Arbitrage and Duality in Nondominated Discrete-Time Models

B. Bouchard

Ceremade - Univ. Paris-Dauphine, and, Crest - EnsaeparisTech

Frontiers in Financial Mathematics, Dublin, June 2013

Joint work M. Nutz (Columbia)
Motivation

- Given \((\Omega, \mathcal{F})\), a family \(\mathcal{P}\) of probability measures and \(S = (S_t)_{t \leq T}\) a \(d\)-dimensional process for stock prices.
Motivation

- Given $(\Omega, \mathcal{F})$, a family $\mathcal{P}$ of probability measures and $S = (S_t)_{t \leq T}$ a $d$-dimensional process for stock prices.

- Give necessary and sufficient conditions for No-Arbitrage in terms of Martingale Measures.
**Motivation**

- Given $(\Omega, \mathcal{F})$, a family $\mathcal{P}$ of probability measures and $S = (S_t)_{t \leq T}$ a $d$-dimensional process for stock prices.

- Give necessary and sufficient conditions for No-Arbitrage in terms of Martingale Measures.

- Show existence of minimal super-hedging strategy.
Motivation

- Given \((\Omega, \mathcal{F})\), a family \(\mathcal{P}\) of probability measures and \(S = (S_t)_{t \leq T}\) a \(d\)-dimensional process for stock prices.

- Give necessary and sufficient conditions for No-Arbitrage in terms of Martingale Measures.

- Show existence of minimal super-hedging strategy.

- Provide a dual formulation for super-hedging prices.
Classical Framework

- Only one reference measure $\mathcal{P} = \{P_o\}$ which fixes the null sets.
- No-Arbitrage $\text{NA}(P_o) : Y_T \geq 0$ $P_o$-a.s. $\Rightarrow Y_T = 0$ $P_o$-a.s.
- $\text{NA}(P_o) \iff \mathcal{Q}(P_o) := \{Q \sim P_o : S$ is a $Q$-mart.$\} \neq \emptyset$.
- Completeness $\iff |\mathcal{Q}(P_o)| = 1$.
- There exists a minimal super-hedging strategy.
- Super-hedging price of $f$ is $\sup\{E_Q[f], \ Q \in \mathcal{Q}(P_o)\}$. 
The non-dominated case

- The family $\mathcal{P}$ is made of (possibly) singular measures $P$ which fix the polar sets: $A \subset A'$ with $P[A'] = 0 \ \forall \ P \in \mathcal{P}$, i.e. $A = \emptyset \ \mathcal{P}$-q.s.

$\Rightarrow$ it stands for model uncertainty.

Example: all Dirac masses on $\Omega = (\mathbb{R}^d)^T \Rightarrow$ Model free point of view.
The non-dominated case

The family $\mathcal{P}$ is made of (possibly) singular measures $P$ which fix the polar sets: $A \subset A'$ with $P[A'] = 0 \ \forall \ P \in \mathcal{P}$, i.e. $A = \emptyset$ $\mathcal{P}$-q.s.

$\Rightarrow$ it stands for model uncertainty.

**Example:** all Dirac masses on $\Omega = (\mathbb{R}^d)^T \Rightarrow$ Model free point of view.

$\square$ A huge related literature: see below.
The non-dominated case

- The family $\mathcal{P}$ is made of (possibly) singular measures $P$ which fix the polar sets: $A \subset A'$ with $P[A'] = 0 \ \forall \ P \in \mathcal{P}$, i.e. $A = \emptyset \ P$-q.s.

$\Rightarrow$ it stands for model uncertainty.

Example: all Dirac masses on $\Omega = (\mathbb{R}^d)^T \Rightarrow$ Model free point of view.

- A huge related literature: see below.

- Questions:
  - What is the good notion of arbitrage?
  - Which duality do we look for?
  - What minimal conditions can we afford?
What is a good notion of no-arbitrage?

Different possibilities:

- $Y_T \geq 0$ $\mathcal{P}$-q.s. and $P[Y_T > 0] > 0 \ \forall \ P \in \mathcal{P}$ is impossible. One has to be lucky whatever the true model is.

- $Y_T(\omega) > 0$ for all $\omega$ is impossible (Acciaio, Beiglböck, Penkner and Schachermayer 2013).
What is a good notion of no-arbitrage?

