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WEAK DYNAMIC PROGRAMMING PRINCIPLE FOR VISCOSITY SOLUTIONS*

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Abstract. We prove a weak version of the dynamic programming principle for standard stochastic control problems and mixed control-stopping problems, which avoids the technical difficulties related to the measurable selection argument. In the Markov case, our result is tailor-made for the derivation of the dynamic programming equation in the sense of viscosity solutions.

Key words. optimal control, dynamic programming, discontinuous viscosity solutions

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1. Introduction.

Consider the standard class of stochastic control problems in the Mayer form

$$V(t,x) := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[f(X_T^\nu)|X_t^\nu = x\right],$$

where $\mathcal{U}$ is the control set, X^ν is the controlled process, f is some given function, $0 < T \leq \infty$ is a given time horizon, $t \in [0,T)$ is the time origin, and $x \in \mathbb{R}^d$ is some given initial condition. This framework includes the general class of stochastic control problems under the so-called Bolza formulation, the corresponding singular versions, and optimal stopping problems.

A key tool for the analysis of such problems is the so-called dynamic programming principle (DPP), which relates the time-t value function $V(t,.)$ to any later time-τ value $V(\tau,.)$ for any stopping time $\tau \in [t,T)$ a.s. A formal statement of the DPP is

$$V(t,x) = v(t,x) \ := \ \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[V(\tau, X_\tau^\nu)|X_t^\nu = x\right]. \tag{1.1}$$

In particular, this result is routinely used in the case of controlled Markov jump-diffusions in order to derive the corresponding dynamic programming equation in the sense of viscosity solutions; see Lions [11, 12], Fleming and Soner [9], and Touzi [15] for the case of controlled diffusions and Øksendal and Sulem [13] for the case of Markov jump-diffusions.

The statement (1.1) of the DPP is very intuitive and can be easily proved in the deterministic framework, or in discrete-time with finite probability space. However, its proof is in general not trivial and requires on the first stage that V be measurable.

When the value function V is known to be continuous, the abstract measurability arguments are not needed and the proof of the DPP is simplified significantly. See,

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e.g., Fleming and Soner [9] or Kabanov and Klueppelberg [10] in the context of a special singular control problem in finance. Our objective is to reduce the proof to this simple context in a general situation where the value function has no a priori regularity.

The inequality $V \leq v$ is the easy one but still requires that V be measurable. Our weak formulation avoids this issue. Namely, under fairly general conditions on the control set and the controlled process, it follows from an easy application of the tower property of conditional expectations that

$$V(t,x) \leq \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[V^*(\tau, X^\nu_\tau)|X^\nu_t = x\right],$$

where V^* is the upper-semicontinuous envelope of the function V.

The proof of the converse inequality $V \geq v$ in a general probability space turns out to be difficult when the function V is not known a priori to satisfy some continuity condition. See, e.g., Bertsekas and Shreve [2], Borkar [3], and El Karoui [8].

Our weak version of the DPP avoids the nontrivial measurable selection argument needed to prove the inequality $V \geq v$ in (1.1). Namely, in the context of a general control problem presented in section 2, we show in section 3 that

$$V(t,x) \ \geq \ \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\varphi(\tau, X^\nu_\tau)|X_t = x\right]$$
for every upper-semicontinuous minorant φ of V.

We also show that an easy consequence of this result is that

$$V(t,x) \geq \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[V_*(\tau^\nu_n, X^\nu_{\tau^\nu_n})|X_t = x\right],$$

where $\tau^\nu_n := \tau \wedge \inf\{s > t : |X^\nu_s - x| > n\}$, and V_* is the lower-semicontinuous envelope of V.

This result is weaker than the classical DPP (1.1). However, in the controlled Markov jump-diffusion case, it turns out to be tailor-made for the derivation of the dynamic programming equation in the sense of viscosity solutions. Section 5 reports this derivation in the context of controlled jump-diffusions.

Finally, section 4 provides an extension of our argument in order to obtain a weak DPP for mixed control-stopping problems.

2. The stochastic control problem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a càdlàg $\mathbb{R}^d$-valued process Z with independent increments. Given $T > 0$, let $\mathbb{F} := \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ be the completion of its natural filtration on $[0,T]$. Note that $\mathbb{F}$ satisfies the usual conditions; see, e.g., [6]. We assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T = \mathcal{F}$.

For every $t \geq 0$, we set $\mathbb{F}^t := (\mathcal{F}^t_s)_{s\geq 0}$, where $\mathcal{F}^t_s$ is the completion of $\sigma(Z_r - Z_t,\ t \leq r \leq s \vee t)$ by null sets of $\mathcal{F}$.

We denote by $\mathcal{T}$ the collection of all $\mathbb{F}$-stopping times. For $\tau_1, \tau_2 \in \mathcal{T}$ with $\tau_1 \leq \tau_2$ a.s., the subset $\mathcal{T}_{[\tau_1,\tau_2]}$ is the collection of all $\tau \in \mathcal{T}$ such that $\tau \in [\tau_1, \tau_2]$ a.s. When $\tau_1 = 0$, we simply write $\mathcal{T}_{\tau_2}$. We use the notation $\mathcal{T}^t_{[\tau_1,\tau_2]}$ and $\mathcal{T}^t_{\tau_2}$ to denote the corresponding sets of $\mathbb{F}^t$-stopping times.

Throughout the paper, the only reason for introducing the filtration $\mathbb{F}$ through the process Z is to guarantee the following property of the filtrations $\mathbb{F}^t$.

Remark 2.1. Notice that $\mathcal{F}^t_s$-measurable random variables are independent of $\mathcal{F}_t$ for all $s, t \leq T$ and that $\mathcal{F}^t_s$ is the trivial degenerate σ-algebra for $s \leq t$. Similarly, all $\mathbb{F}^t$-stopping times are independent of $\mathcal{F}_t$.

For $\tau \in \mathcal{T}$ and a subset A of a finite dimensional space, we denote by $\mathbb{L}^0_\tau(A)$ the collection of all $\mathcal{F}_\tau$-measurable random variables with values in A. $\mathbb{H}^0(A)$ is the collection of all $\mathbb{F}$-progressively measurable processes with values in A, and $\mathbb{H}^0_{\rm rcll}(A)$ is the subset of all processes in $\mathbb{H}^0(A)$ which are right continuous with finite left limits.

In the following, we denote by $B_r(z)$ (resp., $\partial B_r(z)$) the open ball (resp., its boundary) of radius $r > 0$ and center $z \in \mathbb{R}^\ell$, $\ell \in \mathbb{N}$.

Throughout this note, we fix an integer $d \in \mathbb{N}$, and we introduce the sets

$$\mathbf{S} := [0,T] \times \mathbb{R}^d \quad \text{and} \quad \mathcal{S}_o := \left\{(\tau,\xi) : \tau \in \mathcal{T}_T \text{ and } \xi \in \mathbb{L}^0_\tau(\mathbb{R}^d)\right\}.$$

We also denote by USC($\mathbf{S}$) (resp., LSC($\mathbf{S}$)) the collection of all upper-semicontinuous (resp., lower-semicontinuous) functions from $\mathbf{S}$ to $\mathbb{R}$.

