Duality for almost-sure hedging with price impact

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Based on works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich) and Y. Zou (ex Dauphine-PSL) + more recent developments with P. Cardialaguet (Dauphine-PSL) and X. Tan (Dauphine-PSL)

Problem formulation and motivation

Motivation

Construct market models with permanent price impact (possibly with resilience effect) in which hedging is possible :

- Buying pushes up the price, selling pushes it down.
- We pay an illiquidity cost.
- Solve the "running after the delta" effect.
- Avoid hidden transaction costs (fixed or proportional).
- Not at the level of high-frequency level $\rightarrow$ mesoscopic model.

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We will focus on the case of covered options :

- The "premium" is paid at 0 in cash plus delta (number of stocks) asked by the trader.
- The trader delivers at T cash and stocks (evaluated at their current price).

$\Rightarrow$ Avoids jumps at 0 and T, and therefore important impacts on the stock price.

Example

Linear impact rule and covered options : buying Δ_t stocks leads to

- a permanent price move of $X_{t-} \to X_t = X_{t-} + f_t(X_{t-})\Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t-} + X_t)$.

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Consider rebalancing at times t_i^n :

$$X^n = X_0 + \int_0^\cdot \sigma^\circ(X_t^n)dW_t + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} f(X^n_{t_i^n -})\Delta^n_{t_i^n},$$

$$Y^n := \sum_{i=0}^{n-1} \mathbf{1}_{[t_i^n, t_{i+1}^n)} \left(\int_0^\cdot \mathfrak{g}_t dX_t^n + \int_0^\cdot b_t dt \right) , \ \Delta^n_{t_i^n} = Y^n_{t_i^n} - Y^n_{t_{i-1}^n}$$

$$V^n = V_0 + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \frac{1}{2}(\Delta^n_{t_i^n})^2 f(X^n_{t_i^n -}) + \int_0^\cdot Y^n_{t-} dX_t^n,$$

where

$$V^n = \text{ cash part } + Y^n X^n = \text{“portfolio value”}.$$

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$\Rightarrow$ Let $t_{i+1}^n - t_i^n \to 0$:

$$X = \mathrm{x}_{\wedge 0} + \int_0^\cdot \sigma_t^\circ(X_t)dW_t + \int_0^\cdot f_t(X_t)dY_t + \int_0^\cdot \mathfrak{g}_t(f_t'\sigma_t^\circ)(X_t)dt$$

$$Y = y + \int_0^\cdot \mathfrak{g}_t dX_t + \int_0^\cdot b_t dt$$

$$V = V_0 + \int_0^\cdot \frac{1}{2}\mathfrak{g}_t^2 f_t(X_t)dt + \int_0^\cdot Y_t dX_t.$$

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Note that trading impacts the whole dynamics through the flow of the SDE. It will also impact the claim $\Xi(X)$.

Example

Linear impact rule and resilience

$$X = X_0 + \int_0^{\cdot} \sigma_s^{\circ}(X_s) dW_s + R$$

$$R = R_0 + \int_0^{\cdot} f_s(X_s) dY_s + \int_0^{\cdot} (\mathfrak{g}_s(f_s' \sigma_s^{\circ})(X_s) - \rho R_s) ds$$

$$Y = y + \int_0^{\cdot} \mathfrak{g}_t dX_t + \int_0^{\cdot} b_t dt$$

$$V = V_0 + \int_0^{\cdot} Y_t dX_t + \int_0^{\cdot} \frac{1}{2} \mathfrak{g}_t^2 f_t(X_t) dt.$$

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Linear impact rule and resilience

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X &= X_0 + \int_0^{\cdot} \sigma_s^{\circ}(X_s)dW_s + R \\
R &= R_0 + \int_0^{\cdot} f_s(X_s)dY_s + \int_0^{\cdot} (\mathfrak{g}_s(f_s'\sigma_s^{\circ})(X_s) - \rho R_s)ds \\
Y &= y + \int_0^{\cdot} \mathfrak{g}_t dX_t + \int_0^{\cdot} b_t dt \\
V &= V_0 + \int_0^{\cdot} Y_t dX_t + \int_0^{\cdot} \frac{1}{2}\mathfrak{g}_t^2 f_t(X_t)dt.
\end{aligned}
$$

For covered options, resilience does not play any role... we omit it.

