Optimal consumption in discrete time financial models with industrial investment opportunities and non-linear returns*

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Motivation

• Kabanov and Kijima, A consumption-investment problem with production possibilities, preprint 2003.

• Two possibilities :

1. Usual investment in a financial market

2. Industrial investment : Increase the capital of a company which yields a concave return

• Maximize expected utility of consumption in a complete Brownian diffusion model

Motivation

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• Strong condition on the (deterministic) return.

• Use a particular no-bankruptcy constraint which implies a separation principle :

1. First optimize among the industrial investment policies

2. Then find the associated optimal financial investment policy.

Aim of this paper

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• Look at the no-arbitrage conditions and closure property of the set of wealth processes.

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• As a first step : restricted to discrete time models.

Model and notations

- $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}})$, $\mathcal{F}_0$ trivial, $\mathcal{F}_T = \mathcal{F}$, $\mathbb{T} = \{0, \ldots, T\}$.

- d **F**inancial assets (bonds, stocks, currencies,...)

- N "**I**ndustrial" assets (industrial tools - physical assets used for production purposes)

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• d **F**inancial assets (bonds, stocks, currencies,...)

• N "**I**ndustrial" assets (industrial tools - physical assets used for production purposes)

• Initial wealth $x = (x^F, x^I) \in \mathbb{R}^d \times \mathbb{R}^N_+$

Here x^i = # of units of the asset i hold

• Notation : For $x \in \mathbb{R}^{d+N}$, we write $x = (x^F, x^I) \in \mathbb{R}^d \times \mathbb{R}^N$.

$\Rightarrow$ x^F = initial endowment in **F**inancial assets,

x^I = initial endowment in **I**ndustrial assets.

The financial strategies

• Financial strategy : $\xi \in L^0(\mathbb{R}^{d+N};\mathbb{F})$, $\xi^i_s = (\xi^F_s, \xi^I_s)^i$= number of units of asset i bought at time s.

• $\sum_{\tau=0}^{s} \xi_\tau$: cumulated number of units of asset bought between 0 and s.

• $I(\xi)_s = \sum_{\tau=0}^{s} \xi^I_\tau$: cumulated number of units of industrial assets bought between 0 and s.

• $x^I + I(\xi)_s \in L^0(\mathbb{R}^N_+)$: number of units of industrial assets held at s (can not short-sale machine tools or plants)

The financial strategies

• Induces a random return $R_{s+1}(x^I + I(\xi)_s)$ at time $s+1$, taking values in $\underline{\mathbb{R}}^{d+N} := \mathbb{R}^d \times \{0_N\}$.

ex : asset 1= euro, asset 2= dollar and the others are stocks

$\Rightarrow R^i_{s+1} = 0$ for $i > 2$.

The wealth process

- Intial endowment : $x \in \mathbb{R}^d \times \mathbb{R}^N_+$.

- $V_t = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1}(x^I + I(\xi)_s)$ takes values in $\mathbb{R}^{d+N}$.

- V_t^i : position in asset i (in units) at time t.

Admissible exchanges $\xi \in L^0(\mathbb{F})$

1. Case without frictions

• $S = (S^F, S^I)$: assets.

• ξ_t is self-financed if $\quad \xi_t \cdot S_t := \sum_{i=1}^{d+N} \xi_t^i \, S_t^i = 0.$

• If we allow to throw out money : ξ_t is self-financed if it belongs a.s. to

$$-K_t(\omega) \;\; := \;\; \left\{ \xi \in \mathbb{R}^{d+N} \;\; : \;\; \xi \cdot S_t(\omega) \leq 0 \right\} \; .$$

Admissible exchanges $\xi \in L^0(\mathbb{F})$

2. Case with proportional costs

• $S = (S^F, S^I)$: assets

• λ^{ij} : proportional cost paid in units of asset i for a transaction from i to j.

• ξ_t is self-financed if it belongs a.s. to

$$\left\{ \xi \in \mathbb{R}^{d+N} \; : \; \exists a^{ij} \geq 0, \; \sum_{j=1}^{d+N} a^{ji} - (1 + \lambda_t^{ij}(\omega)) a^{ij} = S_t^i(\omega) \, \xi^i \right\} .$$

$\Rightarrow$ $a^{ij} \geq 0$ amount transferred from i to j, $a^{ji} \geq 0$ amount transferred from j to i.