Different possibilities:

• $Y_T \geq 0 \, \mathcal{P}\text{-q.s.}$ and $P[Y_T > 0] > 0 \, \forall \, P \in \mathcal{P}$ is impossible. One has to be lucky whatever the true model is.

• $Y_T \geq 0 \, \mathcal{P}\text{-q.s.}$ and $P[Y_T > 0] > 0$ for some $P \in \mathcal{P}$ is impossible. One has to be lucky on the model as well. (e.g. Deparis and Martini 04 for $\mathcal{P}$ generated by Dirac Mass)
What is a good notion of no-arbitrage?

- Different possibilities:
  
  - $Y_T \geq 0$ $P$-q.s. and $P[Y_T > 0] > 0 \forall P \in \mathcal{P}$ is impossible. One has to be lucky whatever the true model is.

  - $Y_T \geq 0$ $P$-q.s. and $P[Y_T > 0] > 0$ for some $P \in \mathcal{P}$ is impossible. One has to be lucky on the model as well. (e.g. Deparis and Martini 04 for $\mathcal{P}$ generated by Dirac Mass)

  - $Y_T(\omega) > 0$ for all $\omega$ is impossible (Acciaio, Beiglböck, Penkner and Schachermayer 2013).
What is a good notion of no-arbitrage?

- Different possibilities:
  - $Y_T \geq 0$ $P$-q.s. and $P[Y_T > 0] > 0 \forall P \in \mathcal{P}$ is impossible. One has to be lucky whatever the true model is.
  - $Y_T \geq 0$ $P$-q.s. and $P[Y_T > 0] > 0$ for some $P \in \mathcal{P}$ is impossible. One has to be lucky on the model as well. (e.g. Deparis and Martini 04 for $\mathcal{P}$ generated by Dirac Mass)
  - $Y_T(\omega) > 0$ for all $\omega$ is impossible (Acciaio, Beiglböck, Penkner and Schachermayer 2013).
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
- Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...

- Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc.

- If not assumed, there are different possibilities:
  - $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
  - $\exists$ a family $Q$ with the same polar sets: $Q \sim P$.

- One can ask to be consistent with the prices of some options:
  - All calls: Embedding point of view of Hobson, Obloj, Cox, ... and Mass Transport approach.
  - $I$ infinite + a power option (or suitable calls): Acciaio et al. 2013.
  - $I$ finite.
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
- Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
- Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

If not assumed, there are different possibilities:
- $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
- $\exists$ a family $Q$ with the same polar sets: $Q \sim P$.

One can ask to be consistent with the prices of some options:
- All calls: Embedding point of view of Hobson, Obloj, Cox, ... and Mass Transport approach.
- $I$ infinite + a power option (or suitable calls): Acciaio et al. 2013.
- $I$ finite.
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
  - Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
  - Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - All calls: Embedding point of view of Hobson, Obloj, Cox,..., and Mass Transport approach.
  - $I_{\infty}$ infinite + a power option (or suitable calls): Acciaio et al. 2013.
  - $I_{finite}$. 

$\exists Q \text{ on } (\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
$\exists$ a family $Q \sim P$.
No-Arbitrage and martingale measures?

- Most of the time assumed, or \( \mathcal{P} \) is already a set of martingale measures.
  - Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
  - Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - \( \exists Q \) on \((\Omega, \mathcal{F})\) (e.g. Acciaio et al. 2013).
  - \( \exists a \) family \( Q \) with the same polar sets: \( Q \sim P \).

- All calls: Embedding point of view of Hobson, Obloj, Cox, etc., and Mass Transport approach.
- \( I \) infinite + a power option (or suitable calls): Acciaio et al. 2013.
- \( I \) finite.
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
  - Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
  - Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
  - $\exists$ a family $\mathcal{Q}$ with the same polar sets: $\mathcal{Q} \sim \mathcal{P}$. 
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
- Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
- Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
  - $\exists$ a family $Q$ with the same polar sets: $Q \sim \mathcal{P}$.

- One can ask to be consistent with the prices of some options:
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
- Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
- Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
  - $\exists$ a family $Q$ with the same polar sets: $Q \sim \mathcal{P}$.

- One can ask to be consistent with the prices of some options:
  - All calls: Embedding point of view of Hobson, Obloj, Cox, ..., and Mass Transport approach.
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
  - Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
  - Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
  - $\exists$ a family $Q$ with the same polar sets: $Q \sim \mathcal{P}$.

