Optimal consumption in discrete time financial models with industrial investment opportunities and non-linear returns

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Motivation


- Two possibilities:
  1. Usual investment in a financial market
  2. Industrial investment: Increase the capital of a company which yields a concave return

- Maximize expected utility of consumption in a complete Brownian diffusion model
Motivation

- Complete market.
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- Strong condition on the (deterministic) return.
Motivation

- Complete market.
- Strong condition on the (deterministic) return.
- Use a particular no-bankruptcy constraint which implies a separation principle:
  1. First optimize among the industrial investment policies
  2. Then find the associated optimal financial investment policy.
Aim of this paper

- Build up a general model in incomplete market with (possibly) proportional transaction costs
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• Look at the no-arbitrage conditions and closure property of the set of wealth processes.
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- Apply this to optimal consumption problems.
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- Build up a general model in incomplete market with (possibly) proportional transaction costs
- Look at the no-arbitrage conditions and closure property of the set of wealth processes.
- Apply this to optimal consumption problems.
- As a first step: restricted to discrete time models.
Model and notations

• \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}), \mathcal{F}_0 \text{ trivial}, \mathcal{F}_T = \mathcal{F}, \mathbb{T} = \{0, \ldots, T\}\).

• \(d\) Financial assets (bonds, stocks, currencies, ...)

• \(N\) "Industrial" assets (industrial tools - physical assets used for production purposes)
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- \(d\) Financial assets (bonds, stocks, currencies, ...)

- \(N\) "Industrial" assets (industrial tools - physical assets used for production purposes)

- Initial wealth \(x = (x^F, x^I) \in \mathbb{R}^d \times \mathbb{R}^N\)

Here \(x^i\) = \# of units of the asset \(i\) hold

- Notation: For \(x \in \mathbb{R}^{d+N}\), we write \(x = (x^F, x^I) \in \mathbb{R}^d \times \mathbb{R}^N\).

\(\Rightarrow x^F = \) initial endowment in Financial assets,

\(x^I = \) initial endowment in Industrial assets.
The financial strategies

• Financial strategy: $\xi \in L^0(\mathbb{R}^{d+N};\mathbb{F})$, $\xi^i_s = (\xi^F_s, \xi^I_s)^i$ number of units of asset $i$ bought at time $s$.

• $\sum_{\tau=0}^{s} \xi^i_\tau$: cumulated number of units of asset bought between 0 and $s$.

• $I(\xi)_s = \sum_{\tau=0}^{s} \xi^I_\tau$: cumulated number of units of industrial assets bought between 0 and $s$.

• $x^I + I(\xi)_s \in L^0(\mathbb{R}^N_+)$: number of units of industrial assets held at $s$ (can not short-sale machine tools or plants)
The financial strategies

- Induces a random return \( R_{s+1}(x^I + I(\xi)_s) \) at time \( s + 1 \), taking values in \( \mathbb{R}^{d+N} := \mathbb{R}^d \times \{0_N\} \).

ex: asset 1 = euro, asset 2 = dollar and the others are stocks

\[ R^i_{s+1} = 0 \text{ for } i > 2. \]
The wealth process

- Initial endowment: $x \in \mathbb{R}^d \times \mathbb{R}_+^N$.

- $V_t = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1}(x^I + I(\xi)_s)$ takes values in $\mathbb{R}^{d+N}$.

- $V_t^i$: position in asset $i$ (in units) at time $t$. 
Admissible exchanges $\xi \in L^0(\mathcal{F})$

1. Case without frictions

- $S = (S^F, S^I)$: assets.

- $\xi_t$ is self-financed if $\xi_t \cdot S_t := \sum_{i=1}^{d+N} \xi_t^i S_t^i = 0$.

