Almost sure hedging under permanent price impact

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - Ensae-ParisTech

13 mars 2015

Joint work with G. Loeper (BNP-Paribas) and Y. Zou (Paris-Dauphine)

Motivation

Aim of this work

☐ Aim :

- Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).

Aim of this work

- Aim :
  - Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).
  - Not high frequency (no bid-ask spread), but still impact on prices. To be considered as a liquidity model.

Aim of this work

□ Aim :

- Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).
- Not high frequency (no bid-ask spread), but still impact on prices. To be considered as a liquidity model.
- Here, only permanent impact.

Option pricing with illiquidity or impact in the literature (part of)

- Equilibrium dynamics (modified price dynamics) : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98.

- Liquidity curve (but no impact) : Cetin, Jarrow and Protter 04, Cetin, Soner and Touzi 09.

- Illiquidity + impact : Loeper 14 (verification arguments).

- Related works : Liu and Yong 05, Almgren and Li 2013, Millot and Abergel 2011, Guéant and Pu 2013,...

Impact rule and continuous time trading dynamics

Impact rule

□ Basic rule : a small order δ moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta(\frac{1}{2}X_{t-} + \frac{1}{2}X_t).$$

Impact rule

- Basic rule : a small order δ moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta(\frac{1}{2}X_{t-} + \frac{1}{2}X_t).$$

- We just model the curve around $\delta = 0$.

Impact rule

□ Basic rule : a small order δ moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta(\frac{1}{2}X_{t-} + \frac{1}{2}X_t).$$

□ We just model the curve around $\delta = 0$. This should be understood for a "small" order δ.

Impact rule

□ Basic rule : a small order δ moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta(\frac{1}{2}X_{t-} + \frac{1}{2}X_t).$$

□ We just model the curve around $\delta = 0$. This should be understood for a "small" order δ.

Would obtain the same with

$$X_{t-} \longrightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

if $F(x,0) = 0$ and $\partial_\delta F(x,\delta) = f(x) + o(\delta)$.

Trading signal and discrete trading dynamics

- A trading signal is an Itô process of the form

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s.$$

Trading signal and discrete trading dynamics

□ A trading signal is an Itô process of the form

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s.$$

□ Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.

Trading signal and discrete trading dynamics

□ A trading signal is an Itô process of the form

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s.$$

□ Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.

□ Trade at times $t_i^n = iT/n$ the quantity $\delta_{t_i^n}^n = Y_{t_i^n} - Y_{t_{i-1}^n}$.

Trading signal and discrete trading dynamics

□ A trading signal is an Itô process of the form

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s.$$

□ Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.

□ Trade at times $t_i^n = iT/n$ the quantity $\delta_{t_i^n}^n = Y_{t_i^n} - Y_{t_{i-1}^n}$.

□ We assume that the stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades (can add a drift or be multivariate without extra complications).

Trading signal and discrete trading dynamics

- Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s$$

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} a_s \sigma f'(X_s) ds$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds,$$

at a speed $\sqrt{n}$, where

$$V = \text{ cash part } + YS = \text{"portfolio value"}.$$

How to define the super-hedging problem ?

Super-hedging problem

- Fix a claim $g = (g_0, g_1)$ with
  - g_0 = cash part
  - g_1 = # of stocks to deliver.

Super-hedging problem

- Fix a claim $g = (g_0, g_1)$ with
  - g_0 = cash part
  - g_1 = # of stocks to deliver.

- Super-hedging price = minimal initial cash so that

$$V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T).$$

(Recall that $V = \text{cash} + YX$)

Natural definition

- $\hat{w}(0, X_0, Y_0)$ is the min over V_0 such that super-hedge for some (a, b), starting from Y_0.

Natural definition

- $\hat{w}(0, X_0, Y_0)$ is the min over V_0 such that super-hedge for some (a, b), starting from Y_0.

- Problems :

Natural definition

☐ $\hat{w}(0, X_0, Y_0)$ is the min over V_0 such that super-hedge for some (a, b), starting from Y_0.

☐ Problems :

- certainly needs an initial jump of Y at 0 to have "$Y_{0+} = \partial_x \hat{w}(0, X_{0+}, Y_{0+})$"

Natural definition

□ $\hat{w}(0, X_0, Y_0)$ is the min over V_0 such that super-hedge for some (a, b), starting from Y_0.

