On the regularity of solutions of some linear parabolic path-dependent PDEs

Bruno Bouchard *   Xiaolu Tan †

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Abstract

We study a class of linear parabolic path-dependent PDEs (PPDEs) defined on the space of càdlàg paths $\mathrm{x} \in D([0,T])$, in which the coefficient functions at time t depend on $\mathrm{x}(t)$ and $\int_0^t \mathrm{x}(s) dA_s$, for some (deterministic) continuous function A with bounded variations. Under uniform ellipticity and Hölder regularity conditions on the coefficients, together with some technical conditions on A, we obtain the existence of a smooth solution to the PPDE by appealing to the notion of Dupire's derivatives. It provides a generalization to the existing literature studying the case where $A_t = t$, and complements our recent work in [2] on the regularity of approximate viscosity solutions for parabolic PPDEs. As a by-product, we also obtain existence and uniqueness of weak solutions for a class of path-dependent SDEs.

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1 Introduction

We consider linear parabolic path-dependent PDEs (PPDEs) of the form

$$\partial_t \mathrm{v} \; + \; \bar{\mu} \partial_{\mathrm{x}} \mathrm{v} \; + \; \frac{1}{2} \bar{\sigma}^2 \partial^2_{\mathrm{x}} \mathrm{v} \; + \; \bar{\ell} = 0, \text{ on } [0,T) \times D([0,T]) \qquad (1.1)$$
$$\mathrm{v}(T, \cdot) = \bar{g} \text{ on } D([0,T]).$$

In the above, $D([0,T])$ denotes the space of all real-valued càdlàg path $\mathrm{x} = (\mathrm{x}(t))_{t \in [0,T]}$ on $[0,T]$, the derivatives are taken in the sense of Dupire [6, 3] (see Section 2.1 below), and the coefficient functions $(\bar{\mu}, \bar{\sigma}, \bar{\ell}, \bar{g}) : [0,T] \times D([0,T]) \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ are of the form

$$\big(\bar{\mu}_t, \bar{\sigma}_t, \bar{\ell}_t, \bar{g}\big)(\mathrm{x}) \; = \; \big(\mu_t, \sigma_t, \ell_t\big)\big(\mathrm{x}(t), I_t(\mathrm{x})\big), \; \bar{g}(\mathrm{x}) = g(\mathrm{x}_T, I_T(\mathrm{x})), \quad \text{with } I_t(\mathrm{x}) := \int_0^t \mathrm{x}(s) dA_s,$$

for some functions $(\mu, \sigma, \ell, g) : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and a continuous process A with bounded variations. When A is absolutely continuous, say simply $A_t = t$, the above

*CEREMADE, Université Paris-Dauphine, PSL, CNRS. bouchard@ceremade.dauphine.fr.
†Department of Mathematics, The Chinese University of Hong Kong. xiaolu.tan@cuhk.edu.hk.

can be written as a degenerate parabolic PDE

$$\partial_t v + \mu \partial_{x_1} v + x_1 \partial_{x_2} v + \frac{1}{2}\sigma^2 \partial^2_{x_1 x_1} v + \ell = 0 \text{ on } [0,T) \times \mathbb{R}^2,\ v(T,\cdot) = g \text{ on } \mathbb{R}^2, \tag{1.2}$$

in which the derivatives are now taken in the usual sense and

$$\mathrm{v}(t,\mathrm{x}) \ = \ v\big(t, \mathrm{x}_t, I_t(\mathrm{x})\big).$$

Indeed, the Dupire's horizontal derivative $\partial_t \mathrm{v}$ and vertical derivatives $(\partial_{\mathrm{x}} \mathrm{v}, \partial^2_{\mathrm{x}} \mathrm{v})$ are related to the partial derivatives of v through

$$(\partial_t, \partial_{\mathrm{x}}, \partial^2_{\mathrm{x}})\mathrm{v}(t,\mathrm{x}) = \big(\partial_t + \mathrm{x}(t)\partial_{x_2}, \partial_{x_1}, \partial^2_{x_1 x_1}\big)\, v\big(t, \mathrm{x}(t), I_t(\mathrm{x})\big).$$

Various works are devoted to such equations, going back to [10], in more complex multivariate frameworks, see e.g. [5, 8, 11, 13, 15] and the references therein. The latter PDE may not admit a $C^{1,2}$-solution, in the traditional sense, even when $\bar\sigma$ is uniformly elliptic: $\partial_t v$ and $\partial_{x_2} v$ are in general not well-defined and one needs to define $\partial_t v + x_1 \partial_{x_2} v$ jointly, appealing to the notion of Lie derivative, which amounts to considering Dupire's horizontal derivative when the PDE is seen as a PPDE.

