Weak Dynamic Programming for Viscosity Solutions

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Joint work with Nizar Touzi, CMAP, Ecole Polytechnique

Motivations

- Consider the control problem in standard form

$$V(t,x) \quad := \quad \sup_{\nu \in \mathcal{U}} J(t,x;\nu)\ , \quad J(t,x;\nu) := \mathbb{E}\left[f(X_T^\nu) | X_t^\nu = x\right]$$

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- Our aim is to provide a <u>weak version</u>, much <u>easier to prove</u>.

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- Controls :

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- Controlled process :

$$(\tau, \xi; \nu) \in \mathcal{S} \times \mathcal{U}_0 \longmapsto X^{\nu}_{\tau,\xi} \in \mathbb{H}^0_{\mathrm{rcll}}(\mathbb{R}^d)$$

with $[0, T] \times \mathbb{R}^d \subset \mathcal{S} \subset \left\{(\tau, \xi) : \tau \in \mathcal{T}_{[0,T]} \text{ and } \xi \in \mathbb{L}^0_\tau(\mathbb{R}^d)\right\}$.

Reward and Value functions

- Reward function

$$J(t,x;\nu) \;:=\; \mathbb{E}\left[f\left(X^{\nu}_{t,x}(T)\right)\right]$$

defined for controls ν in

$$\mathcal{U} \;:=\; \left\{\nu \in \mathcal{U}_0 : \; \mathbb{E}\left[\left|f(X^{\nu}_{t,x}(T))\right|\right] < \infty \;\forall\,(t,x) \in [0,T] \times \mathbb{R}^d\right\}.$$

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- Admissibility : a control $\nu \in \mathcal{U}$ is *t-admissible* if it is independent of $\mathcal{F}_t$. We denote by $\mathcal{U}_t$ the collection of such processes.

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$$V(t,x) \; := \; \sup_{\nu \in \mathcal{U}_t} J(t,x;\nu) \quad \text{for} \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Assumptions

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A3 (Stability under concatenation) $\forall\ \tilde{\nu} \in \mathcal{U}_t,\ \theta \in \mathcal{T}^t_{[t,T]}$:
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b. $\forall\ t \leq s \leq T$, $\theta \in \mathcal{T}^t_{[t,s]}$, $\tilde{\nu} \in \mathcal{U}_s$, and $\bar{\nu} := \nu \mathbf{1}_{[0,\theta]} + \tilde{\nu} \mathbf{1}_{(\theta,T]}$:

$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu}) \text{ for } \mathbb{P}-\text{a.e. } \omega \in \Omega.$$

The case where $J(\cdot;\nu)$ is l.s.c. and V is continuous

- Aim : Prove the DPP for $\tau \in \mathcal{T}^t_{[t,T]}$ (independent on $\mathcal{F}_t$)

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on $(t_i - r_i, t_i] \times B_{r_i}(x_i)$

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Given $\nu \in \mathcal{U}_t$, define

$$\nu^\varepsilon \quad := \quad \mathbf{1}_{[t,\tau]}\nu + \mathbf{1}_{(\tau,T]}\sum_{i\geq 1}\mathbf{1}_{A_i}(\tau, X^\nu_{t,x}(\tau))\nu^i \ .$$

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Proof : Then,

$$\mathbb{E}\left[f\left(X_{t,x}^{\nu^\varepsilon}(T)\right)|\mathcal{F}_\tau\right] = \sum_{i\geq 1} J(\tau, X_{t,x}^{\nu}(\tau);\nu^i)\mathbf{1}_{A_i}\left(\tau, X_{t,x}^{\nu}(\tau)\right)$$

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&\geq \sum_{i\geq 1}\left(V(\tau, X_{t,x}^{\nu}(\tau)) - 3\varepsilon\right)\mathbf{1}_{A_i}\left(\tau, X_{t,x}^{\nu}(\tau)\right)
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- The lower-semicontinuity of $J(\cdot;\nu)$ is very important in this proof : It is in general not difficult to obtain.
- The upper-semicontinuity of V is also very important : It is much more difficult to obtain, especially when controls are not uniformly bounded (singular control).

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To derive the PDE in the viscosity sense, try to obtain :

$$V(t,x) \geq \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V(\tau, X^{\nu}_{t,x}(\tau))\right]$$

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for all smooth function such that (t,x) achieves a minimum of $V - \varphi$.

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φ being smooth it should be much easier to prove ! !

