Stochastic target game problems

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - Ensae-ParisTech

Université Pierre et Marie Curie, Mai 2012

Joint work with L. Moreau (Paris-Dauphine) and M. Nutz (Columbia)

Problem formulation and Motivations

Problem formulation

Determine the *viability* sets

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

In which :

- $\mathcal{V}$ is a set of admissible adverse controls
- $\mathfrak{U}$ is a set of admissible strategies
- $Z_{t,z}^{\nu[\vartheta],\vartheta}$ is an adapted $\mathbb{R}^d$-valued process s.t. $Z_{t,z}^{\nu[\vartheta],\vartheta}(t) = z$
- ℓ is a given loss/utility function
- p a threshold.

Examples

- $Z^{\nu[\vartheta],\vartheta}_{t,z} = (X^{\vartheta}_{t,x}, Y^{\nu[\vartheta],\vartheta}_{t,x,y})$ where
  - $X^{\nu[\vartheta],\vartheta}_{t,x}$ models financial assets or factors with dynamics depending on ϑ
  - $Y^{\nu[\vartheta],\vartheta}_{t,x,y}$ models a wealth process
  - ϑ is the control of the market : parameter uncertainty (e.g. volatility), adverse players, etc...
  - $\nu[\vartheta]$ is the financial strategy given the past observations of ϑ.
- Flexible enough to embed constraints, transaction costs, market impact, etc...

Examples

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

Examples

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ Almost sure constraint :

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z_{t,z}^{\nu[\vartheta],\vartheta}(T) \in \mathcal{O}\ \mathbb{P}-\text{a.s. } \forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z\in\mathcal{O}}$, $p = 1$.

$\Rightarrow$ Super-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

Examples

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ Almost sure constraint :

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z_{t,z}^{\nu[\vartheta],\vartheta}(T) \in \mathcal{O}\ \mathbb{P}-\text{a.s. } \forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z\in\mathcal{O}}$, $p = 1$.
$\Rightarrow$ Super-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

□ Compare with Peng (G-expectations) and Soner, Touzi and Zhang (2BSDE).

Examples

☐ Constraint in probability :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{P}\left[Z_{t,z}^{\nu[\vartheta],\vartheta}(T) \in \mathcal{O}\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z\in\mathcal{O}}$, $p \in (0,1)$.
$\Rightarrow$ Quantile-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

Examples

$$\Lambda(t) := \left\{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\right\}$$

Examples

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

- Expected loss control for $\ell(z) = -[y - g(x)]^-$

Examples

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

- Expected loss control for $\ell(z) = -[y - g(x)]^-$

- Can impose several constraint : B. and Thanh Nam (discrete P&L constraints).

Examples

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ Expected loss control for $\ell(z) = -[y - g(x)]^-$

□ Can impose several constraint : B. and Thanh Nam (discrete P&L constraints).

□ Give sense to problems that would be degenerate under $\mathbb{P}$ − a.s. constraints : B. and Dang (guaranteed VWAP pricing).

Aim

Provide a **direct** PDE characterization Λ

Aim

Provide a **direct** PDE characterization Λ :
this requires a Dynamic Programming Principle.

Geometric Dynamic Programming Principle I

The case without adverse control

The GDPP for $\mathbb{P}$ – a.s. criteria

- $\mathbb{P}$ – a.s. criteria (Soner and Touzi)

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in \mathcal{O}\}$$

The GDPP for $\mathbb{P}$ – a.s. criteria

- $\mathbb{P}$ – a.s. criteria (Soner and Touzi)

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in \mathcal{O}\}$$

- GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) = \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau)\}$$

The GDPP for $\mathbb{P}$ – a.s. criteria

□ $\mathbb{P}$ – a.s. criteria (Soner and Touzi)

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in \mathcal{O}\}$$

□ GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) = \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau)\}$$

□ Extended by B. and Thanh Nam to constraints on the whole path (obstacle version).

The GDPP for $\mathbb{P}$ – a.s. criteria

□ $\mathbb{P}$ – a.s. criteria (Soner and Touzi)

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in \mathcal{O}\}$$

□ GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) = \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau)\}$$

□ Extended by B. and Thanh Nam to constraints on the whole path (obstacle version).

□ Relies on a measurable selection argument : not very flexible, requires topological properties.

The GDPP for $\mathbb{P}-$ a.s. criteria

□ $\mathbb{P}-$ a.s. criteria (Soner and Touzi)

$$\Lambda(t) := \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in \mathcal{O}\}$$

□ GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) = \{z : \exists\, \nu \in \mathfrak{U} \text{ s.t. } Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau)\}$$

□ Extended by B. and Thanh Nam to constraints on the whole path (obstacle version).

□ Relies on a measurable selection argument : not very flexible, requires topological properties.

□ Is enough to provide a PDE characterization for $(t, z) \mapsto 1_{z\in\Lambda(t)}$.

The GDPP for criteria in expectation

- Controlled loss :

$$\Lambda(t) := \{(z, p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

The GDPP for criteria in expectation

- Controlled loss :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

- GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) = \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } (Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau)\}$$

The GDPP for criteria in expectation

□ Controlled loss :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

□ GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) \neq \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } (Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau)\}$$

since

$$(Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau) \text{``} \Rightarrow \text{''} M^{\nu}(\tau) := \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T)) | \mathcal{F}_{\tau}\right] \geq p.$$

The GDPP for criteria in expectation

□ Controlled loss :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

□ GDPP : for all stopping time τ with values in $[t, T]$

$$\Lambda(t) \neq \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } (Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau)\}$$

since

$$(Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau) \text{``} \Rightarrow\text{''}\ M^{\nu}(\tau) := \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T)) | \mathcal{F}_{\tau}\right] \geq p.$$

But

$$\Lambda(t) = \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } (Z^{\nu}_{t,z}(\tau), M^{\nu}(\tau)) \in \Lambda(\tau)\}$$

The GDPP for criteria in expectation

□ Controlled loss :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

□ GDPP : for all stopping time τ with values in $[t,T]$

$$\Lambda(t) \neq \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } (Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau)\}$$

since

$$(Z^{\nu}_{t,z}(\tau), p) \in \Lambda(\tau) \text{``} \Rightarrow \text{''} M^{\nu}(\tau) := \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T)) | \mathcal{F}_{\tau}\right] \geq p.$$

