A general Doob-Meyer-Mertens decomposition for g-supermartingale systems

B. Bouchard, D. Possamai, X. Tan (and C. Zhou)

Ceremade - Univ. Paris-Dauphine

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Definition (Brownian filtration case) and motivation

- Semi-linear expectation : $\xi \in \mathbf{L}^p(\mathcal{F}_\tau) \mapsto \mathcal{E}^g_{\sigma,\tau}[\xi] := Y_\sigma$ s.t.

$$Y = \xi + \int_{\cdot\wedge\tau}^{\tau} g_s(Y_s, Z_s)ds - \int_{\cdot\wedge\tau}^{\tau} Z_s dW_s \ \text{ with } Z \in \mathbf{L}^2_{\mathcal{P}}.$$

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- g-supermartingale system : $S = \{S(\tau), \tau \in \mathcal{T}\}$ $\mathcal{T}$-system s.t.

$$S(\sigma) \geq \mathcal{E}^g_{\sigma,\tau}[S(\tau)] \quad \text{for } \sigma \leq \tau \in \mathcal{T}.$$

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- g-supersolution : $X \in \mathbf{S}^p$ s.t.

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□ Question : Does the two notions coincide ? Yes, if X aggregated as a cadlag process, Peng (99) for $p = 2$.

Recent and hold problems : 2BSDE of Soner, Touzi and Zhang (10), BSDEs with weak term. cond. of B., Elie and Réveillac (15), BSDE with constraint on Z of Zvitanič, Karatzas and Soner (98),...

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Want to derive a BSDE-type representation : Use DM-type decomposition !

- singular control problems $\Rightarrow$ continuity from the right is very difficult !
- square integrability and quasi left-continuity of the filtration are not necessarily satisfied, e.g. Possamai, Tan and Zhou (15).

Need for a result for ladlag g-supermatingales, in much more general spaces.

Mertens approach ($g \equiv 0$)
Filtration with the usual conditions

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- For $s \le t_0 < t$: use $X_{t_0+} \ge \mathbb{E}_{t_0}[X_t]$ to obtain

$$\mathbb{E}_s[\bar{X}_t] = \mathbb{E}_s[X_t + X_{t_0} - X_{t_0+}] \le \mathbb{E}_s[X_{t_0}] \le X_s = \bar{X}_s. \quad \square$$

Extension to semi-linear conditional expectation operators

□ Definition : A family of maps

$$\mathcal{E}_{\sigma,\tau} : \mathbf{L}^p(\mathcal{F}_\tau) \longmapsto \mathbf{L}^p(\mathcal{F}_\sigma), \quad \text{for } \sigma \le \tau \in \mathcal{T},$$

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(e) There is a family $\mathcal{Q}$ of $\mathbb{P}$-equiv. prob. meas. and $L > 1$ s.t. :

- $\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^q + \left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^{1-q}\right] \le L$ for all $\mathbb{Q} \in \mathcal{Q}$.

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- $\mathbb{Q}^1 \otimes_\tau \mathbb{Q}^2 \in \mathcal{Q}$, for all $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ and $\tau \in \mathcal{T}$.
- For all $\sigma \le \tau \in \mathcal{T}$ and $(\xi, \xi') \in \mathbf{L}^p(\mathcal{F}_\tau) \times \mathbf{L}^p(\mathcal{F}_\tau)$ there exists $\mathbb{Q} \in \mathcal{Q}$ and a $[L^{-1}, 1]$-valued $\beta \in \mathbf{L}^0(\mathcal{F})$ satisfying

$$\mathcal{E}_{\sigma,\tau}[\xi] \le \mathcal{E}_{\sigma,\tau}[\xi'] + \mathbb{E}^{\mathbb{Q}}_\sigma[\beta(\xi - \xi')].$$

□ Definition : For $p > 1$, $\mathbf{X}^p$ (resp. $\mathbf{X}^p_r$, $\mathbf{X}^p_{\ell r}$) is optional processes X s.t. $X_\tau \in \mathbf{L}^p(\mathcal{F}_\tau)$ for $\tau \in \mathcal{T}$ (resp. with right-limits, with right- and left-limits).

