Weak dynamic programming principle toward viscosity solutions

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Joint works with M. Nutz and N. Touzi

Motivation

Provide an easy to prove Dynamic Programming Principle for stochastic optimal control problems in standard form :

$$v(t,x) := \sup_{\nu \in \mathcal{U}} F(t,x;\nu) \quad \text{with} \quad F(t,x,\nu) := \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right] .$$

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Extend it to optimal control problems :
- with constraints in expectation : $\mathbb{E}\left[g(X^{\nu}_{t,x}(T))\right] \leq m$.
- with strong state constraints : $X^{\nu}_{t,x} \in \mathcal{O}$ on $[t,T]$.
(joint work with M. Nutz - preprint)

The case without constraint

$$v(t,x) := \sup_{\nu \in \mathcal{U}} F(t,x;\nu) \quad \textbf{with} \quad F(t,x,\nu) := \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right]$$

Weak Dynamic Programming Principle for Viscosity Solutions, with Nizar Touzi, *SIAM Journal on Control and Optimization*, 49 (3), 2011.

Standard approach for the DPP

Either use a measurable selection argument : $(t, x) \mapsto \nu_\varepsilon(t, x) \in \mathcal{U}$ such that

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Or use the regularity of v and $F(\cdot;\nu)$ to construct one :

$$F(\cdot;\nu_\varepsilon^{s,y}) \underset{lsc}{\geq} F(s,y;\nu_\varepsilon^{s,y}) - \varepsilon \geq v(s,y) - 2\varepsilon \underset{usc}{\geq} v - 3\varepsilon \text{ on } B_{s,y}$$

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$$F(\cdot;\nu_\varepsilon^{t_i,x_i}) \underset{lsc}{\geq} F(t_i,x_i;\nu_\varepsilon^{t_i,x_i}) - \varepsilon \geq v(t_i,x_i) - 2\varepsilon \underset{usc}{\geq} v - 3\varepsilon \text{ on } B_i$$

with $(B_i)_{i\geq 1}$ a partition of the state-space. Then, one constructs a measurable selection by setting

$$\nu_\varepsilon(t,x) := \sum_{i\geq 1} \nu_\varepsilon^{t_i,x_i} \mathbf{1}_{B_i}(t,x) \ .$$

Standard approach for the DPP

In both cases : for $\bar{\nu} = \nu \mathbf{1}_{[0,\theta)} + \mathbf{1}_{[\theta,T]} \nu_\varepsilon(\theta, X^\nu_{t,x}(\theta))$

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\begin{aligned}
v(t,x) &\geq F(t,x;\bar{\nu}) = \mathbb{E}\left[\mathbb{E}\left[f(X^{\nu_\varepsilon(\theta, X^\nu_{t,x}(\theta))}_{\theta, X^\nu_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta\right]\right] \\
&= \mathbb{E}\left[F(\theta, X^\nu_{t,x}(\theta); \nu_\varepsilon(\theta, X^\nu_{t,x}(\theta)))\right] \geq \mathbb{E}\left[v(\theta, X^\nu_{t,x}(\theta))\right] - 3\varepsilon
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on $B_i \ni (t_i, x_i)$, with $(B_i)_{i \geq 1}$ a partition of the state-space.

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In any case a minimum of regularity is required.

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We never use : $v(t,x) = \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[v(\theta, X^{\nu}_{t,x}(\theta))\right]$.

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One can reproduce the usual argument (but on φ) :

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i.e. $v(t,x) \geq \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\varphi(\theta,X_{t,x}^{\nu}(\theta))\right]$.

An abstract setting

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• Controlled process : $\nu \in \mathcal{U} \mapsto X_{t,x}^{\nu}$ a càdlàg adapted process with values in $\mathbb{R}^d$ (could be a separable metric space).

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Structure assumption : Let $(t,x) \in [0,T] \times \mathbb{R}^d$, $\nu \in \mathcal{U}_t$, $\tau \in \mathcal{T}^t$, $\Gamma \in \mathcal{F}^t_\tau$ and $\bar{\nu} \in \mathcal{U}_{\|\tau\|_{L^\infty}}$.