□ Problems :

- certainly needs an initial jump of Y at 0 to have "$Y_{0+} = \partial_x \hat{w}(0, X_{0+}, Y_{0+})$"
- from the pde point of view, will be on a curve $Y = \partial_x \hat{w}(\cdot, X, Y)$!

Another difficulty

- Expending the dynamics leads to

$$
\begin{aligned}
Y &= Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s \\
X &= X_0 + \int_0^{\cdot} (\sigma + a_s f)(X_s) dW_s + \int_0^{\cdot} (a_s \sigma f' + b_s f)(X_s) ds \\
V &= V_0 + \int_0^{\cdot} Y_s (\sigma + a_s f)(X_s) dW_s + \int_0^{\cdot} Y_s (a_s \sigma f' + b_s f)(X_s) ds \\
&\quad + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds,
\end{aligned}
$$

Another difficulty

- Expending the dynamics leads to

$$
\begin{aligned}
Y &= Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s \\
X &= X_0 + \int_0^{\cdot} (\sigma + a_s f)(X_s) dW_s + \int_0^{\cdot} (a_s \sigma f' + b_s f)(X_s) ds \\
V &= V_0 + \int_0^{\cdot} Y_s(\sigma + a_s f)(X_s) dW_s + \int_0^{\cdot} Y_s(a_s \sigma f' + b_s f)(X_s) ds \\
&\quad + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds,
\end{aligned}
$$

- b appears linearly and is not constrained a-priori $\Rightarrow$ singular control problem !

Learning from the above definition

☐ Formally, if $v = \hat{w}(t, x, y)$, we can find (a, b) such that

$$0 = d(V^v - \hat{w}(\cdot, X_\cdot, Y_\cdot))$$

but not better (i.e. with $>$).

Learning from the above definition

☐ Formally, if $v = \hat{w}(t, x, y)$, we can find (a, b) such that

$$0 = d(V^v - \hat{w}(\cdot, X_\cdot, Y_\cdot)) = b[Yf(X) - (f\partial_x \hat{w} + \partial_y \hat{w})(\cdot, X, Y)]dt + \cdots$$

but not better (i.e. with $>$).

Learning from the above definition

☐ Formally, if $v = \hat{w}(t, x, y)$, we can find (a, b) such that

$$0 = d(V^v - \hat{w}(\cdot, X_\cdot, Y_\cdot)) = b[Yf(X) - (f\partial_x \hat{w} + \partial_y \hat{w})(\cdot, X, Y)]dt + \cdots$$

but not better (i.e. with $>$). In particular, we should have

$$yf(x) = f(x)\partial_x \hat{w}(t, x, y) + \partial_y \hat{w}(t, x, y).$$

Learning from the above definition

- The fact that (formally)

$$yf(x) = f(x)\partial_x \hat{w}(t,x,y) + \partial_y \hat{w}(t,x,y)$$

implies

$$\hat{w}(t,x,y) - \mathfrak{I}(\mathrm{x}(x,-y),y) = \hat{w}(t,\mathrm{x}(x,-y),0) =: w(t,\mathrm{x}(x,-y))$$

in which

$$\mathrm{x}(x,\delta) = x + \int_0^\delta f(\mathrm{x}(x,s))ds \ \text{ and } \ \mathfrak{I}(x,\delta) := \int_0^\delta sf(\mathrm{x}(x,s))ds.$$

Learning from the above definition

- The fact that (formally)

$$yf(x) = f(x)\partial_x \hat{w}(t,x,y) + \partial_y \hat{w}(t,x,y)$$

implies

$$\hat{w}(t,x,y) - \mathfrak{I}(\mathrm{x}(x,-y),y) = \hat{w}(t,\mathrm{x}(x,-y),0) =: w(t,\mathrm{x}(x,-y))$$

in which

$$\mathrm{x}(x,\delta) = x + \int_0^\delta f(\mathrm{x}(x,s))ds \text{ and } \mathfrak{I}(x,\delta) := \int_0^\delta sf(\mathrm{x}(x,s))ds.$$

- Interpretation :
  - $\mathrm{x}(x,\delta)$: impact of a jump δ on Y by using the splitting rule,

