Duality for almost-sure hedging with price impact

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Based on works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich) and Y. Zou (ex Dauphine-PSL) + more recent developments with P. Cardialaguët (Dauphine-PSL) and X. Tan (Dauphine-PSL)
Problem formulation and motivation
Motivation

Construct market models with permanent price impact (possibly with resilience effect) in which hedging is possible:

- Buying pushes up the price, selling pushes it down.
- We pay an illiquidity cost.
- Solve the “running after the delta” effect.
- Avoid hidden transaction costs (fixed or proportional).
- Not at the level of high-frequency level $\rightarrow$ mesoscopic model.

We will focus on the case of covered options:

- The “premium” is paid at 0 in cash plus delta (number of stocks) asked by the trader.
- The trader delivers at $T$ cash and stocks (evaluated at their current price).

$\Rightarrow$ Avoids jumps at 0 and $T$, and therefore important impacts on the stock price.
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Example
Linear impact rule and covered options: buying $\Delta_t$ stocks leads to
- a permanent price move of $X_{t-} \rightarrow X_t = X_{t-} + f_t(X_{t-}) \Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t-} + X_t)$.
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When no trading, the stock evolves according to

$$dX_t = \sigma_t^\circ(X_t)dW_t.$$
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Consider rebalancing at times $t_i^n$:

$$X^n = X_0 + \int_0^t \sigma^\circ(X^n_t)dW_t + \sum_{i=1}^n 1_{[t^n_i, T]} f(X^n_{t^n_i-})\Delta^n_{t_i},$$

$$Y^n := \sum_{i=0}^{n-1} 1_{[t^n_i, t^n_{i+1}]} \left( \int_0^t g_t dX_t^n + \int_0^t b_t dt \right), \quad \Delta^n_{t_i} = Y^n_{t_i} - Y^n_{t_i-1},$$

$$V^n = V_0 + \sum_{i=1}^n 1_{[t^n_i, T]} \frac{1}{2}(\Delta^n_{t_i})^2 f(X^n_{t^n_i-}) + \int_0^T Y^n_{t-} dX^n_t,$$

where

$$V^n = \text{cash part} + Y^n X^n = \text{"portfolio value"}.$$
Example

Linear impact rule and covered options : buying $\Delta_t$ stocks leads to

- a permanent price move of $X_{t^-} \rightarrow X_t = X_{t^-} + f_t(X_{t^-})\Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t^-} + X_t)$.

When no trading, the stock evolves according to

$$dX_t = \sigma^\circ_t(X_t)\,dW_t.$$ 

⇒ Let $t_{i+1}^n - t_i^n \rightarrow 0$ :

$$X = x \wedge 0 + \int_0^{t_i^n} \sigma^\circ_t(X_t)\,dW_t + \int_0^{t_i^n} f_t(X_t)\,dY_t + \int_0^{t_i^n} g_t(f'_t \sigma^\circ_t)(X_t)\,dt$$

$$Y = y + \int_0^{t_i^n} g_t\,dX_t + \int_0^{t_i^n} b_t\,dt$$

$$V = V_0 + \int_0^{t_i^n} \frac{1}{2} g_t^2 f_t(X_t)\,dt + \int_0^{t_i^n} Y_t\,dX_t.$$
Example

Linear impact rule and covered options: buying $\Delta_t$ stocks leads to

- a permanent price move of $X_{t-} \to X_t = X_{t-} + f_t(X_{t-})\Delta_t$,
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When no trading, the stock evolves according to

$$dX_t = \sigma_t(X_t) dW_t.$$ 

⇒ Let $t_{i+1}^n - t_i^n \to 0$:

$$X = x^{\wedge 0} + \int_0^t \frac{\sigma_t(X_t)}{1 - f_t(X_t)g_t} dW_t + \int_0^t (\cdots) dt$$

$$Y = y + \int_0^t g_t dX_t + \int_0^t b_t dt$$

$$V = V_0 + \int_0^t \frac{1}{2} g_t^2 f_t(X_t) dt + \int_0^t Y_t dX_t.$$
Example

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$$dX_t = \sigma_t^o(X_t)dW_t.$$ 

⇒ Let $t_{i+1}^n - t_i^n \to 0$:

$$X = x_\wedge 0 + \int_0^{\cdot} \frac{\sigma_t^o(X_t)}{1 - f_t(X_t)g_t}dW_t + \int_0^{\cdot} (\cdots)dt$$

$$Y = y + \int_0^{\cdot} g_t dX_t + \int_0^{\cdot} b_t dt$$

$$V = V_0 + \int_0^{\cdot} \frac{1}{2} g_t^2 f_t(X_t) dt + \int_0^{\cdot} Y_t dX_t.$$

Note that trading impacts the whole dynamics through the flow of the SDE. It will also impact the claim $\Xi(X)$. 
Example

Linear impact rule and resilience

\[ X = X_0 + \int_0^\cdot \sigma_s^\circ(X_s)dW_s + R \]

\[ R = R_0 + \int_0^\cdot f_s(X_s)dY_s + \int_0^\cdot (g_s(f_s\sigma_s^\circ)(X_s) - \rho R_s)ds \]

\[ Y = y + \int_0^\cdot g_t dX_t + \int_0^\cdot b_t dt \]

\[ V = V_0 + \int_0^\cdot Y_t dX_t + \int_0^\cdot \frac{1}{2} g_t^2 f_t(X_t)dt. \]
Example