Abstract hedging of covered options

Given $\mathrm{x} \in C([0,T])$, find $y \in \mathbb{R}$ and $(\mathfrak{g}, \mathfrak{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that (after a change of measure)

$$X = \mathrm{x}_{\wedge 0} + \int_0^{\cdot} \sigma_t(X, \mathfrak{g}_t) dW_t$$

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$$V = V_0 + \int_0^{\cdot} Y_t dX_t + \int_0^{\cdot} F_t(X, \mathfrak{g}_t) dt, \quad V_T = \Xi(X)$$

(possibly weak formulation)

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Interpretation :

- X : stock price,
- Y : number of stocks in the portfolio,
- V : cash value of the portfolio (at the current stock price),
- $F(\cdot, \mathfrak{g})$ and $\sigma(\cdot, \mathfrak{g})$: liquidity cost and price impact.

PDE point of view

B. Bouchard, G. Loeper, M. Soner and C. Zhou.
Second order stochastic target problems with generalized market impact.
arxiv.org/pdf/1806.08533.pdf, 2018.

Markovian setting

Given $x \in \mathbb{R}$, find $y \in \mathbb{R}$ and $\phi := (\mathfrak{g}, \mathfrak{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that

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Assume a solution $V = \mathrm{v}(\cdot, X)$ exists, then $dV = d\mathrm{v}(\cdot, X)$ and therefore :

- $Y = \nabla_x \mathrm{v}(\cdot, X_\cdot)$,
- $F(X, \mathfrak{g}) = \partial_t \mathrm{v}(\cdot, X) + \frac{1}{2}\sigma^2(X, \mathfrak{g}) \nabla_{xx} \mathrm{v}(\cdot, X_\cdot)$

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Moreover, $Y = \nabla_x \mathrm{v}(\cdot, X_\cdot)$ implies $dY = d\nabla_x \mathrm{v}(\cdot, X_\cdot)$ and therefore

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This leads to the PDE :

$$0 = -\partial_t \mathrm{v}(\cdot, x) - \frac{1}{2}\sigma^2(x, \nabla_{xx}\mathrm{v}(\cdot, x))\nabla_{xx}\mathrm{v}(\cdot, x) + F(x, \nabla_{xx}\mathrm{v}(\cdot, x))$$

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= & -\partial_t \mathrm{v}(\cdot, x) - \bar{F}(x, \nabla_{xx}\mathrm{v}(\cdot, x))
\end{aligned}
$$

with

$$\bar{F}(x, g) := \frac{1}{2}\sigma^2(x, g)g - F(x, g).$$

and terminal condition

$$\mathrm{v}(T, \cdot) = \Xi.$$

Markovian setting - Linear impact case

In this case

$$F(x,g) = \frac{1}{2}\left(\frac{\sigma^\circ(x)g}{1-f(x)g}\right)^2 f(x)\mathbf{1}_{\{f(x)g<1\}} + \infty\mathbf{1}_{\{f(x)g\geq 1\}}$$

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Gamma constraint : $\{\bar{F}(x,g) < \infty\} = \{g < \gamma(x)\}$, where $\gamma := 1/f$ in the linear case.

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In general, the correct equation is

$$0 = \min\{-\partial_t \mathrm{v}(\cdot,x) - \bar{F}(x,\nabla_{xx}\mathrm{v}(\cdot,x)); \gamma - \nabla_{xx}\mathrm{v}\}$$

and the terminal condition Ξ is replaced by the smallest function above Ξ satisfying the gamma constraint.