But

$$\Lambda(t) = \{(z,p) : \exists\, (\nu, M) \in \mathfrak{U} \times \mathcal{M}_{t,p} \text{ s.t. } (Z^{\nu}_{t,z}(\tau), M(\tau)) \in \Lambda(\tau)\}$$

The GDPP for criteria in expectation

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

□ Alternative formulation in terms of the set $\mathcal{M}_{t,p}$ of martingales starting from p at t :

$$\Lambda(t) := \{(z,p) : \exists\, (\nu, M) \in \mathfrak{U} \times \mathcal{M}_{t,p} \text{ s.t. } \ell(Z^{\nu}_{t,z}(T)) \geq M(T)\}$$

The GDPP for criteria in expectation

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

□ Alternative formulation in terms of the set $\mathcal{M}_{t,p}$ of martingales starting from p at t :

$$\Lambda(t) := \{(z,p) : \exists\, (\nu, M) \in \mathfrak{U} \times \mathcal{M}_{t,p} \text{ s.t. } \ell(Z^{\nu}_{t,z}(T)) \geq M(T)\}$$

□ In Brownian diffusion settings :

$$\Lambda(t) := \{(z,p) : \exists\, (\nu, \alpha) \in \mathfrak{U} \times \mathcal{A} \text{ s.t. } \ell(Z^{\nu}_{t,z}(T)) \geq M^{\alpha}_{t,p}(T)\}$$

where

$$M^{\alpha}_{t,p} := p + \int_t^{\cdot} \alpha_s dW_s .$$

The GDPP for criteria in expectation

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}$$

□ Alternative formulation in terms of the set $\mathcal{M}_{t,p}$ of martingales starting from p at t :

$$\Lambda(t) := \{(z,p) : \exists\, (\nu, M) \in \mathfrak{U} \times \mathcal{M}_{t,p} \text{ s.t. } \ell(Z^{\nu}_{t,z}(T)) \geq M(T)\}$$

□ In Brownian diffusion settings :

$$\Lambda(t) := \{(z,p) : \exists\, (\nu, \alpha) \in \mathfrak{U} \times \mathcal{A} \text{ s.t. } \ell(Z^{\nu}_{t,z}(T)) \geq M^{\alpha}_{t,p}(T)\}$$

where

$$M^{\alpha}_{t,p} := p + \int_t^{\cdot} \alpha_s dW_s.$$

□ Back to $\mathbb{P}-$a.s. criteria up to an increase of the controls and the system : B., Elie and Touzi.

Geometric Dynamic Programming Principle II

The game formulation

The GDPP for games

- Back to the original problem :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

The GDPP for games

- Back to the original problem :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

- Submartingale property : for ν fixed

$$S_s^{\nu,\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

defines a family of submartingales parameterized by ϑ.

The GDPP for games

□ Back to the original problem :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ Submartingale property : for ν fixed

$$S_s^{\nu,\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

defines a family of submartingales parameterized by ϑ.

□ Doob-Meyer decomposition : $\exists$ a family of martingales $\{M^{\nu,\vartheta},\vartheta\}$ such that $S^{\nu,\vartheta} \geq M^{\nu,\vartheta}$ with $M^{\nu,\vartheta}(t) = p$.

The GDPP for games

□ Back to the original problem :

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ Submartingale property : for ν fixed

$$S_s^{\nu,\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

defines a family of submartingales parameterized by ϑ.

□ Doob-Meyer decomposition : $\exists$ a family of martingales $\{M^{\nu,\vartheta},\vartheta\}$ such that $S^{\nu,\vartheta} \geq M^{\nu,\vartheta}$ with $M^{\nu,\vartheta}(t) = p$.

□ $(z,p) \in \Lambda(t)$ if and only if there exists $\nu \in \mathfrak{U}$ and a family of martingales $\{M^{\vartheta},\vartheta\}$ with $M^{\vartheta}(t) = p$ such that

$$\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T)) \geq M^{\vartheta}(T)\; \mathbb{P}-\text{a.s. } \forall\, \vartheta \in \mathcal{V}.$$

The GDPP for games

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu[\vartheta],\vartheta}_{t,z}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ $(z,p) \in \Lambda(t)$ if and only if there exists $\nu \in \mathfrak{U}$ and a family of martingales $\{M^\vartheta, \vartheta\}$ with $M^\vartheta(t) = p$ such that

$$(Z^{\nu[\vartheta],\vartheta}_{t,z}(\tau), M^\vartheta(\tau)) \in \Lambda(\tau) \;\mathbb{P}-\text{a.s.} \;\forall\, \vartheta \in \mathcal{V}.$$

The GDPP for games

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

□ $(z,p) \in \Lambda(t)$ if and only if there exists $\nu \in \mathfrak{U}$ and a family of martingales $\{M^{\vartheta}, \vartheta\}$ with $M^{\vartheta}(t) = p$ such that

$$(Z_{t,z}^{\nu[\vartheta],\vartheta}(\tau), M^{\vartheta}(\tau)) \in \Lambda(\tau)\ \mathbb{P}-\text{a.s. } \forall\, \vartheta \in \mathcal{V}.$$

□ Main difficulty : how to rely on a measurable selection argument as in the previous cases ?

The GDPP for games

$$\Lambda(t) := \{(z,p) : \exists\, \nu \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta],\vartheta}(T))\right] \geq p\ \forall\, \vartheta \in \mathcal{V}\}$$

□ $(z,p) \in \Lambda(t)$ if and only if there exists $\nu \in \mathfrak{U}$ and a family of martingales $\{M^\vartheta, \vartheta\}$ with $M^\vartheta(t) = p$ such that

$$(Z_{t,z}^{\nu[\vartheta],\vartheta}(\tau), M^\vartheta(\tau)) \in \Lambda(\tau)\ \mathbb{P}-\text{a.s. } \forall\, \vartheta \in \mathcal{V}.$$

□ Main difficulty : how to rely on a measurable selection argument as in the previous cases ?

□ Solution : provide a weak formulation, under a suitable continuity assumption.