The main novelty of this paper is that we do not assume anymore that $(A_t)_{t\in[0,T]}$ is absolutely continuous in t. In this case, the PDE formulation (1.2) is not valide anymore, but the PPDE formulation (1.1) is still adequate. We provide conditions under which (1.1) admits a solution that is smooth in the sense of Dupire's deviratives. It complements [2] in which coefficients are assumed to be $C^{1+\alpha}$, which allows one to construct the so-called approximate viscosity solutions of non-linear path-dependent PDEs with first order Dupire's vertical derivative enjoying some Hölder-type regularity (see [2] for details). As shown in e.g. [1], in many situations, this is already sufficient to derive a Feynman-Kac's representation of the solution by appealing to a version of Itô-Dupire's stochastic calculus for path-dependent functionals that are only vertically differentiable up to the first order. In contrast to [2], we only assume here that the coefficients are Hölder continuous, but require $\bar\sigma$ to be non-degenerate, so as to expect the classical regularization effect to operate.

We rely on the parametrix approach, see e.g. [9, Chapter 1]. For this, we perform a change of variables which allows us to reduce to a PDE of the form

$$\partial_t u + \mu\langle(1,A), Du\rangle + \frac{1}{2}\sigma^2(1,A)D^2u(1,A)^\top = 0,$$

which can be written even if A is not absolutely continuous. The above is again degenerate and (Du, D^2u) may not be well-defined. However, the parametrix approach allows one to show that $\langle(1,A), Du\rangle$ and $(1,A)D^2u(1,A)^\top$ are, which in turn implies that the vertical derivatives $(\partial_{\mathrm{x}}\mathrm{v}, \partial^2_{\mathrm{x}}\mathrm{v})$ of the path-dependent functional v are.

As a by-product, we establish the existence and uniqueness of a weak solution to the path-dependent stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t \mu_s(X_s, I_s)ds + \int_0^t \sigma_t(X_s, I_s)dW_t, \quad I_t = \int_0^t X_s dA_s, \quad t \ge 0,$$

and provide some first properties of the transition density of the Markov process (X, I), as well as the corresponding Feynman-Kac's formula.

These results require structural conditions relating the Hölder regularity of the coefficient (μ, σ) and the path behavior of A. If one knows a priori that the above SDE admits a unique weak solution, then one can prove under weaker conditions that the candidate solution to (1.1), deduced from a formal application of the Feynman-Kac's formula[1], is already C^1 in space, in the sense of Dupire. As mentioned above, this turns out to be enough to deduce its Itô-Dupire's semimartingale decomposition.

All over this paper, we stick to a one-dimensional setting for ease of notations. Extensions to multivariate frameworks can be provided by using similar techniques.

The rest of this paper is organized as follows. Section 2 states our main results. Proofs are collected in Section 3.

In the following, the i-th component of a vector x is denoted by x_i, the (i,j)-component of a matrix M is denoted by M_{ij}. Given $\phi : (t,x) \in [0,T] \times \mathbb{R}^2 \longrightarrow \phi(t,x) \in \mathbb{R}$, we let $D\phi$ and $D^2\phi$ (or $D_x\phi$ and $D^2_{xx}\phi$) be the gradient and the Hessian matrix with respect to x. The space partial derivatives are denoted by $\partial_{x_i}\phi$, $\partial^2_{x_i x_j}\phi$, and so on if we have to consider higher orders.

