Approche probabiliste pour la résolution d'équations paraboliques semi-linéaires

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BSDEs and PDEs: reminder

Semilinear parabolic PDEs

The solution u of

$$\begin{aligned} -\mathcal{L}u - f(\cdot, u, Du'\sigma) &= 0 \quad \text{on } [0,T) \times \mathbb{R}^d \\ u(T,\cdot) &= g \quad \text{on } \mathbb{R}^d \end{aligned}$$

with $\mathcal{L}$ the Dynkin operator

$$\mathcal{L}u = \frac{\partial}{\partial t}u + b(x)'Du + \frac{1}{2}\text{Tr}\left[\sigma\sigma'(x)D^2u\right]$$

is associated to the solution (Y, Z) of

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

where

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s ,$$

through

$$Y_t = u(t, X_t) \quad , \quad Z_t = Du'\sigma(t, X_t)$$

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Remark: if f is independent of u and Du then

$$Y_t = u(t, X_t) = \mathbb{E}\left[g(X_T) + \int_t^T f(X_s)ds \mid \mathcal{F}_t\right]$$

Numerical resolution: first approaches

• Ma, Protter and Yong (94), Douglas, Ma and Protter (96), Ma, Protter, San Martin and Torres (02):
solve the PDE $\Rightarrow (\hat{u}, \widehat{Du})$ and set $(Y^\pi, Z^\pi) = (\hat{u}, \widehat{Du})(\cdot, X^\pi)$.

• Coquet, Mackevicius and Memin (98), Briand, Delyon and Memin (01), Antonelli and Kohatsu (00):
approximate W by a discrete random walk (with values in a finite state-space) and solve the associated discrete time BSDE.

$\Rightarrow$ Curse of dimensionality !

Euler scheme approximation

The forward process X

- Fix a grid of $[0, T]$: $\pi := \{t_i := hi,\ i \le n\}$ with $h = T/n$.

- Set $X_0^\pi = X_0$

- For $i = 1, \ldots, n$, set

$$X_{t_i}^\pi = X_{t_{i-1}}^\pi + b(X_{t_{i-1}}^\pi)h + \sigma(X_{t_{i-1}}^\pi)(W_{t_i} - W_{t_{i-1}})$$

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- Error:

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |X_t - X_{t_i}^\pi|^2\right]^{\frac{1}{2}} \le Ch^{\frac{1}{2}}\,.$$

The BSDE (Y, Z): Adapted backward Euler scheme

- For $i = n-1, \ldots, 0$, write

$$Y_{t_i} \;\sim\; Y_{t_{i+1}} + f(X_{t_i}, Y_{t_i}, Z_{t_i})h - Z_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{1}$$

and take $\mathbb{E}\left[\cdot \mid \mathcal{F}_{t_i}\right]$ to get

$$Y_{t_i} \;\sim\; \mathbb{E}\left[Y_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + f(X_{t_i}, Y_{t_i}, Z_{t_i})h$$

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and take $\mathbb{E}\left[\cdot \mid \mathcal{F}_{t_i}\right]$ to get

$$Y_{t_i} \sim \mathbb{E}\left[Y_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + f(X_{t_i}, Y_{t_i}, Z_{t_i})h$$

multiply (2) by $(W_{t_{i+1}} - W_{t_i})$

$$\begin{aligned} Y_{t_i}(W_{t_{i+1}} - W_{t_i}) &\sim Y_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) + f(X_{t_i}, Y_{t_i}, Z_{t_i})(W_{t_{i+1}} - W_{t_i})h \\ &- Z_{t_i}(W_{t_{i+1}} - W_{t_i})(W_{t_{i+1}} - W_{t_i}) \end{aligned}$$

and take $\mathbb{E}\left[\cdot \mid \mathcal{F}_{t_i}\right]$

$$0 \sim \mathbb{E}\left[Y_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] - Z_{t_i}h$$

The BSDE (Y, Z): Adapted backward Euler scheme (2)