Split δ in δ/n then

$$x + \frac{\delta}{n}f(x) \simeq \mathrm{x}(x,\frac{\delta}{n}) \rightsquigarrow \mathrm{x}(\mathrm{x}(x,\frac{\delta}{n}),\frac{\delta}{n})) = \mathrm{x}(x,\frac{2\delta}{n}) \rightsquigarrow \ldots \simeq \mathrm{x}(x,\delta)$$

Learning from the above definition

□ The fact that (formally)

$$yf(x) = f(x)\partial_x \hat{w}(t,x,y) + \partial_y \hat{w}(t,x,y)$$

implies

$$\hat{w}(t,x,y) - \mathfrak{I}(\mathrm{x}(x,-y),y) = \hat{w}(t,\mathrm{x}(x,-y),0) =: w(t,\mathrm{x}(x,-y))$$

in which

$$\mathrm{x}(x,\delta) = x + \int_0^\delta f(\mathrm{x}(x,s))ds \ \text{ and } \ \mathfrak{I}(x,\delta) := \int_0^\delta sf(\mathrm{x}(x,s))ds.$$

□ Interpretation :

- $\mathrm{x}(x,\delta)$: impact of a jump δ on Y by using the splitting rule,
- $\mathfrak{I}(x,\delta)$: corresponding impact on the portfolio value V if the initial stock position is 0.

Split δ in δ/n then

$$x + \frac{\delta}{n}f(x) \simeq \mathrm{x}(x,\frac{\delta}{n}) \rightsquigarrow \mathrm{x}(\mathrm{x}(x,\frac{\delta}{n}),\frac{\delta}{n})) = \mathrm{x}(x,\frac{2\delta}{n}) \rightsquigarrow \ldots \simeq \mathrm{x}(x,\delta)$$

Specification with jumps

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta \nu(d\delta, ds)$$

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta\nu(d\delta, ds)$$

- Jumps δ_i at time τ_i is passed on $[\tau_i, \tau_i + \varepsilon]$ at a rate δ_i/ε.

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta\nu(d\delta, ds)$$

- Jumps δ_i at time τ_i is passed on $[\tau_i, \tau_i + \varepsilon]$ at a rate δ_i/ε.

- The limit dynamics when $\varepsilon \to 0$ is ($\Delta\mathrm{x}(x,\delta) = \mathrm{x}(x,\delta) - x$)

$$\begin{aligned} X &= X_{0-} + \int_0^{\cdot} \sigma(X_s) dW_s + \int_0^{\cdot} f(X_s) dY_s^c + \int_0^{\cdot} a_s \sigma f'(X_s) ds \\ &\quad + \int_0^{\cdot} \int \Delta\mathrm{x}(X_{s-}, \delta)\nu(d\delta, ds) \\ V &= V_{0-} + \int_0^{\cdot} Y_s dX_s^c + \frac{1}{2}\int_0^{\cdot} a_s^2 f(X_s) ds \\ &\quad + \int_0^{\cdot} \int \left(Y_{s-}\Delta\mathrm{x}(X_{s-}, \delta) + \mathfrak{I}(X_{s-}, \delta)\right) \nu(d\delta, ds). \end{aligned}$$

Geometric dynamic principle

- With this construction, we have the relation

$$w(t, \mathrm{x}(x, -y)) = \hat{w}(t, x, y) - \mathfrak{I}(\mathrm{x}(x, -y), y).$$

Geometric dynamic principle

- With this construction, we have the relation

$$w(t, \mathrm{x}(x, -y)) = \hat{w}(t, x, y) - \mathfrak{I}(\mathrm{x}(x, -y), y).$$

- Geometric dynamic programming transferred from $\hat{w}$ to w.

Geometric dynamic principle

- With this construction, we have the relation

$$w(t, \mathrm{x}(x, -y)) = \hat{w}(t, x, y) - \mathfrak{I}(\mathrm{x}(x, -y), y).$$

- Geometric dynamic programming transferred from $\hat{w}$ to w.

- GDP : (i) If $v > w(t, x)$ then $\exists$ (a, b, ν) and $y \in \mathbb{R}$ s.t.

$$V_\theta \quad \geq \quad w(\theta, \mathrm{x}(X_\theta, -Y_\theta)) + \mathfrak{I}(\mathrm{x}(X_\theta, -Y_\theta), Y_\theta),$$

for all $\theta \geq t$, where $(X_t, Y_t, V_t) = (\mathrm{x}(x, y), y, v + \mathfrak{I}(x, y))$.