2 Dupire's regularity for linear PPDEs depending on the average of the path

2.1 Notations and assumptions

Given $T > 0$, let $D([0,T])$ denote the Skorokhod space of all $\mathbb{R}$–valued càdlàg paths $\mathrm{x} = (\mathrm{x}(t))_{t\in[0,T]}$ on $[0,T]$, and let $C([0,T])$ denote the subspace of continuous paths. Let us equipped $D([0,T])$ with the Skorokhod topology, and $C([0,T])$ with the uniform convergence topology. Let $A = (A_t)_{t\geq 0}$ be a deterministic continuous process with finite variation, and $(\mu, \sigma, \ell) : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be coefficient functions, from which we define path-dependent functionals $(\bar\mu, \bar\sigma, \bar\ell) : [0,T] \times D([0,T]) \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by

$$\big(\bar\mu_t, \bar\sigma_t, \bar\ell_t\big)(\mathrm{x}) \;:=\; \big(\mu_t, \sigma_t, \ell_t\big)\big(\mathrm{x}(t), I_t(\mathrm{x})\big), \quad \text{with } I_t(\mathrm{x}) := \int_0^t \mathrm{x}(s)dA_s.$$

We study the following linear parabolic path-dependent PDE (PPDE):

$$\partial_t \mathrm{v} \;+\; \bar\mu \partial_{\mathrm{x}} \mathrm{v} \;+\; \frac{1}{2}\bar\sigma^2 \partial^2_{\mathrm{x}} \mathrm{v} \;+\; \bar\ell \;=\; 0, \text{ on } [0,T) \times D([0,T]), \tag{2.1}$$

with terminal condition $\mathrm{v}(T,\mathrm{x}) = \bar g(\mathrm{x}) := g\big(\mathrm{x}(T), I_T(\mathrm{x})\big)$ for some function $g : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$. In the above, the derivatives are taken in the sense of Dupire.

[1]Or more rigorously its viscosity solution in the sense of [4, 16], see also e.g. [2, 7, 12] and the references therein for an alternative definition.

Dupire's derivatives for path-dependent functionals To give a precise definition to the PPDE (2.1), let us recall Dupire's [6, 3] notion of horizontal derivative ∂_t and vertical derivatives ∂_{x} and ∂^2_{x} for path-dependent functionals.

Let $F : [0,T] \times D([0,T]) \longrightarrow \mathbb{R}$ be a path-dependent functional, it is said to be non-anticipative if $F(s,\mathrm{x}) = F(s,\mathrm{x}(s \wedge \cdot))$ for all $(s,\mathrm{x}) \in [0,T] \times D([0,T])$. For an non-anticipative map F, its horizontal derivative $\partial_s F(s,\mathrm{x})$ at $(s,\mathrm{x}) \in [0,T] \times D([0,T])$ is defined as

$$\partial_s F(s,\mathrm{x}) \ := \ \lim_{h \searrow 0} \frac{F(s+h, \mathrm{x}(s \wedge \cdot)) - F(s,\mathrm{x})}{h},$$

and its vertical derivative $\partial_{\mathrm{x}} F(s,\mathrm{x})$ is defined as

$$\partial_{\mathrm{x}} F(s,\mathrm{x}) \ := \ \lim_{y \to 0} \frac{F(s, \mathrm{x} + y\mathbf{1}_{[s,T]}) - F(s,\mathrm{x})}{y},$$

whenever the limits exist. In the above, $\mathrm{x} + y\mathbf{1}_{[s,T]}$ denotes the path taking value $\mathrm{x}_t + y\mathbf{1}_{[s,T]}(t)$ at time $t \in [0,T]$. Similarly, one can define the second order vertical derivative $\partial^2_{\mathrm{x}} F$ as the vertical derivative of $\partial_{\mathrm{x}} F$. Given $t \in (0,T]$, we denote by $\mathbb{C}([0,t))$ the space of all continuous non-anticipative functionals $F : [0,t) \times D([0,T]) \longrightarrow \mathbb{R}$, and we set

$$\mathbb{C}^{0,1}([0,t)) := \big\{ F \in \mathbb{C}([0,t)) \ : \partial_{\mathrm{x}} F \text{ is well-defined and belongs to } \mathbb{C}([0,t)) \big\},$$

as well as

$$\mathbb{C}^{1,2}([0,t)) := \big\{ F \in \mathbb{C}^{0,1}([0,t)) \ : \partial_s F \text{ and } \partial^2_{\mathrm{x}} F \text{ are well-defined and belong to } \mathbb{C}([0,t)) \big\}.$$