The set of *control processes* is a given subset $\mathcal{U}_o$ of $\mathbb{H}^0(\mathbb{R}^k)$, for some integer $k \geq 1$, so that the *controlled state process* defined as the mapping

$$(\tau,\xi;\nu) \in \mathcal{S} \times \mathcal{U}_o \longmapsto X^\nu_{\tau,\xi} \in \mathbb{H}^0_{\rm rcll}(\mathbb{R}^d) \quad \text{for some } \mathcal{S} \text{ with } \mathbf{S} \subset \mathcal{S} \subset \mathcal{S}_o$$

is well defined and satisfies

$$\left(\theta, X^\nu_{\tau,\xi}(\theta)\right) \in \mathcal{S} \quad \forall\ (\tau,\xi) \in \mathcal{S} \text{ and } \theta \in \mathcal{T}_{[\tau,T]}.$$

A suitable choice of the set $\mathcal{S}$ in the case of jump-diffusion processes driven by Brownian motion is given in section 5.

Given a Borel function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $(t,x) \in \mathbf{S}$, we introduce the reward function $J : \mathbf{S} \times \mathcal{U} \longrightarrow \mathbb{R}$:

$$J(t,x;\nu) := \mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)\right], \tag{2.1}$$

which is well defined for controls ν in

$$\mathcal{U} := \left\{\nu \in \mathcal{U}_o : \ \mathbb{E}\left[|f(X^\nu_{t,x}(T))|\right] < \infty \quad \forall\ (t,x) \in \mathbf{S}\right\}. \tag{2.2}$$

We say that a control $\nu \in \mathcal{U}$ is *t-admissible* if it is $\mathbb{F}^t$-progressively measurable, and we denote by $\mathcal{U}_t$ the collection of such processes. The stochastic control problem is defined by

$$V(t,x) := \sup_{\nu \in \mathcal{U}_t} J(t,x;\nu) \quad \text{for } (t,x) \in \mathbf{S}. \tag{2.3}$$

Remark 2.2. The restriction to control processes that are $\mathbb{F}^t$-progressively measurable in the definition of $V(t,\cdot)$ is natural and consistent with the case where $t = 0$, since $\mathcal{F}_0$ is assumed to be trivial, and is actually commonly used; compare with, e.g., [16]. It will be technically important in the following. It also seems a priori necessary in order to ensure that Assumption A4 makes sense; see Remark 3.2 and the proof of Proposition 5.4. However, we will show in Remark 5.2 that it is not restrictive.

3. Dynamic programming for stochastic control problems. For the purpose of our weak DPP, the following assumptions are crucial.

Assumption A. For all $(t,x) \in \mathbf{S}$ and $\nu \in \mathcal{U}_t$, the controlled state process satisfies the following:

A1 (independence). The process $X^\nu_{t,x}$ is $\mathbb{F}^t$-progressively measurable.

A2 (causality). For $\tilde\nu \in \mathcal{U}_t$, $\tau \in \mathcal{T}^t_{[t,T]}$, and $A \in \mathcal{F}^t_\tau$, if $\nu = \tilde\nu$ on $[t,\tau]$ and $\nu \mathbf{1}_A = \tilde\nu \mathbf{1}_A$ on $(\tau,T]$, then $X^\nu_{t,x}\mathbf{1}_A = X^{\tilde\nu}_{t,x}\mathbf{1}_A$.

A3 (stability under concatenation). For every $\tilde{\nu} \in \mathcal{U}_t$ and $\theta \in \mathcal{T}^t_{[t,T]}$,

$$\nu \mathbf{1}_{[0,\theta]} + \tilde{\nu} \mathbf{1}_{(\theta,T]} \ \in \ \mathcal{U}_t \ .$$

A4 (consistency with deterministic initial data). For all $\theta \in \mathcal{T}^t_{[t,T]}$, we have the following:

a. For $\mathbb{P}$-a.e $\omega \in \Omega$, there exists $\tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ such that

$$\mathbb{E}\left[f\left(X^\nu_{t,x}(T) \right) | \mathcal{F}_\theta \right](\omega) \le J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}_\omega).$$

b. For $t \le s \le T$, $\theta \in \mathcal{T}^t_{[t,s]}$, $\tilde{\nu} \in \mathcal{U}_s$, and $\bar{\nu} := \nu \mathbf{1}_{[0,\theta]} + \tilde{\nu} \mathbf{1}_{(\theta,T]}$, we have

$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T) \right) | \mathcal{F}_\theta \right](\omega) = J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Remark 3.1. Assumption A2 means that the process $X^\nu_{t,x}$ is defined (caused) by the control ν pathwise.

Remark 3.2. Let θ be equal to a fixed time s in Assumption A4b. If $\tilde{\nu}$ is allowed to depend on $\mathcal{F}_s$, then the left-hand side in Assumption A4b does not coincide with $\mathbb{E}\big[f(X^{\tilde{\nu}}_{s,X^\nu_{t,x}(s)(\omega)}(T)) \big]$. Hence, the above identity cannot hold in this form.

Remark 3.3. In section 5, we show that Assumption A4a holds with equality in the jump-diffusion setting. Although we have no example of a control problem where the equality does not hold, we keep Assumption A4a under this form because the proof needs only this requirement.

Remark 3.4. Assumption A3 implies the following property of the control set which will be needed later:

A5 (stability under bifurcation). For $\nu_1, \nu_2 \in \mathcal{U}_t$, $\tau \in \mathcal{T}^t_{[t,T]}$, and $A \in \mathcal{F}^t_\tau$, we have

$$\bar{\nu} := \nu_1 \mathbf{1}_{[0,\tau]} + (\nu_1 \mathbf{1}_A + \nu_2 \mathbf{1}_{A^c})\, \mathbf{1}_{(\tau,T]} \in \mathcal{U}_t.$$

To see this, observe that $\tau_A := T\mathbf{1}_A + \tau \mathbf{1}_{A^c}$ is a stopping time in $\mathcal{T}^t_{[t,T]}$ (the independence of $\mathcal{F}_t$ follows from Remark 2.1), and $\bar{\nu} = \nu_1 \mathbf{1}_{[0,\tau_A]} + \nu_2 \mathbf{1}_{(\tau_A,T]}$ is the concatenation of ν_1 and ν_2 at the stopping time τ_A.

Given $\bar{\nu}$ as constructed above, it is clear that this control can be concatenated with another control $\nu_3 \in \mathcal{U}_t$ by following the same argument. Iterating the above property, we therefore see that for $0 \le t \le T$ and $\tau \in \mathcal{T}^t_{[t,T]}$, we have the following extension: for a finite sequence $(\nu_1, \dots, \nu_n)$ of controls in $\mathcal{U}_t$ with $\nu_i = \nu_1$ on $[0,\tau]$, and for a partition $(A_i)_{1 \le i \le n}$ of Ω with $A_i \in \mathcal{F}^t_\tau$ for every $i \le n$,

$$\bar{\nu} := \nu_1 \mathbf{1}_{[0,\tau]} + \mathbf{1}_{(\tau,T]} \sum_{i=1}^n \nu_i \mathbf{1}_{A_i} \in \mathcal{U}_t.$$

Our main result is the following weak version of the DPP which uses the following notation:

$$V_*(t,x) := \liminf_{(t',x') \to (t,x)} V(t',x'), \quad V^*(t,x) := \limsup_{(t',x') \to (t,x)} V(t',x'), \ (t,x) \in \mathbf{S}.$$

THEOREM 3.5. *Let Assumption* A *hold true, and assume that* V *is locally bounded. Then, for every* $(t,x) \in \mathbf{S}$*, and for every family of stopping times* $\{\theta^\nu, \nu \in \mathcal{U}_t\} \subset \mathcal{T}^t_{[t,T]}$*, we have*

$$V(t,x) \le \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V^*(\theta^\nu, X^\nu_{t,x}(\theta^\nu)) \right]. \tag{3.1}$$

Assume further that $J(.;\nu) \in \mathrm{LSC}(\mathbf{S})$ *for every* $\nu \in \mathcal{U}_o$. *Then, for any function* $\varphi : \mathbf{S} \longrightarrow \mathbb{R}$,

$$\varphi \in \mathrm{USC}(\mathbf{S}) \quad \text{and} \quad V \geq \varphi \Longrightarrow V(t,x) \;\geq\; \sup_{\nu \in \mathcal{U}_t^{\varphi}} \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right], \tag{3.2}$$

where $\mathcal{U}_t^{\varphi} = \left\{\nu \in \mathcal{U}_t : \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))^+\right] < \infty \text{ or } \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))^-\right] < \infty\right\}$.