Geometric dynamic principle

- With this construction, we have the relation

$$w(t, \mathrm{x}(x, -y)) = \hat{w}(t, x, y) - \mathfrak{I}(\mathrm{x}(x, -y), y).$$

- Geometric dynamic programming transferred from $\hat{w}$ to w.

- GDP : (i) If $v > w(t, x)$ then $\exists$ (a, b, ν) and $y \in \mathbb{R}$ s.t.

$$V_\theta \quad \geq \quad w(\theta, \mathrm{x}(X_\theta, -Y_\theta)) + \mathfrak{I}(\mathrm{x}(X_\theta, -Y_\theta), Y_\theta),$$

for all $\theta \geq t$, where $(X_t, Y_t, V_t) = (\mathrm{x}(x, y), y, v + \mathfrak{I}(x, y))$.

(ii) If $v < w(t, x)$ then $\nexists$ (a, b, ν), y and $\theta \geq t$ s.t.

$$V_\theta \quad > \quad w(\theta, \mathrm{x}(X_\theta, -Y_\theta)) + \mathfrak{I}(\mathrm{x}(X_\theta, -Y_\theta), Y_\theta),$$

with $(X_t, Y_t, V_t) = (\mathrm{x}(x, y), y, v + \mathfrak{I}(x, y))$.

Pricing equation

- If $v = w(t, x)$ the GDP "implies"

$$d\mathcal{E}_t := dV_t - dw(t, \mathrm{x}(X_t, -Y_t)) - d\mathfrak{I}(\mathrm{x}(X_t, -Y_t), Y_t) = 0,$$

where $(X_t, Y_t, V_t) = (\mathrm{x}(x, y), y, v + \mathfrak{I}(x, y))$.

Pricing equation

☐ If $v = w(t,x)$ the GDP "implies"

$$d\mathcal{E}_t := dV_t - dw(t, \mathrm{x}(X_t, -Y_t)) - d\mathfrak{I}(\mathrm{x}(X_t, -Y_t), Y_t) = 0,$$

where $(X_t, Y_t, V_t) = (\mathrm{x}(x,y), y, v + \mathfrak{I}(x,y))$.

☐ Key property :

$$\begin{aligned} d\mathcal{E} &= [\check{Y} - Y]\left[(f'f)(X)a^2/2dt - \sigma(X)dW\right] \\ &\quad + \hat{F}[w](\cdot, \mathrm{x}(X, -Y), Y)dt \end{aligned}$$

in which $\check{Y} = Y$ iff

$$Y = \hat{y}(\cdot, X) := \mathrm{x}^{-1}(X, X + f(X)\partial_x w(\cdot, X)).$$

Pricing equation - viscosity sense

- By identifying the dW and dt terms, we obtain the PDE :

$$0 = \hat{F}[w](\cdot, \hat{y})$$

Pricing equation - viscosity sense

□ By identifying the dW and dt terms, we obtain the PDE :

$$0 = \hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y})\partial_x[w + \mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2\partial^2_{xx}[w + \mathfrak{I}]$$

where

$$\hat{\mu}(\cdot, y) := \frac{1}{2}[\partial^2_{xx}\mathrm{x}\sigma^2](\mathrm{x}(\cdot, y), -y) \;\text{ and }\; \hat{\sigma}(\cdot, y) := (\sigma\partial_x\mathrm{x})(\mathrm{x}(\cdot, y), -y),$$

and

$$\hat{y}(t, x) := \mathrm{x}^{-1}(x, x + f(x)\partial_x w(t, x)).$$

Pricing equation - viscosity sense

- By identifying the dW and dt terms, we obtain the PDE :

$$0 = \hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial^2_{xx}[w+\mathfrak{I}]$$

where

$$\hat{\mu}(\cdot,y) := \frac{1}{2}[\partial^2_{xx}\mathrm{x}\sigma^2](\mathrm{x}(\cdot,y),-y) \text{ and } \hat{\sigma}(\cdot,y) := (\sigma\partial_x\mathrm{x})(\mathrm{x}(\cdot,y),-y),$$

and

$$\hat{y}(t,x) := \mathrm{x}^{-1}(x, x+f(x)\partial_x w(t,x)).$$

- Terminal condition

$$G(x) := \inf\left\{y\mathrm{x}(x,y) + g_0(\mathrm{x}(x,y)) - \mathfrak{I}(x,y) : \; y = g_1(\mathrm{x}(x,y))\right\}.$$