Markovian setting - Linear impact case

Remember the typical example

$$F(x,g) = \frac{1}{2}\left(\frac{\sigma^\circ(x)g}{1-f(x)g}\right)^2 f(x)\mathbf{1}_{\{f(x)g<1\}} + \infty\mathbf{1}_{\{f(x)g\geq 1\}}$$

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If $\nabla_{xx} g \leq \gamma - \varepsilon$, $\varepsilon > 0$, + smooth coefficients, the gamma constraint propagates backward and there exists a smooth solution to

$$0 = -\,\partial_t \mathrm{v}(\cdot,x) - \bar{F}(x, \nabla_{xx}\mathrm{v}(\cdot,x))$$

satisfying $\nabla_{xx}\mathrm{v} < \bar{\gamma}$.

$\Rightarrow$ Perfect hedging strategy with $Y = \nabla_x \mathrm{v}$ + super-hedging price is a hedging price (actually the only, see later).

Markovian setting - Convex case

Assume that : $g \mapsto \bar{F}(x, g)$ is convex (as in the linear impact case).

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Then,

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\begin{aligned}
0 = & -\partial_t \mathrm{v}(\cdot, x) - \bar{F}(x, \nabla_{xx}\mathrm{v}(\cdot, x.)) \\
= & \inf_{\mathrm{s}\in\mathbb{R}} \left(-\partial_t \mathrm{v}(\cdot, x) - \frac{1}{2}\mathrm{s}^2 \nabla_{xx}\mathrm{v}(\cdot, x.) + \bar{F}^*(x, \mathrm{s}) \right)
\end{aligned}
$$

where

$$
\bar{F}^*(\cdot, \mathrm{s}) := \sup_{g<\gamma} \left(\frac{1}{2}\mathrm{s}^2 g - \bar{F}(\cdot, g) \right),
$$

so that

$$
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$$

Markovian setting - Convex case (continued)

If v solves

$$\begin{aligned} 0 &= -\partial_t \mathrm{v}(\cdot, x) - \bar{F}(x, \nabla_{xx}\mathrm{v}(\cdot, x.)) \\ &= \inf_{\mathrm{s}\in\mathbb{R}} \left(-\partial_t \mathrm{v}(\cdot, x) - \frac{1}{2}\mathrm{s}^2 \nabla_{xx}\mathrm{v}(\cdot, x.) + \bar{F}^*(x, \mathrm{s}) \right) \end{aligned}$$

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then

$$\mathrm{v}(0,x) = \bar{\mathrm{v}}(0,x) := \sup_{\mathfrak{s}\in\mathcal{A}_2} \mathbb{E}\left[\Xi(\bar{X}_T^{\mathfrak{s}}) - \int_0^T \bar{F}_t^*(\bar{X}_t^{\mathfrak{s}}, \mathfrak{s}_t) dt \right]$$

with

$$\bar{X}^{\mathfrak{s}} := x + \int_0^{\cdot} \mathfrak{s}_t dW_t .$$

$\Rightarrow$ Dual formulation !

Example

In the linear impact model

$$\bar{F}^*(x, \mathrm{s}) = \frac{1}{2}\gamma(x)|\mathrm{s} - \sigma^\circ|^2, \quad \text{with} \;\; \gamma = 1/f.$$

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(in the following, one can replace W by a martingale M and dt by $d\langle M \rangle$, under the martingale representation property)

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Recall that

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Recall that

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Hedging means :

$$V_0 + \int_0^{\cdot} Y_t dX_t = \Xi(X) - \int_0^{\cdot} F_t(X, \mathfrak{g}_t) dt.$$

Assuming hedging holds...

