A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations

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Aim : a-priori estimates for super-solutions

$$Y_t \;=\; \xi - \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s - \int_t^T dM_s + \int_t^T dK_s$$

when

- $\xi \in \mathbf{L}^p$, $g(0) \in \mathbf{L}^p(dt \times d\mathbb{P})$, $p > 1$, g Lip. in (y, z) (can certainly consider extensions).
- Filtration satisfies usual assumptions.

Here

- $Y \in \mathbf{S}_p$: làdlàg adapted, $\mathbb{E}[\sup_{[0,T]} |Y|^p] < \infty$.
- $Z \in \mathbf{H}_p$: $\mathbb{E}[(\int_0^T \|Z_t\|^2 dt)^{\frac{p}{2}}] < \infty$
- $M \in \mathbf{M}_p$: càdlàg mart., orthogonal to W, $\mathbb{E}[[M]_T^{\frac{p}{2}}] < \infty$
- $K \in \mathbf{I}_p$: non-decreasing predictable, $\mathbb{E}[|K_T|^p] < \infty$.

Motivation

Obtain a general Doob-Meyer type decomposition for g-supermatingale in $\mathbf{L}_p$.

See X. Tan's and Y. Ouknine's talks later in this week.

On $\mathbf{L}^p$ solutions, see : Briand, Delyon, Hu, Pardoux, and Stoica (02), Kruse and Popier (14), Klimsiak (13, 14).

Usual approach goes wrong

For $\mathbf{L}_2$ estimates : apply Itô's Lemma to $e^{\alpha t} Y_t^2$, for some $\alpha > 0$, and use

$$Y_t \;=\; \xi - \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s - \int_t^T dM_s + \int_t^T dK_s.$$

Problem : If $\mathbb{F}$ is not quasi left-continuous, then $[M, K]$ shows up !

The case of (classical) super-martingales

Lemma [Meyer 68] For all $p > 1$, $\exists\ C_p$ s.t. for all strong (làdlàg) supermartingale $X \in \mathbf{S}_p$ with decomposition

$$X_t = X_0 + M_t - A_t,$$

one has

$$\|A\|_{\mathbf{I}_p} \leq C_p \|X\|_{\mathbf{S}_p}.$$

Proof for $p = 2$, A continuous, $X_t = \mathbb{E}[A_T - A_t | \mathcal{F}_t]$.

$$\|A\|_{\mathbf{I}_2}^2 = \mathbb{E}[A_T^2] = \mathbb{E}[2\int_0^T (A_T - A_t) dA_t] = \mathbb{E}[2\int_0^T X_t dA_t] \leq 2\|X\|_{\mathbf{S}_2}\|A\|_{\mathbf{I}_2}.$$

Extension to super-solutions

Assume that

$$Y_t \;=\; \xi - \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s - \int_t^T dM_s + \int_t^T dK_s$$

with

- $\xi \in \mathbf{L}^p$, $g(0) \in \mathbf{L}^p(dt \times d\mathbb{P})$, $p > 1$.
- Filtration satisfies usual assumptions.

Then,

$$\|Z\|^p_{\mathbf{H}_p} + \|M\|^p_{\mathbf{M}_p} + \|K\|^p_{\mathbf{I}_p} \leq C_p\Big(\|\xi\|^p_{\mathbf{L}_b} + \|Y\|^p_{\mathbf{S}_p} + \|g(0)\|^p_{\mathbf{H}_p}\Big).$$

Difference of super-solutions

Consider two super-solutions $(Y^i, Z^i, M^i)_{i=1,2}$ associated to $(g^i)_{i=1,2}$ and $(\xi^i)_{i=1,2}$. Then,

$$\begin{aligned} &\|\delta Z\|^p_{\mathbf{H}_p} + \|\delta(M-K)\|^p_{\mathbf{M}_p} \\ \leq\ & C\left(\|\delta\xi\|^p_{\mathbf{L}_p} + \|\delta Y\|^p_{\mathbf{S}_p} + \|\delta Y\|^{\frac{p}{2}\wedge(p-1)}_{\mathbf{S}_p} + \|\delta g(Y^1_\cdot, Z^1_\cdot)\|^p_{\mathbf{H}_p}\right). \end{aligned}$$

Example : reflected BSDE

Let S be a càdlàg process s.t. $S^+ := S \vee 0 \in \mathbf{S}_p$. Then, existence and uniqueness holds in $\mathbf{S}_p \times \mathbf{H}_p \times \mathbf{M}_p$ for

$$Y_t = \xi - \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s - \int_t^T dM_s + \int_t^T dK_s,$$

s.t.

$$\begin{cases} Y \geq S, \quad \text{on } [0, T], \\ \int_0^T (Y_{s-} - S_{s-})\, dK_s = 0. \end{cases}$$