Linear impact rule and resilience

\[ X = X_0 + \int_0^\cdot \sigma_s^\circ(X_s) dW_s + R \]

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\[ V = V_0 + \int_0^\cdot Y_t dX_t + \int_0^\cdot \frac{1}{2} g_t^2 f_t(X_t) dt. \]

For covered options, resilience does not play any role... we omit it.
Abstract hedging of covered options

Given \( x \in C([0, T]) \), find \( y \in \mathbb{R} \) and \( (g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2 \) such that (after a change of measure)

\[
X = x \wedge 0 + \int_0^\cdot \sigma_t(X, g_t) dW_t \\
Y = y + \int_0^\cdot g_t dX_t + \mathcal{B} \\
V = V_0 + \int_0^\cdot Y_t dX_t + \int_0^\cdot F_t(X, g_t) dt, \quad V_T = \Xi(X)
\]

(possibly weak formulation)
Abstract hedging of covered options

Given $x \in C([0, T])$, find $y \in \mathbb{R}$ and $(g, \mathcal{B}) \in A_2 \times B_2$ such that (after a change of measure)

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(possibly weak formulation)

**Interpretation:**
- $X$: stock price,
- $Y$: number of stocks in the portfolio,
- $V$: cash value of the portfolio (at the current stock price),
- $F(\cdot, g)$ and $\sigma(\cdot, g)$: liquidity cost and price impact.
PDE point of view

Second order stochastic target problems with generalized market impact.
Markovian setting

Given $x \in \mathbb{R}$, find $y \in \mathbb{R}$ and $\phi := (g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that

\begin{align*}
X &= x + \int_0 \cdots \sigma_t(X_t, g_t) dW_t \\
Y &= y + \int_0 \cdots g_t dX_t + \mathcal{B} \quad \text{with} \quad d\mathcal{B}_t = b_t dt \\
V &= \Xi(X_T) - \int . \cdots F_t(X_t, g_t) dt - \int . \cdots Y_t dX_t, \quad (\text{adapted})
\end{align*}
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Given $x \in \mathbb{R}$, find $y \in \mathbb{R}$ and $\phi := (g, \mathcal{B}) \in A_2 \times B_2$ such that

$$X = x + \int_0^\cdot \sigma_t(X_t, g_t) \, dW_t$$

$$Y = y + \int_0^\cdot g_t \, dX_t + \mathcal{B} \quad \text{with} \quad d\mathcal{B}_t = b_t \, dt$$

$$V = \Xi(X_T) - \int_0^T F_t(X_t, g_t) \, dt - \int_0^T Y_t \, dX_t, \quad (\text{adapted})$$

Assume a solution $V = v(\cdot, X)$ exists, then $dV = dv(\cdot, X)$ and therefore:

- $Y = \nabla_x v(\cdot, X)$,
- $F(X, g) = \partial_t v(\cdot, X) + \frac{1}{2} \sigma^2(X, g) \nabla_{xx} v(\cdot, X)$
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$$Y = y + \int_0^\cdot g_t dX_t + \mathcal{B} \quad \text{with } d\mathcal{B}_t = b_t dt$$

$$V = \Xi(X_T) - \int_\cdot^T F_t(X_t, g_t) dt - \int_\cdot^T Y_t dX_t, \quad (adapted)$$

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Moreover, $Y = \nabla_x v(\cdot, X)$ implies $dY = d\nabla_x v(\cdot, X)$ and therefore

- $g = \nabla_{xx} v(\cdot, X)$,
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X = x + \int_0^\cdot \sigma_t(X_t, g_t) dW_t
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Assume a solution $V = v(\cdot, X)$ exists, then $dV = dv(\cdot, X)$ and therefore:

- $Y = \nabla_x v(\cdot, X)$,
- $F(X, \nabla_{xx} v(\cdot, X)) = \partial_t v(\cdot, X) + \frac{1}{2} \sigma^2(X, \nabla_{xx} v(\cdot, X)) \nabla_{xx} v(\cdot, X)$

Moreover, $Y = \nabla_x v(\cdot, X)$ implies $dY = d\nabla_x v(\cdot, X)$ and therefore

- $g = \nabla_{xx} v(\cdot, X)$,
Markovian setting

This leads to the PDE:

\[
0 = - \partial_t v(\cdot, x) - \frac{1}{2} \sigma^2(x, \nabla_{xx}v(\cdot, x)) \nabla_{xx}v(\cdot, x) + F(x, \nabla_{xx}v(\cdot, x))
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Markovian setting

This leads to the PDE:

$$0 = -\partial_t v(\cdot, x) - \frac{1}{2}\sigma^2(x, \nabla_{xx}v(\cdot, x))\nabla_{xx}v(\cdot, x) + F(x, \nabla_{xx}v(\cdot, x))$$

$$= -\partial_t v(\cdot, x) - \tilde{F}(x, \nabla_{xx}v(\cdot, x))$$

with

$$\tilde{F}(x, g) := \frac{1}{2}\sigma^2(x, g)g - F(x, g).$$

and terminal condition

$$v(T, \cdot) = \Xi.$$
Markovian setting - Linear impact case

In this case

\[ F(x, g) = \frac{1}{2} \left( \frac{\sigma^x(x)g}{1 - f(x)g} \right)^2 f(x) \mathbf{1}_{\{f(x)g < 1\}} + \infty \mathbf{1}_{\{f(x)g \geq 1\}} \]

\[ \bar{F}(x, g) = \frac{1}{2} \frac{\sigma^x(x)^2g}{1 - f(x)g} \mathbf{1}_{\{f(x)g < 1\}} + \infty \mathbf{1}_{\{f(x)g \geq 1\}}. \]
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Gamma constraint: \( \{\bar{F}(x, g) < \infty\} = \{g < \gamma(x)\} \), where \( \gamma := 1/f \) in the linear case.
Markovian setting - Linear impact case

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Gamma constraint : \(\{\bar{F}(x, g) < \infty\} = \{g < \gamma(x)\}\), where \(\gamma := 1/f\) in the linear case.