- Recall:

$$
\begin{aligned}
Y_{t_i} &\sim \mathbb{E}\left[Y_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + f(X_{t_i}, Y_{t_i}, Z_{t_i})h \\
0 &\sim \mathbb{E}\left[Y_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] - Z_{t_i}h
\end{aligned}
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- Set $Y_T^\pi = g(X_T^\pi)$ and for $i = n-1, \ldots, 0$

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Y_{t_i}^\pi = \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + f(X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi)h
$$

where

$$
Z_{t_i}^\pi = h^{-1}\mathbb{E}\left[Y_{t_{i+1}}^\pi(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right]
$$

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- Recall:

$$\begin{aligned} Y_{t_i} &\sim \mathbb{E}\left[Y_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + f(X_{t_i}, Y_{t_i}, Z_{t_i})h \\ 0 &\sim \mathbb{E}\left[Y_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] - Z_{t_i}h \end{aligned}$$

- Set $Y_T^\pi = g(X_T^\pi)$ and for $i = n-1, \ldots, 0$

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where

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- Could alternatively set

$$Y_{t_i}^\pi = \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + \mathbb{E}\left[f(X_{t_i}^\pi, Y_{t_{i+1}}^\pi, Z_{t_i}^\pi) \mid \mathcal{F}_{t_i}\right] h$$

Numerical implementation

Quantization

- Bally, Pages and Printems for the case f independent of Z.

- Replace X^π by a quantized version $\hat{X}^\pi$ taking a finite number of possible values.

- Estimate the transition probabilities of $\hat{X}^\pi$.

- Use the algorithm: $\hat{Y}_T^\pi = g(\hat{X}_T^\pi)$ and for $i = n-1, \ldots, 0$

$$\hat{Y}_{t_i}^\pi \ = \ \mathbb{E}\left[\hat{Y}_{t_{i+1}}^\pi \mid \hat{X}_{t_i}^\pi\right] + f(\hat{X}_{t_i}^\pi, \hat{Y}_{t_i}^\pi)h$$

Pure Monte-Carlo approaches

• Simulate $(X^{\pi,j}, W^j, \;,\; j \leq N)$

• Set $\hat{Y}_T^{\pi,j} = g(X_T^{\pi,j})$

• Given $\hat{\mathbb{E}}$ an approximation of $\mathbb{E}$ based on the simulated data, use the induction

$$
\begin{aligned}
\hat{Y}_{t_i}^{\pi,j} &= \hat{\mathbb{E}}\left[\hat{Y}_{t_{i+1}}^{\pi} \mid X_{t_i}^{\pi,j}\right] + f(X_{t_i}^{\pi,j}, \hat{Y}_{t_i}^{\pi,j}, \hat{Z}_{t_i}^{\pi,j})h \\
\hat{Z}_{t_i}^{\pi,j} &= h^{-1}\hat{\mathbb{E}}\left[\hat{Y}_{t_{i+1}}^{\pi}(W_{t_{i+1}} - W_{t_i}) \mid X_{t_i}^{\pi,j}\right]
\end{aligned}
$$

• Two alternatives :

1. Chevance (97), Longstaff and Schwartz (01), Gobet, Lemor and Warin (05): non-parametric regression.

2. Lions and Regnier (01), B., Ekeland and Touzi (04), B. and Touzi (04): Malliavin calculus approach to rewrite conditional expectations in terms of unconditional expectations.

