Stochastic target games

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Joint works with L. Moreau (ETH-Zürich) and M. Nutz (Columbia)

Problem formulation and Motivations

Problem formulation

Provide a PDE characterization of the *viability* sets

$$\Lambda(t) := \{(z, m) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))|\mathcal{F}_t\right] \geq m\ \forall\ \vartheta \in \mathcal{V}\}$$

In which :

- $\mathcal{V}$ is a set of admissible adverse controls
- $\mathfrak{U}$ is a set of admissible strategies
- $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}$ is an adapted $\mathbb{R}^d$-valued process s.t. $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(t) = z$
- ℓ is a given loss/utility function
- m a threshold.

Application in finance

- $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta} = (X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}, Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta})$ where
  - $X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}$ models financial assets or factors with dynamics depending on ϑ
  - $Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}$ models a wealth process
  - ϑ is the control of the market : parameter uncertainty (e.g. volatility), adverse players, etc...
  - $\mathfrak{u}[\vartheta]$ is the financial strategy given the past observations of ϑ.

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Robust partial hedging under uncertainty and related price :

$$\inf\{y : \exists\, \mathfrak{u} \text{ s.t. } \mathbb{E}\left[\Psi\left(Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}(T) \geq g(X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}(T))\right)\right] \geq m \;\forall\, \vartheta\}$$

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Robust hedging under uncertainty and related price :

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□ Flexible enough to embed constraints, transaction costs, market impact, etc...

Setting for this talk

(see the papers for abstract versions)

Brownian diffusion setting

Brownian diffusion setting

- **State process :** $Z^{\mathfrak{u}[\vartheta],\vartheta}$ solves (μ and σ continuous, uniformly Lipschitz in space)

$$Z(s) = z + \int_t^s \mu(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)\, dr + \int_t^s \sigma(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)\, dW_r$$

- The loss function ℓ has polynomial growth and is continuous.

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□ The loss function ℓ has polynomial growth and is continuous.

□ **Controls and strategies :**

- $\mathcal{V}$ is the set of predictable processes with values in $V \subset \mathbb{R}^d$.
- $\mathfrak{U}$ is set of non-anticipating maps $\mathfrak{u} : \vartheta \in \mathcal{V} \mapsto \mathcal{U}$, i.e.

$$\{\omega : \vartheta_1(\omega) =_{[0,s]} \vartheta_2(\omega)\} \subset \{\omega : \mathfrak{u}[\vartheta_1](\omega) =_{[0,s]} \mathfrak{u}[\vartheta_2](\omega)\}.$$

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The game problem

- **The *viability* sets** are given by

$$\Lambda(t) \quad := \quad \{(z,m) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))|\mathcal{F}_t\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

Compare with the formulation of games in Buckdahn and Li (08).

Geometric dynamic programming principle for controlled loss cases

How are the properties

$(z, m) \in \Lambda(t)$ and $(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), ?) \in \Lambda(\theta)$

related ?

$$\Lambda(t) \quad := \quad \{(z, m) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))|\mathcal{F}_t\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

Unformal derivation

- Take $(z, m) \in \Lambda(t)$ and $\mathfrak{u} \in \mathfrak{U}$ such that

$$\operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T)\right) |\mathcal{F}_t\right] \geq m \;\; \mathbb{P}-\text{a.s.}$$

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Take care of the evolution of the worst case scenario conditional expectation :

$$S_r^{\vartheta} := \operatorname{ess\,inf}_{\bar{\vartheta} \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta \oplus_r \bar{\vartheta}],\vartheta \oplus_r \bar{\vartheta}}(T)\right) | \mathcal{F}_r\right],$$

where $\vartheta \oplus_r \bar{\vartheta} = \vartheta \mathbf{1}_{[0,r]} + \mathbf{1}_{(r,T]} \bar{\vartheta}$.

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where $\vartheta \oplus_r \bar{\vartheta} = \vartheta \mathbf{1}_{[0,r]} + \mathbf{1}_{(r,T]}\bar{\vartheta}$.

Then

$$S^{\vartheta} \text{ is a submartingale and } S_t^{\vartheta} \geq m \text{ for all } \vartheta \in \mathcal{V},$$

and we can find a martingale M^{ϑ} such that

$$S^{\vartheta} \geq M^{\vartheta} \text{ and } M_t^{\vartheta} = S_t^{\vartheta} \geq m.$$

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□ Take $(z, m) \in \Lambda(t)$ and $\mathfrak{u} \in \mathfrak{U}$ such that

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where $\vartheta \oplus_r \bar{\vartheta} = \vartheta \mathbf{1}_{[0,r]} + \mathbf{1}_{(r,T]}\bar{\vartheta}$.