- One can ask to be consistent with the prices of some options:
  - All calls: Embedding point of view of Hobson, Obloj, Cox, ..., and Mass Transport approach.
  - $I$ infinite + a power option (or suitable calls): Acciaio et al. 2013.
No-Arbitrage and martingale measures?

- Most of the time assumed, or $\mathcal{P}$ is already a set of martingale measures.
  - Mass transport: Henry-Labordère, Juillet, Galichon, Touzi, Tan, Dolynski, Soner, etc...
  - Uncertain volatility: Denis, Martini, Soner, Touzi, Zhang, Possamaï, Nutz, Neufeld, Kupper, Peng, etc..

- If not assumed, there are different possibilities:
  - $\exists Q$ on $(\Omega, \mathcal{F})$ (e.g. Acciaio et al. 2013).
  - $\exists$ a family $Q$ with the same polar sets: $Q \sim \mathcal{P}$.

- One can ask to be consistent with the prices of some options:
  - All calls: Embedding point of view of Hobson, Obloj, Cox, ..., and Mass Transport approach.
  - $\forall$ infinite + a power option (or suitable calls): Acciaio et al. 2013.
  - $\forall$ finite.
Dual formulation and super-hedging price

\[ \pi(f) := \inf \left\{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|I|} \text{ s.t. } x + (H \cdot S)_T + hg \geq f \text{-q.s.} \right\} \]
\[ = \sup_{Q} \mathbb{E}_Q[f] \]
Dual formulation and super-hedging price

\[ \pi(f) := \inf \left \{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|\mathcal{H}|} \text{ s.t. } x + (H \cdot S)_T + hg \geq f \text{ -q.s.} \right \} \]

\[ = \sup_Q \mathbb{E}_Q[f] \]

\[ \square \text{ On which set do we take the maximum } \sup \{ \mathbb{E}_Q[f], \ Q \in \mathbb{??} \} \]
Dual formulation and super-hedging price

\[ \pi(f) := \inf \left\{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{\mathcal{H}} \text{ s.t. } x + (H \cdot S) + hg \geq f \text{-q.s.} \right\} \]

= \sup_{Q} \mathbb{E}_{Q}[f]

- On which set do we take the maximum \( \sup\{\mathbb{E}_{Q}[f], \ Q \in \text{??}\} \)?
- Martingales measures on \((\Omega, \mathcal{F})\)
Dual formulation and super-hedging price

\[ \pi(f) := \inf \left\{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|I|} \text{ s.t. } x + (H \cdot S)_T + hg \geq f \text{-q.s.} \right\} \]

\[ = \sup_{Q} \mathbb{E}_Q[f] \]

\[ \square \text{ On which set do we take the maximum } \sup\{\mathbb{E}_Q[f], Q \in \mathcal{Q}\} \]
- Martingales measures on \((\Omega, \mathcal{F})\)
- Linear functionals on \(L^1(\mathcal{P})\) generated by \(\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}\) (Nutz 2013).
Dual formulation and super-hedging price

\[
\pi(f) := \inf \{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|I|} \text{ s.t. } x + (H \cdot S)_T + hg \geq f \text{ -q.s.} \} = \sup_Q \mathbb{E}_Q[f]
\]

☐ On which set do we take the maximum \(\sup\{\mathbb{E}_Q[f], \ Q \in \mathbb{P}\}\)?
- Martingales measures on \((\Omega, \mathcal{F})\)
- Linear functionals on \(L^1(\mathcal{P})\) generated by \(\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}\) (Nutz 2013).
- A family of mart. measures \(Q\) with the same polar sets: \(Q \sim \mathcal{P}\).
The one period case

$(\Omega, \mathcal{F})$ a measurable price. $\Delta S$ a random variable. $\mathcal{P}$ a convex set of measures on $(\Omega, \mathcal{F})$. No option for static hedging.
First Fundamental Theorem

- No-Arbitrage condition: Condition $NA(\mathcal{P})$ holds if for all $H \in \mathcal{H}$

$$H \Delta S \geq 0 \quad \mathcal{P}\text{-q.s.} \quad \text{implies} \quad H \Delta S = 0 \quad \mathcal{P}\text{-q.s.}$$
First Fundamental Theorem