Assume we have a hedging strategy $(\hat{\mathfrak{g}}, \hat{\mathfrak{B}})$, then

$$V_0 = \mathbb{E}^{\mathbb{Q}^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}} \left[\Xi(X^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}) - \int_0^T F_t(X^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}, \hat{\mathfrak{g}}_t) dt \right]$$

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We need to retrieve

$$\sup_{\mathfrak{s}} \mathbb{E} \left[\Xi(\bar{X}^{\mathfrak{s}}_T) - \int_0^T \bar{F}^*_t(\bar{X}^{\mathfrak{s}}_t, \mathfrak{s}_t) dt \right]$$

$$\text{with} \quad \bar{X}^{\mathfrak{s}} := x + \int_0^{\cdot} \mathfrak{s}_t dW_t \quad \text{while} \quad X^{\mathfrak{g},\mathfrak{B}} = x + \int_0^{\cdot} \sigma_t(X_t, \mathfrak{g}_t) dW_t^{\mathfrak{g},\mathfrak{B}}$$

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Ok, if (σ^{-1} = inverse w.t. second coordinate)

$$\bar{F}^*(\cdot, \mathrm{s}) = F(\cdot, \sigma^{-1}(\cdot, \mathrm{s})) \text{ i.e. } \frac{1}{2}(\partial_g \sigma^2)g = \partial_g \bar{F}.$$

Assuming hedging holds...

Assume we have a hedging strategy $(\hat{\mathfrak{g}}, \hat{\mathfrak{B}})$, then

$$V_0 = \mathbb{E}^{\mathbb{Q}^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}}\left[\Xi(X^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}) - \int_0^T F_t(X^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}, \hat{\mathfrak{g}}_t)dt\right]$$

$$\leq \sup_{(\mathfrak{g},\mathfrak{B})} \mathbb{E}^{\mathbb{Q}^{\mathfrak{g},\mathfrak{B}}}\left[\Xi(X^{\mathfrak{g},\mathfrak{B}}) - \int_0^T \underbrace{F_t(X^{\mathfrak{g},\mathfrak{B}}, \mathfrak{g}_t)}_{F_t(X^{\mathfrak{g},\mathfrak{B}},\sigma_t^{-1}(X,\mathfrak{s}_t))} dt\right].$$

We need to retrieve

$$\sup_{\mathfrak{s}} \mathbb{E}\left[\Xi(\bar{X}_T^{\mathfrak{s}}) - \int_0^T \bar{F}_t^*(\bar{X}_t^{\mathfrak{s}}, \mathfrak{s}_t)dt\right]$$

with $\bar{X}^{\mathfrak{s}} := x + \int_0^{\cdot} \mathfrak{s}_t dW_t$ while $X^{\mathfrak{g},\mathfrak{B}} = x + \int_0^{\cdot} \underbrace{\sigma_t(X_t, \mathfrak{g}_t)}_{\mathfrak{s}_t} dW_t^{\mathfrak{g},\mathfrak{B}}$

Ok, if (σ^{-1} = inverse w.t. second coordinate)

$$\bar{F}^*(\cdot, \mathrm{s}) = F(\cdot, \sigma^{-1}(\cdot, \mathrm{s})) \text{ i.e. } \frac{1}{2}(\partial_g \sigma^2)g = \partial_g \bar{F}.$$

Assuming hedging holds...

Note that super-hedging does not permit to say anything... :

$$V_0 \geq \mathbb{E}^{\mathbb{Q}^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}}\left[\Xi(X^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}) - \int_0^T F_t(X^{\hat{\mathfrak{g}},\hat{\mathfrak{B}}}, \hat{\mathfrak{g}}_t)dt\right]$$

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$$\not\geq \sup_{(\mathfrak{g},\mathfrak{B})} \mathbb{E}^{\mathbb{Q}^{\mathfrak{g},\mathfrak{B}}}\left[\Xi(X^{\mathfrak{g},\mathfrak{B}}) - \int_0^T F_t(X^{\mathfrak{g},\mathfrak{B}},\mathfrak{g}_t)dt\right].$$

Dupire derivative of the gain function and calculus of variation

Assumption : $\bar{v}(t, \mathrm{x})$ admits a solution $\hat{\mathfrak{s}}[t, \mathrm{x}]$ (need weak...) + smoothness assumptions.