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□ Thm : Let $X \in \mathbf{X}^p_r$ be a $\mathcal{E}$-supermartingale with X^- bounded in $\mathbf{L}^p$. Define

$$I_t := \sum_{s<t} (X_s - X_{s+}),\ t \le T.$$

Then, $I \uparrow$, left-continuous, belongs to $\mathbf{X}^{\frac{1}{p}}$.

Moreover, $\bar{X} := X + I$ is a right-continuous local $\mathcal{E}$-supermatingale.

Application to g-expectations in the Brownian L^2-setting

The case of cadlag processes

□ Thm [Peng 99] : If X is a right-continuous (and lag) g-supermatingale in $\mathbf{S}^2$ then

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Proof : Consider the solution (Y, Z, A) of the reflected BSDE

$$\begin{cases} Y &= X_T + \int_\cdot^T g_s(Y_s, Z_s)ds - \int_\cdot^T Z_s dW_s + A_T - A \\ Y &\geq X \\ 0 &= \int_0^T (Y_{s-} - X_{s-})dA_s. \end{cases}$$

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This is the counterpart of the Snell envelope : the smallest g-supermatingale above $X \Rightarrow Y = X$. □

Mertens strategy for ladlag processes

□ Assume w.l.g. that $g \downarrow$ in y, the general result on $\mathcal{E}$-supermatingale applies. The DM decompo. for $\bar{X} = X + I$:

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Hence

$$X = X_T + \int_\cdot^T g_s(\bar{X}_s, Z_s)ds - \int_\cdot^T Z_s dW_s + (A_T + I_T) - (A + I)$$

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The general result, $\mathbb{F}$ satisfies the usual conditions

Main theorem

Assume that $g(0) \in \mathbf{L}^p(dt \times d\mathbb{P})$. Let S be a $\mathcal{E}^g$-supermatingale system s.t. $\text{esssup}\{S(\tau)\ \tau \in \mathcal{T}\} \in \mathbf{L}^p$.
There exists $(X, Z, A) \in \mathbf{X}^p_{\ell r} \times \mathbf{L}^p_{\mathcal{P}} \times \mathbf{I}^p_{\mathcal{P}}$ s.t. for all $\sigma \leq \tau \in \mathcal{T}$

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Related work : Grigorova, Imkeller, Offen, Ouknine, and Quenez (2015) - in $\mathbf{L}^2$ for the Brownian filtration but have a general result on reflected BSDEs with not right-continuous obstacles.

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☐ Need to show existence of the reflected BSDE with cadlag obstacle in $\mathbf{L}^p$.

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□ Assume quasi left-continuity of $(\mathcal{F}_t)_t$ to avoid jumps of A and N at the same time.

New estimates without quasi left-continuity

□ Thm [Extension of Meyer 68] Let X be a (ladlag) strong supermartingale on $[0, T]$ with decomposition

$$X = X_0 + M - A - I.$$

There exists a universal $C_p > 0$ s.t.

$$\|A\|_{\mathbf{I}^p} + \|I\|_{\mathbf{I}^p} \leq C_p \|X\|_{\mathbf{S}^p}.$$

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⇒ can be "extended" to g-supersolutions.
⇒ general existence and uniqueness for (reflected) BSDEs in $\mathbf{L}^p$.

(see "A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations")

Examples of application

Optional decomposition

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☐ Assume that S is a $\mathcal{E}^{\mathbb{Q},g}$-supermartingale system (the orthogonal mart. N depends on $\mathbb{Q}$), s.t. $\mathrm{esssup}\{|S(\tau)|, \tau \in \mathcal{T}\} \in \mathbf{L}^p(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{M}$.