- If we allow to throw out money: $\xi_t$ is self-financed if it belongs a.s. to

$$-K_t(\omega) := \{\xi \in \mathbb{R}^{d+N} : \xi \cdot S_t(\omega) \leq 0\}.$$
Admissible exchanges $\xi \in L^0(\mathbb{F})$

2. Case with proportional costs

- $S = (S^F, S^I)$: assets
- $\lambda^{ij}$: proportional cost paid in units of asset $i$ for a transaction from $i$ to $j$.
- $\xi_t$ is self-financed if it belongs a.s. to
  \[
  \left\{ \xi \in \mathbb{R}^{d+N} : \exists a^{ij} \geq 0, \sum_{j=1}^{d+N} a^{ji} - (1 + \lambda_t^{ij}(\omega))a^{ij} = S_t^i(\omega) \xi^i \right\}.
  \]

$\Rightarrow a^{ij} \geq 0$ amount transferred from $i$ to $j$, $a^{ji} \geq 0$ amount transferred from $j$ to $i$.

$\Rightarrow S_t^i \xi^i$ net amount transferred from the other accounts to $i$. 
Admissible exchanges $\xi \in L^0(\mathbb{F})$

2. Case with proportional costs (2)

- $S = (S^F, S^I)$ : assets

- $\lambda_{ij}$ : proportional cost paid in units of asset $i$ for a transaction from $i$ to $j$.

- If we allow to throw out money : $\xi_t$ is self-financed if it belongs a.s. to

$$-K_t(\omega) := \left\{ \xi \in \mathbb{R}^{d+N} : \exists a^i \geq 0, \sum_{j=1}^{d+N} a^j - (1 + \lambda^i_t(\omega))a^i \geq S^i_t(\omega) \xi \right\}.$$
Admissible exchanges $\xi \in L^0(\mathbb{F})$

3. General modelization

- $K_t(\omega)$: polyhedral, closed and convex cone such that $\mathbb{R}_+^{d+N} \setminus \{0\} \subset \text{Int}(K_t)$ a.s.

- $\xi = (\xi_t)_{t \in \mathbb{T}}$ is a self-financed strategy if $\xi_t \in -K_t$ a.s. for each $t$.

- $-K_t := \{(\xi^F, 0) \in -K_t\}$, i.e. transaction only on the financial assets.
The wealth process (to sum up)

• $K_t(\omega)$: polyhedral, closed and convex cone such that $\mathbb{R}^{d+N}_+ \setminus \{0\} \subset \text{Int}(K_t)$ a.s.

• Admissibility:

$$\xi_s \in L^0(-K_s; \mathcal{F}_s) \quad \text{and} \quad x^I + I(\xi)_s = x^I + \sum_{\tau=0}^{s} \xi^I_\tau \in L^0(\mathbb{R}^N_+; \mathcal{F}_s)$$

• Wealth process (in units): $V_t = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1}(x^I + I(\xi)_s)$
Assumption on $R$

- For each $t$

  (R1) $R_t(0) = 0$ and $R_t$ is continuous.
Assumption on $R$

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  (R1) $R_t(0) = 0$ and $R_t$ is continuous.

  (R2) If $\lambda \in [0, 1]$ and $(\alpha, \beta) \in (L^0(\mathbb{R}_+^N))^2$, then

  $$\lambda R_t(\alpha) + (1 - \lambda) R_t(\beta) - R_t(\lambda \alpha + (1 - \lambda) \beta) \in -K_t := \{(x^F, 0) \in -K_t\}$$

- (R2) : means $R_t$ is “concave”. Indeed,

  $$\lambda R_t(\alpha) + (1 - \lambda) R_t(\beta) = R_t(\lambda \alpha + (1 - \lambda) \beta) + \underbrace{\xi_t}_{\in -K_t}$$
Assumption on $R$

- For each $t$
  
  $(R1)$ $R_t(0) = 0$ and $R_t$ is continuous.

  $(R2)$ If $\lambda \in [0, 1]$ and $(\alpha, \beta) \in (L^0(\mathbb{R}_+^N))^2$, then

  $$\lambda R_t(\alpha) + (1 - \lambda) R_t(\beta) - R_t(\lambda \alpha + (1 - \lambda) \beta) \in -K_t := \{(x^F, 0) \in -K_t\}$$

  $(R3)$ $R_t$ is bounded from below by an affine (random) map.