Approximation error

Control of the approximation error

- Say $f \equiv 0$, then

$$
\begin{aligned}
Y_{t_i} &= g(X_T) + \int_{t_i}^{T} f(X_s, Y_s, Z_s)ds - \int_{t_i}^{T} Z_s dW_s \\
&= Y_{t_{i+1}} - \int_{t_i}^{t_{i+1}} Z_s dW_s
\end{aligned}
$$

implies

$$
Y_{t_i} = \mathbb{E}\left[Y_{t_{i+1}} \mid \mathcal{F}_{t_i} \right].
$$

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$$= Y_{t_{i+1}} - \int_{t_i}^{t_{i+1}} Z_s dW_s$$

implies

$$Y_{t_i} = \mathbb{E}\left[Y_{t_{i+1}} \mid \mathcal{F}_{t_i}\right].$$

Thus

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |Y_t - Y^\pi_{t_i}|^2\right] \geq \max_{i<n} \mathbb{E}\left[|Y_{t_{i+1}} - Y^\pi_{t_i}|^2\right] \geq \max_{i<n} \mathbb{E}\left[|Y_{t_{i+1}} - Y_{t_i}|^2\right]$$
$$\geq c \max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |Y_t - Y_{t_i}|^2\right] =: c\,\mathcal{R}(Y)_{\mathcal{S}^2}$$

for some $c > 0$.

Control of the approximation error (2)

- Set

$$\tilde{Z}_{t_i} := h^{-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i}\right]$$

then

$$\mathbb{E}\left[\sum_i \int_{t_i}^{t_{i+1}} \|Z_t - Z^{\pi}_{t_i}\|^2 dt\right] \geq \mathbb{E}\left[\sum_i \int_{t_i}^{t_{i+1}} \|Z_t - \tilde{Z}_{t_i}\|^2 dt\right] =: \mathcal{R}(Z)_{\mathcal{H}^2}$$

Control of the approximation error (3)

- Conclusion: up to a constant $c > 0$, the error

$$\mathrm{Err}(h)^2 := \max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |Y_t - Y^{\pi}_{t_i}|^2\right] + \mathbb{E}\left[\sum_i \int_{t_i}^{t_{i+1}} \|Z_t - Z^{\pi}_{t_i}\|^2 dt\right]$$

is bounded from below by

$$\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} = \max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |Y_t - Y_{t_i}|^2\right] + \mathbb{E}\left[\sum_i \int_{t_i}^{t_{i+1}} \|Z_t - \tilde{Z}_{t_i}\|^2 dt\right]$$

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- One can actually show that

$$\text{Err}(h)^2 = O\left(\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} + h\right)$$

Control of the approximation error (4)

- Thus

$$\mathrm{Err}(h)^2 = O\left(\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} + h\right)$$

where (formally)

$$\mathcal{R}(Y)_{\mathcal{S}^2} = \max_{i<n} \mathbb{E}[\sup_{t\in[t_i,t_{i+1}]} |\underbrace{u(t,X_t)}_{Y_t} - \underbrace{u(t_i,X_{t_i})}_{Y_{t_i}}|^2]$$

and

$$\mathcal{R}(Z)_{\mathcal{H}^2} = \mathbb{E}[\sum_i \int_{t_i}^{t_{i+1}} \| \underbrace{Du'\sigma(t,X_t)}_{Z_t} - \underbrace{h^{-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Du'\sigma(s,X_s) \mid \mathcal{F}_{t_i}\right]}_{\tilde{Z}_{t_i}} \|^2 dt]$$

- The error depends on a very weak notion of regularity of (u, Du).

Regularity results

Semilinear PDEs

• Theorem (Ma and Zhang 02, and B. and Touzi 04): Assume all the coefficients are Lipschitz continuous. Then,

$$\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} = O(h) \quad \text{and} \quad \text{Err}(h) = O(h^{\frac{1}{2}})$$

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• Corollary: u is $\frac{1}{2}$-Hölder in t and Lipschitz in x.