Optional decomposition

□ Let $\mathcal{M}$ be the set of $\mathbb{P}$-equiv. proba. meas. s.t. W is a $(\mathbb{Q}, \mathbb{F})$-martingale.

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□ Thm : $\exists$ $(X, Z) \in \mathbf{X}^p_{\ell r} \times \mathbf{H}^p$ s.t. $S(\sigma) = X_\sigma$ for all $\sigma \in \mathcal{T}$, and

$$X_{\cdot} + \int_0^{\cdot} g_s(X_s, Z_s)ds - \int_0^{\cdot} Z_s dW_s \text{ is non-increasing.}$$

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Extension to singular prob. meas. (for 2BSDEs) : Possamai, Tan and Zhou (15).

Constraint on Z

□ Assumptions/notations

- $\mathbb{F}^\circ = (\mathcal{F}_t^\circ)_{t \le T}$ the raw filtration of the canonical process.
- For some $p' > p > 1$, $g(0) \in \mathbf{L}^{p'}(dt \times d\mathbb{P})$, $\xi \in \mathbf{L}^{p'}$.
- $\mathcal{O} = (\mathcal{O}_t(\omega))_{(t,\omega)\in[0,T]\times\Omega}$: closed conv., $\mathbb{F}^\circ$-prog. meas.

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□ Want to characterize the minimal sol. in $\mathbf{X}^p_{\ell r} \times \mathbf{H}^p \times \mathbf{I}^p$ of

$$
\begin{aligned}
& Y = \xi + \int_\cdot^T g_s(Y_s, Z_s) ds + A_T - A - \int_\cdot^T Z_s dW_s, \\
& Z \in \mathcal{O}, \; dt \times d\mathbb{P} - \text{a.e.}
\end{aligned}
$$

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$$Y = \xi + \int_\cdot^T [g_s(Y_s, Z_s) - \delta_s(\nu_s)]ds + A_T - A - \int_\cdot^T Z_s dW_s^\nu$$

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□ Thm : Define

$$S(\tau) := \operatorname*{esssup}_\nu \mathcal{E}^\nu_{\tau,T}[\xi], \ \ \tau \in \mathcal{T}.$$

Assume that $\operatorname{esssup}\{|S(\tau)|,\ \tau \in \mathcal{T}\} \in \mathbf{L}^{p'}$ for some $p' > p$. Then, $\exists\ X \in \mathbf{X}^p_{\ell r}$ s.t. $X_\tau = S(\tau)$ for $\tau \in \mathcal{T}$, and $(Z, A) \in \mathbf{H}^p \times \mathbf{I}^p$ s.t. (X, Z, A) is the minimal solution.

Proof : Use DPP

$$S(\sigma) := \operatorname{ess\,sup}_{\nu} \mathcal{E}^{\nu}_{\sigma,\tau}[S(\tau)] \geq \mathcal{E}^{\nu'}_{\sigma,\tau}[S(\tau)] \ \ \forall\, \nu'.$$

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Use the decomposition associated to each $\mathcal{E}^{\nu}$:

$$X_\sigma = \xi + \int_\sigma^T (g_s(X_s, Z_s^\nu) - \delta_s(\nu_s))ds + A_T^\nu - A_\sigma^\nu - \int_\sigma^T Z_s^\nu dW_s^\nu$$

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By identification of the quadratic variation parts :

$$Z^\nu = Z^0$$

$$\int_0^T (\nu_s \cdot Z_s^0 - \delta(\nu_s))ds + \underbrace{A^\nu_T - A^\nu_0}_{\geq 0} = A^0_T - A^0_0.$$

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$\Rightarrow \sup_u (u \cdot Z^0 - \delta(u)) \leq 0 \ dt \times d\mathbb{P} \Rightarrow Z^0 \in \mathcal{O} \ dt \times d\mathbb{P}$. □