- $(R3)$: In dimension 1 $\Leftrightarrow R_t^l(\infty) > -\infty$ a.s.

- Remark: No monotonicity assumption, need not to be non-negative.
Attainable wealth: $A_t(x; K, R)$

- $A_t(x; K, R) = \left\{ V_t^{x, \xi} = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1} (x^I + I(\xi)_s), \xi \text{ admissible} \right\}$

- Under (R2), $A_t(x; K, R)$ is convex.

Remind (R2): If $\lambda \in [0, 1]$ and $(\alpha, \beta) \in (L^0(\mathbb{R}_+^N))^2$, then

$$\lambda R_{s+1}(\alpha) + (1 - \lambda) R_{s+1}(\beta) = R_{s+1}(\lambda\alpha + (1 - \lambda)\beta) + \xi_{s+1} \in -K_{s+1}$$
**Attainable wealth:** $A_t(x; K, R)$

- $A_t(x; K, R) = \left\{ V_t^{x,\xi} = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1}(x^I + I(\xi)_s), \xi \text{ admissible} \right\}$

- $A_t$ is non-linear with respect to $x$: $A_t(x; K, R) \neq x + A_t(0; K, R)$

We only have $A_t(x; K, R) = x^F + A_t((0, x^I); K, R)$
Remarks on $K_t = -(-K_t)$

- $V \in K_t \iff V - V = 0$ with $-V \in -K_t$ (admissible exchange).

$\Rightarrow$ up to a transfer can transform all the positions in non-negative ones.
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$\Rightarrow K_t$ is the “solvency region” at time $t$. 
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⇒ \( K_t \) is the “solvency region” at time \( t \).

- \( V \in K_t^0 := K_t \cap (-K_t) \iff \) can reach 0 from \( V \) and \( V \) from 0.
Remarks on $K_t = -(-K_t)$

• $V \in K_t \iff V - V = 0$ with $-V \in -K_t$ (admissible exchange).

⇒ up to a transfer can transform all the positions in non-negative ones.

⇒ $K_t$ is the “solvency region” at time $t$.

• $V \in K_t^o := K_t \cap (-K_t) \iff$ can reach 0 from $V$ and $V$ from 0.

⇒ $K_t^o$ is the set of holdings which are equivalent to 0.
No-arbitrage condition: The pure financial case \( N = 0 \)

The robust No-arbitrage condition (S04, KSR01)

1. Weak no-arbitrage property

\[
NA^w(K) : A_T(0; K) \cap L^0(\mathbb{R}^d_+; \mathcal{F}_T) = \{0\}.
\]
No-arbitrage condition: The pure financial case $N = 0$

The robust No-arbitrage condition (S04, KSR01)

1. Weak no-arbitrage property

$$NA^w(K) : A_T(0; K) \cap L^0(\mathbb{R}_+^d; \mathcal{F}_T) = \{0\}.$$ 

2. $\tilde{K}$ dominates $K$ if:

$$\tilde{K}_t \setminus K^0_t \subset \text{ri}(\tilde{K}_t).$$

solvable equivalent to 0 bigger solvency region
No-arbitrage condition: The pure financial case $N = 0$

The robust No-arbitrage condition (S04, KSR01)

1. Weak no-arbitrage property

\[ \text{NA}^w(K) : A_T(0; K) \cap L^0(\mathbb{R}^d_+; \mathcal{F}_T) = \{0\} . \]

2. $\tilde{K}$ dominates $K$ if: $\text{solvable} \backslash \text{equivalent to } 0 \subset \text{ri}(\tilde{K}) \text{ bigger solvency region}$

3. Robust no-arbitrage property

\[ \text{NA}^r(K) : \text{NA}^w(\tilde{K}) \text{ holds for some } \tilde{K} \text{ which dominates } K . \]

$\Rightarrow$ No arbitrage even in a model with slightly lower transaction costs.
No-arbitrage condition: The pure financial case $N = 0$

- Under $NA^r(K)$, $A_T(0; K)$ is closed.