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- Elements of proof for $\mathcal{R}(Z)_{\mathcal{H}^2}$: (case $f = 0$, $d = 1$, smooth coefficients)

$$\begin{aligned} Y_t &= u(t, X_t) = \mathbb{E}\left[g(X_T) \mid \mathcal{F}_t\right] \\ Z_t &= Du(t, X_t)\sigma(X_t) = \frac{\partial}{\partial X_0} u(t, X_t)\left(\frac{\partial}{\partial X_0} X_t\right)^{-1} \sigma(X_t) \end{aligned}$$

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&= \mathbb{E}\left[Dg(X_T)\frac{\partial}{\partial X_0} X_T \mid \mathcal{F}_t\right] \underbrace{\left(\frac{\partial}{\partial X_0} X_t\right)^{-1} \sigma(X_t)}_{\text{say}=1 \text{ for simplicity}}
\end{aligned}$$

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$$\mathbb{E}\left[|Z_t - Z_{t_i}|^2\right] \leq \mathbb{E}\left[Z_{t_{i+1}}^2 - Z_{t_i}^2\right] \quad , \; t \in [t_i, t_{i+1}]$$

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and

$$\int_{t_i}^{t_i+1} \mathbb{E}\left[|Z_t - \tilde{Z}_{t_i}|^2\right] dt \le \int_{t_i}^{t_i+1} \mathbb{E}\left[|Z_t - Z_{t_i}|^2\right] dt \le h\mathbb{E}\left[Z^2_{t_{i+1}} - Z^2_{t_i}\right] \; .$$

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We thus obtain a $O(h)$ behavior for

$$\mathcal{R}(Y)_{\mathcal{S}^2} = \max_{i<n} \mathbb{E}[\sup_{t\in[t_i,t_{i+1}]} |\underbrace{u(t,X_t)}_{Y_t} - \underbrace{u(t_i,X_{t_i})}_{Y_{t_i}}|^2]$$

and

$$\mathcal{R}(Z)_{\mathcal{H}^2} = \mathbb{E}[\sum_i \int_{t_i}^{t_{i+1}} \| \underbrace{Du'\sigma(t,X_t)}_{Z_t} - \underbrace{h^{-1}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Du'\sigma(s,X_s) \mid \mathcal{F}_{t_i}\right]}_{\tilde{Z}_{t_i}} \|^2 dt]$$

with the only assumption that the coefficients are Lipschitz continuous. No ellipticity condition.

Extension 1:

Semilinear parabolic IPDEs

and systems

B. and Elie (05)

PDEs with integral term

The solution u of

$$-\mathcal{L}u - f(\cdot, u, Du'\sigma, \mathcal{I}[u](t,x)) \;=\; 0 \quad \text{on } [0,T) \times \mathbb{R}^d \,,\; u(T,\cdot){=}g \quad \text{on } \mathbb{R}^d$$

with the non local term

$$\mathcal{I}[u](t,x) := \int_E \{u(t, x+\beta(x,e)) - u(t,x)\}\, \rho(e)\, \lambda(de)$$

and $\mathcal{L}$ the non local Dynkin operator

$$\mathcal{L}u = \frac{\partial}{\partial t}u + b(x)'Du + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma'(x)D^2u\right] + \int_E \{u(t,x+\beta(x,e)) - u(t,x) - Du(t,x)\beta(x,e)\}\, \lambda(de)$$

is associated to the solution (Y,Z,U) of

$$Y_t{=}g(X_T) + \int_t^T f(X_s, Y_s, Z_s, \int_E \rho(e)U_s(e)\lambda(de))ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\bar{\mu}(de,ds)$$

where

$$X_t \;=\; X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_E \beta(X_{s-},e)\bar{\mu}(de,ds)$$

through

$$Y_t = u(t,X_t) \quad , \quad Z_t = Du'\sigma(t,X_t)\,,\; U_t(e) = u(t, X_{t-} + \beta(X_{t-},e)) - u(t,X_{t-})$$

Systems of PDEs

Pardoux, Pradeilles and Rao (97), Sow and Pardoux (04).