$\Rightarrow$ $S_t^i \, \xi^i$ net amount transferred from the other accounts to i.

Admissible exchanges $\xi \in L^0(\mathbb{F})$

2. Case with proportional costs (2)

- $S = (S^F, S^I)$: assets

- λ^{ij} : proportional cost paid in units of asset i for a transaction from i to j.

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$$-K_t(\omega) \ := \ \left\{ \xi \in \mathbb{R}^{d+N} \ : \ \exists a^{ij} \geq 0, \ \sum_{j=1}^{d+N} a^{ji} - (1 + \lambda_t^{ij}(\omega)) a^{ij} \geq S_t^i(\omega) \, \xi^i \right\} .$$

Admissible exchanges $\xi \in L^0(\mathbb{F})$

3. General modelization

• $K_t(\omega)$: polyhedral, closed and convex cone such that $\mathbb{R}_+^{d+N} \setminus \{0\} \subset \text{Int}(K_t)$ a.s.

• $\xi = (\xi_t)_{t \in \mathbb{T}}$ is a self-financed strategy if $\xi_t \in -K_t$ a.s. for each t.

• $-\underline{K}_t := \{(\xi^F, 0) \in -K_t\}$, i.e. transaction only on the financial assets.

The wealth process (to sum up)

- $K_t(\omega)$: polyhedral, closed and convex cone such that $\mathbb{R}_+^{d+N} \setminus \{0\} \subset \text{Int}(K_t)$ a.s.

- Admissibility :

$$\xi_s \in L^0(-K_s; \mathcal{F}_s) \ \text{ and } \ x^I + I(\xi)_s = x^I + \sum_{\tau=0}^{s} \xi_\tau^I \ \in L^0(\mathbb{R}_+^N; \mathcal{F}_s)$$

- Wealth process (in units) : $V_t = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1}(x^I + I(\xi)_s)$

Assumption on R

- For each t

  (R1) $R_t(0) = 0$ and R_t is continuous.

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  (R2) If $\lambda \in [0,1]$ and $(\alpha, \beta) \in (L^0(\mathbb{R}^N_+))^2$, then

$$\lambda R_t(\alpha) + (1-\lambda)R_t(\beta) - R_t\left(\lambda\alpha + (1-\lambda)\beta\right) \ \in \ -\underline{K}_t := \{(x^F, 0) \in -K_t\}$$

- (R2) : means R_t is "concave". Indeed,

$$\lambda R_t(\alpha) + (1-\lambda)R_t(\beta) = R_t\left(\lambda\alpha + (1-\lambda)\beta\right) + \underbrace{\xi_t}_{\in -\underline{K}_t}$$

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  (R3) R_t is bounded from below by an affine (random) map.

- (R3) : In dimension 1 $\Leftrightarrow R'_t(\infty) > -\infty \ a.s.$

- Remark : No monotonicity assumption, need not to be non-negative.

Attainable wealth : $A_t(x; K, R)$

- $A_t(x; K, R) = \left\{ V_t^{x,\xi} = x + \sum_{s=0}^{t} \xi_s + \sum_{s=0}^{t-1} R_{s+1}(x^I + I(\xi)_s),\ \xi \text{ admissible} \right\}$

- Under (R2), $A_t(x; K, R)$ is convex.

Remind (R2) : If $\lambda \in [0,1]$ and $(\alpha, \beta) \in (L^0(\mathbb{R}_+^N))^2$, then

$$\lambda R_{s+1}(\alpha) + (1-\lambda) R_{s+1}(\beta) = R_{s+1}\left(\lambda\alpha + (1-\lambda)\beta\right) + \underbrace{\xi_{s+1}}_{\in -\underline{K}_{s+1}}$$

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- A_t is non-linear with respect to x : $A_t(x; K, R) \neq x + A_t(0; K, R)$

We only have $A_t(x; K, R) = x^F + A_t((0, x^I); K, R)$

Remarks on $K_t = -(-K_t)$

- $V \in K_t \Leftrightarrow V - V = 0$ with $-V \in -K_t$ (admissible exchange).

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• $V \in K_t^o := K_t \cap (-K_t) \Leftrightarrow$ can reach 0 from V and V from 0.

$\Rightarrow$ K_t^o is the set of holdings which are equivalent to 0.