Before proceeding to the proof of this result, we report the following consequence.

COROLLARY 3.6. *Let the conditions of Theorem* 3.5 *hold. For* $(t,x) \in \mathbf{S}$, *let* $\{\theta^\nu, \nu \in \mathcal{U}_t\} \subset \mathcal{T}_{[t,T]}^t$ *be a family of stopping times such that* $X_{t,x}^\nu \mathbf{1}_{[t,\theta^\nu]}$ *is* $\mathbb{L}^\infty$*-bounded for all* $\nu \in \mathcal{U}_t$. *Then,*

$$\sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V_*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] \leq V(t,x) \leq \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V^*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right]. \tag{3.3}$$

Proof. The right-hand side inequality is already provided in Theorem 3.5. Fix $r > 0$. It follows from standard arguments (see, e.g., Lemma 3.5 in [14]) that we can find a sequence of continuous functions $(\varphi_n)_n$ such that $\varphi_n \leq V_* \leq V$ for all $n \geq 1$ and such that φ_n converges pointwise to V_* on $[0,T] \times B_r(0)$. Set $\phi_N := \min_{n \geq N} \varphi_n$ for $N \geq 1$, and observe that the sequence $(\phi_N)_N$ is nondecreasing and converges pointwise to V_* on $[0,T] \times B_r(0)$. Applying (3.2) of Theorem 3.5 and using the monotone convergence theorem, we then obtain

$$V(t,x) \geq \lim_{N \to \infty} \mathbb{E}\left[\phi_N(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] = \mathbb{E}\left[V_*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right]. \qquad \square$$

Remark 3.7. Notice that the value function $V(t,x)$ is defined by means of $\mathcal{U}_t$ as the set of controls. Because of this, the lower semicontinuity of $J(.,\nu)$ required in the second part of Theorem 3.5 does not imply that V is lower semicontinuous in its t-variable. See, however, Remark 5.3.

Proof of Theorem 3.5. 1. Let $\nu \in \mathcal{U}_t$ be arbitrary, and set $\theta := \theta^\nu$. The first assertion is a direct consequence of Assumption A4a. Indeed, it implies that, for $\mathbb{P}$-almost all $\omega \in \Omega$, there exists $\tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ such that

$$\mathbb{E}\left[f\left(X_{t,x}^\nu(T)\right)|\mathcal{F}_\theta\right](\omega) \leq J(\theta(\omega), X_{t,x}^\nu(\theta)(\omega); \tilde{\nu}_\omega).$$

Since, by definition, $J(\theta(\omega), X_{t,x}^\nu(\theta)(\omega); \tilde{\nu}_\omega) \leq V^*(\theta(\omega), X_{t,x}^\nu(\theta)(\omega))$, it follows from the tower property of conditional expectations that

$$\mathbb{E}\left[f\left(X_{t,x}^\nu(T)\right)\right] = \mathbb{E}\left[\mathbb{E}\left[f\left(X_{t,x}^\nu(T)\right)|\mathcal{F}_\theta\right]\right] \;\leq\; \mathbb{E}\left[V^*\left(\theta, X_{t,x}^\nu(\theta)\right)\right].$$

2. Let $\varepsilon > 0$ be given. Then there is a family $(\nu^{(s,y),\varepsilon})_{(s,y)\in\mathbf{S}} \subset \mathcal{U}_o$ such that

$$\nu^{(s,y),\varepsilon} \in \mathcal{U}_s \quad \text{and} \quad J(s,y;\nu^{(s,y),\varepsilon}) \geq V(s,y) - \varepsilon \text{ for every } (s,y) \in \mathbf{S}. \tag{3.4}$$

By the lower semicontinuity of $(t',x') \mapsto J(t',x';\nu^{(s,y),\varepsilon})$, for fixed $(s,y) \in \mathbf{S}$, together with the upper semicontinuity of φ, we may find a family $(r_{(s,y)})_{(s,y)\in\mathbf{S}}$ of positive scalars so that, for any $(s,y) \in \mathbf{S}$,

$$\begin{aligned}\varphi(s,y) - \varphi(t',x') \geq -\varepsilon \;\; \text{and} \;\; & J(s,y;\nu^{(s,y),\varepsilon}) - J(t',x';\nu^{(s,y),\varepsilon}) \\ & \leq \varepsilon \quad \text{for } (t',x') \in B(s,y;r_{(s,y)}),\end{aligned} \tag{3.5}$$

where, for $r > 0$ and $(s,y) \in \mathbf{S}$,

$$B(s,y;r) := \{(t',x') \in \mathbf{S} \;:\; t' \in (s-r,s],\; |x'-y| < r\}. \tag{3.6}$$

Note that here we do not use balls of the usual form $B_r(s,y)$ and consider the topology induced by half-closed intervals on $[0,T]$. The fact that $t' \le s$ for $(t',x') \in B(s,y;r)$ will play an important role when appealing to Assumption A4b in step 3. Clearly, $\{B(s,y;r) : (s,y) \in \mathbf{S},\, 0 < r \le r_{(s,y)}\}$ forms an open covering of $(0,T] \times \mathbb{R}^d$. It then follows from the Lindelöf covering theorem (see, e.g., [7, Theorem 6.3, Chapter VIII]) that we can find a countable sequence $(t_i, x_i, r_i)_{i\ge1}$ of elements of $\mathbf{S}\times\mathbb{R}$, with $0 < r_i \le r_{(t_i,x_i)}$ for all $i \ge 1$, such that $\mathbf{S} \subset \{0\} \times \mathbb{R}^d \cup (\cup_{i\ge1} B(t_i, x_i; r_i))$. Set $A_0 := \{T\} \times \mathbb{R}^d$, $C_{-1} := \emptyset$, and define the sequence

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i, \quad \text{where } C_i := C_{i-1} \cup A_i, \quad i \ge 0.$$

With this construction, it follows from (3.4), (3.5), together with the fact that $V \ge \varphi$, that the countable family $(A_i)_{i\ge0}$ satisfies

$$\begin{aligned} &(\theta, X^{\nu}_{t,x}(\theta)) \in \bigcup_{i\ge0} A_i \ \mathbb{P}\text{-a.s.}, \quad A_i \cap A_j = \emptyset \quad \text{for } i \ne j \in \mathbb{N}, \\ &\text{and } J(\cdot;\nu^{i,\varepsilon}) \ge \varphi - 3\varepsilon \quad \text{on } A_i \text{ for } i \ge 1, \end{aligned} \tag{3.7}$$

where $\nu^{i,\varepsilon} := \nu^{(t_i,x_i),\varepsilon}$ for $i \ge 1$.