- **System of κ PDE's ($m = 0, \ldots, \kappa - 1$)**

$$\begin{aligned} 0 &= u_t^m + b_m' D u^m + \frac{1}{2}\mathrm{Tr}[\sigma_m \sigma_m' D^2 u^m] + f_m(\cdot, u, (Du^m)'\sigma_m) \\ g_m &= u^m(T, \cdot)\ . \end{aligned}$$

- Define for $i = 0, \ldots, \kappa - 1$

$$\tilde{f}(m, x, y, \gamma, z) \;=\; f_m\left(x, (\ldots, y + \gamma^{\kappa-2}, y + \gamma^{\kappa-1}, \underbrace{y}_{i}, y + \gamma^1, y + \gamma^2, \ldots), z\right)$$

- Set $E = \{1, \ldots, \kappa - 1\}$, $\lambda(de) = \lambda \sum_{k=1}^{\kappa-1} \delta_k(e)$ and

$$M_t \;=\; \int_0^t \int_E e\mu(de, ds) \ \ [\kappa]$$

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$\Rightarrow u^{M_t}(t, X_t) = Y_t$ where

$$\begin{aligned} dX_t &= b_{M_t}(X_t)dt + \sigma_{M_t}(X_t)dW_t \\ -dY_t &= \tilde{f}(M_t, X_t, Y_t, U_t, Z_t)dt - \lambda \sum_{k=1}^{\kappa-1} U(k)_t dt - Z_t dW_t - \int_E U_t(e)\bar{\mu}(de, dt) \\ Y_T &= g_{M_T}(X_T) \end{aligned}$$

Regularity result

- Theorem (B. and Elie 05): Assume all the coefficients are Lipschitz continuous and that $\mathbf{H}$: For each $e \in E$, the map $x \in \mathbb{R}^d \mapsto \beta(x,e)$ admits a Jacobian matrix $\nabla\beta(x,e)$ such that the function

$$(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto a(x,\xi;e) := \xi'(\nabla\beta(x,e) + I_d)\xi$$

satisfies one of the following condition uniformly in $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$

$$a(x,\xi;e) \ \geq \ |\xi|^2 K^{-1} \quad \text{or} \quad a(x,\xi;e) \ \leq \ -|\xi|^2 K^{-1} \ .$$

Then,

$$\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} = O(h) \quad \text{and} \quad \mathrm{Err}(h) = O(h^{\frac{1}{2}})$$

Remark: Same result without $\mathbf{H}$ if the coefficients are C^1_b with Lipschitz first derivatives.

Extension 2:

Free boundary problems

B. and J.-F. Chassagneux (06)

Representation

The solution u of

$$\min\left\{-\mathcal{L}u - f(\cdot, u, Du'\sigma)\ ,\ u-g\right\} \;=\; 0 \quad \text{on } [0,T)\times\mathbb{R}^d\ ,\ u(T,\cdot)=g \quad \text{on } \mathbb{R}^d$$

is associated to the solution (Y, Z, K) of

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t \\ Y_t &\geq g(X_t)\ ,\ t \leq T\ ,\ \int_0^T (Y_s - g(X_s))\mathrm{d}K_s = 0 \quad \text{and} \quad K\uparrow , \end{aligned}$$

through

$$Y_t = u(t, X_t)\quad ,\quad Z_t = Du'\sigma(t, X_t)$$

Approximation scheme

- Backward "American" scheme:

$$
\begin{aligned}
Z^\pi_{t_i} &= h^{-1}\,\mathbb{E}\left[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\
\tilde{Y}^\pi_{t_i} &= \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + h\, f(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}) \\
Y^\pi_{t_i} &= \max\left\{g(X^\pi_{t_i})\ ,\ \tilde{Y}^\pi_{t_i}\right\}\ ,\ i \le n-1\ .
\end{aligned}
$$

with the terminal condition

$$
Y^\pi_T = g(X^\pi_T)\ .
$$

Formulation for Z ?