A Weak version of the Geometric Dynamic Programming Principle for game formulations

Abstract setting

- To $(\nu, \vartheta) \in \mathfrak{U} \times \mathcal{V}$ associate a $\mathbb{R}^d$-valued càdlàg process $Z^{\nu,\vartheta}_{t,z}(\cdot) = Z^{\nu[\vartheta],\vartheta}_{t,z}(\cdot)$ with values in $\mathbb{R}^d$ such that $Z^{\nu,\vartheta}_{t,z}(t) = z$.

Abstract setting

- To $(\nu, \vartheta) \in \mathfrak{U} \times \mathcal{V}$ associate a $\mathbb{R}^d$-valued càdlàg process $Z_{t,z}^{\nu,\vartheta}(\cdot) = Z_{t,z}^{\nu[\vartheta],\vartheta}(\cdot)$ with values in $\mathbb{R}^d$ such that $Z_{t,z}^{\nu,\vartheta}(t) = z$.

- Set

$$I(t, z, \nu, \vartheta) := \mathbb{E}\left[\ell\left(Z_{t,z}^{\nu,\vartheta}(T)\right) | \mathcal{F}_t\right] \text{ and } J(t, z, \nu) := \operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} I(t, z, \nu, \vartheta).$$

Abstract setting

- To $(\nu, \vartheta) \in \mathfrak{U} \times \mathcal{V}$ associate a $\mathbb{R}^d$-valued càdlàg process $Z^{\nu,\vartheta}_{t,z}(\cdot) = Z^{\nu[\vartheta],\vartheta}_{t,z}(\cdot)$ with values in $\mathbb{R}^d$ such that $Z^{\nu,\vartheta}_{t,z}(t) = z$.

- Set

$$I(t,z,\nu,\vartheta) := \mathbb{E}\left[\ell\left(Z^{\nu,\vartheta}_{t,z}(T)\right)|\mathcal{F}_t\right] \text{ and } J(t,z,\nu) := \operatorname*{ess\,inf}_{\vartheta\in\mathcal{V}} I(t,z,\nu,\vartheta).$$

- We consider

$$\Lambda(t) := \left\{(z,p) \in \mathbb{R}^d \times \mathbb{R} : \exists\, \nu \in \mathfrak{U} \text{ s.t. } J(t,z,\nu) \geq p \ \mathbb{P}-\text{a.s.}\right\}.$$

Concatenation possibilities

- **Adverse controls :** $\vartheta := \vartheta_0 \oplus_t (\vartheta_1 \mathbf{1}_A + \vartheta_2 \mathbf{1}_{A^c}) \in \mathcal{V}$,
for $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathcal{V}$, and $A \in \mathcal{F}_t$

Concatenation possibilities

□ **Adverse controls :** $\vartheta := \vartheta_0 \oplus_t (\vartheta_1 \mathbf{1}_A + \vartheta_2 \mathbf{1}_{A^c}) \in \mathcal{V}$,
for $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathcal{V}$, and $A \in \mathcal{F}_t$

□ **Non-anticipating strategies :**
$\nu : \vartheta \in \mathcal{V} \mapsto \nu[\vartheta] := \nu_0[\vartheta] \oplus_t \sum_{j \geq 1} \nu_j[\vartheta] \mathbf{1}_{A_j^\vartheta} \in \mathfrak{U}$,
for $\{A_j^\vartheta, \vartheta \in \mathcal{V}\}_{j \geq 1} \subset \mathfrak{F}_t$ such that $\{A_j^\vartheta, j \geq 1\}$ forms a partition of Ω for each $\vartheta \in \mathcal{V}$.

Concatenation possibilities

□ **Adverse controls :** $\vartheta := \vartheta_0 \oplus_t (\vartheta_1 \mathbf{1}_A + \vartheta_2 \mathbf{1}_{A^c}) \in \mathcal{V}$,
for $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathcal{V}$, and $A \in \mathcal{F}_t$

□ **Non-anticipating strategies :**
$\nu : \vartheta \in \mathcal{V} \mapsto \nu[\vartheta] := \nu_0[\vartheta] \oplus_t \sum_{j \geq 1} \nu_j[\vartheta] \mathbf{1}_{A_j^\vartheta} \in \mathfrak{U}$,
for $\{A_j^\vartheta, \vartheta \in \mathcal{V}\}_{j \geq 1} \subset \mathfrak{F}_t$ such that $\{A_j^\vartheta, j \geq 1\}$ forms a partition of Ω for each $\vartheta \in \mathcal{V}$.

□ **Non-anticipating event sets :** $\{A^\vartheta, \vartheta \in \mathcal{V}\} \in \mathfrak{F}_t$ if
$\{A^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathcal{F}_t$ is such that
$A^{\vartheta_1} \cap \{\vartheta_1 =_{(0,t]} \vartheta_2\} = A^{\vartheta_2} \cap \{\vartheta_1 =_{(0,t]} \vartheta_2\}$ for $\vartheta_1, \vartheta_2 \in \mathcal{V}$.

Concatenation possibilities

□ **Adverse controls :** $\vartheta := \vartheta_0 \oplus_t (\vartheta_1 \mathbf{1}_A + \vartheta_2 \mathbf{1}_{A^c}) \in \mathcal{V}$, for $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathcal{V}$, and $A \in \mathcal{F}_t$

□ **Non-anticipating strategies :**
$\nu : \vartheta \in \mathcal{V} \mapsto \nu[\vartheta] := \nu_0[\vartheta] \oplus_t \sum_{j\geq 1} \nu_j[\vartheta] \mathbf{1}_{A_j^\vartheta} \in \mathfrak{U}$,
for $\{A_j^\vartheta, \vartheta \in \mathcal{V}\}_{j\geq 1} \subset \mathfrak{F}_t$ such that $\{A_j^\vartheta, j \geq 1\}$ forms a partition of Ω for each $\vartheta \in \mathcal{V}$.

□ **Non-anticipating event sets :** $\{A^\vartheta, \vartheta \in \mathcal{V}\} \in \mathfrak{F}_t$ if $\{A^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathcal{F}_t$ is such that
$A^{\vartheta_1} \cap \{\vartheta_1 =_{(0,t]} \vartheta_2\} = A^{\vartheta_2} \cap \{\vartheta_1 =_{(0,t]} \vartheta_2\}$ for $\vartheta_1, \vartheta_2 \in \mathcal{V}$.