Hence,

$$\operatorname*{ess\,inf}_{\bar{\vartheta} \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta \oplus_\theta \bar{\vartheta}],\vartheta \oplus_\theta \bar{\vartheta}}(T)\right) |\mathcal{F}_\theta\right] = S_\theta^{\vartheta} \geq M_\theta^{\vartheta} \;\; \mathbb{P} - \text{a.s.}$$

and therefore there exists a martingale M^{ϑ} such that $M_t^{\vartheta} = m$ and

$$\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_\theta^{\vartheta}\right) \in \Lambda(\theta) \; \mathbb{P} - \text{a.s.}$$

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and therefore there exists a predictable $\alpha^{\vartheta} \in \mathcal{A}$ such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_{t,m}^{\alpha^{\vartheta}}(\theta)) \in \Lambda(\theta)\; \mathbb{P}-\text{a.s.}\;, \;\; M_{t,m}^{\alpha^{\vartheta}} := m + \int_t^{\cdot} \alpha_s^{\vartheta} dW_s$$

The geometric dynamic programming principle

(GDP1) : If $(z,m)\in\Lambda(t)$, then $\exists\ \mathfrak{u}\in\mathfrak{U}$ and $\{\alpha^{\vartheta},\vartheta\in\mathcal{V}\}\subset\mathcal{A}$ such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_{t,m}^{\alpha^{\vartheta}}(\theta))\in\Lambda(\theta)\ \mathbb{P}-\text{a.s.}\ \forall\ \vartheta\in\mathcal{V}.$$

(GDP2) : If $(\mathfrak{u},\mathfrak{a})\in\mathfrak{U}\times\mathfrak{A}$ are such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta[\vartheta]), M_{t,m}^{\mathfrak{a}[\vartheta]}(\theta[\vartheta]))\in\Lambda(\theta[\vartheta])\ \mathbb{P}-\text{a.s.}\ \forall\ \vartheta\in\mathcal{V}$$

for some family $(\theta[\vartheta],\vartheta\in\mathcal{V})$ of non-anticipating stopping times, then

$$(z,m)\in\Lambda(t).$$

The geometric dynamic programming principle

(GDP1) : If $(z,m) \in \Lambda(t)$, then $\exists\ \mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^{\vartheta}, \vartheta \in \mathcal{V}\} \subset \mathcal{A}$ such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_{t,m}^{\alpha^{\vartheta}}(\theta)) \in \Lambda(\theta)\ \mathbb{P}-\text{a.s.}\ \forall\ \vartheta \in \mathcal{V}.$$

(GDP2) : If $(\mathfrak{u},\mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ are such that

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for some family $(\theta[\vartheta], \vartheta \in \mathcal{V})$ of non-anticipating stopping times, then

$$(z,m) \in \Lambda(t).$$

Rem : Use heavily the regularity of the constraint in expectation (ℓ continuous + unif. Lipschitz coefficients). Exact statement requires an extra relaxation, which does not alter the pde derivation. See Bouchard, Moreau and Nutz, AAP to appear.

PDE Characterization

□ **Monotone case :** $Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y} = (X^{\mathfrak{u}[\vartheta],\vartheta}_{t,x}, Y^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y})$ with values in $\mathbb{R}^d \times \mathbb{R}$ with $X^{\mathfrak{u}[\vartheta],\vartheta}_{t,x}$ independent of y and $\ell \uparrow y$.

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□ **The value function is :**

$$\varpi(t,x,m) := \inf\{y : (x,y,m) \in \Lambda(t)\}.$$

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$$\varpi(t, x, m) := \inf\{y : (x, y, m) \in \Lambda(t)\}.$$

☐ We have the "characterization"

$$y > \varpi(t, x, m) \Rightarrow (z, m) \in \Lambda(t) \Rightarrow y \geq \varpi(t, x, m)$$

PDE characterization - "waving hands" version

- Assuming smoothness, existence of optimal strategies...

- $y = \varpi(t, x, m)$ implies
$Y^{\mathfrak{u}[\vartheta],\vartheta}(t+) \geq \varpi(t+, X^{\mathfrak{u}[\vartheta],\vartheta}(t+), M^{\mathfrak{a}[\vartheta]}(t+))$ for all ϑ.