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Result #1 : The gain function

$$J(t, \mathrm{x}; \mathfrak{s}) := \mathbb{E}\left[\Xi(\bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \int_t^T \bar{F}_r^*(\bar{X}^{t,\mathrm{x},\mathfrak{s}}, \mathfrak{s}_r)dr\right],$$

$$\bar{X}^{t,\mathrm{x},\mathfrak{s}} := \mathrm{x}_{\wedge t} + \int_t^{\cdot} \mathfrak{s}_r dW_r,$$

admits a Dupire vertical derivative

$$\nabla_{\mathrm{x}} J(t, \mathrm{x}; \mathfrak{s}) := \mathbb{E}\left[\mathfrak{B}_T^{\mathrm{x},\mathfrak{s}} - \mathfrak{B}_t^{\mathrm{x},\mathfrak{s}}\right]$$

where $\mathfrak{B}^{\mathrm{x},\mathfrak{s}}$ is an adapted BV process.

Example

Recall

$$\bar{X}^{t,\mathrm{x},\mathfrak{s}} := \mathrm{x}_{\wedge t} + \int_t^{\cdot} \mathfrak{s}_r dW_r.$$

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then

$$\nabla_x J(t,\mathrm{x};\mathfrak{s}) := \mathbb{E}\left[\int_t^T \lambda_{\Xi}^{\circ}(dr;\bar{X}^{t,\mathrm{x},\mathfrak{s}})\right],$$

where $\lambda_{\Xi}^{\circ}(\cdot;\bar{X}^{t,\mathrm{x},\mathfrak{s}})$ is the dual predictable projection of the Fréchet derivative of Ξ at $\bar{X}^{t,\mathrm{x},\mathfrak{s}}$.

Dupire derivative of the gain function and calculus of variation (continued)

Result $\#2$: By a simple calculus of variations argument,

$$\partial_{\mathrm{s}} \bar{F}^*(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}, \hat{\mathfrak{s}}[t,\mathrm{x}]) = \beta[t,\mathrm{x}]$$

where $(m[t,\mathrm{x}], \beta[t,\mathrm{x}])$ is the element of $\in \mathbb{R} \times \mathcal{A}_2$ such that

$$m[t,\mathrm{x}] + \int_t^T \beta[t,\mathrm{x}]_u dW_u = \mathfrak{B}_T^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]} - \mathfrak{B}_t^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}.$$

Dupire derivative of the gain function and calculus of variation (continued)

Result #2 : By a simple calculus of variations argument,

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Recall that

$$\nabla_{\mathrm{x}} J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\mathfrak{B}_T^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]} - \mathfrak{B}_t^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}\right].$$

Dupire derivative of the gain function and calculus of variation (continued)

Example for

$$J(t, \mathrm{x}; \hat{\mathfrak{s}}[t, \mathrm{x}]) := \mathbb{E}\left[\Xi(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) - \int_t^T \bar{F}_r^*(\hat{\mathfrak{s}}[t, \mathrm{x}]_r)dr\right],$$

the first order condition implies (for all δ adapted bounded) :

$$\begin{aligned} 0 =& \mathbb{E}\left[\int_t^T (\int_t^r \delta_s dW_s)\lambda_{\Xi}^{\circ}(dr; \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) - \int_t^T \delta_r \partial_{\mathrm{s}} \bar{F}_r^*(\hat{\mathfrak{s}}[t, \mathrm{x}]_r)dr\right] \\ =& \mathbb{E}\left[\int_t^T \lambda_{\Xi}^{\circ}(dr; \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) \int_t^T \delta_r dW_r - \int_t^T \delta_r \partial_{\mathrm{s}} \bar{F}_r^*(\hat{\mathfrak{s}}[t, \mathrm{x}]_r)dr\right] \end{aligned}$$

Dupire derivative of the gain function and calculus of variation (continued)

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Set $\int_t^T \lambda_{\Xi}^{\circ}(dr;\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) = m + \int_t^T \beta_r dW_r$,

Dupire derivative of the gain function and calculus of variation (continued)

Example for

$$J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\Xi(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) - \int_t^T \bar{F}_r^*(\hat{\mathfrak{s}}[t,\mathrm{x}]_r)dr\right],$$