3. We now prove (3.2). We fix $\nu \in \mathcal{U}_t$ and $\theta \in \mathcal{T}^t_{[t,T]}$. Set $A^n := \cup_{0\le i\le n} A_i$, $n \ge 1$. Given $\nu \in \mathcal{U}_t$, we define

$$\nu^{\varepsilon,n}_s := \mathbf{1}_{[t,\theta]}(s)\nu_s + \mathbf{1}_{(\theta,T]}(s)\left(\nu_s \mathbf{1}_{(A^n)^c}(\theta, X^{\nu}_{t,x}(\theta)) + \sum_{i=1}^{n} \mathbf{1}_{A_i}(\theta, X^{\nu}_{t,x}(\theta))\nu^{i,\varepsilon}_s\right)$$
$$\text{for } s \in [t,T].$$

Notice that $\{(\theta, X^{\nu}_{t,x}(\theta)) \in A_i\} \in \mathcal{F}^t_\theta$ as a consequence of Assumption A1. Then, it follows from Assumption A3 and Remark 3.4 that $\nu^{\varepsilon,n} \in \mathcal{U}_t$. By the definition of the neighborhood (3.6), notice that $\theta = \theta \wedge t_i \le t_i$ on $\{(\theta, X^{\nu}_{t,x}(\theta)) \in A_i\}$. Then, using Assumptions A4b and A2 and (3.7), we deduce that

$$\begin{aligned} \mathbb{E}\left[f\left(X^{\nu^{\varepsilon,n}}_{t,x}(T)\right)|\mathcal{F}_\theta\right]\mathbf{1}_{A^n}\left(\theta, X^{\nu}_{t,x}(\theta)\right) &= \mathbb{E}\left[f\left(X^{\nu^{\varepsilon,n}}_{t,x}(T)\right)|\mathcal{F}_\theta\right]\mathbf{1}_{A_0}\left(\theta, X^{\nu}_{t,x}(\theta)\right) \\ &\quad + \sum_{i=1}^{n} \mathbb{E}\left[f\left(X^{\nu^{\varepsilon,n}}_{t,x}(T)\right)|\mathcal{F}_{\theta\wedge t_i}\right]\mathbf{1}_{A_i}\left(\theta, X^{\nu}_{t,x}(\theta)\right) \\ &= V\left(T, X^{\nu^{\varepsilon,n}}_{t,x}(T)\right)\mathbf{1}_{A_0}\left(\theta, X^{\nu}_{t,x}(\theta)\right) \\ &\quad + \sum_{i=1}^{n} J(\theta\wedge t_i, X^{\nu}_{t,x}(\theta\wedge t_i); \nu^{i,\varepsilon})\mathbf{1}_{A_i}\left(\theta, X^{\nu}_{t,x}(\theta)\right) \\ &\ge \sum_{i=0}^{n}\left(\varphi(\theta, X^{\nu}_{t,x}(\theta)) - 3\varepsilon\right)\mathbf{1}_{A_i}\left(\theta, X^{\nu}_{t,x}(\theta)\right) \\ &= \left(\varphi(\theta, X^{\nu}_{t,x}(\theta)) - 3\varepsilon\right)\mathbf{1}_{A^n}\left(\theta, X^{\nu}_{t,x}(\theta)\right), \end{aligned}$$

which, by definition of V and the tower property of conditional expectations, implies

$$\begin{aligned} V(t,x) &\ge J(t,x;\nu^{\varepsilon,n}) \\ &= \mathbb{E}\left[\mathbb{E}\left[f\left(X^{\nu^{\varepsilon,n}}_{t,x}(T)\right)|\mathcal{F}_\theta\right]\right] \\ &\ge \mathbb{E}\left[\left(\varphi\left(\theta, X^{\nu}_{t,x}(\theta)\right) - 3\varepsilon\right)\mathbf{1}_{A^n}\left(\theta, X^{\nu}_{t,x}(\theta)\right)\right] + \mathbb{E}\left[f\left(X^{\nu}_{t,x}(T)\right)\mathbf{1}_{(A^n)^c}\left(\theta, X^{\nu}_{t,x}(\theta)\right)\right]. \end{aligned}$$

Since $f\left(X^{\nu}_{t,x}(T)\right) \in \mathbb{L}^1$, it follows from the dominated convergence theorem that

$$\begin{aligned}
V(t,x) &\geq -3\varepsilon + \liminf_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta))\mathbf{1}_{A^n}\left(\theta, X^{\nu}_{t,x}(\theta)\right)\right] \\
&= -3\varepsilon + \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta))^+\mathbf{1}_{A^n}\left(\theta, X^{\nu}_{t,x}(\theta)\right)\right] \\
&\qquad - \lim_{n\to\infty} \mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta))^-\mathbf{1}_{A^n}\left(\theta, X^{\nu}_{t,x}(\theta)\right)\right] \\
&= -3\varepsilon + \mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta))\right],
\end{aligned}$$

where the last equality follows from the left-hand side of (3.7) and from the monotone convergence theorem, due to the fact that either $\mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta))^+\right] < \infty$ or $\mathbb{E}\left[\varphi(\theta, X^{\nu}_{t,x}(\theta))^-\right] < \infty$. The proof of (3.2) is completed by the arbitrariness of $\nu \in \mathcal{U}_t$ and $\varepsilon > 0$. $\square$

Remark 3.8 (lower-semicontinuity condition I). It is clear from the above proof that it suffices to prove the lower semicontinuity of $(t,x) \mapsto J(t,x;\nu)$ for ν in a subset $\tilde{\mathcal{U}}_o$ of $\mathcal{U}_o$ such that $\sup_{\nu\in\tilde{\mathcal{U}}_t} J(t,x;\nu) = V(t,x)$. Here $\tilde{\mathcal{U}}_t$ is the subset of $\tilde{\mathcal{U}}_o$ whose elements are $\mathbb{F}^t$-progressively measurable. In most applications, this allows one to reduce to the case where the controls are essentially bounded or satisfy a strong integrability condition.

Remark 3.9 (lower-semicontinuity condition II). In the above proof, the lower-semicontinuity assumption is used only to construct the balls B_i on which $J(t_i,x_i;\nu^{i,\varepsilon}) - J(\cdot;\nu^{i,\varepsilon}) \leq \varepsilon$. Clearly, it can be alleviated, and it suffices that the lower semicontinuity holds in time from the left, i.e.,

$$\liminf_{(t',x')\to(t_i,x_i),\ t'\leq t_i} J(t',x';\nu^{i,\varepsilon}) \geq J(t_i,x_i;\nu^{i,\varepsilon}).$$

Remark 3.10 (the Bolza and Lagrange formulations). Consider the stochastic control problem under the so-called Lagrange formulation:

$$V(t,x) := \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\int_t^T Y^{\nu}_{t,x,1}(s) g\left(s, X^{\nu}_{t,x}(s), \nu_s\right) ds + Y^{\nu}_{t,x,1}(T) f\left(X^{\nu}_{t,x}(T)\right)\right],$$

where

$$dY^{\nu}_{t,x,y}(s) = -Y^{\nu}_{t,x,y}(s) k\left(s, X^{\nu}_{t,x}(s), \nu_s\right) ds, \quad Y^{\nu}_{t,x,y}(t) = y > 0.$$

Then, it is well known that this problem can be converted into the Mayer formulation (2.3) by augmenting the state process to (X,Y,Z), where

$$dZ^{\nu}_{t,x,y,z}(s) = Y^{\nu}_{t,x,y}(s) g\left(s, X^{\nu}_{t,x}(s), \nu_s\right) ds, \quad Z^{\nu}_{t,x,y,z}(t) = z \in \mathbb{R},$$

and considering the value function

$$\bar{V}(t,x,y,z) := \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[Z^{\nu}_{t,x,y,z}(T) + Y^{\nu}_{t,x,y}(T) f\left(X^{\nu}_{t,x}(T)\right)\right] = yV(t,x) + z.$$

In particular, $V(t,x) = \bar{V}(t,x,1,0)$. The first assertion of Theorem 3.5 implies