There exists a control $\tilde{\nu} \in \mathcal{U}_t$, denoted by $\nu \otimes_{(\tau,\Gamma)} \bar{\nu}$, such that

$$\begin{aligned} X^{\tilde{\nu}}_{t,x}(\cdot) &= X^{\nu}_{t,x}(\cdot) && \text{on } [t,T] \times (\Omega \setminus \Gamma); \\ X^{\tilde{\nu}}_{t,x}(\cdot) &= X^{\bar{\nu}}_{\tau, X^{\nu}_{t,x}(\tau)}(\cdot) && \text{on } [\tau,T] \times \Gamma; \\ \mathbb{E}\left[f(X^{\tilde{\nu}}_{t,x}(T)) \mid \mathcal{F}_\tau\right] &= F(\tau, X^{\nu}_{t,x}(\tau); \bar{\nu}) && \text{on } \Gamma. \end{aligned}$$

Easy part of the DPP

- Fix $(t,x) \in [0,T] \times \mathbb{R}^d$, $\nu \in \mathcal{U}_t$. Let $\{\theta^{\nu'}, \nu' \in \mathcal{U}\} \subset \mathcal{T}^t$ and let $\varphi : [0,T] \times S \to \mathbb{R}$ be a measurable function such that $v \leq \varphi$.Then,

$$F(t,x;\nu) \leq \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right].$$

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Corollary :

$$v(t, x) \leq \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right]$$

Proof :

$$F(t,x;\nu) = \mathbb{E}\left[\mathbb{E}\left[f(X^{\nu}_{\theta,X^{\nu}_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta\right]\right]$$

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where

$$\begin{aligned}\mathbb{E}\left[f(X^{\nu}_{\theta,X^{\nu}_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta\right](\omega) &= F(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \nu(\omega^{\theta(\omega)}, \cdot)) \\ &\leq v(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega)) \\ &\leq \varphi(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega))\end{aligned}$$

because $\nu(\omega^{\theta(\omega)}, \cdot) \in \mathcal{U}_{\theta(\omega)}$.

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By the Lindelöf property : we can find $(s_i, y_i)_{i \geq 1}$ such that $\cup_i B_{s_i,y_i} = (0, T] \times \mathbb{R}^d$.

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Take a disjoint sub-covering $(A_i)_i$ and set $\Gamma_i := \{(\theta, X^{\nu}_{t,x}(\theta)) \in B_i\}$, $\Gamma^n := \cup_{i \leq n} \Gamma_i$.

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Set

$$\nu_n := (((\nu \otimes_{\theta,\Gamma_1} \nu_{s_1,y_1}) \otimes_{\theta,\Gamma_2} \nu_{s_2,y_2}) \otimes \cdots) \otimes_{\theta,\Gamma_n} \nu_{s_n,y_n}$$

Then

$$F(t,x;\nu_n) = \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\mathbf{1}_{\Gamma^{nc}} + \sum_{i\le n} \mathbb{E}\left[f(X^{\nu_{s_i,y_i}}_{\theta, X^{\nu}_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta \right] \mathbf{1}_{\Gamma_i} \right]$$

where, on Γ_i (with $A_i \subset B_i$),

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Then

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Pass to the limit in $n \to \infty$ and $\varepsilon \to 0$.

The weak DPP (summing up)

Theorem : Assume that $F(\cdot;\nu)$ is lsc for all $\nu \in \mathcal{U}$ (on the left in time). Let $\varphi_- \leq v \leq \varphi_+$ be two smooth functions. Then

$$\sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\varphi_-(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right] \leq v(t,x) \leq \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\varphi_+(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right]$$

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Remark : if v is locally bounded and $\{(\theta^\nu, X^\nu_{t,x}(\theta^\nu)),\ \nu \in \mathcal{U}_t\}$ is bounded then

$$\sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[v_*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right] \leq v(t,x) \leq \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[v^*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right]$$

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Remark : $v = v_*$ if $F(\cdot;\nu)$ is lsc. If v is usc, then one retrieves the usual DPP.