□ **Non-anticipating stopping times :** $\{\tau^\vartheta \in (s_1, s_2]\}, \vartheta \in \mathcal{V}\}$ and $\{\{\tau^\vartheta \notin (s_1, s_2]\}, \vartheta \in \mathcal{V}\}$ belong to $\mathfrak{F}_{s_2}$ for $\{\tau^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathfrak{T}_t$.

Path dependency

□ $Z_{t,z}^{\nu,\vartheta_1}(s)(\omega) = Z_{t,z}^{\nu,\vartheta_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\vartheta_1 =_{(0,s]} \vartheta_2\}$, for all $\nu \in \mathcal{U}$ and $\vartheta_1, \vartheta_2 \in \mathcal{V}$.

Path dependency

- $Z_{t,z}^{\nu,\vartheta_1}(s)(\omega) = Z_{t,z}^{\nu,\vartheta_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\vartheta_1 =_{(0,s]} \vartheta_2\}$, for all $\nu \in \mathcal{U}$ and $\vartheta_1, \vartheta_2 \in \mathcal{V}$.

- $Z_{t,z}^{\nu_1,\vartheta}(s)(\omega) = Z_{t,z}^{\nu_2,\vartheta}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\nu_1[\vartheta] =_{(t,s]} \nu_2[\vartheta]\}$, for all $\vartheta \in \mathcal{V}$ and $\nu_1, \nu_2 \in \mathfrak{U}$.

Path dependency

□ $Z_{t,z}^{\nu,\vartheta_1}(s)(\omega) = Z_{t,z}^{\nu,\vartheta_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\vartheta_1 =_{(0,s]} \vartheta_2\}$, for all $\nu \in \mathcal{U}$ and $\vartheta_1, \vartheta_2 \in \mathcal{V}$.

□ $Z_{t,z}^{\nu_1,\vartheta}(s)(\omega) = Z_{t,z}^{\nu_2,\vartheta}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\nu_1[\vartheta] =_{(t,s]} \nu_2[\vartheta]\}$, for all $\vartheta \in \mathcal{V}$ and $\nu_1, \nu_2 \in \mathfrak{U}$.

□ There exists a constant $K(t,z) \in \mathbb{R}$ such that

$$\operatorname*{ess\,sup}_{\nu \in \mathfrak{U}} \operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta}(T)) | \mathcal{F}_t\right] = K(t,z) \;\; \mathbb{P} - \text{a.s.}$$

Path regularity and growth conditions

□ There exists $C > 0$, $q \geq 0, \varepsilon > 0$ and a continuous map ϱ such that

$$|\ell(z)| \leq C(1 + |z|^q) \quad , \quad \operatorname*{ess\,sup}_{(\bar{\nu},\bar{\vartheta})\in\mathcal{U}\times\mathcal{V}} \mathbb{E}\left[|Z_{t,z}^{\bar{\nu},\bar{\vartheta}}(T)|^{(q\vee 1)+\varepsilon}|\mathcal{F}_t\right] \leq \varrho(z)$$

and

$$\operatorname*{ess\,sup}_{(\bar{\nu},\bar{\vartheta})\in\mathcal{U}\times\mathcal{V}} \mathbb{E}\left[|Z_{t,z}^{\nu\oplus_s\bar{\nu},\vartheta\oplus_s\bar{\vartheta}}(T) - Z_{s,z'}^{\bar{\nu},\vartheta\oplus_s\bar{\vartheta}}(T)|\ |\mathcal{F}_s\right] \leq C|Z_{t,z}^{\nu,\vartheta}(s) - z'|.$$

Path regularity and growth conditions

□ There exists $C > 0$, $q \geq 0, \varepsilon > 0$ and a continuous map ϱ such that

$$|\ell(z)| \leq C(1 + |z|^q) \quad , \quad \operatorname{ess} \sup_{(\bar\nu,\bar\vartheta)\in\mathcal{U}\times\mathcal{V}} \mathbb{E}\left[|Z^{\bar\nu,\bar\vartheta}_{t,z}(T)|^{(q\vee 1)+\varepsilon}|\mathcal{F}_t\right] \leq \varrho(z)$$

and

$$\operatorname{ess} \sup_{(\bar\nu,\bar\vartheta)\in\mathcal{U}\times\mathcal{V}} \mathbb{E}\left[|Z^{\nu\oplus_s\bar\nu,\vartheta\oplus_s\bar\vartheta}_{t,z}(T) - Z^{\bar\nu,\vartheta\oplus_s\bar\vartheta}_{s,z'}(T)| \; |\mathcal{F}_s\right] \leq C|Z^{\nu,\vartheta}_{t,z}(s) - z'|.$$

□ As $\delta \to 0$

$$\sup_{\bar\vartheta\in\mathcal{V},\tau\in\mathcal{T}_t} \mathbb{P}\left[\sup_{0\leq h\leq\delta} \left|Z^{\nu,\bar\vartheta}_{t,z}(\tau + h) - Z^{\nu,\bar\vartheta}_{t,z}(\tau)\right| \geq \iota\right] \to 0.$$

Non-anticipating admissible martingale strategies

☐ $M^{\vartheta_1}(s)(\omega) = M^{\vartheta_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\vartheta_1 =_{(0,s]} \vartheta_2\}$, for all $\{M^{\vartheta},\ \vartheta \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ and $\vartheta_1, \vartheta_2 \in \mathcal{V}$.