(3.8)

$$V(t,x) \leq \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[Y^{\nu}_{t,x,1}(\theta^\nu) V^*(\theta^\nu, X^{\nu}_{t,x}(\theta^\nu)) + \int_t^{\theta^\nu} Y^{\nu}_{t,x,1}(s) g\left(s, X^{\nu}_{t,x}(s), \nu_s\right) ds\right].$$

Given an upper-semicontinuous minorant φ of V, the function $\bar\varphi$ defined by $\bar\varphi(t,x,y,z) := y\varphi(t,x)+z$ is an upper-semicontinuous minorant of $\bar V$. From the second assertion of Theorem 3.5, we see that for a family $\{\theta^\nu,\ \nu\in\mathcal{U}_t\}\subset\mathcal{T}^t_{[t,T]}$,

$$
\begin{aligned}
V(t,x) &\geq \sup_{\nu\in\mathcal{U}_t^{\bar\varphi}} \mathbb{E}\left[\bar\varphi\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu), Y^\nu_{t,x,1}(\theta^\nu), Z^\nu_{t,x,1,0}(\theta^\nu)\right)\right] \\
&= \sup_{\nu\in\mathcal{U}_t^{\bar\varphi}} \mathbb{E}\left[Y^\nu_{t,x,1}(\theta^\nu)\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu)) + \int_t^{\theta^\nu} Y^\nu_{t,x,1}(s) g\left(s, X^\nu_{t,x}(s),\nu_s\right) ds\right]. \qquad (3.9)
\end{aligned}
$$

Remark 3.11 (infinite horizon). Infinite horizon problems can be handled similarly. Following the notation of Remark 3.10, we introduce the infinite horizon stochastic control problem:

$$
V^\infty(t,x) := \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[\int_t^\infty Y^\nu_{t,x,1}(s) g\left(s, X^\nu_{t,x}(s),\nu_s\right) ds\right].
$$

Then, it is immediately seen that V^∞ satisfies the weak DPP (3.8)–(3.9).

4. Dynamic programming for mixed control-stopping problems. In this section, we provide a direct extension of the DPP of Theorem 3.5 to the larger class of mixed control and stopping problems.

In the context of the previous section, we consider a Borel function $f:\mathbb{R}^d\longrightarrow\mathbb{R}$, and we assume $|f|\leq\bar f$ for some continuous function $\bar f$. For $(t,x)\in\mathbf{S}$ we introduce the reward function $\bar J:\mathbf{S}\times\bar{\mathcal{U}}\times\mathcal{T}_{[t,T]}\longrightarrow\mathbb{R}$:

$$
\bar J(t,x;\nu,\tau) := \mathbb{E}\left[f\left(X^\nu_{t,x}(\tau)\right)\right], \qquad (4.1)
$$

which is well defined for every control ν in

$$
\bar{\mathcal{U}} := \left\{\nu\in\mathcal{U}_o:\ \mathbb{E}\left[\sup_{t\leq s\leq T}\bar f(X^\nu_{t,x}(s))\right]<\infty \quad \forall\,(t,x)\in\mathbf{S}\right\}.
$$

The mixed control-stopping problem is defined by

$$
\bar V(t,x) := \sup_{(\nu,\tau)\in\bar{\mathcal{U}}_t\times\mathcal{T}^t_{[t,T]}} \bar J(t,x;\nu,\tau), \qquad (4.2)
$$

where $\bar{\mathcal{U}}_t$ is the subset of elements of $\bar{\mathcal{U}}$ that are $\mathbb{F}^t$-progressively measurable.

The key ingredient for the proof of (4.6) is the following property of the set of stopping times $\mathcal{T}_T$:

$$
\text{For all } \theta,\tau_1\in\mathcal{T}^t_T \text{ and } \tau_2\in\mathcal{T}^t_{[\theta,T]},\ \text{we have } \tau_1\mathbf{1}_{\{\tau_1<\theta\}}+\tau_2\mathbf{1}_{\{\tau_1\geq\theta\}}\in\mathcal{T}^t_T. \qquad (4.3)
$$

In order to extend the result of Theorem 3.5, we shall assume that the following version of A4 holds.

Assumption A4'. For all $(t,x)\in\mathbf{S}$, $(\nu,\tau)\in\bar{\mathcal{U}}_t\times\mathcal{T}^t_{[t,T]}$, and $\theta\in\mathcal{T}^t_{[t,T]}$, we have the following:

a. For $\mathbb{P}$-a.e $\omega\in\Omega$, there exists $(\tilde\nu_\omega,\tilde\tau_\omega)\in\bar{\mathcal{U}}_{\theta(\omega)}\times\mathcal{T}^{\theta(\omega)}_{[\theta(\omega),T]}$ such that

$$
\mathbf{1}_{\{\tau\geq\theta\}}(\omega)\mathbb{E}\left[f\left(X^\nu_{t,x}(\tau)\right)|\mathcal{F}_\theta\right](\omega) \leq \mathbf{1}_{\{\tau\geq\theta\}}(\omega) J\left(\theta(\omega), X^\nu_{t,x}(\theta)(\omega);\tilde\nu_\omega,\tilde\tau_\omega\right).
$$

b. For $t \le s \le T$, $\theta \in \mathcal{T}^t_{[t,s]}$, $(\tilde\nu, \tilde\tau) \in \bar{\mathcal{U}}_s \times \mathcal{T}^s_{[s,T]}$, $\bar\tau := \tau \mathbf{1}_{\{\tau<\theta\}} + \tilde\tau \mathbf{1}_{\{\tau\ge\theta\}}$, and $\bar\nu := \nu \mathbf{1}_{[0,\theta]} + \tilde\nu \mathbf{1}_{(\theta,T]}$, we have for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\mathbf{1}_{\{\tau\ge\theta\}}(\omega)\mathbb{E}\left[f\left(X^{\bar\nu}_{t,x}(\bar\tau)\right)|\mathcal{F}_\theta\right](\omega) = \mathbf{1}_{\{\tau\ge\theta\}}(\omega) J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde\nu, \tilde\tau).$$

THEOREM 4.1. *Let Assumptions* A1, A2, A3, *and* A4' *hold true. Then, for every* $(t,x) \in \mathbf{S}$, *and for all family of stopping times* $\{\theta^\nu,\ \nu \in \bar{\mathcal{U}}_t\} \subset \mathcal{T}^t_{[t,T]}$,

$$\bar V(t,x) \le \sup_{(\nu,\tau)\in\bar{\mathcal{U}}_t\times\mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta^\nu\}} f(X^\nu_{t,x}(\tau)) + \mathbf{1}_{\{\tau\ge\theta^\nu\}} \bar V^*(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right]. \tag{4.4}$$

Assume further that the map $(t,x) \longmapsto \bar J(t,x;\nu,\tau)$ *satisfies the following lower-semicontinuity property:*

$$\liminf_{t'\uparrow t, x'\to x} \bar J(t',x';\nu,\tau) \ge \bar J(t,x;\nu,\tau) \quad \textit{for every } (t,x)\in\mathbf{S} \textit{ and } (\nu,\tau)\in\bar{\mathcal{U}}\times\mathcal{T}. \tag{4.5}$$

Then, for any function $\varphi \in \mathrm{USC}(\mathbf{S})$ *with* $\bar V \ge \varphi$,

$$\bar V(t,x) \ge \sup_{(\nu,\tau)\in\bar{\mathcal{U}}^\varphi_t\times\mathcal{T}^t_{[t,T]}} \mathbb{E}\left[\mathbf{1}_{\{\tau<\theta^\nu\}} f(X^\nu_{t,x}(\tau)) + \mathbf{1}_{\{\tau\ge\theta^\nu\}} \varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right], \tag{4.6}$$

where $\bar{\mathcal{U}}^\varphi_t = \left\{\nu\in\bar{\mathcal{U}}_t : \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))^+\right] < \infty \textit{ or } \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))^-\right] < \infty\right\}$.