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This implies $dY^{\mathfrak{u}[\vartheta],\vartheta}(t) \geq d\varpi(t, X^{\mathfrak{u}[\vartheta],\vartheta}(t), M^{\mathfrak{a}[\vartheta]}(t))$ for all ϑ

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Hence, for all ϑ,

$$
\begin{aligned}
\mu_Y(x, y, \mathfrak{u}[\vartheta]_t, \vartheta_t) &\geq \mathcal{L}_{X,M}^{\mathfrak{u}[\vartheta]_t, \vartheta_t, \mathfrak{a}[\vartheta]_t} \varpi(t, x, m) \\
\sigma_Y(x, y, \mathfrak{u}[\vartheta]_t, \vartheta_t) &= \sigma_X(x, \mathfrak{u}[\vartheta]_t, \vartheta_t) D_x \varpi(t, x, m) \\
&\quad + \mathfrak{a}[\vartheta]_t D_m \varpi(t, x, m)
\end{aligned}
$$

with $y = \varpi(t, x, m)$

PDE characterization - "waving hands" version

□

$$\sup_{(u,a)\in\mathcal{N}^v\varpi}\left(\mu_Y(\cdot,\varpi,u,v)-\mathcal{L}^{u,v,a}_{X,M}\varpi\right)\geq 0$$

where

$$\mathcal{N}^v\varpi := \{(u,a)\in U\times\mathbb{R}^d : \sigma_Y(\cdot,\varpi,u,v)=\sigma_X(\cdot,u,v)D_x\varpi + aD_m\varpi\}.$$

PDE characterization - "waving hands" version

□

$$\inf_{v \in V} \sup_{(u,a) \in \mathcal{N}^v \varpi} \left(\mu_Y(\cdot, \varpi, u, v) - \mathcal{L}^{u,v,a}_{X,M} \varpi \right) \geq 0$$

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PDE characterization - “waving hands” version

- **Supersolution property**

$$\inf_{v\in V}\ \sup_{(u,a)\in\mathcal{N}^v\varpi}\left(\mu_Y(\cdot,\varpi,u,v)-\mathcal{L}^{u,v,a}_{X,M}\varpi\right)\geq 0$$

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Subsolution property

$$\sup_{(u[\cdot],a[\cdot])\in\mathcal{N}^{[\cdot]}\varpi}\ \inf_{v\in V}\left(\mu_Y(\cdot,\varpi,u[v],v)-\mathcal{L}^{u[v],v,a[v]}_{X,M}\varpi\right)\leq 0$$

where

$$\mathcal{N}^{[\cdot]}\varpi := \{\text{loc. Lip. } (u[\cdot],a[\cdot]) \text{ s.t. } (u[\cdot],a[\cdot])\in\mathcal{N}^{\cdot}\varpi(\cdot)\}.$$

Geometric dynamic programming principle for the a.s. constraint case

Prove that

$$y \geq \varpi(t,x) \Leftrightarrow Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}(\theta) \geq \varpi(\theta, X_{t,x}^{\vartheta}(\theta))$$

(when X does not depend on $\mathfrak{u}$)

$$\varpi(t,x) := \inf\{y : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } Y_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T) \geq g(X_{t,x}^{\vartheta}(T)) \; a.s. \; \forall\, \vartheta \in \mathcal{V}\}$$

Geometric dynamic programming principle for the a.s. constraint case

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Expected loss case : play with the regularity of the constraint in expectation form. Not possible here.

Geometric dynamic programming principle for the a.s. constraint case

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No adverse control case : measurable selection argument which requires to have a Polish space structure. Not possible here.

Geometric dynamic programming principle for the a.s. constraint case

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Main difficulty : no smoothness and no measurable selection argument possible.

GDP1 - "Easy part"

□ **GDP1** Assume that $y > \varpi(t,x)$. Then, there exists $\mathfrak{u} \in \mathfrak{U}$ such that

$$Y^{\mathfrak{u},\vartheta}_{t,x,y}(\theta) \geq \varpi_*(\theta, X^{\vartheta}_{t,x}(\theta)) \; a.e. \;\; \forall\, \vartheta \in \mathcal{V}.$$

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This implies as before that ϖ_* is a supersolution of

$$H\varpi_* := \inf_{v \in V} \sup_{u \in \mathcal{N}^v \varpi_*} \left(\mu_Y(\cdot, \varpi_*, u, v) - \mathcal{L}^v_X \varpi_*\right) \geq 0$$

where

$$\mathcal{N}^v \varpi_* := \{u \in U : \sigma_Y(\cdot, \varpi_*, u, v) = \sigma_X(\cdot, v) D\varpi_*\}.$$

GDP2 - "Difficult part"

- Assume that : $\varpi_*(T,\cdot) \geq g$ and that the operator

$$H\varphi := \inf_{v \in V} \sup_{u \in \mathcal{N}^v \varphi} \left(\mu_Y(\cdot, \varphi, u, v) - \mathcal{L}_X^v \varphi \right)$$

is concave in φ (as a function of φ, $D\varphi$ and $D^2\varphi$).