Pricing equation - viscosity sense

- By identifying the dW and dt terms, we obtain the PDE :

$$0 = \hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y})\partial_x[w + \mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2 \partial^2_{xx}[w + \mathfrak{I}]$$

where

$$\hat{\mu}(\cdot, y) := \frac{1}{2}[\partial^2_{xx}\mathrm{x}\sigma^2](\mathrm{x}(\cdot, y), -y) \text{ and } \hat{\sigma}(\cdot, y) := (\sigma\partial_x\mathrm{x})(\mathrm{x}(\cdot, y), -y),$$

and

$$\hat{y}(t, x) := \mathrm{x}^{-1}(x, x + f(x)\partial_x w(t, x)).$$

- Terminal condition

$$G(x) := \inf\left\{y\mathrm{x}(x, y) + g_0(\mathrm{x}(x, y)) - \mathfrak{I}(x, y) : \; y = g_1(\mathrm{x}(x, y))\right\}.$$

- To be first taken in the discontinuous viscosity sense for the relaxed semi-limits associated to problems with bounded controls (comparison holds-> uniqueness + numerical schemes / smooth solution).

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-,\cdot) = G$ of

$$\hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial_{xx}^2[w+\mathfrak{I}] = 0.$$

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-,\cdot) = G$ of

$$\hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial^2_{xx}[w+\mathfrak{I}] = 0.$$

- Then $\mathcal{E}_t := V - w(\cdot,\mathrm{x}(X,-Y)) - \mathfrak{I}(\mathrm{x}(X,-Y),Y)$ satisfies

$$d\mathcal{E} = [\check{Y} - Y]\,[(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot,\hat{X},Y)dt$$

with $\hat{X} = \mathrm{x}(X,-Y)$

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-,\cdot) = G$ of

$$\hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial^2_{xx}[w+\mathfrak{I}] = 0.$$

- Then $\mathcal{E}_t := V - w(\cdot,\mathrm{x}(X,-Y)) - \mathfrak{I}(\mathrm{x}(X,-Y),Y)$ satisfies

$$d\mathcal{E} = [\check{Y} - Y]\,[(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot,\hat{X},Y)dt$$

with $\hat{X} = \mathrm{x}(X,-Y)$

- We can use a strategy ensuring $Y = \check{Y} = \hat{y}(\cdot,\hat{X})$:

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-,\cdot) = G$ of

$$\hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial^2_{xx}[w+\mathfrak{I}] = 0.$$

- Then $\mathcal{E}_t := V - w(\cdot,\mathrm{x}(X,-Y)) - \mathfrak{I}(\mathrm{x}(X,-Y),Y)$ satisfies

$$d\mathcal{E} = [\check{Y} - Y]\,[(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot,\hat{X},Y)dt$$

with $\hat{X} = \mathrm{x}(X,-Y)$

- We can use a strategy ensuring $Y = \check{Y} = \hat{y}(\cdot,\hat{X})$:
  - Make an initial jump of size $Y_0 = \mathrm{x}^{-1}(x,\mathrm{x}(x,\partial_x w(0,X_{0-})))$.

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-,\cdot) = G$ of

$$\hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial^2_{xx}[w+\mathfrak{I}] = 0.$$

- Then $\mathcal{E}_t := V - w(\cdot,\mathrm{x}(X,-Y)) - \mathfrak{I}(\mathrm{x}(X,-Y),Y)$ satisfies

$$d\mathcal{E} = [\check{Y} - Y]\,[(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot,\hat{X},Y)dt$$

with $\hat{X} = \mathrm{x}(X,-Y)$

- We can use a strategy ensuring $Y = \check{Y} = \hat{y}(\cdot,\hat{X})$:
  - Make an initial jump of size $Y_0 = \mathrm{x}^{-1}(x,\mathrm{x}(x,\partial_x w(0,X_{0-})))$.
  - Follow (a,b) such that $Y = \mathrm{x}^{-1}(\hat{X},\mathrm{x}(\hat{X},\partial_x w(t,\hat{X})))$.