Weak formulation

□ We consider relaxed versions of Λ

$$\bar{\Lambda}(t) := \left\{ \begin{array}{l} (z,p) \in \mathbb{R}^d \times \mathbb{R} : \text{ there exist } (t_n, z_n, p_n) \to (t,z,p) \\ \text{such that } (z_n,p_n) \in \Lambda(t_n) \text{ and } t_n \geq t \text{ for all } n \geq 1 \end{array} \right\}$$

and

$$\mathring{\Lambda}_\iota(t) := \left\{ \begin{array}{l} (z,p) \in \mathbb{R}^d \times \mathbb{R} : \ (t',z',p') \in B_\iota(t,z,p) \\ \text{ implies } (z',p') \in \Lambda(t') \end{array} \right\}$$

Weak formulation

Theorem

(GDP1) : If $(z, p+\varepsilon) \in \Lambda(t)$ for some $\varepsilon > 0$, then $\exists\ \nu \in \mathfrak{U}$ and $\{M^{\vartheta}, \vartheta \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ s.t.

$$\left(Z_{t,z}^{\nu,\vartheta}(\tau), M^{\vartheta}(\tau)\right) \in \bar{\Lambda}(\tau) \quad \mathbb{P}-\text{a.s. } \forall\ \vartheta \in \mathcal{V},\ \tau \in \mathcal{T}_t.$$

Weak formulation

Theorem

(GDP1) : If $(z, p+\varepsilon) \in \Lambda(t)$ for some $\varepsilon > 0$, then $\exists\ \nu \in \mathfrak{U}$ and $\{M^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ s.t.

$$\left(Z_{t,z}^{\nu,\vartheta}(\tau), M^\vartheta(\tau)\right) \in \bar{\Lambda}(\tau) \quad \mathbb{P}-\text{a.s. } \forall\ \vartheta \in \mathcal{V},\ \tau \in \mathcal{T}_t.$$

(GDP2) : Let $\iota > 0$, $\nu \in \mathfrak{U}$, $\{M^\vartheta, \vartheta \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ and $\{\tau^\vartheta, \vartheta \in \mathcal{V}\} \in \mathfrak{T}_t$ be s.t.

$$\left(Z_{t,z}^{\nu,\vartheta}(\tau^\vartheta), M^\vartheta(\tau^\vartheta)\right) \in \mathring{\Lambda}_\iota(\tau^\vartheta) \quad \mathbb{P}-\text{a.s. for all } \vartheta \in \mathcal{V}.$$

Assume further that $\{Z_{t,z}^{\nu,\vartheta}(\tau^\vartheta), \vartheta \in \mathcal{V}\}$ is uniformly bounded in L^∞. Then $(z, p-\varepsilon) \in \Lambda(t)$ for all $\varepsilon > 0$.

More abstract setting

- $\mathbb{R}^d$ replaced by a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$

More abstract setting

□ $\mathbb{R}^d$ replaced by a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$
□ Lipschitz continuity, and growth estimates on Z and ℓ replaced by :
For all $s \in [t, T]$, $\varepsilon > 0$, there exists a partition $(B_j)_j$ of $\mathcal{Z}$ into Borel sets and a sequence $(z_j)_{j \geq 1} \subset \mathcal{Z}$ such that, for all $\nu \in \mathfrak{U}$, $\vartheta \in \mathcal{V}$ and $j \geq 1$:

More abstract setting

□ $\mathbb{R}^d$ replaced by a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$

□ Lipschitz continuity, and growth estimates on Z and ℓ replaced by :

For all $s \in [t, T]$, $\varepsilon > 0$, there exists a partition $(B_j)_j$ of $\mathcal{Z}$ into Borel sets and a sequence $(z_j)_{j\geq 1} \subset \mathcal{Z}$ such that, for all $\nu \in \mathfrak{U}$, $\vartheta \in \mathcal{V}$ and $j \geq 1$:

- $\mathbb{E}\left[\ell(Z^{\nu,\vartheta}_{t,z}(T))|\mathcal{F}_s\right] \geq I(s, z_j, \nu, \vartheta) - \varepsilon$ on $C^{\nu,\vartheta}_j := \{Z^{\nu,\vartheta}_{t,z}(s) \in B_j\}$.

More abstract setting

□ $\mathbb{R}^d$ replaced by a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$

□ Lipschitz continuity, and growth estimates on Z and ℓ replaced by :

For all $s \in [t, T]$, $\varepsilon > 0$, there exists a partition $(B_j)_j$ of $\mathcal{Z}$ into Borel sets and a sequence $(z_j)_{j\geq 1} \subset \mathcal{Z}$ such that, for all $\nu \in \mathfrak{U}$, $\vartheta \in \mathcal{V}$ and $j \geq 1$:

- $\mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta}(T))|\mathcal{F}_s\right] \geq I(s, z_j, \nu, \vartheta) - \varepsilon$ on $C_j^{\nu,\vartheta} := \{Z_{t,z}^{\nu,\vartheta}(s) \in B_j\}$.
- $\operatorname{ess\,inf}_{\bar{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta\oplus_s\bar{\vartheta}}(T))|\mathcal{F}_s\right] \leq J(s, z_j, \nu(\vartheta \oplus_s \cdot)) + \varepsilon$ on $C_j^{\nu,\vartheta}$.

More abstract setting

□ $\mathbb{R}^d$ replaced by a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$
□ Lipschitz continuity, and growth estimates on Z and ℓ replaced by :
For all $s \in [t, T]$, $\varepsilon > 0$, there exists a partition $(B_j)_j$ of $\mathcal{Z}$ into Borel sets and a sequence $(z_j)_{j\geq 1} \subset \mathcal{Z}$ such that, for all $\nu \in \mathfrak{U}$, $\vartheta \in \mathcal{V}$ and $j \geq 1$:

- $\mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta}(T))|\mathcal{F}_s\right] \geq I(s, z_j, \nu, \vartheta) - \varepsilon$ on $C_j^{\nu,\vartheta} := \{Z_{t,z}^{\nu,\vartheta}(s) \in B_j\}$.
- $\operatorname{ess\,inf}_{\bar\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta\oplus_s\bar\vartheta}(T))|\mathcal{F}_s\right] \leq J(s, z_j, \nu(\vartheta \oplus_s \cdot)) + \varepsilon$ on $C_j^{\nu,\vartheta}$.
- $|K(s, z_j) - K(s, Z_{t,z}^{\nu,\vartheta}(s))| \leq \varepsilon$ on $C_j^{\nu,\vartheta}$.