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is concave in φ (as a function of φ, $D\varphi$ and $D^2\varphi$).

Then, for each compact set B and $\eta > 0$, one can construct (under Lipschitz continuity assumptions) a smooth supersolution w of

$$Hw \geq 0 \text{ on } [0,T) \times \mathbb{R}^d \text{ and } w \geq g \text{ on } \{T\} \times \mathbb{R}^d$$

satisfying

$$w \leq \varpi + \eta \text{ on } B.$$

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satisfying

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Slight extension of the smoothing technic of Krylov : Provide a supersolution with shaked coefficients obtained by studying a suitable optimal control of BSDEs problem. Then integrate with a smooth kernel as in Ishii. Need stability for the family of BSDEs.

GDP2 - "Difficult part"

- Assume further that : There exist a unique solution $\hat{u}(x, y, \rho, v)$ to $\sigma_Y(x, y, u, v) = \rho$ for all y, v, ρ.

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Then, (under Lipschitz continuity assumptions) use a verification ensuring that if

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq w(\theta, X^{\vartheta}_{t,x}(\theta)) \;\; a.e. \; \forall \, \vartheta \in \mathcal{V},$$

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then since

$$\mu_Y(\cdot, w, \hat{u}(\cdot, w, \sigma_X(\cdot, v)Dw, v), v) - \mathcal{L}^v_X w \geq 0 \ \text{ and } \ w(T, \cdot) \geq g,$$

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□ Assume further that : There exist a unique solution $\hat{u}(x, y, \rho, v)$ to $\sigma_Y(x, y, u, v) = \rho$ for all y, v, ρ.

Then, (under Lipschitz continuity assumptions) use a verification ensuring that if

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq w(\theta, X^{\vartheta}_{t,x}(\theta)) \ \ a.e. \ \forall \ \vartheta \in \mathcal{V},$$

then since

$$\mu_Y(\cdot, w, \hat{u}(\cdot, w, \sigma_X(\cdot, v)Dw, v), v) - \mathcal{L}^v_X w \geq 0 \ \ \text{and} \ \ w(T, \cdot) \geq g,$$

the Markovian strategy defined by

$$\bar{\mathfrak{u}}[\vartheta] := \mathfrak{u}_o \mathbf{1}_{[t,\theta)} + 1_{[\theta,T]} \hat{u}(Z^{\bar{\mathfrak{u}},\vartheta}_{t,x,y}, [\sigma_X(\cdot, \vartheta) D_X w](\cdot, X^{\vartheta}_{t,x}), \vartheta)$$

GDP2 - "Difficult part"

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is such that

$$Y_{t,x,y}^{\bar{\mathfrak{u}},\vartheta}(T) \geq g(X_{t,x}^{\vartheta}(T)) \;\; a.e. \; \forall \, \vartheta \in \mathcal{V}.$$

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☐ **GDP2** Let ϕ be a test function for ϖ^* at (t,x). Let $\eta > 0$ be such that

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq \varpi(\theta, X^{\vartheta}_{t,x}(\theta)) + \eta \;\; a.e. \; \forall\, \vartheta \in \mathcal{V},$$

where θ is the first exit time from an open ball $\mathcal{O} \ni (t,x)$. Then, $y \geq \varpi(t,x)$.

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Then,

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq w(\theta, X^{\vartheta}_{t,x}(\theta)) \;\; a.e. \; \forall \, \vartheta \in \mathcal{V}.$$

From this, we can use the Markovian strategy based on $\hat{u}$ to reach the target at T for all $\vartheta \in \mathcal{V}$.

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Essentially only needs the concavity of the operator which is related to the fact that controlling the volatility imposes the choice of the control.

References

☐ B., L. Moreau and M. Nutz. *Stochastic Target Games with Controlled Loss*, AAP, to appear.

☐ B. and M. Nutz. *Dynamic programming for a class of stochastic target games - Application to hedging under model uncertainty*, in preparation.

☐ H.M. Soner and N. Touzi. *Dynamic programming for stochastic target problems and geometric flows*, JEMS, 4, 201-236, 2002.

☐ B. , R. Elie and N. Touzi. *Stochastic Target problems with controlled loss*, SIAM SICON, 48 (5), 3123-3150, 2009.

☐ L. Moreau. *Stochastic target problems with controlled loss in a jump diffusion model*, SIAM SICON, 49, 2577-2607, 2011.