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-, \cdot) = G$ of

$$\hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y})\partial_x[w + \mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2\partial^2_{xx}[w + \mathfrak{I}] = 0.$$

- Then $\mathcal{E}_t := V - w(\cdot, \mathrm{x}(X, -Y)) - \mathfrak{I}(\mathrm{x}(X, -Y), Y)$ satisfies

$$d\mathcal{E} = [\check{Y} - Y]\,[(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot, \hat{X}, Y)dt$$

with $\hat{X} = \mathrm{x}(X, -Y)$

- We can use a strategy ensuring $Y = \check{Y} = \hat{y}(\cdot, \hat{X})$:
  - Make an initial jump of size $Y_0 = \mathrm{x}^{-1}(x, \mathrm{x}(x, \partial_x w(0, X_{0-})))$.
  - Follow (a, b) such that $Y = \mathrm{x}^{-1}(\hat{X}, \mathrm{x}(\hat{X}, \partial_x w(t, \hat{X})))$.
- Then
  - $V_{T-} = G(\mathrm{x}(X_{T-}, -Y_{T-})) + \mathfrak{I}(\mathrm{x}(X_{T-}, -Y_{T-}), Y_{T-})$.

Pricing equation - verification

- Assume that w is a smooth solution with $w(T-,\cdot) = G$ of

$$\hat{F}[w](\cdot,\hat{y}) = -\partial_t w - \hat{\mu}(\cdot,\hat{y})\partial_x[w+\mathfrak{I}] - \frac{1}{2}\hat{\sigma}(\cdot,\hat{y})^2\partial^2_{xx}[w+\mathfrak{I}] = 0.$$

- Then $\mathcal{E}_t := V - w(\cdot,\mathrm{x}(X,-Y)) - \mathfrak{I}(\mathrm{x}(X,-Y),Y)$ satisfies

$$d\mathcal{E} = [\check{Y} - Y]\,[(\cdots)dt + (\cdots)dW] + \hat{F}[w](\cdot,\hat{X},Y)dt$$

with $\hat{X} = \mathrm{x}(X,-Y)$

- We can use a strategy ensuring $Y = \check{Y} = \hat{y}(\cdot,\hat{X})$:
  - Make an initial jump of size $Y_0 = \mathrm{x}^{-1}(x,\mathrm{x}(x,\partial_x w(0,X_{0-})))$.
  - Follow (a,b) such that $Y = \mathrm{x}^{-1}(\hat{X},\mathrm{x}(\hat{X},\partial_x w(t,\hat{X})))$.
- Then
  - $V_{T-} = G(\mathrm{x}(X_{T-},-Y_{T-})) + \mathfrak{I}(\mathrm{x}(X_{T-},-Y_{T-}),Y_{T-})$.
  - Liquidate Y_{T-} : $V_T = G(X_T)$ and $Y_T = 0$.

Constant impact

- Model : $dX_t = \sigma(X_t)dW_t$ (between trades) and $f(X) = \lambda$.

Constant impact

☐ Model : $dX_t = \sigma(X_t)dW_t$ (between trades) and $f(X) = \lambda$.

☐ In this case, $\mathrm{x}(x,\delta) = x + \lambda\delta$, $\mathfrak{I}(x,\delta) = \frac{1}{2}\delta^2\lambda$, and the pde is

$$-\partial_t w - \frac{1}{2}\sigma^2(x + \lambda\partial_x w)\partial^2_{xx} w = 0$$

For $\lambda = 0$ or $\sigma =$cst, this is the usual heat equation ! ! !

Constant impact

☐ Model : $dX_t = \sigma(X_t)dW_t$ (between trades) and $f(X) = \lambda$.

☐ In this case, $\mathrm{x}(x,\delta) = x + \lambda\delta$, $\mathfrak{I}(x,\delta) = \frac{1}{2}\delta^2\lambda$, and the pde is

$$-\partial_t w - \frac{1}{2}\sigma^2(x + \lambda\partial_x w)\partial^2_{xx} w = 0$$

For $\lambda = 0$ or $\sigma =$cst, this is the usual heat equation ! ! !

☐ We can use the strategy : $Y = \partial_x w(\cdot, \hat{X}) = \partial_x w(\cdot, X - \lambda Y)$.

Thank you very much !