- Important property: Under $NA^r(K)$

  $$\xi_t \in -K_t \text{ and } \sum_{t=0}^{T} \xi_t = 0 \Rightarrow \xi_t \in K_t^o$$

- The closure property is a consequence of this property.
No-arbitrage condition: The general case

1. Weak no-arbitrage property:

\[ NA^w(K, R) : A_T(0; K, R) \cap L^0(\mathbb{R}_+^{d+N}) = \{0\} \]

2. Set \( K = \{(x^F, 0) \in K\} \). \((\tilde{K}, \tilde{R})\) dominates \((K, R)\) if

(D1) \( K_t \setminus K_t^o \subset \text{ri}(\tilde{K}_t) \) and \( K_t \subset \tilde{K}_t \)

(D2) \( \tilde{R}_t(0) \in K_t \) and \( \tilde{R}_t(\alpha) - R_t(\alpha) \in \text{ri}(K_t) \), \( \alpha \in \mathbb{R}_+^N \setminus \{0\} \).

(D1) : Slight reduction of transaction costs for the exchanges involving only Financial assets.

(D2) : Slight increase of the return of Industrial assets.
No-arbitrage condition: The general case

1. Weak no-arbitrage property:

\[ NA^w(K, R) : A_T(0; K, R) \cap L^0(\mathbb{R}_+^d) = \{0\} \]

2. Set \( K = \{(x^K, 0) \in K\} \). \((\tilde{K}, \tilde{R})\) dominates \((K, R)\) if

(D1) \( K_t \setminus K_t^o \subset \text{ri}(\tilde{K}_t) \) and \( K_t \subset \tilde{K}_t \)

(D2) \( \tilde{R}_t(0) \in K_t \) and \( \tilde{R}_t(\alpha) - R_t(\alpha) \in \text{ri}(K_t), \alpha \in \mathbb{R}_+^N \setminus \{0\} \).

3. Robust no-arbitrage property

\[ NA^r(K, R) : \exists (\tilde{K}, \tilde{R}) \text{ which dominates } (K, R) \text{ such that } NA^w(\tilde{K}, \tilde{R}) \text{ holds} \]
No-arbitrage condition: The general case

- Under $NA^r(K, R)$

$$\xi_t \in -K_t \text{ and } \sum_{t=0}^{T} \xi_t + \sum_{t=0}^{T-1} R_{t+1}(I(\xi)_t) = 0 \Rightarrow \xi_t \in K_t^0 \ (= K_t \cap -K_t).$$

$\Rightarrow$ Under $NA^r(K, R)$: $A_T(x; K, R)$ is closed $\forall x$. 
Dual formulation for $A_T(x; K, R)$

- $(K_t)^*(\omega) := \{ y \in \mathbb{R}^{d+N} : x \cdot y \geq 0 \ \forall x \in K_t(\omega) \}$.

- Let $\mathcal{Z}(K, Q)$ be the set of $Z = (Z^F, Z^I) \in L^\infty(\text{Int}(\mathbb{R}^{d+N}_+))$ such that $(\mathbb{E}^Q [Z^F | \mathcal{F}_t], 0_N) \in \text{ri}((K_t)^*)$.

- Under $NA^r(K, R)$, for all $Q \sim \mathbb{P}$ there is $Z \in \mathcal{Z}(K, Q)$ such that

$$a(x; Z, Q) := \sup_{g \in A_T(x; K, R) \cap L^1(Q)} \mathbb{E}^Q[Z \cdot g] < \infty$$
Dual formulation for $A_T(x; K, R)$

• Dual formulation for $A_T(x; K, R) \cap L^1(Q)$:

\[ g \in A_T(x; K, R) \cap L^1(Q) \iff \mathbb{E}^Q[Z \cdot g] \leq a(x; Z, Q) \quad \forall Z \in \mathcal{Z}(K, Q). \]

• Can drop the integrability condition on $g$ if it is uniformly bounded from below for the natural partial order induced by $K_T$. 
Remark on $Z(K, Q)$: The case $N = 0$

- In the case with no transaction costs:

  $$K_t(\omega) = \{x \in \mathbb{R}^d : x \cdot S_t(\omega) \geq 0\}$$

  $$K_t^*(\omega) = \{\lambda S_t(\omega), \lambda \in \mathbb{R}_+\}$$

- $Z_t := \mathbb{E}[Z | \mathcal{F}_t] \in \text{ri}(K_t^*)$ implies $Z_t = H_t S_t$ which is a $\mathbb{P}$-martingale.