Other examples of application

• Set of controls depending on the state process : Bouchard, Dang and Lehall, Optimal control of trading algorithms : a general impulse control approach, to appear in *SIAM Journal on Financial Mathematics*.

• Game problem : Bayraktar and Hang, On the Multi-dimensional controller and stopper games, preprint 2010.

The case with constraint in expectation

$$v(t,x) := \sup_{\nu \in \mathcal{U}(t,x,m)} F(t,x;\nu) \quad \textbf{with} \quad F(t,x,\nu) := \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right]$$

and

$$\mathcal{U}(t,x,m) := \{\nu \in \mathcal{U}_t \ : \ G(t,x;\nu) := \mathbb{E}\left[g(X^{\nu}_{t,x}(T))\right] \leq m\}$$

Weak Dynamic Programming for Generalized State Constraints, with Marcel Nutz, preprint 2011.

(compare with Bouchard, Elie, Imbert, *SIAM Journal on Control and Optimization, 48 (5), 2010.*)

Problem reformulation towards DPP

- State space augmentation : Let $\mathcal{M}_{t,m}$ be a set of càdlàg martingales $M = \{M(s), s \in [t, T]\}$ with initial value $M(t) = m$, adapted to $\mathbb{F}^t$.

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- **Martingale representation assumption** : We assume that, for all $(t, x) \in [0,\, T] \times \mathbb{R}^d$ and $\nu \in \mathcal{U}_t$:

$$\exists\ M_t^\nu[x] \in \mathcal{M}_{t,m} \text{ such that } M_t^\nu[x](T) = g(X_{t,x}^\nu(T)),$$

with $m := \mathbb{E}\left[g(X_{t,x}^\nu(T))\right]$.

Problem reformulation towards DPP

- State space augmentation : Let $\mathcal{M}_{t,m}$ be a set of càdlàg martingales $M = \{M(s),\, s \in [t,\, T]\}$ with initial value $M(t) = m$, adapted to $\mathbb{F}^t$.

- **Martingale representation assumption** : We assume that, for all $(t, x) \in [0,\, T] \times \mathbb{R}^d$ and $\nu \in \mathcal{U}_t$:

$$\exists\ M_t^\nu[x] \in \mathcal{M}_{t,m} \text{ such that } M_t^\nu[x](T) = g(X_{t,x}^\nu(T)),$$

with $m := \mathbb{E}\left[g(X_{t,x}^\nu(T))\right]$.

- Reformulation : We set

$$\mathcal{M}_{t,x,m}^+(\nu) := \{M \in \mathcal{M}_{t,m} :\ M(T) \geq g(X_{t,x}^\nu(T))\}$$

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Example : In a Brownian filtration, we can take

$$\mathcal{M}_{t,m} = m + \{M^\alpha_{t,0}(T) := \int_t^T \alpha_s dW_s, \ \alpha \in \mathcal{A}_t\}$$

where $\mathcal{A}_t$ is the set of predictable $\mathbb{R}^d$-valued processes such that $M^\alpha_{t,0}$ is a $\mathbb{F}^t$-adapted martingale.

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Heuristic DPP in the Brownian setting

- In the case $\mathcal{M}_{t,m} = m + \{M^{\alpha}_{t,0}(T) := \int_t^T \alpha_s dW_s,\ \alpha \in \mathcal{A}_t\}$:

$$v(t,x,m) \stackrel{?}{=} \sup_{(\nu,\alpha)\in\Theta(t,x,m)} \mathbb{E}\left[v(\theta, X^{\nu}_{t,x}(\theta), M^{\alpha}_{t,m}(\theta))\right]$$

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Guess : for all $\delta > 0$

$$v(t,x,m+\delta) \geq \sup_{(\nu,\alpha)\in\Theta(t,x,m)} \mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta), M^{\alpha}_{t,m}(\theta))\right].$$