- No-Arbitrage condition: Condition $NA(\mathcal{P})$ holds if for all $H \in \mathcal{H}
  \[ H \Delta S \geq 0 \quad \mathcal{P}\text{-q.s.} \quad \text{implies} \quad H \Delta S = 0 \quad \mathcal{P}\text{-q.s.} \]

- Martingale measures:
  \[ Q = \{ Q \ll \mathcal{P} : Q \text{ is a martingale measure} \} . \]
First Fundamental Theorem

- No-Arbitrage condition: Condition $NA(P)$ holds if for all $H \in \mathcal{H}$

  \[ H \Delta S \geq 0 \quad \text{P-q.s.} \quad \text{implies} \quad H \Delta S = 0 \quad \text{P-q.s.} \]

- Martingale measures:

  \[ Q = \{ Q \ll P : Q \text{ is a martingale measure} \}. \]

- First Fundamental Theorem: The following are equivalent:
  (i) $NA(P)$ holds.
  (ii) For all $P \in \mathcal{P}$ there exists $Q \in Q$ such that $P \ll Q$.
  (ii') $\mathcal{P}$ and $Q$ have the same polar sets.

Rem: These are the usual equivalent conditions when $\mathcal{P} = \{P_o\}$. 
One can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).
One cannot use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).

Finite dimensional separation on $\mathbb{R}^d$:
One can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).

Finite dimensional separation on $\mathbb{R}^d$:

Step 1: Assume $d = 1$ and that $\mathbb{E}_P[\Delta S] > 0$. 
One can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).

Finite dimensional separation on $\mathbb{R}^d$:
Step 1: Assume $d = 1$ and that $\mathbb{E}_P[\Delta S] > 0$. $NA(\mathcal{P})$ implies that $\exists P' \ll \mathcal{P}$ s.t. $\mathbb{E}_{P'}[\Delta S] < 0$. 

One can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).

Finite dimensional separation on $\mathbb{R}^d$:
Step 1: Assume $d = 1$ and that $\mathbb{E}_P[\Delta S] > 0$. $NA(\mathcal{P})$ implies that $\exists P' \ll P$ s.t. $\mathbb{E}_{P'}[\Delta S] < 0$. Do a convex combination to find $P \ll Q \ll P + P'$.
One cannot use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(P)$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(P))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).

Finite dimensional separation on $\mathbb{R}^d$:

Step 1: Assume $d = 1$ and that $\mathbb{E}_P[\Delta S] > 0$. $NA(\mathcal{P})$ implies that $\exists P' \ll P$ s.t. $\mathbb{E}_{P'}[\Delta S] < 0$. Do a convex combination to find $P \ll Q \ll P + P'$.

Step 2: For $d > 1$. Show that $0 \in ri\{E_R[\Delta S] : P \ll R \ll \mathcal{P}, E_R[|\Delta S|] < \infty\}$. 
One cannot use the usual separation argument based on the closedness of the set of super-hedgeable claims. Could show closedness in $L^1(\mathcal{P})$ (generated by $\sup\{\mathbb{E}_P[|\cdot|], P \in \mathcal{P}\}$) but would have to work with $(L^1(\mathcal{P}))^*$ (Nutz 2013 and talk of M. Kupper).

Our approach is close to Dalang, Morton and Willinger (90) and Rasonyi (09).

Finite dimensional separation on $\mathbb{R}^d$:

Step 1: Assume $d = 1$ and that $\mathbb{E}_P[\Delta S] > 0$. $NA(\mathcal{P})$ implies that $\exists P' \ll \mathcal{P}$ s.t. $\mathbb{E}_{P'}[\Delta S] < 0$. Do a convex combination to find $P \ll Q \ll P + P'$.

Step 2: For $d > 1$. Show that $0 \in ri\{E_R[\Delta S] : P \ll R \ll \mathcal{P}, E_R[|\Delta S|] < \infty\}$. If not: $0 \leq y\Delta S \Rightarrow 0 = y\Delta S$. And reduce the dimension by one until the case $d = 1$ is reached.
Super-hedging Theorem

Theorem: Let $NA(\mathcal{P})$ hold and let $f$ be a random variable. Then

$$\sup_{Q \in \mathcal{Q}} E_Q[f] = \pi(f) := \inf \{ x : \exists H \in \mathbb{R}^d \text{ s.t. } x + H \Delta S \geq f \ \mathcal{P}\text{-q.s.} \}.$$ 