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Set $\int_t^T \lambda_{\Xi}^{\circ}(dr;\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) = m + \int_t^T \beta_r dW_r$, then

$$0 = \mathbb{E}[\int_t^T \delta_s \beta_r dr - \int_t^T \delta_r \partial_{\mathrm{s}} \bar{F}_r^*(\hat{\mathfrak{s}}[t,\mathrm{x}]_r)dr] = \mathbb{E}[\int_t^T \delta_s(\beta_r - \partial_{\mathrm{s}} \bar{F}_r^*(\hat{\mathfrak{s}}[t,\mathrm{x}]_r))dr].$$

Dupire derivative of the gain function and calculus of variation (continued)

Result #2 : By a simple calculus of variations argument,

$$\partial_{\mathrm{s}} \bar{F}^*(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}, \hat{\mathfrak{s}}[t,\mathrm{x}]) = \beta[t,\mathrm{x}]$$

where $(m[t,\mathrm{x}], \beta[t,\mathrm{x}])$ is the element of $\mathbb{R} \times \mathcal{A}_2$ such that

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Since, $\nabla_{\mathrm{x}} J(\cdot, \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}; \hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\mathfrak{B}_T^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]} - \mathfrak{B}_\cdot^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]} | \mathcal{F}_\cdot\right]$,

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$$\hat{Y}[t,\mathrm{x}] := m[t,\mathrm{x}] + \int_t^\cdot \beta[t,\mathrm{x}]_u dW_u - (\mathfrak{B}^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]} - \mathfrak{B}_t^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]})$$

satisfies

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Dupire derivative of the gain function and calculus of variation (continued)

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Regularity of the value function

Assumption : $\bar{F}$ is bounded from below (by a map with linear growth in x).

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Result #3 : Set

$$\Gamma(t,\mathrm{x}) = \int_0^{\mathrm{x}_t} \int_0^{y^1} \gamma_t(\mathrm{x}_{\wedge t} + \mathbf{1}_{\{t\}}(y^2 - \mathrm{x}_t)) dy^2 dy^1,$$

then $y \mapsto (\bar{\mathrm{v}} - \Gamma)(t, \mathrm{x} + \mathbf{1}_{\{t\}} y)$ is concave ($\bar{\mathrm{v}} - \Gamma$ is Dupire concave).

Recall that :

$$J(t,\mathrm{x};\mathfrak{s}) := \mathbb{E}\left[\Xi(\bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \int_t^T \bar{F}_r^*(\bar{X}^{t,\mathrm{x},\mathfrak{s}}, \mathfrak{s}_r) dr\right],$$

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Result #4 : $\bar{\mathrm{v}}$ admits a continuous vertical Dupire derivative given by

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because (t,x) maximizes $(t',\mathrm{x}') \mapsto \bar{\mathrm{v}}(t',\mathrm{x}') - J(t',\mathrm{x}';\hat{\mathfrak{s}}[t,\mathrm{x}])$

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and (Meyer-Tanaka + martingale property - just need $\mathrm{C}_{\mathrm{r}}^{0,1}$)

$$\bar{\mathrm{v}}(t', \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) = \bar{\mathrm{v}}(t,\mathrm{x}) + \int_t^{t'} \nabla_{\mathrm{x}}\bar{\mathrm{v}}(r, \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]})d\bar{X}_r^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]}$$
$$+ \int_t^{t'} \bar{F}^*(r, \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}, \hat{\mathfrak{s}}[t,\mathrm{x}]_r)dr.$$

More generally

Let Z be a $(\mathbb{F}, \mathbb{P})$-continuous adapted process such that $\mathbb{E}^{\mathbb{P}}[\|Z\|^2] < \infty$. Let ϕ be a non-anticipative map in $\mathrm{C}_{\mathrm{r}}^{0,1}$. Assume that there exists $R \in \mathrm{C}_{\mathrm{r}}^{1,2}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that :

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, \mathrm{x} + \mathbf{1}_{\{t\}} y)$ is concave for all t),
2. $\phi - \ell$ is non-increasing in time.

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2. $\phi - \ell$ is non-increasing in time.