Assumptions on the process A: Recall that $A = (A_t)_{t \in [0,T]}$ is a deterministic process with finite variation. For $0 \le s < t \le T$, let us define $\overline{A}_{s,t} := \frac{1}{t-s} \int_s^t A_r dr$ and

$$m_{s,t} := \frac{1}{t-s} \int_s^t \big(A_r - \overline{A}_{s,t} \big)^2 dr, \quad \tilde{m}_{s,t} := \frac{1}{t-s} \int_s^t (A_r - A_s)^2 dr.$$

The above will play a major role in our analysis, as they will drive the behavior of the parametrix density on small time intervals.

Assumption 2.1. (i) *There exist constants $\beta_0, \beta_1, \beta_2, \beta_3 \ge 0$ and $C_{(2.2)}, C_{(2.3)} > 0$ such that, for all $0 \le s < t \le T$,*

$$\frac{1}{C_{(2.2)}} (t-s)^{-\beta_1} \ \le \ \frac{\tilde{m}_{s,t}}{m_{s,t}} \ \le \ C_{(2.2)} (t-s)^{-\beta_0}, \tag{2.2}$$

$$\frac{1}{C_{(2.3)}} (t-s)^{-\beta_2} \ \le \ \frac{1}{m_{s,t}} \ \le \ C_{(2.3)} (t-s)^{-\beta_3}. \tag{2.3}$$

(ii) *There exist constants $\beta_4 \ge 0$ and $C_{(2.4)} > 0$ such that*

$$|A_t - A_s| \ \le \ C_{(2.4)} (t-s)^{\beta_4}, \quad \textit{for all } 0 \le s < t \le T. \tag{2.4}$$

Remark 2.2. *Notice that, $\lim_{t \downarrow s} \tilde{m}_{s,t} = 0$ by continuity of A. Without loss of generality, one can therefore assume that*

$$\beta_1 \le \beta_0 \le \beta_3 \ \textit{and} \ \beta_1 \le \beta_2 \le \beta_3.$$

Moreover, one can always choose $\beta_0 = \beta_4 = 0$ since $m_{s,t} \le \tilde{m}_{s,t}$ by their definitions.

Let us provide some typical examples.

Example 2.3. (i) *Let A be defined by $A_t = \int_0^t \rho(s)ds$, $t \geq 0$, with $\varepsilon \leq \rho \leq 1/\varepsilon$ a.e. for some $\varepsilon > 0$. Then, it is easy to check that Assumption 2.1 holds with*

$$\beta_0 = 0, \quad \beta_1 = 0, \quad \beta_2 = 2, \quad \beta_3 = 2, \quad \beta_4 = 1.$$

In this setting, our main results are similar to those in [8], which studied a multivariate version of the case where ρ is constant.

(ii) *Let $A_t = t^\gamma$ for some $\gamma \in (0,1)$. Then,*

$$m_{s,t} \;=\; \frac{t^{2\gamma+1} - s^{2\gamma+1}}{(2\gamma+1)(t-s)} - \frac{|t^{\gamma+1} - s^{\gamma+1}|^2}{(\gamma+1)^2(t-s)^2},$$

and

$$\tilde{m}_{s,t} \;=\; \frac{\frac{1}{2\gamma+1}(t^{2\gamma+1} - s^{2\gamma+1}) - 2\frac{1}{\gamma+1}(t^{\gamma+1} - s^{\gamma+1})s^\gamma + (t-s)s^{2\gamma}}{t-s}.$$

In this setting, Assumption 2.1 holds true with

$$\beta_0 = 0, \quad \beta_1 = 0, \quad \beta_2 = 2\gamma, \quad \beta_3 = 2\gamma, \quad \beta_4 = \gamma.$$

(iii) *Assume that there exists $1 \geq \gamma_1 \geq \gamma_2 > 0$ and $C_1, C_2 > 0$ such that*

$$C_1|t-s|^{\gamma_1} \leq A_t - A_s \leq C_2|t-s|^{\gamma_2}, \text{ for all } s \leq t \leq T.$$

