Arbitrage and Duality in Nondominated Discrete-Time Models

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Joint work M. Nutz (Columbia)

Motivation

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- Show existence of minimal super-hedging strategy.

- Provide a dual formulation for super-hedging prices.

Classical Framework

- Only one reference measure $\mathcal{P} = \{P_o\}$ which fixes the null sets.
- No-Arbitrage NA(P_o) : $Y_T \geq 0$ P_o-a.s. $\Rightarrow$ $Y_T = 0$ P_o-a.s.
- NA$(P_o)\Leftrightarrow$ $\mathcal{Q}(P_o) := \{Q \sim P_o : S \text{ is a } Q\text{-mart.}\} \neq \emptyset$.
- Completeness $\Leftrightarrow$ $|\mathcal{Q}(P_o)| = 1$.
- There exists a minimal super-hedging strategy.
- Super-hedging price of f is $\sup\{\mathbb{E}_Q[f],\ Q \in \mathcal{Q}(P_o)\}$.

The non-dominated case

☐ The family $\mathcal{P}$ is made of (possibly) singular measures P which fix the polar sets : $A \subset A'$ with $P[A'] = 0 \; \forall \; P \in \mathcal{P}$, i.e. $A = \emptyset \; \mathcal{P}$-q.s.

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□ Questions :

- What is the good notion of arbitrage ?
- Which duality do we look for ?
- What minimal conditions can we afford ?

What is a good notion of no-arbitrage ?

□ Different possibilities :

• $Y_T \geq 0$ $\mathcal{P}$-q.s. and $P[Y_T > 0] > 0 \ \forall \ P \in \mathcal{P}$ is impossible. One has to be lucky whatever the true model is.

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Dual formulation and super-hedging price

$$\begin{aligned}\pi(f) &:= \inf\left\{x \in \mathbb{R} : \exists\,(H,h) \in \mathcal{H} \times \mathbb{R}^{|\mathrm{I}|} \text{ s.t. } x + (H \bullet S)_T + hg \geq f \text{ -q.s.}\right\} \\ &= \sup_Q \mathbb{E}_Q[f]\end{aligned}$$

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- A family of mart. measures $\mathcal{Q}$ with the same polar sets : $\mathcal{Q} \sim \mathcal{P}$.

The one period case

$(\Omega, \mathcal{F})$ a measurable price. ΔS a random variable. $\mathcal{P}$ a convex set of measures on $(\Omega, \mathcal{F})$. No option for static hedging.

First Fundamental Theorem

- No-Arbitrage condition : Condition $NA(\mathcal{P})$ holds if for all $H \in \mathcal{H}$

$$H\Delta S \geq 0 \quad \mathcal{P}\text{-q.s.} \qquad \text{implies} \qquad H\Delta S = 0 \quad \mathcal{P}\text{-q.s.}$$

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□ First Fundamental Theorem : The following are equivalent :
(i) $NA(\mathcal{P})$ holds.
(ii) For all $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that $P \ll Q$.
(ii') $\mathcal{P}$ and $\mathcal{Q}$ have the same polar sets.

Rem : These are the usual equivalent conditions when $\mathcal{P} = \{P_o\}$.

☐ One can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $\mathbf{L}^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(\mathbf{L}^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

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If not : $0 \le y\Delta S \ \Rightarrow\ 0 = y\Delta S$.
And reduce the dimension by one until the case $d = 1$ is reached.

Super-hedging Theorem

- Theorem : Let $NA(\mathcal{P})$ hold and let f be a random variable. Then

$$\sup_{Q \in \mathcal{Q}} E_Q[f] = \pi(f) := \inf \left\{ x : \exists\, H \in \mathbb{R}^d \text{ s.t. } x + H\Delta S \geq f \ \mathcal{P}\text{-q.s.} \right\}.$$

Moreover, $\pi(f) > -\infty$ and $\exists$ H s.t. $\pi(f) + H\Delta S \geq f$ $\mathcal{P}$-q.s.

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□ Again, one can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. We do neither have compactness on $\mathcal{Q}$ (role plaid by the *power option* in Acciaio et al. 2013).