In general, the correct equation is

\[
0 = \min \{-\partial_t v(\cdot, x) - \bar{F}(x, \nabla_{xx} v(\cdot, x)); \gamma - \nabla_{xx} v\}
\]

and the terminal condition \(\Xi\) is replaced by the smallest function above \(\Xi\) satisfying the gamma constraint.
Markovian setting - Linear impact case

Remember the typical example

\[ F(x, g) = \frac{1}{2} \left( \frac{\sigma^\circ(x)g}{1 - f(x)g} \right)^2 f(x) \mathbf{1}_{\{f(x)g < 1\}} + \infty \mathbf{1}_{\{f(x)g \geq 1\}} \]

\[ \tilde{F}(x, g) = \frac{1}{2} \frac{\sigma^\circ(x)^2 g}{1 - f(x)g} \mathbf{1}_{\{f(x)g < 1\}} + \infty \mathbf{1}_{\{f(x)g \geq 1\}}. \]
Markovian setting - Linear impact case

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\[ \bar{F}(x, g) = \frac{1}{2} \frac{\sigma(x)^2g}{2 - f(x)g} 1\{f(x)g < 1\} + \infty 1\{f(x)g \geq 1\}. \]

If \( \nabla_{xx} g \leq \gamma - \varepsilon, \varepsilon > 0 \), + smooth coefficients, the gamma constraint propagates backward and there exists a smooth solution to

\[ 0 = -\partial_t v(\cdot, x) - \bar{F}(x, \nabla_{xx} v(\cdot, x)) \]

satisfying \( \nabla_{xx} v < \bar{\gamma} \).

\[ \Rightarrow \] Perfect hedging strategy with \( Y = \nabla_x v + \) super-hedging price is a hedging price (actually the only, see later).
Markovian setting - Convex case

Assume that: \( g \mapsto \bar{F}(x, g) \) is convex (as in the linear impact case).
Markovian setting - Convex case

Assume that: \( g \mapsto \bar{F}(x, g) \) is convex (as in the linear impact case).

Then,

\[
0 = -\partial_t v(\cdot, x) - \bar{F}(x, \nabla_{xx} v(\cdot, x)) \\
= \inf_{s \in \mathbb{R}} \left( -\partial_t v(\cdot, x) - \frac{1}{2} s^2 \nabla_{xx} v(\cdot, x) + \bar{F}^*(x, s) \right)
\]

where

\[
\bar{F}^*(\cdot, s) := \sup_{g < \gamma} \left( \frac{1}{2} s^2 g - \bar{F}(\cdot, g) \right),
\]

so that

\[
\bar{F}(\cdot, g) := \sup_{s \in \mathbb{R}} \left( \frac{1}{2} s^2 g - \bar{F}^*(\cdot, s) \right).
\]
Markovian setting - Convex case (continued)

If \( v \) solves

\[
0 = -\partial_t v(\cdot, x) - \bar{F}(x, \nabla_{xx} v(\cdot, x))
\]

\[
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\]

then

\[
v(0, x) = \bar{v}(0, x) := \sup_{s \in A_2} \mathbb{E} \left[ \Xi(\bar{X}^s_T) - \int_0^T \bar{F}^*_t(\bar{X}^s_t, s_t) dt \right]
\]

with

\[
\bar{X}^s := x + \int_0^t s_t dW_t.
\]

\( \Rightarrow \) Dual formulation!
Example

In the linear impact model

\[ \bar{F}^*(x, s) = \frac{1}{2} \gamma(x)|s - \sigma^\circ|^2, \quad \text{with} \quad \gamma = 1/f. \]
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In the linear impact model

\[ \bar{F}^*(x, s) = \frac{1}{2} \gamma(x) |s - \sigma^o|^2, \quad \text{with } \gamma = 1/f. \]

then

\[ v(0, x) = \bar{v}(0, x) := \sup_{s \in A} \mathbb{E} \left[ \Xi(\bar{X}^s_T) - \int_0^T \frac{1}{2} \gamma(\bar{X}^s_t) |s_t - \sigma^o_t(\bar{X}^s_t)|^2 dt \right] \]

with

\[ \bar{X}^s := x + \int_0^s s_t dW_t. \]
How can one retrieve this in a general Path Dependent case?

(in the following, one can replace $W$ by a martingale $M$ and $dt$ by $d\langle M \rangle$, under the martingale representation property)
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(in the following, one can replace \( W \) by a martingale \( M \) and \( dt \) by \( d\langle M\rangle \), under the martingale representation property)

Recall that

\[
V = V_0 + \int_0^\cdot Y_t dX_t + \int_0^\cdot F_t(X, g_t) dt.
\]
How can one retrieve this in a general Path Dependent case?