- Previous approach ($d = 1,\ f = 0$)

$$
\begin{aligned}
Y_t &= u(t, X_t) = \mathbb{E}\left[g(X_{\tau^t}) \mid \mathcal{F}_t\right] \text{ with } \tau^t := \inf\{s \geq t \ :\ Y_s = g(X_s)\} \\
Z_t &= Du(t, X_t)\sigma(X_t) = \frac{\partial}{\partial X_0} u(t, X_t)(\frac{\partial}{\partial X_0} X_t)^{-1}\sigma(X_t) \\
&= \mathbb{E}\left[Dg(X_{\tau^t})\frac{\partial}{\partial X_0} X_{\tau^t} \mid \mathcal{F}_t\right](\frac{\partial}{\partial X_0} X_t)^{-1}\sigma(X_t)
\end{aligned}
$$

$\Rightarrow$ Problem...

Discretely reflected BSDE

- (Y, Z, K) solution of

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + K_T - K_t \\ Y_t &\geq g(X_t) \quad , \quad t \in \pi \, , \end{aligned}$$

with $K_{t_{i+1}} = K_{t_i} + [Y_{t_{i+1}-} - g(X_{t_{i+1}})]^- \; .$

- Then (for $f = 0$)

$$Z_t = \mathbb{E}\left[Dg(X_T) \nabla X_T + \sum_{t_{i+1} > t} \frac{\partial}{\partial X_0} [Y_{t_{i+1}-} - g(X_{t_{i+1}})]^- \mid \mathcal{F}_t \right] (\nabla X_t)^{-1} \sigma(X_t)$$

- IPP in the Malliavin sens

$$Z_t = \mathbb{E}\left[g(X_T) N_T^t + \sum_{t_{i+1} > t} [Y_{t_{i+1}-} - g(X_{t_{i+1}})]^- N_{t_{i+1}}^t \mid \mathcal{F}_t \right] (\nabla X_t)^{-1} \sigma(X_t)$$

with

$$N_s^t \; := \; (s-t)^{-1} \int_t^s \sigma(X_r)^{-1} \nabla X_r dW_r$$

Regularity result and convergence rate (1)

Take the limit

$$Z_t = \mathbb{E}\left[g(X_T)N_T^t + \int_t^T f(\Theta_s)N_s^t ds + \int_t^T N_s^t dK_s \mid \mathcal{F}_t\right](\nabla X_t)^{-1}\sigma(X_t)$$

with

$$N_s^t \ := \ (s-t)^{-1}\int_t^s \sigma(X_r)^{-1}\nabla X_r dW_r$$

Theorem (Ma and Zhang 05): Assume that all the coefficients are Lipschitz, b and $\sigma \in C_b^1$, $g \in C_b^{1,2}$ and σ is uniformly elliptic. Then,

$$\mathcal{R}(Y)_{\mathcal{S}^2} = O(h)\ ,\ \mathcal{R}(Z)_{\mathcal{H}^2} = O(h^{\frac{1}{2}}) \quad \text{and} \quad \text{Err}(h) = O(h^{\frac{1}{4}})$$

Regularity result and convergence rate (2)

Alternative representation (written formally in the case $f = 0$, u smooth and $Du = Dg$ on $\{u = g\}$): Use the martingale property of $Du(t, X_t)\nabla X_t$ to get

$$Z_t = \mathbb{E}\left[Dg(X_{\tau^t})\nabla X_{\tau^t} \mid \mathcal{F}_t\right](\nabla X_t)^{-1}\sigma(X_t)$$

Theorem (B. and Chassagneux 06): Assume that all the coefficients are Lipschitz, $g \in C_b^1$ with Lipschitz derivatives. Then,

$$\mathcal{R}(Y)_{\mathcal{S}^2} = O(h)\ ,\ \mathcal{R}(Z)_{\mathcal{H}^2} = O(h^{\frac{1}{2}}) \quad \text{and} \quad \text{Err}(h) = O(h^{\frac{1}{4}})$$

If moreover, $\sigma \in C_b^1$ with Lipschitz derivatives and $g \in C_b^2$ with Lipschitz first and second derivatives, then

$$\max_{i<n} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} |Y_t - Y_{t_i}^\pi|^2\right] = O(h^{\frac{1}{2}})\ .$$