- If we take $S^1$ as a numeraire and set $(\hat{H}, \hat{S}) = (H S^1, S/S^1)$ then $\hat{H}$ is a martingale as well as $\hat{H} \hat{S}$.

- $\hat{S}$ is a martingale under $Q = (\hat{H}_T/\mathbb{E}[\hat{H}_T]) \cdot \mathbb{P}$. 

Additional remarks on the separating measures

• In general, there is no $Z$ in $\mathcal{Z}(K, \mathbb{Q})$ such that

$$a(0; Z, \mathbb{Q}) := \sup_{g \in A_T(0; K, R) \cap L^1(\mathbb{Q})} \mathbb{E}^{\mathbb{Q}}[Z \cdot g] \leq 0$$

• In particular, $\text{NA}^r$ does not imply the absence of arbitrage opportunity in the “tangent” model:

$$\lim_{\varepsilon \to 0} \sum_{t=0}^{T} \frac{\varepsilon \xi_t}{\varepsilon} + \sum_{t=0}^{T-1} R_{t+1}(\varepsilon I(\xi)_t)/\varepsilon = \sum_{t=0}^{T} \xi_t + \sum_{t=0}^{T-1} R'_{t+1}(0) I(\xi)_t$$

• However, under $\text{NA}^r$, for all $g \in A_T(0; K, R)$ there is $\mathbb{Q}^g$ and $Z^g$ in $\mathcal{Z}(K, \mathbb{Q}^g)$ such that $\mathbb{E}^{\mathbb{Q}^g}[Z^g \cdot g] \leq 0$. 

Admissible consumption plans

- $C_T(x; K, R) := \left\{ (c_t)_{t \leq T} \in L^0(\mathbb{R}^d_+; \mathbb{F}) : \left( \sum_{t \leq T} c_t, 0_N \right) \in A_T(x; K, R) \right\}$

- Under $NA^r : C_T(x; K, R)$ is closed (and convex).

Utility maximization problem

$$\text{Max } \mathbb{E} \left[ \sum_{t \leq T} U_t(c_t) \right] \rightarrow u(x)$$

over $C_T^U(x; K, R) = \{ c \in C_T(x; K, R) : \mathbb{E}[\left( \sum_{t \leq T} U_t(c_t) \right)^-] < \infty \}$. 
Assumptions on $U_t$

- Concave, non-decreasing for the natural partial order on $\mathbb{R}^d$, and $\text{cl}(\text{dom}(U_t)) = \mathbb{R}^d_+$

- Non-smooth Inada’s conditions: The Fenchel transform
  $$\tilde{U}_t(y) = \sup_{x \in \mathbb{R}^d_+} U_t(x) - x \cdot y$$
  satisfies $\text{int}(\mathbb{R}^d_+) \subset \text{dom}(\tilde{U}_t)$.

- Need not to be smooth.
Additional assumptions on $U_t$

- **Asymptotic elasticity** condition

\[
\limsup_{\ell(y) \to 0} \left( \sup_{q \in -\partial \tilde{U}_t(y)} q \cdot y \right) / \tilde{U}_t(y) < \infty
\]  

where $\partial \tilde{U}_t(y)$ denotes the subgradient of $\tilde{U}_t$ at $y$ in the sense of convex analysis and

\[
\ell(y) := \inf_{x \in \mathbb{R}_+^d, \|x\|=1} x \cdot y.
\]

See KS (99) and compare with DPT (02) and BTZ (04).
**Additional assumptions on** $U_t$