The weak DPP

Assume that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\nu \in \mathcal{U}_t$

$$\liminf_{(t',x') \to (t,x),\ t' \leq t} J(t',x';\nu) \geq J(t,x;\nu).$$

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$$\liminf_{(t',x')\to(t,x),\ t'\le t} J(t',x';\nu) \ge J(t,x;\nu).$$

Theorem : Fix $\{\theta^\nu,\ \nu \in \mathcal{U}_t\} \subset \mathcal{T}^t_{[t,T]}$ a family of stopping times. Then, for any upper-semicontinuous function φ such that $V \ge \varphi$ on $[t,T] \times \mathbb{R}^d$, we have

$$V(t,x) \ \ge \ \sup_{\nu \in \mathcal{U}_t^\varphi} \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right],$$

where $\mathcal{U}_t^\varphi =$
$\left\{\nu \in \mathcal{U}_t :\ \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))^+\right] < \infty \text{ or } \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))^-\right] < \infty\right\}$.

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Remark : The minimum can be taken to be local if $\{(\theta^\nu, X^\nu_{t,x}(\theta^\nu)),\ \nu \in \mathcal{U}_t\}$ is bounded in $\mathbb{L}^\infty$. In practice φ is taken to be $C^{1,2}$.

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Given $\nu \in \mathcal{U}_t$, define

$$\nu^\varepsilon \quad := \quad \mathbf{1}_{[t,\theta^\nu]}\nu + \mathbf{1}_{(\theta^\nu,T]} \sum_{i \geq 1} \mathbf{1}_{A_i}(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\nu^i \ .$$

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$$\mathbb{E}\left[f\left(X^{\nu^\varepsilon}_{t,x}(T)\right)|\mathcal{F}^\nu_\theta\right] \quad = \quad \sum_{i\geq 1} J(\theta^\nu, X^\nu_{t,x}(\theta^\nu);\nu^i)\mathbf{1}_{A_i}\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)$$

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&\geq \sum_{i\geq 1}\left(\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu)) - 3\varepsilon\right)\mathbf{1}_{A_i}\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)
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Proof. For $i \geq 1$, fix $r_i > 0$ and $\nu^i \in \mathcal{U}_{t_i}$ such that on $A_i := (t_i - r_i, t_i] \times B_i$, disjoint sets that cover $[t, T] \times \mathbb{R}^d$. Given $\nu \in \mathcal{U}_t$, define

$$\nu^\varepsilon \quad := \quad \mathbf{1}_{[t,\theta^\nu]}\nu + \mathbf{1}_{(\theta^\nu,T]} \sum_{i \geq 1} \mathbf{1}_{A_i}(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\nu^i \ .$$

and

$$\begin{aligned} V(t,x) \quad &\geq \quad J(t,x;\nu^\varepsilon) \\ &= \quad \mathbb{E}\left[\mathbb{E}\left[f\left(X^{\nu^\varepsilon}_{t,x}(T)\right)|\mathcal{F}^\nu_\theta\right]\right] \\ &\geq \quad \mathbb{E}\left[\varphi\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)\right] - 3\varepsilon \ . \end{aligned}$$

The weak DPP

Using test functions makes the proof straightforward :

$$\sup_{\nu \in \mathcal{U}_t^{\varphi}} \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] \leq V(t,x)$$

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Remark : If $\{X_{t,x}^\nu(\theta^\nu),\ \nu \in \mathcal{U}_t\}$ is bounded in $\mathbb{L}^\infty$, one can approximate V_* from below by smooth functions and obtain :

$$\sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V_*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] \leq V(t,x) \leq \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V^*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right]$$

Example : Framework

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- $\mathcal{U}$ = square integrable progressively measurable processes with values in $U \subset \mathbb{R}^d$
- f is l.s.c with f^- with linear growth, μ and σ Lipschitz continuous.

Example : Verification of the assumptions

For all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\nu \in \mathcal{U}_t$:

L.s.c. $(t', x') \to (t, x) \Rightarrow X^{\nu}_{t',x'}(T) \to X^{\nu}_{t,x}(T)$ in $\mathbb{L}^2$

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A3 (Stability under concatenation) $\forall\ \tilde{\nu} \in \mathcal{U}_t,\ \theta \in \mathcal{T}^t_{[t,T]}$:
$\nu \mathbf{1}_{[0,\theta]} + \tilde{\nu} \mathbf{1}_{(\theta,T]} \in \mathcal{U}_t$.