If in addition to the previous condition $X = X^\pi$ on π, then

$$\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} = O(h) \quad \text{and} \quad \text{Err}(h) = O(h^{\frac{1}{2}})\ .$$

Extension 3:

Cauchy-Dirichlet problems

B. and S. Menozzi (07)

Representation

The solution u of

$$\begin{aligned} -\mathcal{L}u - f(\cdot, u, Du'\sigma) &= 0 \quad \text{on } D := [0,T) \times \mathcal{O} \\ u &= g \quad \text{on } \partial_P D := ([0,T) \times \partial\mathcal{O}) \cup (\{T\} \times \bar{\mathcal{O}}) \end{aligned}$$

is associated to the solution (Y, Z) of

$$Y_t = g(X_\tau) + \int_t^\tau f(X_s, Y_s, Z_s)ds - \int_t^\tau Z_s dW_s$$

where

$$\tau = \inf \{t \geq 0 \ : \ (t, X_t) \notin [0,T) \times \mathcal{O}\},$$

through

$$Y_t = u(t \wedge \tau, X_{t\wedge\tau}) \quad , \quad Z_t = Du'\sigma(t, X_t)\mathbf{1}_{t\leq\tau}$$

Approximation scheme

We approximate the first exit time τ by

$$\tau^\pi := \inf\{t \in \pi \ : \ (t, X_t^\pi) \notin D\} \ .$$

The Euler scheme is defined as previously with $Y_{\tau^\pi}^\pi = g(X_{\tau^\pi}^\pi)$ and

$$\begin{aligned} Z_{t_i}^\pi &= h^{-1}\,\mathbb{E}\left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ Y_{t_i}^\pi &= \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + h\, f(X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi) \end{aligned}$$

Representation in the smooth case

For ease of notations ($d = 1$ and $f = 0$): martingale property of $Du(t, X_t)\nabla X_t$ gives

$$\begin{aligned} Z_t &= Du'\sigma(t, X_t)\mathbf{1}_{t\leq\tau} \\ &= \mathbb{E}\left[Du(\tau, X_\tau)\nabla X_\tau/\nabla X_t \mid \mathcal{F}_t\right]\sigma(X_t)\mathbf{1}_{t\leq\tau} \end{aligned}$$

If Du bounded, we can use the same technique as in the first case to bound $\mathcal{R}(Z)^{\pi}_{\mathcal{H}^2}$!

Gradient bound on the boundary

HL: All coefficients are Lipschitz.
D1: $\mathcal{O} := \bigcap_{\ell=1}^m \mathcal{O}^\ell$ where $\mathcal{O}^\ell$ is a C^2 domain of $\mathbb{R}^d$ with a compact boundary.
D2. For all $x \in \partial\mathcal{O}$, there is $y(x) \in \mathcal{O}^c$, $r(x) \in [L^{-1}, L]$ and $\delta(x) \in B(0,1)$ such that $\bar{B}(y(x), r(x)) \cap \bar{\mathcal{O}} = \{x\}$ and

$$\{x' \in B(x, L^{-1}) \ : \ \langle x' - x, \delta(x) \rangle \geq (1 - L^{-1}) \|x' - x\| \} \subset \bar{\mathcal{O}} \, .$$

C. The boundary satisfies a non characteristic condition outside a neighborhood of $\mathcal{C} := \bigcap_{\ell \neq k=1}^m \partial\mathcal{O}^\ell \cap \partial\mathcal{O}^k$ and σ is uniformly elliptic on a neighborhood of $\mathcal{C}$.
Hg: $g \in C^{1,2}(\bar{D})$ and $\|\partial_t g\| + \|Dg\| + \|D^2 g\| \ \leq \ L \quad$ on $\bar{D}$.

Theorem: Assume that the above conditions hold. Then, u is uniformly Lipschitz continuous and $|Z| \leq \xi$ a.e. for some $\xi \in L^p$ for all $p \geq 2$.