Then, Assumption 2.1 holds with

$$\beta_0 = 2(\gamma_1 - \gamma_2), \quad \beta_1 = 0, \quad \beta_2 = 2\gamma_2, \quad \beta_3 = 2\gamma_1, \quad \beta_4 = \gamma_2.$$

Indeed, let us choose $t_0 \in [0,T]$ such that $A_{t_0} = \overline{A}_{s,t}$. Assume that $t - t_0 \geq (t-s)/2$ (otherwise $t_0 - s \geq (t-s)/2$ and we can use similar computations), then

$$(t-s)m_{s,t} \;=\; \int_s^t |A_r - A_{t_0}|^2 \;\geq\; \int_{t_0}^t C_1^2|r - t_0|^{2\gamma_1}dr \;\geq\; \frac{C_1^2}{2^{2\gamma_1+1}(2\gamma_1+1)}|t-s|^{2\gamma_1+1}.$$

On the other hand, $(t-s)m_{s,t} \leq \int_s^t |A_r - A_s|^2 \leq \frac{C_2^2}{2\gamma_2+1}|t-s|^{2\gamma_2+1}$. Thus,

$$1 \;\leq\; \frac{\tilde{m}_{s,t}}{m_{s,t}} \;\leq\; 2^{2\gamma_1+1}\frac{C_2^2(2\gamma_1+1)}{C_1^2(2\gamma_2+1)}|t-s|^{-2(\gamma_1-\gamma_2)}.$$

Assumptions on the coefficient functions μ and σ: As in e.g. [8], the following Hölder regularity assumption on the coefficient functions $(\mu,\sigma) : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R} \times \mathbb{R}$ is calibrated to match with the explosion rate of the quadratic form entering the parametrix. It does not impose smoothness conditions on μ and σ as in e.g. [13], and facilitate the analysis, see Remark 2.6 below. Let us set

$$\Theta \;:=\; \big\{(s,x,t,y) \in [0,T] \times \mathbb{R}^2 \times [0,T] \times \mathbb{R}^2 \;:\; s < t\big\},$$

and, for $(s,x,t,y) \in \Theta$,

$$\mathrm{w}_{s,t}(x,y) := x - E_{s,t}(y) \in \mathbb{R}^2, \quad \text{with} \quad E_{s,t}(y) := \begin{pmatrix} 1 & 0 \\ -(A_t - A_s) & 1 \end{pmatrix} y \in \mathbb{R}^2. \tag{2.5}$$

Assumption 2.4. *Let $\beta_0, \beta_1, \beta_2 \geq 0$ be the constants in Assumption 2.1. Then,*
(i) *We have*

$$\beta_1' := \beta_1 - \beta_0 > -1, \qquad \beta_2' := \beta_2 - \beta_0 > -1.$$

Moreover, the coefficients μ and σ are continuous, and there exist constants $(\underline{\mathfrak{a}}, \bar{\mathfrak{a}}) \in \mathbb{R}^2$, $\mathfrak{b} \in \mathbb{R}$, $C_{(2.7)} > 0$ and $\alpha > 0$ such that

$$|\mu| \leq \mathfrak{b}, \quad 0 < \underline{\mathfrak{a}} \leq \sigma^2 \leq \bar{\mathfrak{a}}, \quad \text{on } [0,T] \times \mathbb{R}^2, \tag{2.6}$$

and

$$\big|\sigma_s(x) - \sigma_t(y)\big| \;\leq\; C_{(2.7)}\Big(|t-s|^{\alpha} + \big|\mathrm{w}_{s,t}(x,y)\big|^{\frac{2\alpha}{1+\beta_1'}} + \big|\mathrm{w}_{s,t}(x,y)\big|^{\frac{2\alpha}{1+\beta_2'}}\Big), \tag{2.7}$$

for all $(s,x,t,y) \in \Theta$.
(ii) *There exists a constant $C_{(2.8)} > 0$ such that*

$$\big|\mu_t(x) - \mu_t(y)\big| \leq C_{(2.8)}\Big(\big|x_1 - y_1\big|^{\frac{2\alpha}{1+\beta_1'}} + \big|x_2 - y_2\big|^{\frac{2\alpha}{1+\beta_2'}}\Big), \text{ for all } (t,x,y) \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2. \tag{2.8}$$