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a. For $\mathbb{P}$-a.e $\omega \in \Omega$, $\exists\ \tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ s.t.

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Proof. Canonical space : $W(\omega) = \omega$. Set $\mathbf{T}_s(\omega) := (\omega_r - \omega_s)_{r \ge s}$ and $\omega^s := (\omega_{r\wedge s})_{r\ge 0}$.

$$\begin{aligned}
\mathbb{E}\left[f\left(X^\nu_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X^{\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\omega))}_{\theta(\omega),X^\nu_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\omega))\right) d\mathbb{P}(\mathbf{T}_{\theta(\omega)}(\omega)) \\
&= \int f\left(X^{\nu(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))}_{\theta(\omega),X^\nu_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega}) \\
&= J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}_\omega)
\end{aligned}$$

where, $\tilde{\nu}_\omega(\tilde{\omega}) := \nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega})) \in \mathcal{U}_{\theta(\omega)}$.

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$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu}) \text{ for } \mathbb{P}-\text{a.e. } \omega \in \Omega.$$

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Proof.

$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = \int f\left(X^{\tilde{\nu}(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))}_{\theta(\omega),X^{\nu}_{\mathbf{t},\mathbf{x}}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega}),$$

and therefore

$$\begin{aligned}\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X^{\tilde{\nu}(\mathbf{T}_{\mathbf{s}}(\tilde{\omega}))}_{\theta(\omega),X^{\nu}_{\mathbf{t},\mathbf{x}}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega}) \\ &= J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu})\ .\end{aligned}$$

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$$\inf_{u \in U} (-\mathcal{L}^u V_*) \geq 0$$

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- Set $\theta_n := \inf \left\{ s \geq t_n \ : \ (s, X^u_{t_n, x_n}(s)) \notin B_r \right\}$ so that

$$V(t_n, x_n) + \iota_n = \tilde{\varphi}(t_n, x_n) \quad \leq \quad \mathbb{E}\left[\tilde{\varphi}(\theta_n, X^u_{t_n, x_n}(\theta_n)) \right]$$

for some $\iota_n \longrightarrow 0$

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$$
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&\leq \ \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4
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- Set $\tilde{\varphi}(t, x) := \varphi(t, x) - |t - t_0|^4 - |x - x_0|^4$.
- Then, $-\mathcal{L}^u \tilde{\varphi}(t, x) \ \leq \ 0$ on $B_r := \{|t - t_0| \leq r, \ |x - x_0| \leq r\}$.
- Fix $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V_*(t_0, x_0))$.
- Set $\theta_n := \inf \left\{ s \geq t_n \ : \ (s, X^u_{t_n, x_n}(s)) \notin B_r \right\}$ so that

$$\begin{aligned} V(t_n, x_n) + \iota_n = \tilde{\varphi}(t_n, x_n) \ &\leq \ \mathbb{E}\left[\tilde{\varphi}(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] \\ &\leq \ \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4 \end{aligned}$$

for some $\iota_n \to 0$, i.e. $V(t_n, x_n) \leq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4/2$.

Example : Super-solution property

- $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ s.t. $0 = (V_* - \varphi)(t_0, x_0) = \min(V_* - \varphi)$
- Assume that $-\mathcal{L}^u \varphi(t_0, x_0) < 0$, for some $u \in U$.
- Set $\tilde{\varphi}(t, x) := \varphi(t, x) - |t - t_0|^4 - |x - x_0|^4$.
- Then, $-\mathcal{L}^u \tilde{\varphi}(t, x) \ \leq \ 0$ on $B_r := \{|t - t_0| \leq r, \ |x - x_0| \leq r\}$.
- Fix $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V_*(t_0, x_0))$.
- Set $\theta_n := \inf \left\{ s \geq t_n \ : \ (s, X^u_{t_n, x_n}(s)) \notin B_r \right\}$ so that

$$
\begin{aligned}
V(t_n, x_n) + \iota_n = \tilde{\varphi}(t_n, x_n) &\leq \mathbb{E}\left[\tilde{\varphi}(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] \\
&\leq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4
\end{aligned}
$$

  for some $\iota_n \to 0$, i.e. $V(t_n, x_n) \leq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4/2$.
- While $V(t_n, x_n) \geq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right]$ by the weak DPP.

Extensions

- Optimal control with running gain

Extensions

- Optimal control with running gain
- Optimal stopping

Extensions

- Optimal control with running gain
- Optimal stopping
- Mixed optimal control/stopping, impulse control, ...