For simplicity, we provide the proof of Theorem 4.1 only for optimal stopping problems, i.e., in the case where $\bar{\mathcal{U}}$ is reduced to a singleton. The DPP for mixed control-stopping problems is easily proved by combining the arguments below with those of the proof of Theorem 3.5.

Proof (for optimal stopping problems). We omit the control ν from all notation, thus simply writing $X_{t,x}(\cdot)$ and $\bar J(t,x;\tau)$. Inequality (4.4) follows immediately from the tower property together with Assumptions A4'a; recall that $\bar J \le \bar V^*$.

We next prove (4.6). Arguing as in step 2 of the proof of Theorem 3.5, we first observe that, for every $\varepsilon > 0$, we can find a countable family $\bar A_i \subset (t_i - r_i, t_i] \times A_i \subset \mathbf{S}$, together with a sequence of stopping times $\tau^{i,\varepsilon}$ in $\mathcal{T}^{t_i}_{[t_i,T]}$, $i \ge 1$, satisfying $\bar A_0 = \{T\}\times\mathbb{R}^d$ and

$$\bigcup_{i\ge0} \bar A_i = \mathbf{S}, \quad \bar A_i \cap \bar A_j = \emptyset \quad \text{for } i \ne j \in \mathbb{N}, \tag{4.7}$$
$$\text{and } \bar J(\cdot;\tau^{i,\varepsilon}) \ge \varphi - 3\varepsilon \quad \text{on } \bar A_i \text{ for } i \ge 1.$$

Set $\bar A^n := \cup_{i\le n} \bar A_i$, $n \ge 1$. Given two stopping times $\theta, \tau \in \mathcal{T}^t_{[t,T]}$, it follows from (4.3) (and Assumption A1 in the general mixed control case) that

$$\tau^{n,\varepsilon} := \tau \mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\ge\theta\}} \left(T \mathbf{1}_{(\bar A^n)^c}(\theta, X_{t,x}(\theta)) + \sum_{i=1}^n \tau^{i,\varepsilon} \mathbf{1}_{\bar A_i}(\theta, X_{t,x}(\theta))\right)$$

is a stopping time in $\mathcal{T}^t_{[t,T]}$. We then deduce from the tower property together with Assumption A4'b and (4.7) that

$$\begin{aligned}
\bar V(t,x) &\ge \bar J(t,x;\tau^{n,\varepsilon}) \\
&\ge \mathbb{E}\left[f\left(X^\nu_{t,x}(\tau)\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\ge\theta\}}\left(\varphi(\theta, X_{t,x}(\theta)) - 3\varepsilon\right)\mathbf{1}_{\bar A^n}(\theta, X_{t,x}(\theta))\right] \\
&\quad + \mathbb{E}\left[\mathbf{1}_{\{\tau\ge\theta\}} f(X_{t,x}(T)) \mathbf{1}_{(\bar A^n)^c}(\theta, X_{t,x}(\theta))\right].
\end{aligned}$$

By sending $n \to \infty$ and arguing as in the end of the proof of Theorem 3.5, we deduce that

$$\bar{V}(t,x) \geq \mathbb{E}\left[f\left(X_{t,x}(\tau)\right)\mathbf{1}_{\{\tau<\theta\}} + \mathbf{1}_{\{\tau\geq\theta\}}\varphi(\theta, X_{t,x}(\theta))\right] - 3\varepsilon,$$

and the result follows from the arbitrariness of $\varepsilon > 0$ and $\tau \in \mathcal{T}^t_{[t,T]}$. ◻

5. Application to controlled Markov jump-diffusions. In this section, we show how the weak DPP of Theorem 3.5 allows us to derive the corresponding dynamic programming equation in the sense of viscosity solutions. We refer the reader to Crandall, Ishii, and Lions [5] and Fleming and Soner [9] for a presentation of the general theory of viscosity solutions.

For simplicity, we specialize the discussion to the context of controlled Markov jump-diffusions driven by a Brownian motion and a compound Poisson process. The same technology can be adapted to optimal stopping and impulse control or mixed problems; see, e.g., [4].

5.1. Problem formulation and verification of Assumption A. We shall work on the product space $\Omega := \Omega_W \times \Omega_N$, where Ω_W is the set of continuous functions from $[0,T]$ into $\mathbb{R}^d$, and Ω_N is the set of integer-valued measures on $[0,T] \times E$ with $E := \mathbb{R}^m$ for some $m \geq 1$. For $\omega = (\omega^1, \omega^2) \in \Omega$, we set $W(\omega) = \omega^1$ and $N(\omega) = \omega^2$ and define $\mathbb{F}^W = (\mathcal{F}^W_t)_{t\leq T}$ (resp., $\mathbb{F}^N = (\mathcal{F}^N_t)_{t\leq T}$) as the smallest right-continuous filtration on Ω_W (resp., Ω_N) such that W (resp., N) is optional. We let $\mathbb{P}_W$ be the Wiener measure on $(\Omega_W, \mathcal{F}^W_T)$ and $\mathbb{P}_N$ be the measure on $(\Omega_N, \mathcal{F}^N_T)$ under which N is a compound Poisson measure with intensity $\tilde{N}(de,dt) = \lambda(de)dt$, for some finite measure λ on E, endowed with its Borel tribe $\mathcal{E}$. We then define the probability measure $\mathbb{P} := \mathbb{P}_W \otimes \mathbb{P}_N$ on $(\Omega, \mathcal{F}^W_T \otimes \mathcal{F}^N_T)$. With this construction, W and N are independent under $\mathbb{P}$. Without loss of generality, we can assume that the natural right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$ induced by (W,N) is complete. In the following, we shall slightly abuse notation and sometimes write $N_t(\cdot)$ for $N(\cdot,(0,t])$ for simplicity.

We let U be a closed subset of $\mathbb{R}^k$, $k \geq 1$, let $\mu : \mathbf{S} \times U \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbf{S} \times U \longrightarrow \mathbb{M}^d$ be two Lipschitz continuous functions, and let $\beta : \mathbf{S} \times U \times E \longrightarrow \mathbb{R}^d$ be a measurable function, Lipschitz continuous with linear growth in (t,x,u) uniformly in $e \in E$. Here $\mathbb{M}^d$ denotes the set of d-dimensional square matrices.

By $\mathcal{U}_o$, we denote the collection of all square integrable predictable processes with values U. For every $\nu \in \mathcal{U}_o$, the stochastic differential equation

$$dX(r) = \mu\left(r, X(r), \nu_r\right)dr + \sigma\left(r, X(r), \nu_r\right)dW_r + \int_E \beta(r, X(r-), \nu_r, e)N(de,dr),$$

$$t \leq r \leq T, \tag{5.1}$$

has a unique strong solution $X^\nu_{\tau,\xi}$ such that $X^\nu_{\tau,\xi}(\tau) = \xi$ for any initial condition $(\tau,\xi) \in \mathcal{S} := \left\{(\tau,\xi) \in \mathcal{S}_o : \xi \text{ is } \mathcal{F}_\tau\text{-measurable, and } \mathbb{E}\left[|\xi|^2\right] < \infty\right\}$. Moreover, this solution satisfies

$$\mathbb{E}\left[\sup_{\tau\leq r\leq T}|X^\nu_{\tau,\xi}(r)|^2\right] < C(1 + \mathbb{E}\left[|\xi|^2\right]) \tag{5.2}$$

for some constant C which may depend on ν.