(in the following, one can replace \( W \) by a martingale \( M \) and \( dt \) by \( d\langle M\rangle \), under the martingale representation property)

Recall that

\[
V = V_0 + \int_0^\cdot Y_t \,dX_t + \int_0^\cdot F_t(X, g_t) \,dt.
\]

Hedging means:

\[
V_0 + \int_0^\cdot Y_t \,dX_t = \Xi(X) - \int_0^\cdot F_t(X, g_t) \,dt.
\]
Assuming hedging holds...

Assume we have a hedging strategy \((\hat{g}, \hat{B})\), then

\[
V_0 = \mathbb{E}^{Q, \hat{g}, \hat{B}} \left[ \Xi(\hat{X}, \hat{B}) - \int_0^T F_t(\hat{X}, \hat{B}, \hat{g}_t) dt \right]
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\]

\[
\leq \sup_{(g, B)} \mathbb{E}^{Q, g, B} \left[ \Xi(X, B) - \int_0^T F_t(X, B, g_t) \, dt \right].
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\]

\[
\leq \sup_{(g, B)} \mathbb{E}^{Q^g, \mathfrak{B}} \left[ \Xi(X^g, \mathfrak{B}) - \int_0^T F_t(X^g, \mathfrak{B}, g_t) dt \right].
\]

We need to retrieve

\[
\sup_{\bar{s}} \mathbb{E} \left[ \Xi(\bar{X}_T^\bar{s}) - \int_0^T \bar{F}_t^*(\bar{X}_t^\bar{s}, s_t) dt \right]
\]

with \(\bar{X}_t^\bar{s} := x + \int_0^t s_t dW_t\) while \(X^g, \mathfrak{B} = x + \int_0^T \sigma_t(X_t, g_t) dW^g_t\).
Assuming hedging holds...

Assume we have a hedging strategy \((\hat{g}, \hat{B})\), then

\[
V_0 = \mathbb{E}^Q_{\hat{g}, \hat{B}} \left[ \Xi(X_{\hat{g}, \hat{B}}) - \int_0^T F_t(X_{\hat{g}, \hat{B}}, \hat{g}_t) \, dt \right]
\]

\[
\leq \sup_{(g, B)} \mathbb{E}^Q_{g, B} \left[ \Xi(X_{g, B}) - \int_0^T F_t(X_{g, B}, g_t) \, dt \right].
\]

We need to retrieve

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\sup_s \mathbb{E} \left[ \Xi(\bar{X}_T^s) - \int_0^T \bar{F}^*_t(\bar{X}_T^s, s_t) \, dt \right]
\]

with \(\bar{X}^s := x + \int_0^\cdot s_t \, dW_t\) while \(X_{g, B} = x + \int_0^\cdot \sigma_t(X_t, g_t) \, dW_t^{g, B}\)

Ok, if \((\sigma^{-1} = \text{inverse w.t. second coordinate})\)

\[
\bar{F}^*(\cdot, s) = F(\cdot, \sigma^{-1}(\cdot, s)) \quad \text{i.e.} \quad \frac{1}{2} (\partial_g \sigma^2) g = \partial_g \bar{F}.
\]
Assuming hedging holds...

Assume we have a hedging strategy \((\hat{g}, \hat{B})\), then

\[
V_0 = \mathbb{E}^{Q_{\hat{g}}, \hat{B}} \left[ \Xi(\hat{X}, \hat{B}) - \int_0^T F_t(\hat{X}^g, \hat{B}^g, \hat{g}_t) dt \right] 
\]

\[
\leq \sup_{(g, B)} \mathbb{E}^{Q_g, B} \left[ \Xi(X^g, B) - \int_0^T \left( F_t(X^g, B, g_t) - F_t(X^g, B, \sigma^{-1}_t(X, s_t)) \right) dt \right].
\]

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\sup_{s} \mathbb{E} \left[ \Xi(\bar{X}^s_T) - \int_0^T \bar{F}_t^*(\bar{X}^s_t, s_t) dt \right]
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with \(\bar{X}^s := x + \int_0^t s_t dW_t\) while \(X^{g, B} = x + \int_0^t \sigma_t(X, g_t) dW^{g, B}_t\)

Ok, if \((\sigma^{-1} = \text{inverse w.r.t. second coordinate})\)

\[
\bar{F}^*(\cdot, s) = F(\cdot, \sigma^{-1}(\cdot, s)) \quad \text{i.e.} \quad \frac{1}{2} (\partial_g \sigma^2) g = \partial_g \bar{F}.
\]
Assuming hedging holds...

Note that super-hedging does not permit to say anything... :

\[ V_0 \geq \mathbb{E}^\mathbb{Q}_{\hat{\theta}, \hat{\mathbb{B}}} \left[ \Xi(\hat{X}, \hat{\mathbb{B}}) - \int_0^T F_t(\hat{X}, \hat{\mathbb{B}}, \hat{\theta}_t)dt \right] \]
Assuming hedging holds...