Additional assumption

Assumption : Let $(t,x) \in [0,T] \times \mathbb{R}^d$, $\nu \in \mathcal{U}_t$, $\tau \in \mathcal{T}^t$, $\Gamma \in \mathcal{F}^t_\tau$, $\bar{\nu} \in \mathcal{U}_{\|\tau\|_{L^\infty}}$, and $M \in \mathcal{M}_{t,0}$. Then, there exists a process $\bar{M} = \{\bar{M}(r),\ r \in [\tau,T]\}$ such that

$$\bar{M}(\cdot)(\omega) = \left(M^{\bar{\nu}}_{\tau(\omega)}[X^{\nu}_{t,x}(\tau)(\omega)](\cdot)\right)(\omega) \quad \text{on } [\tau,T] \quad \mathbb{P}-\text{a.s.}$$

and

$$M\mathbf{1}_{[t,\tau)} + \mathbf{1}_{[\tau,T]}\Big(M\mathbf{1}_{\Omega\setminus\Gamma} + \left[\bar{M} - \bar{M}(\tau) + M(\tau)\right]\mathbf{1}_\Gamma\Big) \in \mathcal{M}_{t,0}.$$

General result

Theorem : Assume the above holds.
(i) Let $\varphi_+ \geq v$ be a measurable function. Then

$$v(t,x,m) \leq \mathbb{E}\left[\varphi_+(\theta^\nu, X^\nu_{t,x}(\theta^\nu), M(\theta^\nu))\right]$$

for some $\nu \in \mathcal{U}(t,x,m)$ and $M \in \mathcal{M}^+_{t,x,m}(\nu)$.

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for some $\nu \in \mathcal{U}(t,x,m)$ and $M \in \mathcal{M}^+_{t,x,m}(\nu)$.
(ii) Assume that $F(\cdot;\nu)$ and $-G(\cdot;\nu)$ are lsc for all $\nu \in \mathcal{U}$ (on the left in time). Let $\varphi_- \leq v$ be a usc function and fix $\delta > 0$. Then

$$v(t,x,m+\delta) \geq \mathbb{E}\left[\varphi_-(\theta^\nu, X^\nu_{t,x}(\theta^\nu), M(\theta^\nu))\right]$$

for all $\nu \in \mathcal{U}(t,x,m)$ and $M \in \mathcal{M}^+_{t,x,m}(\nu)$.

The Brownian setting

- $\mathcal{U}$: set of square integrable predictable processes with values in $U \subset \mathbb{R}^d$.

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• The set of martingales is given by :

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Domain of definition

- The natural domain $D := \{(t, x, m) \; : \; \mathcal{U}(t, x, m) \neq \emptyset\}$ is associated to

$$w(t, x) := \inf_{\nu \in \mathcal{U}_t} E[g(X^{\nu}_{t,x}(T))],$$

through

$$\mathrm{int} D = \left\{(t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} : \; m > w(t, x), \; t < T\right\},$$

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- One has

$$\begin{aligned} D &\subseteq \left\{(t,x,m) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} : \ m \geq w_*(t,x)\right\} \\ &= \overline{\mathrm{int} D}, \end{aligned}$$

where w_* is the lower semicontinuous envelope of v on $[0,T] \times \mathbb{R}^d$.

DPP for viscosity super-solution

Corollary : Assume the above holds. Let $\theta_B^{\nu,\alpha}$ be the first exist time of $(\cdot, X_{t,x}^{\nu}, M_{t,m}^{\alpha})$ from a ball B around $(t,x,m) \in \mathrm{int} D$. Then, for all $\delta > 0$ and $\theta^{\nu,\alpha} \leq \theta_B^{\nu,\alpha}$,

$$v(t,x,m+\delta) \geq \sup_{(\nu,\alpha)\in\mathcal{U}_t\times\mathcal{A}_t} \mathbb{E}\left[\varphi_-(\theta^{\nu,\alpha}, X_{t,x}^{\nu}(\theta^{\nu,\alpha}), M_{t,m}^{\alpha}(\theta^{\nu,\alpha}))\right]$$

Viscosity super-solution property derivation

Let φ_- be a test function for v_* at (t, x, m) Fix $(t_\varepsilon, x_\varepsilon, m_\varepsilon, \delta_\varepsilon) \to (t, x, m, 0)$ such that

$$|v(t_\varepsilon, x_\varepsilon, m_\varepsilon + \delta_\varepsilon) - \varphi_-(t_\varepsilon, x_\varepsilon, m_\varepsilon)| \leq \varepsilon^2 \to 0$$