Remark 5.1. Clearly, less restrictive conditions could be imposed on β and N. We deliberately restrict ourselves here to this simple case, in order to avoid standard technicalities related to the definition of viscosity solutions for integro-differential operators; see, e.g., [1] and the references therein.

The following remark shows that in the present case, it is not necessary to restrict the control processes ν to $\mathcal{U}_t$ in the definition of the value function $V(t,x)$.

Remark 5.2. Let $\tilde V$ be defined by

$$\tilde V(t,x) := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right].$$

The difference between $\tilde V(t,\cdot)$ and $V(t,\cdot)$ comes from the fact that all controls in $\mathcal{U}$ are considered in the former, while we restrict ourselves to controls independent of $\mathcal{F}_t$ in the latter. We claim that

$$\tilde V = V,$$

so that both problems are indeed equivalent. Clearly, $\tilde V \geq V$. To see that the converse holds true, fix $(t,x)\in[0,T)\times\mathbb{R}^d$ and $\nu\in\mathcal{U}$. Then, ν can be written as a measurable function of the canonical process $\nu((\omega_s)_{0\leq s\leq t},(\omega_s-\omega_t)_{t\leq s\leq T})$, where, for fixed $(\omega_s)_{0\leq s\leq t}$, the map $\nu_{(\omega_s)_{0\leq s\leq t}}:(\omega_s-\omega_t)_{t\leq s\leq T}\mapsto \nu((\omega_s)_{0\leq s\leq t},(\omega_s-\omega_t)_{t\leq s\leq T})$ can be viewed as a control independent of $\mathcal{F}_t$. Using the independence of the increments of the Brownian motion and the compound Poisson process, and Fubini's lemma, it thus follows that

$$J(t,x;\nu) = \int \mathbb{E}\left[f(X^{\nu_{(\omega_s)_{0\leq s\leq t}}}_{t,x}(T))\right] d\mathbb{P}((\omega_s)_{0\leq s\leq t}) \leq \int V(t,x)d\mathbb{P}((\omega_s)_{0\leq s\leq t}),$$

where the latter equals $V(t,x)$. By arbitrariness of $\nu\in\mathcal{U}$, this implies that $\tilde V(t,x)\leq V(t,x)$.

Remark 5.3. By the previous remark, it follows that the value function V inherits the lower semicontinuity of the performance criterion required in the second part of Theorem 3.5; compare with Remark 3.7. This simplification is specific to the simple stochastic control problem considered in this section and may not hold in other control problems; see, e.g., [4]. Consequently, we shall deliberately ignore the lower semicontinuity of V in the subsequent analysis in order to show how to derive the dynamic programming equation in a general setting.

Let $f:\mathbb{R}^d\longrightarrow\mathbb{R}$ be a lower-semicontinuous function with linear growth, and define the performance criterion J by (2.1). Then, it follows that $\mathcal{U}=\mathcal{U}_o$ and, from (5.2) and the almost sure continuity of $(t,x)\mapsto X^{\nu}_{t,x}(T)$, that $J(.,\nu)$ is lower semicontinuous, as required in the second part of Theorem 3.5.

The value function V is defined by (2.3). Various types of conditions can be formulated in order to guarantee that V is locally bounded. For instance, if f is bounded from above, this condition is satisfied trivially. Alternatively, one may restrict the set U to be bounded, so that the linear growth of f implies corresponding bounds for V. We do not want to impose such a constraint because we would like to highlight the fact that our methodology applies to general singular control problems. We then leave this issue as a condition which is to be checked by arguments specific to the case in hand.

PROPOSITION 5.4. *In the above controlled diffusion context, assume further that V is locally bounded. Then, the value function V satisfies the weak DPP* (3.1)–(3.2).

Proof. Conditions A1, A2, and A3 from Assumption A are obviously satisfied in the present context. It remains to check that A4 holds true. For $\omega\in\Omega$ and $r\geq 0$, we denote $\omega^r_\cdot := \omega_{\cdot\wedge r}$ and $\mathbf{T}_r(\omega)(\cdot) := \omega_{\cdot\vee r} - \omega_r$ so that $\omega_\cdot = \omega^r_\cdot + \mathbf{T}_r(\omega)(\cdot)$. Fix $(t,x) \in \mathbf{S}$, $\nu \in \mathcal{U}_t$, $\theta \in \mathcal{T}^t_{[t,T]}$, and observe that, by the flow property,

$$\begin{aligned}
\mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X^{\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\omega))}_{\theta(\omega),X^\nu_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\omega))\right) d\mathbb{P}(\mathbf{T}_{\theta(\omega)}(\omega))\\
&= \int f\left(X^{\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde\omega))}_{\theta(\omega),X^\nu_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde\omega))\right) d\mathbb{P}(\tilde\omega)\\
&= J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde\nu_\omega),
\end{aligned}$$

where $\tilde\nu_\omega(\tilde\omega) := \nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde\omega))$ is an element of $\mathcal{U}_{\theta(\omega)}$. This already proves A4a. As for A4b, note that if $\bar\nu := \nu\mathbf{1}_{[0,\theta]} + \tilde\nu\mathbf{1}_{(\theta,T]}$ with $\tilde\nu \in \mathcal{U}_s$ and $\theta \in \mathcal{T}^t_{[t,s]}$, then the same computations imply

$$\mathbb{E}\left[f\left(X^{\bar\nu}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = \int f\left(X^{\tilde\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde\omega))}_{\theta(\omega),X^\nu_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde\omega))\right) d\mathbb{P}(\tilde\omega),$$

where we used the flow property together with the fact that $X^\nu_{t,x} = X^{\bar\nu}_{t,x}$ on $[t,\theta]$ and that the dynamics of $X^{\bar\nu}_{t,x}$ depends only on $\tilde\nu$ after θ. Now observe that $\tilde\nu$ is independent of $\mathcal{F}_s$ and therefore of $\omega^{\theta(\omega)}$ since $\theta \le s$ $\mathbb{P}$-a.s. It follows that

$$\begin{aligned}
\mathbb{E}\left[f\left(X^{\bar\nu}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X^{\tilde\nu(\mathbf{T}_s(\tilde\omega))}_{\theta(\omega),X^\nu_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde\omega))\right) d\mathbb{P}(\tilde\omega)\\
&= J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde\nu)\ . \qquad \square
\end{aligned}$$

Remark 5.5. It can be similarly proved that A4' holds true in the context of mixed control-stopping problems.

5.2. PDE derivation. We can now show how our weak formulation of the DPP allows us to characterize the value function as a discontinuous viscosity solution of a suitable Hamilton–Jacobi–Bellman equation.

Let C^0 denote the set of continuous maps on $[0,T]\times\mathbb{R}^d$ endowed with the topology of uniform convergence on compact sets. With $(t,x,p,A,\varphi) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{M}^d\times C^0$ we associate the Hamiltonian of the control problem:

$$H(t,x,p,A,\varphi) \ := \ \inf_{u\in U} H^u(t,x,p,A,\varphi),$$

where, for $u \in U$,

$$\begin{aligned}
H^u(t,x,p,A,\varphi) := &-\langle\mu(t,x,u),p\rangle - \frac{1}{2}\mathrm{Tr}\left[(\sigma\sigma')(t,x,u)A\right]\\
&- \int_E \left(\varphi(t,x+\beta(t,x,u,e)) - \varphi(t,x)\right)\lambda(de),
\end{aligned}$$

and σ' is the transpose of the matrix σ.