Moreover, $\pi(f) > -\infty$ and $\exists H$ s.t. $\pi(f) + H \Delta S \geq f \ \mathcal{P}\text{-q.s.}$
Super-hedging Theorem

Theorem: Let $NA(\mathcal{P})$ hold and let $f$ be a random variable. Then

$$\sup_{Q \in \mathcal{Q}} E_Q[f] = \pi(f) := \inf\{x : \exists H \in \mathbb{R}^d \text{ s.t. } x + H\Delta S \geq f \text{ } \mathcal{P}-\text{q.s.}\}.$$ 

Moreover, $\pi(f) > -\infty$ and $\exists H \text{ s.t. } \pi(f) + H\Delta S \geq f \text{ } \mathcal{P}-\text{q.s.}$.

Existence of the cheapest super-hedging strategy holds by the argument in Kabanov and Stricker’s *Teacher's Note* (even with finitely many options and $T$ periods). One has the closure property for the $\mathcal{P}$-q.s.-convergence. Not true with infinitely many options in general.
Super-hedging Theorem

□ Theorem: Let $NA(\mathcal{P})$ hold and let $f$ be a random variable. Then

$$\sup_{Q \in \mathcal{Q}} E_Q[f] = \pi(f) := \inf \{ x : \exists H \in \mathbb{R}^d \text{ s.t. } x + H \Delta S \geq f \ \mathcal{P}\text{-q.s.} \}.$$ 

Moreover, $\pi(f) > -\infty$ and $\exists H$ s.t. $\pi(f) + H \Delta S \geq f$ $\mathcal{P}\text{-q.s.}$.

□ Existence of the cheapest super-hedging strategy holds by the argument in Kabanov and Stricker’s *Teacher’s Note* (even with finitely many options and $T$ periods). One has the closure property for the $\mathcal{P}\text{-q.s.-convergence}$. Not true with infinitely many options in general.

□ Again, one can not use the usual separation argument based on the closedness of the set of super-hedgeable claims. We do neither have compactness on $Q$ (role plaid by the *power option* in Acciaio et al. 2013).
Step 1: Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists R_n \ll P \text{ s.t. } E_{R_n}[\Delta S] \to 0 \quad \text{and} \quad E_{R_n}[f] \to 0.$$
Step 1: Construct approximating martingale measures
Assume \( \pi(f) = 0 \) and show that

\[
\exists R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \quad \text{and} \quad E_{R_n}[f] \to 0.
\]

1. If not: \( 0 \not\in \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\} \)
Step 1: Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists \ R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \text{ and } E_{R_n}[f] \to 0.$$  

1. If not: $0 \notin \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\}$
2. This implies $0 < \alpha \leq y\Delta S + zf$. 

Step 2: Correct the approximating martingale measures
1. Choose $R_n \ll \mathcal{P}$ s.t. $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$.
2. One has $0 \in \mathring{\text{cl}}\{E_{R_n}[\Delta S] : P \ll R \ll P, E_{R_n}[|\Delta S| + |f|] < \infty\}$.
3. We can correct in $\tilde{R}_n = (1 - \lambda_n)R_n + \lambda_n R'_n$ s.t. $E_{\tilde{R}_n}[\Delta S] = 0$ and $E_{\tilde{R}_n}[f] \to 0 = \pi(f)$. 

Step 1: Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \text{ and } E_{R_n}[f] \to 0.$$  

1. If not: $0 \not\in \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\}$
2. This implies $0 < \alpha \leq y\Delta S + zf$. If $z = -1 : f \leq -\alpha + y\Delta S$
Step 1 : Construct approximating martingale measures
Assume \( \pi(f) = 0 \) and show that

\[ \exists \, R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \quad \text{and} \quad E_{R_n}[f] \to 0. \]

1. If not : \( 0 \notin \text{cl}\{ E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty \} \)
2. This implies \( 0 < \alpha \leq y\Delta S + zf. \) If \( z \in \{0, 1\} : 0 < y\Delta S + H\Delta S \)
Step 1 : Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists \, R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \text{ and } E_{R_n}[f] \to 0.$$ 

1. If not : $0 \notin \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\}$
2. This implies $0 < \alpha \leq y\Delta S + zf$.

Step 2 : Correct the approximating martingale measures
Step 1 : Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \text{ and } E_{R_n}[f] \to 0.$$ 

1. If not : $0 \notin \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\}$
2. This implies $0 < \alpha \leq y\Delta S + zf$.

