y) + \partial_y \hat{w}(t, x, y) \]

implies

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- Interpretation:
  - \( x(x, \delta) \): impact of a jump \( \delta \) on \( Y \) by using the splitting rule,
  - \( \mathcal{I}(x, \delta) \): corresponding impact on the portfolio value \( V \) if the initial stock position is 0.

Split \( \delta \) in \( \delta/n \) then

\[ x + \frac{\delta}{n} f(x) \sim x(x, \frac{\delta}{n}) \sim x(x(x, \frac{\delta}{n}), \frac{\delta}{n})) = x(x, \frac{2\delta}{n}) \sim \ldots \sim x(x, \delta) \]
Specification with jumps
Adding jumps and splitting of large orders

We now consider a trading signal of the form

\[ Y = Y_0 - \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s + \int_0^\cdot \delta \nu (d\delta, ds) \]
Adding jumps and splitting of large orders

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Y = Y_0 - \int_0^T b_s ds + \int_0^T a_s dW_s + \int_0^T \delta \nu(d\delta, ds)
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- Jumps \( \delta_i \) at time \( \tau_i \) is passed on \([\tau_i, \tau_i + \varepsilon]\) at a rate \( \delta_i / \varepsilon \).
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The limit dynamics when \( \varepsilon \to 0 \) is \( (\Delta x(x, \delta) = x(x, \delta) - x) \)

\[ X = X_0 + \int_0^\cdot \sigma (X_s) dW_s + \int_0^\cdot f(X_s) dY_s^c + \int_0^\cdot a_s \sigma f'(X_s) ds \]
\[ + \int_0^\cdot \int_0^\cdot \Delta x(X_{s-}, \delta) \nu (d\delta, ds) \]

\[ V = V_0 + \int_0^\cdot Y_s dX_s^c + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s) ds \]
\[ + \int_0^\cdot \int_0^\cdot \left( Y_{s-} \Delta x(X_{s-}, \delta) + I(X_{s-}, \delta) \right) \nu (d\delta, ds). \]
Geometric dynamic principle

- With this construction, we have the relation

\[ w(t, x(x, -y)) = \hat{w}(t, x, y) - \mathcal{I}(x(x, -y), y). \]
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- Geometric dynamic programming transferred from \( \hat{w} \) to \( w \).
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- GDP: (i) If \( v > w(t, x) \) then \( \exists (a, b, \nu) \) and \( y \in \mathbb{R} \) s.t.
  \[ V_\theta \geq w(\theta, x(X_\theta, -Y_\theta)) + \mathcal{I}(x(X_\theta, -Y_\theta), Y_\theta), \]
  for all \( \theta \geq t \), where \( (X_t, Y_t, V_t) = (x(x, y), y, v + \mathcal{I}(x, y)) \).
Geometric dynamic principle

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for all \( \theta \geq t \), where \( (X_t, Y_t, V_t) = (x(x, y), y, v + J(x, y)) \).

(ii) If \( v < w(t, x) \) then \( \not\exists (a, b, \nu), y \) and \( \theta \geq t \) s.t.

\[ V_\theta > w(\theta, x(X_\theta, -Y_\theta)) + J(x(X_\theta, -Y_\theta), Y_\theta), \]

with \( (X_t, Y_t, V_t) = (x(x, y), y, v + J(x, y)) \).
If \( v = w(t, x) \) the GDP “implies”

\[
d\mathcal{E}_t := dV_t - dw(t, x(X_t, -Y_t)) - d\mathcal{I}(x(X_t, -Y_t), Y_t) = 0,
\]

where \( (X_t, Y_t, V_t) = (x(x, y), y, v + \mathcal{I}(x, y)) \).
Pricing equation

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$$dE_t := dV_t - dw(t, x(X_t, - Y_t)) - d\mathcal{I}(x(X_t, - Y_t), Y_t) = 0,$$

where $(X_t, Y_t, V_t) = (x(x, y), y, v + \mathcal{I}(x, y))$.

- Key property:

$$dE = [\dot{Y} - Y] \left[ (f'f)(X) a^2 / 2 \, dt - \sigma(X) dW \right] + \hat{F}[w](\cdot, x(X, - Y), Y) dt$$

in which $\dot{Y} = Y$ iff

$$Y = \hat{y}(\cdot, X) := x^{-1}(X, X + f(X) \partial_x w(\cdot, X)).$$
Pricing equation - viscosity sense

- By identifying the $dW$ and $dt$ terms, we obtain the PDE:

$$0 = \hat{F}[w](\cdot, \hat{y})$$
Pricing equation - viscosity sense

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\[
0 = \hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y}) \partial_x [w + \mathcal{I}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx}^2 [w + \mathcal{I}]
\]

where

\[
\hat{\mu}(\cdot, y) := \frac{1}{2} \left[ \partial_{xx}^2 x \sigma^2 \right](x(\cdot, y), -y) \quad \text{and} \quad \hat{\sigma}(\cdot, y) := (\sigma \partial_x x)(x(\cdot, y), -y),
\]

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\hat{y}(t, x) := x^{-1}(x, x + f(x) \partial_x w(t, x)).
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□ By identifying the $dW$ and $dt$ terms, we obtain the PDE:

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where

$$\hat{\mu}(\cdot, y) := \frac{1}{2} [\partial_{xx}^2 x \sigma^2](x(\cdot, y), -y) \quad \text{and} \quad \hat{\sigma}(\cdot, y) := (\sigma \partial_x x)(x(\cdot, y), -y),$$

and

$$\hat{y}(t, x) := x^{-1}(x, x + f(x)\partial_x w(t, x)).$$

□ Terminal condition

$$G(x) := \inf \{yx(x, y) + g_0(x(x, y)) - \mathcal{I}(x, y) : y = g_1(x(x, y))\}.$$
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- To be first taken in the discontinuous viscosity sense for the relaxed semi-limits associated to problems with bounded controls (comparison holds -> uniqueness + numerical schemes / smooth solution).
Pricing equation - verification

Assume that $w$ is a smooth solution with $w(T-, \cdot) = G$ of

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\hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y}) \partial_x [w + \mathcal{J}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial^2_{xx} [w + \mathcal{J}] = 0.
$$
Assume that $w$ is a smooth solution with $w(T-\cdot) = G$ of

$$\hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y}) \partial_x [w + \mathcal{I}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx}^2 [w + \mathcal{I}] = 0.$$ 

Then $\mathcal{E}_t := V - w(\cdot, x(X, -Y)) - \mathcal{I}(x(X, -Y), Y)$ satisfies

$$d\mathcal{E} = [\mathcal{Y} - Y][\cdots]dt + (\cdots)dW] + \hat{F}[w](\cdot, \hat{X}, Y)dt$$

with $\hat{X} = x(X, -Y)$
Pricing equation - verification

Assume that \( w \) is a smooth solution with \( w(T-, \cdot) = G \) of

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\hat{F}[w](\cdot, \hat{y}) = -\partial_tw - \hat{\mu}(\cdot, \hat{y})\partial_x[w + I] - \frac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2\partial_{xx}[w + I] = 0.
\]

Then \( \mathcal{E}_t := V - w(\cdot, x(X, -Y)) - I(x(X, -Y), Y) \) satisfies

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with \( \hat{X} = x(X, -Y) \)

We can use a strategy ensuring \( Y = \hat{Y} = \hat{y}(\cdot, \hat{X}) \) :
Pricing equation - verification

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- We can use a strategy ensuring $Y = \hat{Y} = \hat{y}(\cdot, \hat{X})$:
  - Make an initial jump of size $Y_0 = x^{-1}(x, x(x, \partial_x w(0, X_0-)))$. 
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with \( \hat{X} = x(X, -Y) \)

We can use a strategy ensuring \( Y = \hat{Y} = \hat{y}(\cdot, \hat{X}) \):

- Make an initial jump of size \( Y_0 = x^{-1}(x, x(x, \partial_x w(0, X_0-))) \).
- Follow \((a, b)\) such that \( Y = x^{-1}(\hat{X}, x(\hat{X}, \partial_x w(t, \hat{X}))) \).
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Then $E_t := V - w(\cdot, x(X, -Y)) - \mathcal{I}(x(X, -Y), Y)$ satisfies

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We can use a strategy ensuring $Y = \hat{Y} = \hat{y}(\cdot, \hat{X})$:

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- Follow $(a, b)$ such that $Y = x^{-1}(\hat{X}, x(\hat{X}, \partial_x w(t, \hat{X}))).$

Then

- $V_{T-} = G(x(X_{T-}, -Y_{T-})) + \mathcal{I}(x(X_{T-}, -Y_{T-}), Y_{T-}).$
Pricing equation - verification

- Assume that \( w \) is a smooth solution with \( w(T-, \cdot) = G \) of
  \[
  \hat{F}[w](\cdot, \hat{y}) = -\partial_tw - \hat{\mu}(\cdot, \hat{y})\partial_x[w + \mathcal{I}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx}^2[w + \mathcal{I}] = 0.
  \]

- Then \( \mathcal{E}_t := V - w(\cdot, x(X, -Y)) - \mathcal{I}(x(X, -Y), Y) \) satisfies
  \[
  d\mathcal{E} = [\hat{Y} - Y] [(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot, \hat{X}, Y)dt
  \]
  with \( \hat{X} = x(X, -Y) \)

- We can use a strategy ensuring \( Y = \hat{Y} = \hat{y}(\cdot, \hat{X}) \):
  - Make an initial jump of size \( Y_0 = x^{-1}(x, x, \partial_xw(0, X_{0-})) \).
  - Follow \((a, b)\) such that \( Y = x^{-1}(\hat{X}, x(\hat{X}, \partial_xw(t, \hat{X})) \).

- Then
  - \( V_{T-} = G(x(X_{T-}, -Y_{T-})) + \mathcal{I}(x(X_{T-}, -Y_{T-}), Y_{T-}) \).
  - Liquidate \( Y_{T-} : V_T = G(X_T) \) and \( Y_T = 0 \).
Constant impact

Model: $dX_t = \sigma(X_t) dW_t$ (between trades) and $f(X) = \lambda$. 
\[ dX_t = \sigma(X_t) dW_t \] (between trades) and \( f(X) = \lambda \).

In this case, \( x(x, \delta) = x + \lambda \delta, \mathcal{I}(x, \delta) = \frac{1}{2} \delta^2 \lambda, \) and the pde is

\[
-\partial_t w - \frac{1}{2} \sigma^2 (x + \lambda \partial_x w) \partial_{xx}^2 w = 0
\]

For \( \lambda = 0 \) or \( \sigma = \text{cst} \), this is the usual heat equation!!!
Constant impact

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For \( \lambda = 0 \) or \( \sigma = \text{cst} \), this is the usual heat equation!!!

- We can use the strategy: \( Y = \partial_x w(\cdot, \hat{X}) = \partial_x w(\cdot, X - \lambda Y) \).
Thank you very much!