Note that super-hedging does not permit to say anything... :

\[ V_0 \geq \mathbb{E}^{Q, \hat{\mathcal{B}}} \left[ \Xi(\hat{\mathbf{X}}, \hat{\mathbf{B}}) - \int_0^T F_t(\hat{\mathbf{X}}, \hat{\mathbf{B}}, \hat{g}_t) dt \right] \]

\[ \not\geq \sup_{(g, \mathcal{B})} \mathbb{E}^{Q^g, \mathcal{B}_t} \left[ \Xi(\mathbf{X}^g, \mathcal{B}) - \int_0^T F_t(\mathbf{X}^g, \mathcal{B}, g_t) dt \right]. \]
Dupire derivative of the gain function and calculus of variation

Assumption: \( \bar{v}(t, x) \) admits a solution \( \hat{s}[t, x] \) (need weak...) + smoothness assumptions.
Dupire derivative of the gain function and calculus of variation

**Assumption**: $\bar{v}(t, x)$ admits a solution $\hat{s}[t, x]$ (need weak...) + smoothness assumptions.

**Result #1**: The gain function

$$J(t, x; s) := \mathbb{E} \left[ \Xi(\bar{X}^{t, x, s}) - \int_t^T \bar{F}_r^{*}(\bar{X}^{t, x, s}, s_r)dr \right],$$

$$\bar{X}^{t, x, s} := x \wedge t + \int_t^* s_r dW_r,$$

admits a Dupire vertical derivative

$$\nabla_x J(t, x; s) := \mathbb{E} \left[ \mathcal{B}^{x, s} - \mathcal{B}^{x, s}_T \right]$$

where $\mathcal{B}^{x, s}$ is an adapted BV process.
Example

Recall

\[ \bar{X}^{t,x,s} := x \wedge t + \int_t^\cdot s_r dW_r. \]
Recall

\[ \bar{X}^{t,x,s} := x \wedge t + \int_t^s s_r dW_r. \]

If

\[ J(t, x; s) := \mathbb{E} \left[ \Xi(\bar{X}^{t,x,s}) - \int_t^T \bar{F}_r(s_r) dr \right], \]
Example

Recall

\[ \bar{X}^{t, x, s} := x \wedge t + \int_t \cdot s_r \, dW_r. \]

If

\[ J(t, x; s) := \mathbb{E} \left[ \Xi(\bar{X}^{t, x, s}) - \int_t^T \tilde{F}_r^*(s_r) \, dr \right], \]

then

\[ \nabla_x J(t, x; s) := \mathbb{E} \left[ \int_t^T \lambda^\circ_{\Xi}(dr; \bar{X}^{t, x, s}) \right], \]

where \( \lambda^\circ_{\Xi}(\cdot; \bar{X}^{t, x, s}) \) is the dual predictable projection of the Fréchet derivative of \( \Xi \) at \( \bar{X}^{t, x, s} \).
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2**: By a simple calculus of variations argument,

\[ \partial_s \bar{F}^*(\bar{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t, x]) = \beta[t, x] \]

where \((m[t, x], \beta[t, x])\) is the element of \(\in \mathbb{R} \times A_2\) such that

\[ m[t, x] + \int_t^T \beta[t, x]_u dW_u = \mathcal{B}^{x,\hat{s}[t,x]}_T - \mathcal{B}^{x,\hat{s}[t,x]}_t. \]
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2**: By a simple calculus of variations argument,

\[ \partial_s \bar{F}^*(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t, x] \]

where \((m[t, x], \beta[t, x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[ m[t, x] + \int_t^T \beta[t, x] u dW_u = \mathcal{B}_T^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} . \]

Recall that

\[ \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \mathcal{B}_T^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} \right] . \]
Dupire derivative of the gain function and calculus of variation (continued)

Example for

\[ J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \Xi(\tilde{X}^{t, x, \hat{s}[t, x]} - \int_t^T \tilde{F}_r^*(\hat{s}[t, x]_r)dr \right], \]

the first order condition implies (for all \( \delta \) adapted bounded):

\[ 0 = \mathbb{E} \left[ \int_t^T (\int_t^r \delta_s dW_s) \lambda_\Xi^\circ(dr; \tilde{X}^{t, x, \hat{s}[t, x]} - \int_t^T \delta_r \partial_s \tilde{F}_r^*(\hat{s}[t, x]_r)dr \right] = \mathbb{E} \left[ \int_t^T \lambda_\Xi^\circ(dr; \tilde{X}^{t, x, \hat{s}[t, x]} \int_t^T \delta_r dW_r - \int_t^T \partial_s \tilde{F}_r^*(\hat{s}[t, x]_r)dr \right] \]
Dupire derivative of the gain function and calculus of variation (continued)

Example for

$$J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \Xi(\bar{X}^{t,x,\hat{s}[t,x]} - \int_t^T \bar{F}_r^*(\hat{s}[t,x]_r) dr \right],$$

the first order condition implies (for all $\delta$ adapted bounded):

$$0 = \mathbb{E} \left[ \int_t^T (\int_t^r \delta_s dW_s) \lambda^\circ_{\Xi}(dr; \bar{X}^{t,x,\hat{s}[t,x]} - \int_t^T \delta_r \partial_s \bar{F}_r^*(\hat{s}[t,x]_r) dr \right]$$

$$= \mathbb{E} \left[ \int_t^T \lambda^\circ_{\Xi}(dr; \bar{X}^{t,x,\hat{s}[t,x]} - \int_t^T \delta_r \partial_s \bar{F}_r^*(\hat{s}[t,x]_r) dr \right]$$

Set $\int_t^T \lambda^\circ_{\Xi}(dr; \bar{X}^{t,x,\hat{s}[t,x]}) = m + \int_t^T \beta_r dW_r,$
Dupire derivative of the gain function and calculus of variation (continued)