Step 2 : Correct the approximating martingale measures
1. Choose $R_n \ll \mathcal{P}$ s.t. $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$. 
Step 1: Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \text{ and } E_{R_n}[f] \to 0.$$ 

1. If not: $0 \notin \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\}$
2. This implies $0 < \alpha \leq y\Delta S + zf$.

Step 2: Correct the approximating martingale measures
1. Choose $R_n \ll \mathcal{P}$ s.t. $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$.
2. One has $0 \in \text{ri}\{E_R[\Delta S] : P \ll R \ll \mathcal{P}, E_R[|\Delta S| + |f|] < \infty\}$. 
Step 1: Construct approximating martingale measures
Assume $\pi(f) = 0$ and show that

$$\exists \, R_n \ll \mathcal{P} \text{ s.t. } E_{R_n}[\Delta S] \to 0 \quad \text{and} \quad E_{R_n}[f] \to 0.$$  

1. If not: $0 \notin \text{cl}\{E_R[(\Delta S, f)] : R \ll \mathcal{P}, \, E_R[|\Delta S| + |f|] < \infty\}$  
2. This implies $0 < \alpha \leq y\Delta S + zf$.

Step 2: Correct the approximating martingale measures
1. Choose $R_n \ll \mathcal{P}$ s.t. $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$.  
2. One has $0 \in \text{ri}\{E_R[\Delta S] : P \ll R \ll \mathcal{P}, \, E_R[|\Delta S| + |f|] < \infty\}$.  
3. We can correct in $\tilde{R}_n = (1 - \lambda_n)R_n + \lambda_n R'_n$ s.t.

$$E_{\tilde{R}_n}[\Delta S] = 0 \quad \text{and} \quad E_{\tilde{R}_n}[f] \to 0 = \pi(f).$$
The multiperiod case with options for static hedging

\( Q = \{ Q \ll P : Q \text{ is a mart. measure and } E_Q[g^i] = 0 \text{ for } i = 1, \ldots, |I| \} \).

**Theorem:** The following are equivalent:

(i) \( NA(P) \) holds.

(ii) For all \( P \in P \) there exists \( Q \in Q \) such that \( P \ll Q \).

(ii’) \( P \) and \( Q \) have the same polar sets.

**Theorem:** Let \( NA(P) \) hold and let \( f : \Omega \to \mathbb{R} \) be upper semianalytic. Then,

\[ \pi(f) := \inf \{ x \in \mathbb{R} : \exists (H, h) \in \mathcal{H} \times \mathbb{R}^{|I|} \text{ s.t. } x + (H \cdot S)_T + hg \geq f \ P\text{-q.s.} \} \]

admits existence and satisfies

\[ \pi(f) = \sup_{Q \in Q} E_Q[f] \in (-\infty, \infty] \]
Strategy of proof and Assumptions

- One argue on one step models and then try to glue the steps together. This requires some measurable selection arguments.
Strategy of proof and Assumptions

- One argue on one step models and then try to glue the steps together. This requires some measurable selection arguments.

This is feasible under the assumptions:

- $\Omega = \Omega_1$ with $\Omega_1$ a Polish space.
- $F_t$ is the universal completion of $B(\Omega_t)$.
- $(S_t)_{t \leq T}$ are Borel, possibly not adapted.
- $P = \{P = P_0 \otimes \cdots \otimes P_{T-1} : P_t(\omega) \in P_t(\omega)\}$.
- The $\omega \mapsto P_t(\omega)$ have analytic graphs.
- Options for static hedging are assumed Borel.
- Claims to super-hedge are upper-semianalytic.
Strategy of proof and Assumptions

- One argue on one step models and then try to glue the steps together. This requires some measurable selection arguments.

This is feasible under the assumptions:

- $\Omega = \Omega^T_1$ with $\Omega_1$ a Polish space.
- $\mathcal{F}_t$ is the universal completion of $\mathcal{B}(\Omega^t_1)$. $\mathcal{F} = \mathcal{F}_T$.
- $(S_t)_{t \leq T}$ are Borel, possibly not adapted.
Strategy of proof and Assumptions

- One argue on one step models and then try to glue the steps together. This requires some measurable selection arguments.