Notice that the operator H is upper semicontinuous as an infimum over a family of continuous maps (note that β is locally bounded uniformly with respect to its last argument and that λ is finite, by assumption). However, since the set U may be unbounded, it may fail to be continuous. We therefore introduce the corresponding lower-semicontinuous envelope:

$$H_*(z) \ := \ \liminf_{z'\to z} H(z') \quad \text{ for } z = (t,x,p,A,\varphi) \in \mathbf{S}\times\mathbb{R}^d\times\mathbb{M}^d\times C^0.$$

COROLLARY 5.6. *Assume that V is locally bounded. Then the following hold:*
(i) V^* *is a viscosity subsolution of*

$$-\partial_t V^* + H_*(., DV^*, D^2V^*, V^*) \;\leq\; 0 \quad \text{ on } [0,T) \times \mathbb{R}^d.$$

(ii) V_* *is a viscosity supersolution of*

$$-\partial_t V_* + H(., DV_*, D^2V_*, V_*) \;\geq\; 0 \quad \text{ on } [0,T) \times \mathbb{R}^d.$$

Proof. 1. We start with the supersolution property. Assume to the contrary that there is $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$ together with a smooth function $\varphi : [0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying

$$0 = (V_* - \varphi)(t_0, x_0) < (V_* - \varphi)(t, x) \quad \forall\, (t,x) \in [0,T) \times \mathbb{R}^d, \quad (t,x) \neq (t_0, x_0),$$

such that

$$\left(-\partial_t \varphi + H(., D\varphi, D^2\varphi, \varphi)\right)(t_0, x_0) < 0. \tag{5.3}$$

For $\varepsilon > 0$, let ϕ be defined by

$$\phi(t,x) := \varphi(t,x) - \varepsilon(|t - t_0|^2 + |x - x_0|^4),$$

and note that ϕ converges uniformly on compact sets to φ as $\varepsilon \to 0$. Since H is upper semicontinuous and $(\phi, \partial_t\phi, D\phi, D^2\phi)(t_0, x_0) = (\varphi, \partial_t\varphi, D\varphi, D^2\varphi)(t_0, x_0)$, we can choose $\varepsilon > 0$ small enough so that there exist $u \in U$ and $r > 0$, with $t_0 + r < T$, satisfying

$$\left(-\partial_t \phi + H^u(., D\phi, D^2\phi, \phi)\right)(t,x) \;<\; 0 \quad \forall\, (t,x) \in B_r(t_0, x_0), \tag{5.4}$$

where we recall that $B_r(t_0, x_0)$ denotes the ball of radius r and center (t_0, x_0). Let $(t_n, x_n)_n$ be a sequence in $B_r(t_0, x_0)$ such that $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V_*(t_0, x_0))$, let $X^n_\cdot := X^u_{t_n, x_n}(\cdot)$ denote the solution of (5.1) with constant control $\nu = u$ and initial condition $X^n_{t_n} = x_n$, and consider the stopping time

$$\theta_n := \inf\left\{s \geq t_n \;:\; (s, X^n_s) \notin B_r(t_0, x_0)\right\}.$$

Note that $\theta_n < T$ since $t_0 + r < T$. Applying Itô's formula to $\phi(\cdot, X^n)$ and using (5.4) and (5.2), we see that

$$\begin{aligned}\phi(t_n, x_n) &= \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \left[\partial_t\phi - H^u(., D\phi, D^2\phi, \phi)\right](s, X^n_s)ds\right] \\ &\leq \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n})\right].\end{aligned}$$

Now observe that $\varphi \geq \phi + \eta$ on $([0,T] \times \mathbb{R}^d) \setminus B_r(t_0, x_0)$ for some $\eta > 0$. Hence, the above inequality implies that $\phi(t_n, x_n) \leq \mathbb{E}\left[\varphi(\theta_n, X^n_{\theta_n})\right] - \eta$. Since $(\phi - V)(t_n, x_n) \to 0$, we can then find n large enough so that

$$V(t_n, x_n) \;\leq\; \mathbb{E}\left[\varphi(\theta_n, X^n_{\theta_n})\right] - \eta/2 \quad \text{for sufficiently large } n \geq 1.$$

On the other hand, it follows from (3.2) that

$$V(t_n, x_n) \;\geq\; \sup_{\nu \in \mathcal{U}_{t_n}} \mathbb{E}\left[\varphi(\theta_n, X^\nu_{t_n, x_n}(\theta_n))\right] \geq \mathbb{E}\left[\varphi(\theta_n, X^n_{\theta_n})\right],$$

which is the required contradiction.

2. We now prove the subsolution property. Assume to the contrary that there is $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$ together with a smooth function $\varphi : [0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying

$$0 = (V^* - \varphi)(t_0, x_0) > (V^* - \varphi)(t, x) \quad \forall\, (t,x) \in [0,T) \times \mathbb{R}^d, \quad (t,x) \neq (t_0, x_0),$$

such that

$$\left(-\partial_t \varphi + H_*(., D\varphi, D^2\varphi, \varphi)\right)(t_0, x_0) > 0. \tag{5.5}$$

For $\varepsilon > 0$, let ϕ be defined by

$$\phi(t,x) := \varphi(t,x) + \varepsilon(|t - t_0|^2 + |x - x_0|^4),$$

and note that ϕ converges uniformly on compact sets to φ as $\varepsilon \to 0$. By the lower semicontinuity of H_*, we can then find $\varepsilon, r > 0$ such that $t_0 + r < T$ and

$$\left(-\partial_t \phi + H^u(., D\phi, D^2\phi, \phi)\right)(t,x) \;>\; 0 \quad \text{for every } u \in U \text{ and } (t,x) \in B_r(t_0, x_0). \tag{5.6}$$

Since (t_0, x_0) is a strict maximizer of the difference $V^* - \phi$, it follows that

$$\sup_{([0,T]\times\mathbb{R}^d)\setminus B_r(t_0,x_0)} (V^* - \phi) \leq -2\eta \quad \text{ for some } \ \eta > 0. \tag{5.7}$$

Let $(t_n, x_n)_n$ be a sequence in $B_r(t_0, x_0)$ so that $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V^*(t_0, x_0))$. For an arbitrary control $\nu^n \in \mathcal{U}_{t_n}$, let $X^n := X^{\nu^n}_{t_n, x_n}$ denote the solution of (5.1) with initial condition $X^n_{t_n} = x_n$, and set

$$\theta_n := \inf\left\{s \geq t_n \ : \ (s, X^n_s) \notin B_r(t_0, x_0)\right\}.$$

Notice that $\theta_n < T$ as a consequence of the fact that $t_0 + r < T$. We may assume without loss of generality that

$$|(V - \phi)(t_n, x_n)| \;\leq\; \eta \quad \forall\, n \geq 1. \tag{5.8}$$

Applying Itô's formula to $\phi(\cdot, X^n)$ and using (5.6) leads to

$$\begin{aligned}
\phi(t_n, x_n) &= \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \left[\partial_t \phi - H^{\nu^n}(., D\phi, D^2\phi, \phi)\right](s, X^n_s)ds\right] \\
&\geq \; \mathbb{E}\left[\phi(\theta_n, X^n_{\theta_n})\right].
\end{aligned}$$

In view of (5.7), the above inequality implies that $\phi(t_n, x_n) \geq \mathbb{E}\left[V^*(\theta_n, X^n_{\theta_n})\right] + 2\eta$, which implies by (5.8) that

$$V(t_n, x_n) \;\geq\; \mathbb{E}\left[V^*(\theta_n, X^n_{\theta_n})\right] + \eta \quad \text{for } n \geq 1.$$

Since $\nu^n \in \mathcal{U}_{t_n}$ is arbitrary, this contradicts (3.1) for $n \geq 1$ fixed. ∎

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