Example for

\[ J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \Xi(\bar{X}^{t,x,\hat{s}[t,x]}_t) - \int_t^T \bar{F}_r^*(\hat{s}[t, x]_r) dr \right] , \]

the first order condition implies (for all \( \delta \) adapted bounded) :

\[ 0 = \mathbb{E} \left[ \int_t^T \left( \int_t^r \delta_s dW_s \right) \lambda^0_\Xi (dr; \bar{X}^{t,x,\hat{s}[t,x]}_t) - \int_t^T \delta_r \partial_s \bar{F}_r^*(\hat{s}[t, x]_r) dr \right] \]

\[ = \mathbb{E} \left[ \int_t^T \lambda^0_\Xi (dr; \bar{X}^{t,x,\hat{s}[t,x]}_t) \int_t^T \delta_r dW_r - \int_t^T \delta_r \partial_s \bar{F}_r^*(\hat{s}[t, x]_r) dr \right] \]

Set \( \int_t^T \lambda^0_\Xi (dr; \bar{X}^{t,x,\hat{s}[t,x]}_t) = m + \int_t^T \beta_r dW_r \), then

\[ 0 = \mathbb{E} \left[ \int_t^T \delta_s \beta_r dr - \int_t^T \delta_r \partial_s \bar{F}_r^*(\hat{s}[t, x]_r) dr \right] = \mathbb{E} \left[ \int_t^T \delta_s (\beta_r - \partial_s \bar{F}_r^*(\hat{s}[t, x]_r)) dr \right] . \]
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2**: By a simple calculus of variations argument,

\[
\frac{\partial}{\partial s} \tilde{F}^\ast(\tilde{X}^{t,x},\hat{s}[t,x],\hat{s}[t,x]) = \beta[t,x]
\]

where \((m[t,x],\beta[t,x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[
m[t,x] + \int_t^T \beta[t,x]_u d\mathcal{W}_u = \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]}.
\]
Dupire derivative of the gain function and calculus of variation (continued)

Result #2: By a simple calculus of variations argument,

\[ \partial_s \bar{F}^* (\bar{X}^t, x, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t,x] \]

where \((m[t,x], \beta[t,x])\) is the element of \(\mathbb{R} \times A_2\) such that

\[ m[t,x] + \int_t^T \beta[t,x] u dW_u = \mathcal{B}_T^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} \]

Since, \(\nabla_x J(\cdot, \bar{X}^t, x, \hat{s}[t,x], \hat{s}[t,x]) := \mathbb{E} \left[ \mathcal{B}_T^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} | \mathcal{F} \right] \),
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2**: By a simple calculus of variations argument,

\[ \partial_s \bar{F}^* (\bar{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t, x]) = \beta[t, x] \]

where \((m[t, x], \beta[t, x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[ m[t, x] + \int_t^T \beta[t, x] u d\mathcal{W}_u = \mathcal{B}^{x,\hat{s}[t,x]}_{T} - \mathcal{B}^{x,\hat{s}[t,x]}_{t} \]

Since, \(\nabla_x J(\cdot, \bar{X}^{t,x,\hat{s}[t,x]}; \hat{s}[t, x]) := \mathbb{E} \left[ \mathcal{B}^{x,\hat{s}[t,x]}_{T} - \mathcal{B}^{x,\hat{s}[t,x]}_{t} \mid \mathcal{F} \right] \),

\[ \hat{Y}[t, x] := m[t, x] + \int_t^T \beta[t, x] u d\mathcal{W}_u - (\mathcal{B}^{x,\hat{s}[t,x]}_{T} - \mathcal{B}^{x,\hat{s}[t,x]}_{t}) \]

satisfies

\[ \hat{Y}[t, x] = \nabla_x J(\cdot, \bar{X}^{t,x,\hat{s}[t,x]}; \hat{s}[t, x]) \]
**Result #2**: By a simple calculus of variations argument,

\[
\partial_s \bar{F}^*(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t,x]
\]

where \((m[t,x], \beta[t,x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[
m[t,x] + \int_t^T \beta[t,x]_u d\mathcal{W}_u = \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]}.
\]

Since, \(\nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) := \mathbb{E} \left[ \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]} \mid \mathcal{F} \right],\)

\[
\hat{Y}[t,x] := m[t,x] + \int_t^T \beta[t,x]_u d\mathcal{W}_u - (\mathcal{B}_t^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]})
\]

satisfies

\[
\hat{Y}[t,x] = \nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]).
\]
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2**: By a simple calculus of variations argument,

\[ \partial_s \bar{F}^*(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t,x] \]

where \((m[t,x], \beta[t,x])\) is the element of \(\mathbb{R} \times A_2\) such that

\[ m[t,x] + \int_t^T \beta[t,x]_u \, dW_u = \mathcal{B}^{x,\hat{s}[t,x]}_T - \mathcal{B}^{x,\hat{s}[t,x]}_t. \]

Since, \(\nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) := \mathbb{E} \left[ \mathcal{B}^{x,\hat{s}[t,x]}_T - \mathcal{B}^{x,\hat{s}[t,x]}_t | \mathcal{F} \right],\)

\[ \hat{Y}[t,x] := m[t,x] + \int_t^T \partial_s \bar{F}^*_u(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]_u) \, dW_u - (\mathcal{B}^{x,\hat{s}[t,x]}_T - \mathcal{B}^{x,\hat{s}[t,x]}_t) \]

satisfies

\[ \hat{Y}[t,x] = \nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]). \]
Regularity of the value function

**Assumption**: $\bar{F}$ is bounded from below (by a map with linear growth in $x$).
Regularity of the value function

Assumption: \( \bar{F} \) is bounded from below (by a map with linear growth in \( x \)).