Regularity under general conditions

Recall that (formally) for $d = 1$ and $f = 0$:

$$\begin{aligned} Z_t &= Du'\sigma(t, X_t)\mathbf{1}_{t \le \tau} \\ &= \mathbb{E}\left[Du(\tau, X_\tau)\nabla X_\tau / \nabla X_t \mid \mathcal{F}_t\right] \sigma(t, X_t)\mathbf{1}_{t \le \tau} \end{aligned}$$

Corollary: Assume that the above conditions hold. Then,

$$\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} = O(h) \ .$$

Abstract error and exit time approximation

Proposition: Assume that **HL** and **Hg** hold. Then,

$$\text{Err}(h)_T^2 \;\leq\; C\left(h + \mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} + \mathbb{E}\left[\xi|\tau - \tau^\pi|\right]\right)$$

and

$$\text{Err}(h)_{\tau\wedge\tau^\pi}^2 \;\leq\; C\left(h + \mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} + \mathbb{E}\left[\mathbb{E}\left[\xi|\tau - \tau^\pi| \mid \mathcal{F}_{\tau_+\wedge\tau^\pi}\right]^2\right]\right)$$

where τ_+ is the next time after τ in the grid π:

$$\tau_+ := \inf\{t \in \pi \;:\; \tau \leq t\}\,.$$

Abstract error and exit time approximation

Proposition: Assume that **HL** and **Hg** hold. Then,

$$\text{Err}(h)_T^2 \;\le\; C\left(h + \mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} + \mathbb{E}\left[\xi|\tau-\tau^\pi|\right]\right)$$

and

$$\text{Err}(h)_{\tau\wedge\tau^\pi}^2 \;\le\; C\left(h + \mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2} + \mathbb{E}\left[\mathbb{E}\left[\xi|\tau-\tau^\pi| \mid \mathcal{F}_{\tau_+\wedge\tau^\pi}\right]^2\right]\right)$$

where τ_+ is the next time after τ in the grid π:

$$\tau_+ := \inf\{t\in\pi \;:\; \tau\le t\}\,.$$

Theorem: Assume that **HL**, **D1** and **C** hold. Then, for $\varepsilon\in(0,1)$ and each positive random variable $\xi\in\cap_p L^p$ there is $C^\varepsilon>0$ such that

$$\mathbb{E}\left[\xi\,\mathbb{E}\left[\xi\,|\tau-\tau^\pi| \mid \mathcal{F}_{\tau_+\wedge\tau^\pi}\right]^2\right] \;\le\; C^\varepsilon h^{1-\varepsilon}\,.$$

In particular, for each $\varepsilon\in(0,1/2)$,

$$\mathbb{E}\left[|\tau-\tau^\pi|\right] \;\le\; C^\varepsilon h^{1/2-\varepsilon}\,.$$

Global approximation error

Theorem: Assume that **HL** and **Hg** hold. Then,

$$\text{Err}(h)^2_T \leq C(h + \underbrace{\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2}}_{O(h)} + \underbrace{\mathbb{E}\left[\xi|\tau - \tau^\pi|\right]}_{O(h^{\frac{1}{2}-\varepsilon})})$$

and

$$\text{Err}(h)^2_{\tau\wedge\tau^\pi} \leq C(h + \underbrace{\mathcal{R}(Y)_{\mathcal{S}^2} + \mathcal{R}(Z)_{\mathcal{H}^2}}_{O(h)} + \underbrace{\mathbb{E}\left[\mathbb{E}\left[\xi|\tau - \tau^\pi| \mid \mathcal{F}_{\tau_+\wedge\tau^\pi}\right]^2\right]}_{O(h^{1-\varepsilon})})$$

In particular: $u(0, X_0) - Y_0^\pi = O(h^{\frac{1}{2}-\varepsilon})$ (weak error).

Remaining questions

Semilinear PDEs with quadratic driver ?

Elliptic semilinear PDEs ?

FBSDEs and quasilinear PDEs ?