No-arbitrage condition : The pure financial case $N = 0$

The robust No-arbitrage condition (S04, KSR01)

1. Weak no-arbitrage property

$$NA^w(K) \;\; : \;\; A_T(0;K) \cap L^0(\mathbb{R}^d_+;\mathcal{F}_T) \; = \; \{0\} \, .$$

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2. $\tilde{K}$ dominates K if :

$$\underbrace{K_t}_{\text{solvable}} \setminus \underbrace{K_t^o}_{\text{equivalent to 0}} \subset \mathrm{ri}(\underbrace{\tilde{K}_t}_{\text{bigger solvency region}}) \, .$$

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3. Robust no-arbitrage property

$$NA^r(K) \ : \ NA^w(\tilde{K}) \text{ holds for some } \tilde{K} \text{ which dominates } K \, .$$

$\Rightarrow$ No arbitrage even in a model with slightly lower transaction costs.

No-arbitrage condition : The pure financial case $N = 0$

- Under $NA^r(K)$, $A_T(0; K)$ is closed.

- Important property : Under $NA^r(K)$

$$\xi_t \in -K_t \text{ and } \sum_{t=0}^{T} \xi_t = 0 \Rightarrow \xi_t \in K_t^o$$

- The closure property is a consequence of this property.

No-arbitrage condition : The general case

1. Weak no-arbitrage property :

$$NA^w(K,R) \ : \ A_T(0;K,R) \cap L^0(\mathbb{R}_+^{d+N}) = \{0\}$$

2. Set $\underline{K} = \{(x^F,0) \in K\}$. $(\tilde{K},\tilde{R})$ dominates (K,R) if

(D1) $\underline{K}_t \setminus \underline{K}_t^o \ \subset \ \mathrm{ri}(\underline{\tilde{K}}_t)$ and $K_t \subset \tilde{K}_t$

(D2) $\tilde{R}_t(0) \in \underline{K}_t$ and $\tilde{R}_t(\alpha) - R_t(\alpha) \in \mathrm{ri}(\underline{K}_t)$, $\alpha \in \mathbb{R}_+^N \setminus \{0\}$.

(D1) : Slight reduction of transaction costs for the exchanges involving only Financial assets.

(D2) : Slight increase of the return of Industrial assets.

No-arbitrage condition : The general case

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(D1) $\underline{K}_t \setminus \underline{K}_t^o \;\subset\;$ ri$(\underline{\tilde{K}}_t)$ and $K_t \subset \tilde{K}_t$

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3. Robust no-arbitrage property

$NA^r(K,R)$: $\exists\, (\tilde{K}, \tilde{R})$ which dominates (K,R) such that $NA^w(\tilde{K}, \tilde{R})$ holds

No-arbitrage condition : The general case

• Under $NA^r(K,R)$

$$\xi_t \in -K_t \text{ and } \sum_{t=0}^{T} \xi_t + \sum_{t=0}^{T-1} R_{t+1}(I(\xi)_t) \;=\; 0 \;\Rightarrow\; \xi_t \in \underline{K}_t^o \;\; (= \underline{K}_t \cap -\underline{K}_t)\,.$$

$\Rightarrow$ Under $NA^r(K,R)$: $A_T(x;K,R)$ is closed $\forall\ x$.

Dual formulation for $A_T(x; K, R)$

• $(\underline{K}_t)^*(\omega) := \{y \in \mathbb{R}^{d+N} \ : \ x \cdot y \geq 0 \ \ \forall\, x \in \underline{K}_t(\omega)\}$.

• Let $\mathcal{Z}(K, \mathbb{Q})$ be the set of $Z = (Z^F, Z^I) \in L^\infty(\text{Int}(\mathbb{R}^{d+N}_+))$ such that $(\mathbb{E}^{\mathbb{Q}}\left[Z^F \mid \mathcal{F}_t\right], 0_N) \in \text{ri}((\underline{K}_t)^*)$.

• Under $NA^r(K, R)$, for all $\mathbb{Q} \sim \mathbb{P}$ there is $Z \in \mathcal{Z}(K, \mathbb{Q})$ such that

$$a(x; Z, \mathbb{Q}) \ := \sup_{g \in A_T(x;K,R) \cap L^1(\mathbb{Q})} \mathbb{E}^{\mathbb{Q}}[Z \cdot g] < \infty$$

Dual formulation for $A_T(x; K, R)$

• Dual formulation for $A_T(x; K, R) \cap L^1(\mathbb{Q})$:

$$g \in A_T(x; K, R) \cap L^1(\mathbb{Q}) \iff \mathbb{E}^{\mathbb{Q}}[Z \cdot g] \leq a(x; Z, \mathbb{Q}) \ \forall \, Z \in \mathcal{Z}(K, \mathbb{Q}) \ .$$

• Can drop the integrability condition on g if it is uniformly bounded from below for the natural partial order induced by K_T.