Result #3: Set

\[
\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x_{\land t} + 1_{\{t\}}(y^2 - x_t)) dy^2 dy^1,
\]

then \( y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y) \) is concave (\( \bar{v} - \Gamma \) is Dupire concave).

Recall that:

\[
J(t, x; s) := \mathbb{E} \left[ \Xi(\bar{X}^{t, x, s}) - \int_t^T \bar{F}^*_r(\bar{X}^{t, x, s}, s_r) dr \right] ,
\]

\[
\bar{X}^{t, x, s} := x_{\land t} + \int_t^r s_r dW_r ,
\]
Regularity of the value function

Assumption: $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

Result #3: Set

$$\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x \wedge t + 1_{\{t\}}(y^2 - x_t)) dy^2 dy^1,$$
then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y)$ is concave ($\bar{v} - \Gamma$ is Dupire concave).

Result #4: $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) = \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}$$
Regularity of the value function

Assumption: $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

Result #3: Set

$$\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x \wedge t + 1_{\{t\}}(y^2 - x_t)) dy^2 dy^1,$$

then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y)$ is concave ($\bar{v} - \Gamma$ is Dupire concave).

Result #4: $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) = \mathbb{E} \left[ \hat{B}[t, x]_T - \hat{B}[t, x]_t \right], \quad \hat{B}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}$$

because $(t, x)$ maximizes $(t', x') \mapsto \bar{v}(t', x') - J(t', x'; \hat{s}[t, x])$
Regularity of the value function

**Assumption:** $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

**Result #3:** Set

$$\Gamma(t, x) = \int_0^{\gamma_t(x \wedge t + 1_{\{t\}})} \int_0^{y^1} \gamma_t(x \wedge t + 1_{\{t\}}) (y^2 - x_t)) dy^2 dy^1,$$

then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}} y)$ is concave ($\bar{v} - \Gamma$ is Dupire concave).

**Result #4:** $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{\mathcal{S}}[t, x]) = \mathbb{E} \left[ \mathcal{B}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{\mathcal{S}}[t, x]}$$

and (Meyer-Tanaka + martingale property - just need $C^{0,1}_r$)

$$\bar{v}(t', X^{t,x,\hat{\mathcal{S}}[t,x]}) = \bar{v}(t, x) + \int_t^{t'} \nabla_x \bar{v}(r, X^{t,x,\hat{\mathcal{S}}[t,x]}) dX^{t,x,\hat{\mathcal{S}}[t,x]} + \int_t^{t'} \bar{F}^*(r, X^{t,x,\hat{\mathcal{S}}[t,x]}, \mathcal{S}[t, x]_r) dr.$$
More generally

Let $Z$ be a $(\mathcal{F}, \mathbb{P})$-continuous adapted process such that $\mathbb{E}^\mathbb{P}[\|Z\|^2] < \infty$. Let $\phi$ be a non-anticipative map in $C_r^{0,1}$. Assume that there exists $R \in C_r^{1,2}$ and a continuous function $\ell : [0, T] \rightarrow \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1_{\{t\}}y)$ is concave for all $t$),
2. $\phi - \ell$ is non-increasing in time.

Moreover, if $Z$ and $\phi \cdot (Z)$ are $(\mathcal{F}, \mathbb{P})$-martingales, for some predictable bounded variation process $B$, then $\phi \cdot (Z) = \phi_0(Z_0) + \int_0^\cdot \nabla x \phi_t(Z_t) dZ_t + B$, on $[0, T]$. Compare with Cont and Fournier (2013), Saporito (2017) for the Functional Itô-Meyer-Tanaka, Russo and Vallois (1996), and Gozzi and Russo (2006) for $C^1$ functionals of semimartingales.
More generally

Let $Z$ be a $(\mathcal{F}, \mathbb{P})$-continuous adapted process such that $\mathbb{E}^\mathbb{P}[\|Z\|^2] < \infty$. Let $\phi$ be a non-anticipative map in $C^{0,1}_r$. Assume that there exists $R \in C^{1,2}_r$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1_{\{t\}}y)$ is concave for all $t$),

2. $\phi - \ell$ is non-increasing in time.

Then, there exists a non-increasing predictable process $A$ starting at 0 such that

$$\phi \cdot (Z) - \int_0^\cdot \frac{1}{2} \nabla^2_x R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^\cdot \nabla_x \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).$$
More generally

Let $Z$ be a $(\mathbb{F}, \mathbb{P})$-continuous adapted process such that $\mathbb{E}^{\mathbb{P}}[\|Z\|^2] < \infty$.

Let $\phi$ be a non-anticipative map in $C^{0,1}_{r}$. Assume that there exists $R \in C^{1,2}_{r}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1_{\{t\}}y)$ is concave for all $t$),
2. $\phi - \ell$ is non-increasing in time.