More abstract setting

□ $\mathbb{R}^d$ replaced by a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$

□ Lipschitz continuity, and growth estimates on Z and ℓ replaced by :

For all $s \in [t, T]$, $\varepsilon > 0$, there exists a partition $(B_j)_j$ of $\mathcal{Z}$ into Borel sets and a sequence $(z_j)_{j\geq 1} \subset \mathcal{Z}$ such that, for all $\nu \in \mathfrak{U}$, $\vartheta \in \mathcal{V}$ and $j \geq 1$:

- $\mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta}(T))|\mathcal{F}_s\right] \geq I(s, z_j, \nu, \vartheta) - \varepsilon$ on $C_j^{\nu,\vartheta} := \{Z_{t,z}^{\nu,\vartheta}(s) \in B_j\}$.
- $\operatorname{ess\,inf}_{\bar{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\nu,\vartheta\oplus_s\bar{\vartheta}}(T))|\mathcal{F}_s\right] \leq J(s, z_j, \nu(\vartheta \oplus_s \cdot)) + \varepsilon$ on $C_j^{\nu,\vartheta}$.
- $|K(s, z_j) - K(s, Z_{t,z}^{\nu,\vartheta}(s))| \leq \varepsilon$ on $C_j^{\nu,\vartheta}$.

□ Plus some additional control on the negative part of I.

Sketch of proof for GDP1

- Assume

$$S_s^{\nu,\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}_s} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

is such that $S_t^{\nu,\vartheta} \geq p$.

Sketch of proof for GDP1

- Assume

$$S_s^{\nu,\vartheta} := \operatorname*{ess\,inf}_{\tilde\vartheta\in\mathcal{V}_s} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde\vartheta],\vartheta\oplus_s\tilde\vartheta}(T))|\mathcal{F}_s\right]$$

is such that $S_t^{\nu,\vartheta} \geq p$.

- It admits a càlàg decomposition (up to a modification),+ Doob-Meyer-type decomposition : $S^{\nu,\vartheta} \geq M^{\nu,\vartheta}$ a càdlàg martingale.

Sketch of proof for GDP1

□ Assume

$$S_s^{\nu,\vartheta} := \operatorname{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}_s} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

is such that $S_t^{\nu,\vartheta} \geq p$.

□ It admits a càlàg decomposition (up to a modification),+
Doob-Meyer-type decomposition : $S^{\nu,\vartheta} \geq M^{\nu,\vartheta}$ a càdlàg martingale.

□

$$K(\tau, Z_{t,z}^{\nu,\vartheta}(\tau)) \geq M^{\nu,\vartheta}(\tau)$$

Sketch of proof for GDP1

□ Assume

$$S_s^{\nu,\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}_s} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

is such that $S_t^{\nu,\vartheta} \geq p$.

□ It admits a càlàg decomposition (up to a modification),+ Doob-Meyer-type decomposition : $S^{\nu,\vartheta} \geq M^{\nu,\vartheta}$ a càdlàg martingale.

□

$$K(\tau, Z_{t,z}^{\nu,\vartheta}(\tau)) \geq M^{\nu,\vartheta}(\tau)$$

which leads to

$$\left(Z_{t,z}^{\nu,\vartheta}(\tau), M^{\nu,\vartheta}(\tau) - \varepsilon\right) \in \Lambda(\tau) \quad \mathbb{P} - \text{a.s.}$$

Sketch of proof for GDP1

- Assume

$$S_s^{\nu,\vartheta} := \operatorname{ess\,inf}_{\tilde\vartheta \in \mathcal{V}_s} \mathbb{E}\left[\ell(Z_{t,z}^{\nu[\vartheta\oplus_s\tilde\vartheta],\vartheta\oplus_s\tilde\vartheta}(T))|\mathcal{F}_s\right]$$

is such that $S_t^{\nu,\vartheta} \geq p$.

- It admits a càlàg decomposition (up to a modification),+ Doob-Meyer-type decomposition : $S^{\nu,\vartheta} \geq M^{\nu,\vartheta}$ a càdlàg martingale.

-

$$K(\tau, Z_{t,z}^{\nu,\vartheta}(\tau)) \geq M^{\nu,\vartheta}(\tau)$$

which leads to

$$\left(Z_{t,z}^{\nu,\vartheta}(\tau), M^{\nu,\vartheta}(\tau) - \varepsilon\right) \in \Lambda(\tau) \quad \mathbb{P} - \text{a.s.}$$

- Ok for stopping times τ with values in a countable set. Pass to the limit.

Sketch of proof for GDP2

- τ_n^{ϑ} an approximation of τ^{ϑ} on a sequence of finite grids π_n.

Sketch of proof for GDP2

- τ_n^ϑ an approximation of τ^ϑ on a sequence of finite grids π_n.

- If

$$\left(Z_{t,z}^{\nu,\vartheta}\left(\tau^\vartheta\right), M^\vartheta\left(\tau^\vartheta\right)\right) \in \mathring{\Lambda}_\iota\left(\tau^\vartheta\right) \quad \mathbb{P}-\text{a.s.}$$

then

$$\left(Z_{t,z}^{\nu,\vartheta}\left(\tau_n^\vartheta\right), M^\vartheta\left(\tau_n^\vartheta\right)\right) \in \Lambda\left(\tau_n^\vartheta\right) \text{ on } E_n^\vartheta \text{ for all } \vartheta \in \mathcal{V}$$

with $\mathbb{P}\left[E_n^\vartheta\right] \to 1$ as $n \to \infty$ (uniformly in ϑ).

Sketch of proof for GDP2

- τ_n^ϑ an approximation of τ^ϑ on a sequence of finite grids π_n.

- If
$$\left(Z_{t,z}^{\nu,\vartheta}\left(\tau^\vartheta\right), M^\vartheta\left(\tau^\vartheta\right)\right) \in \mathring{\Lambda}_\iota\left(\tau^\vartheta\right) \quad \mathbb{P}-\text{a.s.}$$
then
$$\left(Z_{t,z}^{\nu,\vartheta}\left(\tau_n^\vartheta\right), M^\vartheta\left(\tau_n^\vartheta\right)\right) \in \Lambda\left(\tau_n^\vartheta\right) \text{ on } E_n^\vartheta \text{ for all } \vartheta \in \mathcal{V}$$
with $\mathbb{P}\left[E_n^\vartheta\right] \to 1$ as $n \to \infty$ (uniformly in ϑ).

- Hence,
$$K(\tau_n^\vartheta, Z_{t,z}^{\nu,\vartheta}(\tau_n^\vartheta)) \geq M^\vartheta(\tau_n^\vartheta).$$

Sketch of proof for GDP2

- τ_n^ϑ an approximation of τ^ϑ on a sequence of finite grids π_n.