Remark on $\mathcal{Z}(K,\mathbb{Q})$: The case $N=0$

- In the case with no transaction costs :

$$
\begin{aligned}
K_t(\omega) &= \{x \in \mathbb{R}^d \ : \ x \cdot S_t(\omega) \geq 0\} \\
K_t^*(\omega) &= \{\lambda S_t(\omega),\ \lambda \in \mathbb{R}_+\}
\end{aligned}
$$

- $Z_t := \mathbb{E}\left[Z \mid \mathcal{F}_t\right] \in \mathrm{ri}(K_t^*)$ implies $Z_t = H_t S_t$ which is a $\mathbb{P}$-martingale.

- If we take S^1 as a numeraire and set $(\hat{H},\hat{S}) = (HS^1, S/S^1)$ then $\hat{H}$ is a martingale as well as $\hat{H}\hat{S}$.

- $\hat{S}$ is a martingale under $\mathbb{Q} = (\hat{H}_T/\mathbb{E}\left[\hat{H}_T\right]) \cdot \mathbb{P}$.

Additional remarks on the separating measures

• In general, there is **no** Z in $\mathcal{Z}(K,\mathbb{Q})$ such that

$$a(0;Z,\mathbb{Q}) \;:=\; \sup_{g\in A_T(0;K,R)\cap L^1(\mathbb{Q})} \mathbb{E}^{\mathbb{Q}}[Z\cdot g] \le 0$$

• In particular, NA^r does not imply the absence of arbitrage opportunity in the “tangent” model :

$$\lim_{\varepsilon\to 0} \sum_{t=0}^{T} \varepsilon\xi_t/\varepsilon + \sum_{t=0}^{T-1} R_{t+1}(\varepsilon I(\xi)_t)/\varepsilon \text{ " = " } \sum_{t=0}^{T}\xi_t + \sum_{t=0}^{T-1} R'_{t+1}(0)I(\xi)_t$$

• However, under NA^r, for all $g\in A_T(0;K,R)$ there is $\mathbb{Q}^g$ and Z^g in $\mathcal{Z}(K,\mathbb{Q}^g)$ such that $\mathbb{E}^{\mathbb{Q}^g}[Z^g\cdot g]\le 0$.

Admissible consumption plans

- $C_T(x;K,R) := \left\{(c_t)_{t\leq T} \in L^0(\mathbb{R}^d_+;\mathbb{F}) \ : \ \left(\sum_{t\leq T} c_t, 0_N\right) \ \in \ A_T(x;K,R)\right\}$

- Under NA^r : $C_T(x;K,R)$ is closed (and convex).

Utility maximization problem

$$\text{Max } \mathbb{E}\left[\sum_{t\leq T} U_t(c_t)\right] \rightarrow u(x)$$

$$\text{over } C_T^U(x;K,R) = \{c \in C_T(x;K,R) \ : \ \mathbb{E}[(\sum_{t\leq T} U_t(c_t))^-] < \infty\}.$$

Assumptions on U_t

• Concave, non-decreasing for the natural partial order on $\mathbb{R}^d$, and $\mathrm{cl}\,(\mathrm{dom}(U_t)) = \mathbb{R}^d_+$

• Non-smooth Inada's conditions : The Fenchel transform

$\tilde{U}_t(y) = \sup_{x \in \mathbb{R}^d_+} U_t(x) - x \cdot y$ satisfies $\mathrm{int}(\mathbb{R}^d_+) \subset \mathrm{dom}(\tilde{U}_t)$.

• Need not to be smooth.

Additional assumptions on U_t

• *Asymptotic elasticity* condition

$$\limsup_{\ell(y)\to 0} \left(\sup_{q\in -\partial \tilde{U}_t(y)} q\cdot y \right) / \tilde{U}_t(y) \;<\; \infty \tag{1}$$

where $\partial \tilde{U}_t(y)$ denotes the subgradient of $\tilde{U}_t$ at y in the sense of convex analysis and

$$\ell(y) \;:=\; \inf_{x\in\mathbb{R}^d_+\,,\ \|x\|=1} x\cdot y\ .$$

See KS (99) and compare with DPT (02) and BTZ (04).