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Set $(\nu, \alpha) = (u, a) \in U \times \mathbb{R}^d$, $\theta_\varepsilon := \theta_B^{u,a} \wedge (t_\varepsilon + \varepsilon)$. Then,

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and therefore

$$0 \geq \varepsilon^{-1}\mathbb{E}\left[\int_{t_\varepsilon}^{\theta_B^{u,a}\wedge(t_\varepsilon+\varepsilon)} (\partial_t + L^{u,a}_{X,M})\varphi_-(s, X^u_{t_\varepsilon,x_\varepsilon}(s), M^a_{t_\varepsilon,m_\varepsilon}(s))ds\right] - \varepsilon$$

and pass to the limit $\varepsilon \to 0$.

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and pass to the limit $\varepsilon \to 0$.
[In practice use a proof by contradiction to avoid passages to the limit]

PDE characterization

Theorem : Assume the above holds. Then,
(i) v_* is a viscosity super-solution on $\mathrm{int}D$ of

$$-\partial_t \varphi + H(\cdot, D\varphi, D^2\varphi) = 0.$$

(ii) v^* is a viscosity sub-solution on $\mathrm{cl}D$ of

$$-\partial_t \varphi + H_*(\cdot, D\varphi, D^2\varphi) = 0$$

where

$$H(\cdot, D\varphi, D^2\varphi) := - \sup_{(u,a) \in U \times \mathbb{R}^d} L^{u,a}_{X,M}\varphi.$$

(See Bouchard, Elie and Imbert 2010 for a discussion on the boundary conditions)

The case with $\mathbb{P}$ – a.s. state constraint

$$v(t,x) := \sup_{\nu \in \mathcal{U}(t,x)} F(t,x;\nu) \;\; \textbf{with} \;\; F(t,x,\nu) := \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right]$$

and

$$\mathcal{U}(t,x) := \{\nu \in \mathcal{U}_t \; : \; X^{\nu}_{t,x} \in \mathcal{O} \;\; \textbf{on } [t,T]\}$$

with $\mathcal{O}$ an open subset.

Weak Dynamic Programming for Generalized State Constraints, with Marcel Nutz, preprint 2011.

A-priori difficulty

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• Can in fact almost do this if $\mathcal{O}$ is open : if exits, it should be with small probability.

Approximation by constraints in expectations

- Additional dimension

$$Y^{\nu}_{t,x,y}(s) := y \wedge \inf_{r \in [t,s]} d(X^{\nu}_{t,x}(r)), \quad s \in [t,T], \quad y > 0.$$

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By continuity, each trajectory $\{X^{\nu}_{t,x}(r)(\omega),\ r \in [t,T]\}$ has strictly positive distance to $\mathcal{O}^c$ whenever it is contained in $\mathcal{O}$:

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- Equivalent control problem

$$v(t,x) = \bar{v}(t,x,y,0)$$

where

$$\bar{v}(t,x,y,m) := \sup_{\nu\in\mathcal{U}(t,x,y,m)} F(t,x;\nu)$$

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$$\mathcal{U}(t,x,y,m) := \{\nu \in \mathcal{U}_t \ : \ \mathbb{P}\left[Y^{\nu}_{t,x,1}(T) \le 0\right] \le m\}.$$

DPP in the state constraint case

- Apply the DPP to $\bar{v}$: for all $(\nu, \alpha) \in \Theta(t, x, y, 0)$

$$\bar{v}(t, x, y, 0 + \delta) \geq \mathbb{E}\left[\varphi_-(\theta^\nu, X^\nu_{t,x}(\theta^\nu), Y^\nu_{t,x,y}(\theta^\nu), M^\alpha_{t,0}(\theta^\nu))\right]$$

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If $\bar{v}(t, x, y, 0+) = \bar{v}(t, x, y, 0)$, then

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Hence

$$v(t, x) \geq \sup_{\nu \in \mathcal{U}(t,x)} \mathbb{E}\left[\phi_{-}(\theta^{\nu}, X^{\nu}_{t,x}(\theta^{\nu}))\right]$$

for $\phi_{-} \leq v$ usc.