- If
$$\left(Z_{t,z}^{\nu,\vartheta}\left(\tau^\vartheta\right), M^\vartheta\left(\tau^\vartheta\right)\right) \in \mathring{\Lambda}_\iota\left(\tau^\vartheta\right) \quad \mathbb{P}-\text{a.s.}$$
then
$$\left(Z_{t,z}^{\nu,\vartheta}\left(\tau_n^\vartheta\right), M^\vartheta\left(\tau_n^\vartheta\right)\right) \in \Lambda\left(\tau_n^\vartheta\right) \text{ on } E_n^\vartheta \text{ for all } \vartheta \in \mathcal{V}$$
with $\mathbb{P}\left[E_n^\vartheta\right] \to 1$ as $n \to \infty$ (uniformly in ϑ).

- Hence,
$$K(\tau_n^\vartheta, Z_{t,z}^{\nu,\vartheta}(\tau_n^\vartheta)) \geq M^\vartheta(\tau_n^\vartheta).$$

- Regularity + covering : there exists $\nu_\varepsilon \in \mathfrak{U}$ such that
$$\mathbb{E}\left[\ell(Z_{t,z}^{\nu_\varepsilon,\vartheta}(T))|\mathcal{F}_{\tau_n^\vartheta}\right] \geq M^\vartheta(\tau_n^\vartheta) - \varepsilon.$$
which implies
$$\mathbb{E}\left[\ell(Z_{t,z}^{\nu_\varepsilon,\vartheta}(T))|\mathcal{F}_t\right] \geq p - \varepsilon.$$

Application to Brownian controlled SDEs

Framework

□ $\mathfrak{U}$ is the set of maps $\nu : \mathcal{V} \to \mathcal{U}$ such that $\{\vartheta_1 =_{(0,\tau]} \vartheta_2\} \subset \{\nu[\vartheta_1] =_{(0,\tau]} \nu[\vartheta_2]\}$ for all $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\tau \in \mathcal{T}$.

Framework

□ $\mathfrak{U}$ is the set of maps $\nu : \mathcal{V} \to \mathcal{U}$ such that $\{\vartheta_1 =_{(0,\tau]} \vartheta_2\} \subset \{\nu[\vartheta_1] =_{(0,\tau]} \nu[\vartheta_2]\}$ for all $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\tau \in \mathcal{T}$.

□ $Z_{t,z}^{\nu,\vartheta} = (X_{t,x}^{\nu,\vartheta}, Y_{t,x,y}^{\nu,\vartheta})$ is the strong solution of

$$Z(s) = z + \int_t^s \mu(Z(r), \nu[\vartheta]_r, \vartheta_r)dr + \int_t^s \sigma(Z(r), \nu[\vartheta]_r, \vartheta_r)dW_r.$$

Framework

- $\mathfrak{U}$ is the set of maps $\nu : \mathcal{V} \to \mathcal{U}$ such that $\{\vartheta_1 =_{(0,\tau]} \vartheta_2\} \subset \{\nu[\vartheta_1] =_{(0,\tau]} \nu[\vartheta_2]\}$ for all $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\tau \in \mathcal{T}$.

- $Z^{\nu,\vartheta}_{t,z} = (X^{\nu,\vartheta}_{t,x}, Y^{\nu,\vartheta}_{t,x,y})$ is the strong solution of

$$Z(s) = z + \int_t^s \mu(Z(r), \nu[\vartheta]_r, \vartheta_r)dr + \int_t^s \sigma(Z(r), \nu[\vartheta]_r, \vartheta_r)dW_r.$$

- $\mathcal{M}_{t,p}$ is identified to $\{M^\alpha_{t,p}, \alpha \in \mathcal{A}\}$ with

$$M^\alpha_{t,p} := p + \int_t^\cdot \alpha_s dW_s$$

Framework

□ $\mathfrak{U}$ is the set of maps $\nu : \mathcal{V} \to \mathcal{U}$ such that $\{\vartheta_1 =_{(0,\tau]} \vartheta_2\} \subset \{\nu[\vartheta_1] =_{(0,\tau]} \nu[\vartheta_2]\}$ for all $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\tau \in \mathcal{T}$.

□ $Z^{\nu,\vartheta}_{t,z} = (X^{\nu,\vartheta}_{t,x}, Y^{\nu,\vartheta}_{t,x,y})$ is the strong solution of

$$Z(s) = z + \int_t^s \mu(Z(r), \nu[\vartheta]_r, \vartheta_r)dr + \int_t^s \sigma(Z(r), \nu[\vartheta]_r, \vartheta_r)dW_r.$$

□ $\mathcal{M}_{t,p}$ is identified to $\{M^{\alpha}_{t,p}, \alpha \in \mathcal{A}\}$ with

$$M^{\alpha}_{t,p} := p + \int_t^{\cdot} \alpha_s dW_s$$

□ $\mathfrak{X}$ the set of maps $\alpha[\cdot] : \mathcal{V} \mapsto \mathcal{A}$ such that $\{\vartheta_1 =_{(0,\tau]} \vartheta_2\} \subset \{\alpha[\vartheta_1] =_{(0,\tau]} \alpha[\vartheta_2]\}$ for $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\tau \in \mathcal{T}$.

Framework

☐ Assume $\ell : (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mapsto \ell(x, y) \in \mathbb{R}$ is non-decreasing in its y-variable.

Framework

- Assume $\ell : (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mapsto \ell(x, y) \in \mathbb{R}$ is non-decreasing in its y-variable.

- This allows to associate to Λ the real valued function :

$$\mathbf{y}(t, x) := \inf\{y : (x, y) \in \Lambda(t)\}.$$

GDP

Corollary

(GDP1)$_\mathbf{y}$ If $y > \mathbf{y}(t, x, p + \varepsilon)$ with $\varepsilon > 0$, then $\exists\ \nu \in \mathfrak{U}$ and $\{\alpha^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathcal{A}$ s.t.

$$Y^{\nu,\vartheta}_{t,x,y}(\tau) \geq \mathbf{y}_* \left(\tau, X^{\nu,\vartheta}_{t,x}(\tau), M^{\alpha^\vartheta}_{t,p}(\tau)\right) \quad \mathbb{P}\text{-a.s.}$$

for all $\vartheta \in \mathcal{V}$, $\tau \in \mathcal{T}_t$.