- For each $t \in T$, one of the above conditions hold:
  
  $(\tilde{U}1)$ there is $e_t \in \text{int}(\mathbb{R}^d_+)$ such that $V_t : r \in \mathbb{R}_+ \mapsto \tilde{U}_t(re_t)$ is strictly convex and $\lim_{r \to +\infty} V_t'(r) = 0$.

  or

  $(\tilde{U}2)$ $\tilde{U}^n_t(y) = \sup_{x \in \mathbb{R}^d_+, \|x\| \leq n} U_t(x) - x \cdot y$ is uniformly bounded from below in $y \in \mathbb{R}^d_+$ and $n \geq M_t$. 

Abstract duality

• Problem reduction

\[ u_1(x^1) := u(x^1, 0_{d-1+N}), \quad x^1 \in \mathbb{R}_+ , \]

• Dual variables

\[ D(y^1) = \left\{ (Y, \alpha) \in L^1(\Omega \times T, \mathbb{R}_+^d) \times \mathbb{R}_+ : \forall x^1 \in \mathbb{R}_+, \forall c \in C_T((x^1, 0); K, R) \right\} , \quad y^1 \in \mathbb{R}_+ \]

\[ \mathbb{E} \left[ \sum_{t \in T} Y_t \cdot c_t - y^1 x^1 \right] \leq \alpha, \quad y^1 \in \mathbb{R}_+ \]

• Dual problem

\[ \tilde{u}_1(y^1) = \inf_{(Y, \alpha) \in D(y^1)} \mathbb{E} \left[ \sum_{t \in T} \tilde{U}_t(Y_t) + \alpha \right] , \quad y^1 \in \mathbb{R}_+ . \]
Abstract duality

\[ \tilde{u}_1(y^1) = \sup_{x^1 \in \mathbb{R}_+} \left[ u_1(x^1) - x^1 y^1 \right], \quad y^1 \in \mathbb{R}_+ \]

\[ u_1(x^1) = \inf_{y^1 \in \mathbb{R}_+} \left[ \tilde{u}_1(x^1) - x^1 y^1 \right], \quad x^1 \in \mathbb{R}_+ . \]
Existence result

• If there is an initial wealth $x \in \text{int}(K_0)$ such that $u(x) < \infty$, then

(i) $u(x) < \infty$ for all $x \in \mathbb{R}^d \times \mathbb{R}^N_+$

(ii) for all $x \in \mathbb{R}^d \times \mathbb{R}^N_+$ such that $C_{U}^{T}(x; K, R) \neq \emptyset$, there is some $c^* \in C_{U}^{T}(x; K, R)$ for which

$$u(x) = \mathbb{E} \left[ \sum_{t \in T} U_t(c^*_t) \right].$$

• Proof: adaptation of the direct argument of Kramkov et Schachermayer AAP 13(4) 2003 to this multivariate setting.
Final comment

- We used the $NA^r$ condition, i.e.

There is $(\tilde{K}, \tilde{R})$ such that

\[(D1) \quad \tilde{K}_t \setminus \mathcal{K}_t^o \subset \text{ri}(\tilde{K}_t) \quad \text{and} \quad K_t \subset \tilde{K}_t\]

\[(D2) \quad \tilde{R}_t(0) \in K_t \quad \text{and} \quad \tilde{R}_t(\alpha) - R_t(\alpha) \in \text{ri}(K_t), \quad \alpha \in \mathbb{R}_+^N \setminus \{0\}\]

for which $NA^w(\tilde{K}, \tilde{R})$ holds.
Final comment

- Under the additional conditions on $R$

(i) $R_t \in K_t$

(ii) $R_t(\alpha) \in ri(K_t)$ for $\alpha \neq 0$

(iii) $R_t$ bounded

all the results holds if there is some $\tilde{K}$ satisfying

\[(D1) \quad K_t \setminus K_t^o \subset ri(\tilde{K}_t) \text{ and } K_t \subset \tilde{K}_t\]

such that $NA^w(\tilde{K}, R)$ holds.