This is feasible under the assumptions:

- $\Omega = \Omega^T_1$ with $\Omega_1$ a Polish space.
- $\mathcal{F}_t$ is the universal completion of $\mathcal{B}(\Omega^t_1)$. $\mathcal{F} = \mathcal{F}_T$.
- $(S_t)_{t \leq T}$ are Borel, possibly not adapted.

- $\mathcal{P} = \{ P = P_0 \otimes \cdots \otimes P_{T-1} : P_t(\omega) \in \mathcal{P}_t(\omega) \}$.
- The $\omega \mapsto \mathcal{P}_t(\omega)$ have analytic graphs.
Strategy of proof and Assumptions

- One argue on one step models and then try to glue the steps together. This requires some measurable selection arguments.

This is feasible under the assumptions:

- $\Omega = \Omega_1^T$ with $\Omega_1$ a Polish space.
- $\mathcal{F}_t$ is the universal completion of $\mathcal{B}(\Omega_1^t)$. $\mathcal{F} = \mathcal{F}_T$.
- $(S_t)_{t \leq T}$ are Borel, possibly not adapted.

- $\mathcal{P} = \{P = P_0 \otimes \cdots \otimes P_{T-1} : P_t(\omega) \in \mathcal{P}_t(\omega)\}$.
- The $\omega \mapsto \mathcal{P}_t(\omega)$ have analytic graphs.

- Options for static hedging are assumed Borel.
- Claims to super-hedge are upper-semianalytic.
Second Fundamental Theorem

As in the dominated setting it follows from the super-hedging theorem.
Second Fundamental Theorem

As in the dominated setting it follows from the super-hedging theorem.

Theorem: Let $NA(P)$ hold and let $f : \Omega \rightarrow \mathbb{R}$ be upper semianalytic. The following are equivalent:

(i) $f$ is replicable, i.e. $\pi(f) + (H \cdot S)_T = f$ $P$-q.s.
(ii) $Q \mapsto E_Q[f]$ is constant (and finite) on $Q$.
(iii) $\forall P \in \mathcal{P} \exists Q \in Q$ s.t. $P \ll Q$ and $E_Q[f] = \pi(f)$.

Moreover, the market is complete (for Borel claims) if and only if $Q$ is a singleton.
Application to Optional Decomposition

□ Theorem : Let $NA(\mathcal{P})$ hold and let $V$ be an adapted process such that $V_t$ is upper semianalytic and in $L^1(Q) \forall Q \in \mathcal{Q}$. The following are equivalent:
- $V$ is a supermartingale under each $Q \in \mathcal{Q}$.
- There exist a predictable $H$ and an adapted increasing process $K$ with $K_0 = 0$ such that

$$V_t = V_0 + (H \cdot S)_t - K_t \quad \mathcal{P}\text{-}q.s., \quad t \in \{0, 1, \ldots, T\}.$$

Rem : The decomposition can not be obtained by hand as for continuous processes, but we have discrete time (measurable selection).
Connection to Martingale Inequalities

- Take $\Omega_1 = \mathbb{R}^d$, $\mathcal{P}$ be generated by all Dirac Mass and let $S$ be the canonical process.
Connection to Martingale Inequalities

- Take $\Omega_1 = \mathbb{R}^d$, $\mathcal{P}$ be generated by all Dirac Mass and let $S$ be the canonical process.

- Then, $NA(\mathcal{P})$ holds for the universal completion of the raw filtration.

Compare with Acciaio, Beiglböck, Penkner and Schachermayer (2013).
Connection to Martingale Inequalities

- Take $\Omega_1 = \mathbb{R}^d$, $\mathcal{P}$ be generated by all Dirac Mass and let $S$ be the canonical process.

- Then, $NA(\mathcal{P})$ holds for the universal completion of the raw filtration.

- One can apply the super-hedging theorem:
  Assume that
  \[ \mathbb{E}_P[f(S_1, \cdots, S_T)] \leq 0 \text{ for all martingale measure } P \text{ on } \Omega_T. \]

Then, there exists universally measurable maps $H_1, \ldots, H_T$ such that
\[ f(x_1, \cdots, x_T) \leq \sum_{t=0}^{T-1} H_{t+1}(x_0, \ldots, x_t)(x_{t+1} - x_t) \quad \forall \ x \in (\mathbb{R}^d)^{T+1}. \]

Compare with Acciaio, Beiglböck, Penkner and Schachermayer (2013).