Then, there exists a non-increasing predictable process A starting at 0 such that

$$\phi_{\cdot}(Z) - \int_0^{\cdot} \frac{1}{2} \nabla_{\mathrm{x}}^2 R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^{\cdot} \nabla_{\mathrm{x}} \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).$$

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$$\phi_\cdot(Z) - \int_0^\cdot \frac{1}{2} \nabla_{\mathrm{x}}^2 R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^\cdot \nabla_{\mathrm{x}} \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).$$

Moreover, if Z and $\phi_\cdot(Z) - B$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process B, then

$$\phi_\cdot(Z) = \phi_0(Z_0) + \int_0^\cdot \nabla_{\mathrm{x}} \phi_t(Z) dZ_t + B \ , \quad \text{on } [0, T].$$

More generally

Let Z be a $(\mathbb{F},\mathbb{P})$-continuous adapted process such that $\mathbb{E}^{\mathbb{P}}[\|Z\|^2] < \infty$. Let ϕ be a non-anticipative map in $\mathrm{C}^{0,1}_{\mathrm{r}}$. Assume that there exists $R \in \mathrm{C}^{1,2}_{\mathrm{r}}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that :

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2. $\phi - \ell$ is non-increasing in time.

Then, there exists a non-increasing predictable process A starting at 0 such that

$$\phi_\cdot(Z) - \int_0^\cdot \frac{1}{2} \nabla^2_{\mathrm{x}} R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^\cdot \nabla_{\mathrm{x}} \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).$$

Moreover, if Z and $\phi_\cdot(Z) - B$ are $(\mathbb{P},\mathbb{F})$-martingales, for some predictable bounded variation process B, then

$$\phi_\cdot(Z) = \phi_0(Z_0) + \int_0^\cdot \nabla_{\mathrm{x}} \phi_t(Z) dZ_t + B \ , \quad \text{on } [0, T].$$

Compare with Cont and Fournier (2013), Saporito (2017) for the Functional Itô-Meyer-Tanaka, Russo and Vallois (1996), and Gozzi and Russo (2006) for C^1 functionals of semimartingales.

Regularity of the value function

Result #4 : $\bar{\mathrm{v}}$ admits a continuous vertical Dupire derivative given by

$$\nabla_{\mathrm{x}}\bar{\mathrm{v}}(t,\mathrm{x}) = \nabla_{\mathrm{x}}J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_t\right], \quad \hat{\mathfrak{B}}[t,\mathrm{x}] := \mathfrak{B}^{\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}$$

and (Meyer-Tanaka + martingale property - just need $C^{0,1}$)

$$\begin{aligned}\bar{\mathrm{v}}(t',\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) =& \bar{\mathrm{v}}(t,\mathrm{x}) + \int_t^{t'} \nabla_{\mathrm{x}}\bar{\mathrm{v}}(r,\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]})d\bar{X}_r^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]} \\ &+ \int_t^{t'} \bar{F}^*(r,\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]},\hat{\mathfrak{s}}[t,\mathrm{x}]_r)dr.\end{aligned}$$

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Assumption : $\frac{1}{2}(\partial_g \sigma^2) g = \partial_g \bar{F}$ (satisfied in the linear impact model).

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$\Rightarrow$ $\hat{\mathfrak{s}}[\mathrm{x}]$ **provides** $(\hat{\mathfrak{g}}[\mathrm{x}], -\hat{\mathfrak{B}}[\mathrm{x}])$ **which is the hedging strategy starting from** $V_0 = \bar{\mathrm{v}}(0,\mathrm{x})$ **and** $Y_0 = \nabla_{\mathrm{x}} \bar{\mathrm{v}}(0,\mathrm{x})$. □

Conclusion and open question

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□ **Open question :** In the Markovian setting, and under smoothness conditions, the super-hedging price is the only hedging price. How to prove this in the path-dependent case by simply using probabilistic arguments ?

Main issue : the terminal condition $\Xi(X)$ depends on the hedging strategy -> standard comparison does not hold.

Thank you !

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