Then, there exists a non-increasing predictable process $A$ starting at 0 such that

$$
\phi(Z) - \int_0^T \frac{1}{2} \nabla^2_x R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^T \nabla_x \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).
$$

Moreover, if $Z$ and $\phi(Z) - B$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process $B$, then

$$
\phi(Z) = \phi_0(Z_0) + \int_0^T \nabla_x \phi_t(Z) dZ_t + B, \text{ on } [0, T].
$$
More generally

Let \( Z \) be a \((\mathbb{F}, \mathbb{P})\)-continuous adapted process such that \( \mathbb{E}^\mathbb{P}[\|Z\|^2] < \infty \). Let \( \phi \) be a non-anticipative map in \( C_r^{0,1} \). Assume that there exists \( R \in C_r^{1,2} \) and a continuous function \( \ell : [0, T] \to \mathbb{R} \) such that:

1. \( \phi - R \) is Dupire-concave (i.e. \( y \mapsto (\phi - R)(t, x + 1_{\{t\}}y) \) is concave for all \( t \)),
2. \( \phi - \ell \) is non-increasing in time.

Then, there exists a non-increasing predictable process \( A \) starting at 0 such that

\[
\phi(Z) - \int_0^\cdot \frac{1}{2} \nabla_x^2 R_r(Z)d\langle Z \rangle_r = \phi_0(Z) + \int_0^\cdot \nabla_x \phi_t(Z)dZ_r + A + \ell(\cdot) - \ell(0).
\]

Moreover, if \( Z \) and \( \phi.(Z) - B \) are \((\mathbb{P}, \mathbb{F})\)-martingales, for some predictable bounded variation process \( B \), then

\[
\phi.(Z) = \phi_0(Z_0) + \int_0^\cdot \nabla_x \phi_t(Z)dZ_t + B, \text{ on } [0, T].
\]

Regularity of the value function

Result #4: \( \tilde{v} \) admits a continuous vertical Dupire derivative given by

\[
\nabla_x \tilde{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}
\]

and (Meyer-Tanaka + martingale property - just need \( C^{0, 1} \))

\[
\tilde{v}(t', \bar{X}^{t, x, \hat{s}[t, x]}) = \tilde{v}(t, x) + \int_t^{t'} \nabla_x \tilde{v}(r, \bar{X}^{t, x, \hat{s}[t, x]}) d\bar{X}^{t, x, \hat{s}[t, x]}_r + \int_t^{t'} \bar{F}^*(r, \bar{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]_r) dr.
\]
**Regularity of the value function**

**Result #4:** $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$
\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}
$$

and (Meyer-Tanaka + martingale property - just need $C^{0,1}$)

$$
\bar{v}(t', \bar{X}^{t, x, \hat{s}[t, x]}) = \bar{v}(t, x) + \int_t^{t'} \hat{Y}[t, x]_r d\bar{X}^{t, x, \hat{s}[t, x]}_r + \int_t^{t'} \hat{F}^* (r, \bar{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]_r, \hat{s}[t, x]_r) dr.
$$
Construction of the hedging strategy

Assumption: \( \frac{1}{2} (\partial_s \sigma^2) g = \partial_s \bar{F} \) (satisfied in the linear impact model).
Construction of the hedging strategy

**Assumption :** \( \frac{1}{2}(\partial_g \sigma^2)g = \partial_g \bar{F} \) (satisfied in the linear impact model).

Recall that \( \bar{v}(T, \cdot) = \Xi \) and that

\[
\bar{v}(T, \bar{X}^x, \hat{s}[t,x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}_r^x, \hat{s}[x] + \int_0^T \bar{F}_r^*(\bar{X}_r^x, \hat{s}[x], \hat{s}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \partial_s \bar{F}_t^*(\bar{X}_t^x, \hat{s}[x], \hat{s}[x]_t) dW_t - (\hat{B}[x] - \hat{B}[x]_0).
\]
Construction of the hedging strategy

**Assumption:** \( \frac{1}{2}(\partial_g \sigma^2)g = \partial_g \bar{F} \) (satisfied in the linear impact model).

Recall that \( \bar{v}(T, \cdot) = \Xi \) and that

\[
\bar{v}(T, \bar{X}^x, \hat{s}[t,x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}[x] + \int_0^T \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]_t) dW_t - (\hat{B}[x] - \hat{B}[x]_0).
\]

Under the above assumption, for \( \hat{g}[x]\hat{s}[x] := \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]_t) \),

\[
\bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]) = F(\bar{X}^x, \hat{s}[x], \hat{g}[x]), \quad \hat{s}[x] = \sigma(\cdot, \hat{g}[x])
\]
Construction of the hedging strategy

Assumption: \(\frac{1}{2} (\partial g \sigma^2) g = \partial g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v} (T, \cdot) = \Xi\) and that

\[
\bar{v}(T, \tilde{X}^x, \hat{s}[t, x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\tilde{X}^x_r, \hat{s}[x] + \int_0^T \bar{F}^*(\tilde{X}^x, \hat{s}[x], \hat{s}[x]_r) dr,
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\]

\( \Rightarrow \hat{s}[x] \) provides \((\hat{g}[x], -\hat{\mathcal{B}}[x])\) which is the hedging strategy starting from \( V_0 = \bar{v}(0, x) \) and \( Y_0 = \nabla_x \bar{v}(0, x) \). \( \square \)
Conclusion and open question

- **Conclusion**: In a fairly general path-dependent setting, solving the dual problem provides one solution to the hedging problem.

- **Open question**: In the Markovian setting, and under smoothness conditions, the super-hedging price is the only hedging price. How to prove this in the path-dependent case by simply using probabilistic arguments?

  **Main issue**: the terminal condition $\Xi(X)$ depends on the hedging strategy -> standard comparison does not hold.
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Main issue: the terminal condition $\Xi(X)$ depends on the hedging strategy -> standard comparison does not hold.
Thank you!

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