GDP

Corollary

(GDP1)$_{\mathbf{y}}$ If $y > \mathbf{y}(t,x,p+\varepsilon)$ with $\varepsilon > 0$, then $\exists\ \nu \in \mathfrak{U}$ and $\{\alpha^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathcal{A}$ s.t.

$$Y^{\nu,\vartheta}_{t,x,y}(\tau) \geq \mathbf{y}_* \left(\tau, X^{\nu,\vartheta}_{t,x}(\tau), M^{\alpha^\vartheta}_{t,p}(\tau)\right) \quad \mathbb{P}\text{-a.s.}$$

for all $\vartheta \in \mathcal{V}$, $\tau \in \mathcal{T}_t$.

(GDP2)$_{\mathbf{y}}$ Fix a bounded open set $O \ni (t,x,y,p)$, $(\nu,\alpha) \in \mathfrak{U} \times \mathfrak{X}$ and let τ^ϑ denote the first exit time of $(\cdot, X^{\nu,\vartheta}_{t,x}, Y^{\nu,\vartheta}_{t,x,y}, M^{\alpha[\vartheta]}_{t,p})$, $\vartheta \in \mathcal{V}$. Assume that there exists $\eta > 0$ and a continuous function $\varphi \geq \mathbf{y}$ such that

$$Y^{\nu,\vartheta}_{t,x,y}(\tau^\vartheta) \geq \varphi\left(\tau, X^{\nu,\vartheta}_{t,x}(\tau^\vartheta), M^{\alpha[\vartheta]}_{t,p}(\tau^\vartheta)\right) + \eta\, \mathbb{P} - \text{a.s.} \quad \text{for all } \vartheta \in \mathcal{V}.$$

Then, $y \geq \mathbf{y}(t,x,p-\varepsilon)$ for all $\varepsilon > 0$.

PDE characterization - “waving hands” version

- Assuming smoothness, existence of optimal strategies...

- $y = \mathbf{y}(t, x, p)$ implies
$Y^{\nu[\vartheta],\vartheta}(t+) \geq \mathbf{y}(t+, X^{\nu[\vartheta],\vartheta}(t+), M^{\alpha[\vartheta]}(t+))$ for all ϑ.

PDE characterization - “waving hands” version

- Assuming smoothness, existence of optimal strategies...

- $y = \mathbf{y}(t, x, p)$ implies
$Y^{\nu[\vartheta],\vartheta}(t+) \geq \mathbf{y}(t+, X^{\nu[\vartheta],\vartheta}(t+), M^{\alpha[\vartheta]}(t+))$ for all ϑ.

- This implies $dY^{\nu[\vartheta],\vartheta}(t) \geq d\mathbf{y}(t, X^{\nu[\vartheta],\vartheta}(t), M^{\alpha[\vartheta]}(t))$ for all ϑ

PDE characterization - “waving hands” version

- Assuming smoothness, existence of optimal strategies...
- $y = \mathbf{y}(t,x,p)$ implies
$Y^{\nu[\vartheta],\vartheta}(t+) \geq \mathbf{y}(t+, X^{\nu[\vartheta],\vartheta}(t+), M^{\alpha[\vartheta]}(t+))$ for all ϑ.
- This implies $dY^{\nu[\vartheta],\vartheta}(t) \geq d\mathbf{y}(t, X^{\nu[\vartheta],\vartheta}(t), M^{\alpha[\vartheta]}(t))$ for all ϑ
- Hence, for all ϑ,

$$
\begin{aligned}
\mu_Y(x,y,\nu[\vartheta]_t,\vartheta_t) &\geq \mathcal{L}_{X,M}^{\nu[\vartheta]_t,\vartheta_t,\alpha[\vartheta]_t}\mathbf{y}(t,x,p) \\
\sigma_Y(x,y,\nu[\vartheta]_t,\vartheta_t) &= \sigma_X(x,\nu[\vartheta]_t,\vartheta_t)D_x\mathbf{y}(t,x,p) + \alpha[\vartheta]_t D_p\mathbf{y}(t,x,p)
\end{aligned}
$$

with $y = \mathbf{y}(t,x,p)$

PDE characterization - "waving hands" version

- PDE characterization

$$\inf_{v\in V}\sup_{(u,a)\in\mathcal{N}^v\mathbf{y}}\Big(\mu_Y(\cdot,\mathbf{y},u,v)-\mathcal{L}^{u,v,a}_{X,M}\mathbf{y}\Big)=0$$

where

$$\mathcal{N}^v\mathbf{y}:=\{(u,a)\in U\times\mathbb{R}^d:\sigma_Y(\cdot,\mathbf{y},u,v)=\sigma_X(\cdot,u,v)D_x\mathbf{y}+aD_p\mathbf{y}\}.$$

PDE characterization - “waving hands” version

- PDE characterization

$$\inf_{v \in V} \sup_{(u,a) \in \mathcal{N}^v \mathbf{y}} \left(\mu_Y(\cdot, \mathbf{y}, u, v) - \mathcal{L}^{u,v,a}_{X,M} \mathbf{y} \right) = 0$$

where

$$\mathcal{N}^v \mathbf{y} := \{(u, a) \in U \times \mathbb{R}^d : \sigma_Y(\cdot, \mathbf{y}, u, v) = \sigma_X(\cdot, u, v) D_x \mathbf{y} + a D_p \mathbf{y}\}.$$

- Need to be relaxed...

References

□ H.M. Soner and N. Touzi (2002). *Dynamic programming for stochastic target problems and geometric flows*, JEMS, 4, 201-236.

□ B. , R. Elie and N. Touzi (2009). *Stochastic Target problems with controlled loss*, SIAM SICON, 48 (5), 3123-3150.

□ L. Moreau (2011), *Stochastic target problems with controlled loss in a jump diffusion model*, SIAM SICON, 49, 2577-2607, 2011.

□ B. and N. M. Dang (2010). *Generalized stochastic target problems for pricing and partial hedging under loss constraints - Application in optimal book liquidation*, to appear in F&S.

□ B. and V. Thanh Nam (2011). *A stochastic target approach for P&L matching problems*, to appear in MOR.

□ R. Buckdahn and J. Li. *Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations*. SIAM SICON, 47(1) :444-475, 2008.