DPP in the state constraint case : sufficient condition

Measurable selection assumption : There exists a Lipschitz continuous mapping $\hat{u} : \mathcal{O} \to U$ such that, for all $(t, x) \in [0, T] \times \mathcal{O}$, the solution $\hat{X}_{t,x}$ of

$$\hat{X}(s) = x + \int_t^s b\big(\hat{X}(r), \hat{u}(\hat{X}(r))\big)\, dr + \int_t^s \sigma\big(\hat{X}(r), \hat{u}(\hat{X}(r))\big)\, dW_r$$

satisfies $\hat{X}_{t,x}(s) \in \mathcal{O}$ for all $s \in [t, T]$, $\mathbb{P}$ − a.s.

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Proposition : Under the above assumption, $\bar{v}(t, x, y, 0+) = \bar{v}(t, x, y, 0)$.

DPP in the state constraint case

Lemma : Let B be an open neighborhood of $(t,x) \in [0,T] \times \mathcal{O}$ such that $v(t,x)$ is finite.
(i) Let $\varphi : \mathrm{cl}B \to \mathbb{R}$ be a continuous function such that $v \leq \varphi$ on $\mathrm{cl}B$. For all $\varepsilon > 0$, there exists $\nu \in \mathcal{U}(t,x)$ such that

$$v(t,x) \leq E\left[\varphi(\tau, X_{t,x}^{\nu}(\tau))\right] + \varepsilon,$$

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(ii) For any $\nu \in \mathcal{U}_t$ and any continuous function φ s.t. $v \geq \varphi$ on $\mathrm{cl}B$

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The case of closed domain

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  a. the sub-solution property still holds for $v^*_{\mathrm{cl}\mathcal{O}}$,
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  c. but $v_{\mathrm{cl}\mathcal{O}} \geq v_{\mathcal{O}}$ by definition.

Sufficient condition for comparison

Sufficient condition : Comparison holds if the super-solution is of class $\mathcal{R}(\mathcal{O})$ (for functions with polynomial growth).

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Definition : w is *of class* $\mathcal{R}(\mathcal{O})$ if

1. $\exists\, r > 0$, an open neighborhood B of x in $\mathbb{R}^d$ and a function $\ell : \mathbb{R}_+ \to \mathbb{R}^d$ such that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}|\ell(\varepsilon)| < \infty \quad \text{and}$$
$$y + \ell(\varepsilon) + o(\varepsilon) \in \mathcal{O} \quad \text{for all } y \in \mathrm{cl}\mathcal{O} \cap B \text{ and } \varepsilon \in (0, r).$$

2. $\exists\, \lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{\varepsilon\to 0} \lambda(\varepsilon) = 0 \quad \text{and}$$
$$\lim_{\varepsilon\to 0} w\big(t + \lambda(\varepsilon), x + \ell(\varepsilon)\big) = w(t, x).$$

Sufficient condition for comparison

Example : - There exists a C^1-function δ such that $D\delta$ is locally Lipschitz continuous and

$$\delta > 0 \text{ on } \mathcal{O}, \quad \delta = 0 \text{ on } \partial\mathcal{O}, \quad \delta < 0 \text{ outside } \mathrm{cl}\mathcal{O}.$$

- There exists a locally Lipschitz continuous mapping $\check{u} : \mathbb{R}^d \to U$ s.t. for all $x \in \mathrm{cl}\mathcal{O}$ $\exists$ open neighborhood B of x and $\iota > 0$ satisfying

$$\mu(z, \check{u}(z))^\top D\delta(y) \geq \iota \text{ and } \sigma(y, \check{u}(y)) = 0 \ \forall\, y \in B \cap \mathrm{cl}\mathcal{O} \ , \, z \in B.$$

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Similar conditions from the literature : Soner (1986), Katsoulakis (1994), Ishii and Loreti (2002).