Additional assumptions on U_t

• For each $t \in \mathbb{T}$, one of the above conditions hold :

($\tilde{U}1$) there is $e_t \in \text{int}(\mathbb{R}^d_+)$ such that $V_t : r \in \mathbb{R}_+ \mapsto \tilde{U}_t(re_t)$ is strictly convex and $\lim_{r \to +\infty} V_t'(r) = 0$.

or

($\tilde{U}2$) $\tilde{U}^n_t(y) = \sup_{x \in \mathbb{R}^d_+,\, \|x\| \le n} U_t(x) - x \cdot y$ is uniformly bounded from below in $y \in \mathbb{R}^d_+$ and $n \ge M_t$.

Abstract duality

- Problem reduction

$$u_1(x^1) \ := \ u(x^1, 0_{d-1+N}), \quad x^1 \in \mathbb{R}_+ \ ,$$

- Dual variables

$$\mathcal{D}(y^1) \ = \ \Big\{(Y,\alpha) \in L^1(\Omega \times \mathbb{T}, \mathbb{R}^d_+) \times \mathbb{R}_+ \ : \forall x^1 \in \mathbb{R}_+, \ \forall c \in \mathcal{C}_T((x^1,0); K, R)$$

$$\mathbb{E}\left[\sum_{t\in\mathbb{T}} Y_t \cdot c_t - y^1 x^1\right] \ \le \ \alpha, \Bigg\}, \quad y^1 \in \mathbb{R}_+$$

- Dual problem

$$\tilde{u}_1(y^1) \ = \ \inf_{(Y,\alpha)\in\mathcal{D}(y^1)} \mathbb{E}\left[\sum_{t\in\mathbb{T}} \tilde{U}_t(Y_t) + \alpha\right], \quad y^1 \in \mathbb{R}_+ \ .$$

Abstract duality

$$
\begin{aligned}
\tilde{u}_1(y^1) &= \sup_{x^1 \in \mathbb{R}_+} \left[u_1(x^1) - x^1 y^1 \right], \quad y^1 \in \mathbb{R}_+ \\
u_1(x^1) &= \inf_{y^1 \in \mathbb{R}_+} \left[\tilde{u}_1(x^1) - x^1 y^1 \right], \quad x^1 \in \mathbb{R}_+ \,.
\end{aligned}
$$

Existence result

• If there is an initial wealth $x \in \text{int}(K_0)$ such that $u(x) < \infty$, then

(i) $u(x) < \infty$ for all $x \in \mathbb{R}^d \times \mathbb{R}_+^N$

(ii) for all $x \in \mathbb{R}^d \times \mathbb{R}_+^N$ such that $\mathcal{C}_T^U(x; K, R) \neq \emptyset$, there is some $c^* \in \mathcal{C}_T^U(x; K, R)$ for which

$$u(x) \;=\; \mathbb{E}\left[\sum_{t \in \mathbb{T}} U_t(c_t^*)\right] \,.$$

• Proof : adaptation of the direct argument of Kramkov et Schachermayer AAP 13(4) 2003 to this multivariate setting.

Final comment

• We used the NA^r condition, i.e.

There is $(\tilde{K}, \tilde{R})$ such that

(D1) $\underline{K}_t \setminus \underline{K}_t^o \subset$ ri$(\underline{\tilde{K}}_t)$ and $K_t \subset \tilde{K}_t$

(D2) $\tilde{R}_t(0) \in \underline{K}_t$ and $\tilde{R}_t(\alpha) - R_t(\alpha) \in$ ri$(\underline{K}_t)$, $\alpha \in \mathbb{R}_+^N \setminus \{0\}$

for which $NA^w(\tilde{K}, \tilde{R})$ holds.

Final comment

• Under the additional conditions on R

(i) $R_t \in \underline{K}_t$

(ii) $R_t(\alpha) \in ri(\underline{K}_t)$ for $\alpha \neq 0$

(iii) R_t bounded

all the results holds if there is some $\tilde{K}$ satisfying

$$\text{(D1)} \quad \underline{K}_t \setminus \underline{K}_t^o \subset \text{ri}(\underline{\tilde{K}}_t) \text{ and } K_t \subset \tilde{K}_t$$

such that $NA^w(\tilde{K}, R)$ holds.