Step 1 : Construct approximating martingale measures

Assume $\pi(f) = 0$ and show that

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Step 1 : Construct approximating martingale measures

Assume $\pi(f) = 0$ and show that

$$\exists\ R_n \lll \mathcal{P} \text{ s.t. } \ E_{R_n}[\Delta S] \to 0 \quad \text{and} \quad E_{R_n}[f] \to 0.$$

1. If not : $0 \notin \mathrm{cl}\big\{E_R[(\Delta S, f)] : R \lll \mathcal{P},\ E_R[|\Delta S| + |f|] < \infty\big\}$
2. This implies $0 < \alpha \leq y\Delta S + zf$.

Step 2 : Correct the approximating martingale measures

1. Choose $R_n \lll \mathcal{P}$ s.t. $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$.
2. One has $0 \in ri\big\{E_R[\Delta S] :\ P \ll R \lll \mathcal{P},\ E_R[|\Delta S| + |f|] < \infty\big\}$.
3. We can correct in $\tilde{R}_n = (1 - \lambda_n)R_n + \lambda_n R'_n$ s.t.

$$E_{\tilde{R}_n}[\Delta S] = 0 \quad \text{and} \quad E_{\tilde{R}_n}[f] \to 0 = \pi(f).$$

The multiperiod case with options for static hedging

$\mathcal{Q} = \{ Q \lll \mathcal{P} : Q$ is a mart. measure and $E_Q[g^i] = 0$ for $i = 1, \dots, |\mathrm{I}| \}$.

Theorem : The following are equivalent :
(i) $NA(\mathcal{P})$ holds.
(ii) For all $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that $P \ll Q$.
(ii') $\mathcal{P}$ and $\mathcal{Q}$ have the same polar sets.

Theorem : Let $NA(\mathcal{P})$ hold and let $f : \Omega \to \mathbb{R}$ be upper semianalytic. Then,

$$\pi(f) := \inf \{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|\mathrm{I}|} \text{ s.t. } x + (H \bullet S)_T + hg \geq f \ \mathcal{P}\text{-q.s.} \}$$

admits existence and satisfies

$$\pi(f) = \sup_{Q \in \mathcal{Q}} E_Q[f] \in (-\infty, \infty]$$

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- Options for static hedging are assumed Borel.
- Claims to super-hedge are upper-semianalytic.

Second Fundamental Theorem

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Theorem : Let $NA(\mathcal{P})$ hold and let $f : \Omega \to \mathbb{R}$ be upper semianalytic. The following are equivalent :

(i) f is replicable, i.e. $\pi(f) + (H \bullet S)_T = f$ $\mathcal{P}$-q.s.
(ii) $Q \mapsto E_Q[f]$ is constant (and finite) on $\mathcal{Q}$.
(ii) $\forall\ P \in \mathcal{P}\ \exists\ Q \in \mathcal{Q}$ s.t. $P \ll Q$ and $E_Q[f] = \pi(f)$.

Moreover, the market is complete (for Borel claims) if and only if $\mathcal{Q}$ is a singleton.

Application to Optional Decomposition

□ Theorem : Let $NA(\mathcal{P})$ hold and let V be an adapted process such that V_t is upper semianalytic and in $L^1(Q)$ $\forall\ Q \in \mathcal{Q}$.
The following are equivalent :

- V is a supermartingale under each $Q \in \mathcal{Q}$.
- There exist a predictable H and an adapted increasing process K with $K_0 = 0$ such that

$$V_t = V_0 + (H \bullet S)_t - K_t \quad \mathcal{P}\text{-q.s.}, \quad t \in \{0, 1, \dots, T\}.$$

Rem : The decomposition can not be obtained by hand as for continuous processes, but we have discrete time (measurable selection).

Connection to Martingale Inequalities

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☐ Then, $NA(\mathcal{P})$ holds for the universal completion of the raw filtration.

☐ One can apply the super-hedging theorem :
Assume that

$$\mathbb{E}_P[f(S_1, \cdots, S_T)] \leq 0 \text{ for all martingale measure } P \text{ on } \Omega_T.$$

Then, there exists universally measurable maps $H_1, \ldots, H_T$ such that

$$f(x_1, \cdots, x_T) \leq \sum_{t=0}^{T-1} H_{t+1}(x_0, \ldots, x_t)(x_{t+1} - x_t) \;\; \forall\, x \in (\mathbb{R}^d)^{T+1}.$$

Compare with Acciaio, Beiglböck, Penkner and Schachermayer (2013).