Example 2.5. *The condition* (2.7) *holds for instance if $\sigma_s(x_1,x_2)$ depends only on (s,x_1) and is Hölder with respect to (s,x_1). It would also hold if it is of the form $\sigma_s(x) = \tilde{\sigma}_s(x_1, x_2 - A_s x_1)$ for some Hölder continuous map $\tilde{\sigma}$. Indeed, one has*

$$\begin{aligned}
\big|x_2 - A_s x_1 - (y_2 - A_t y_1)\big| &= \big|x_2 - A_s x_1 - (y_2 - A_t y_1) - A_s(y_1 - x_1) + A_s(y_1 - x_1)\big| \\
&\leq \big|x_2 - y_2 + (A_t - A_s) y_1)\big| + |A_s|\big|y_1 - x_1\big| \\
&\leq \Big(1 + \max_{[0,T]} |A|\Big)|\mathrm{w}_{s,t}(x,y)|.
\end{aligned}$$

Remark 2.6. *The case where the coefficient σ is Hölder in the classical sense, i.e.*

$$\big|\sigma_s(x) - \sigma_t(y)\big| \;\leq\; C\Big(|t-s|^{\alpha} + \big|x_1 - y_1\big|^{\frac{2\alpha}{1+\beta_1'}} + \big|x_2 - y_2\big|^{\frac{2\alpha}{1+\beta_2'}}\Big)$$

can be tackled by combining the arguments below with those of e.g. [13]. This will add additional exponentially growing terms in the estimates on $\widetilde{\Phi}$ in Proposition 3.5 below, which can be handled, to the price of adapted restrictions on the coefficients $(\beta_i)_{0 \leq i \leq 4}$. We chose the formulation of the conditions in (2.7) *for sake of simplicity.*

2.2 Heuristic derivation using a change of variables and the parametrix method

Let us consider the path-dependent SDE

$$X_t = X_0 + \int_0^t \mu_s(X_s, I_s) ds + \int_0^t \sigma_t(X_s, I_s) dW_s, \quad I_t = \int_0^t X_s dA_s, \quad t \in [0,T], \tag{2.9}$$

where W is a Brownian motion. Assume that the above SDE has a solution X such that (X, I) is Markov. Then, to deduce a solution to the PPDE (2.1), it suffices to find the transition probability (density) function $f(s,x;t,y)$ of the Markov process (X,I) from (s,x)

to (t,y). When $t \longmapsto A_t$ is absolutely continuous, it is well-known that $(s,x) \longmapsto f(s,x;t,y)$ solves a Kolmogorov's backward PDE and that $(t,y) \longmapsto f(s,x;t,y)$ solves a Kolmogorov's forward PDE. One can then apply the classical parametrix method as in [8, Section 4] to guess the expression of $f(s,x;t,y)$.

In our setting where A is not necessarily absolutely continuous, it is no more possible to write the Kolmogorov's PDE for the transition probability (density) function of (X,I). We therefore perform a change of variable and set

$$\widetilde{X}_t := \mathbf{A}_t \begin{pmatrix} X_t \\ I_t \end{pmatrix}, \text{ with } \mathbf{A}_t := \begin{pmatrix} 1 & 0 \\ A_t & -1 \end{pmatrix}, \quad t \in [0,T]. \tag{2.10}$$

Notice that $\mathbf{A}_t^{-1} = \mathbf{A}_t$ and that $\widetilde{X}$ is a diffusion process with dynamics

$$\widetilde{X}_t \; = \; \widetilde{X}_0 + \int_0^t \tilde{\mu}_s(\widetilde{X}_s)\overrightarrow{\mathrm{A}}_s ds + \int_0^t \tilde{\sigma}_s(\widetilde{X}_s)\overrightarrow{\mathrm{A}}_s dW_s, \quad t \in [0,T], \tag{2.11}$$

where $\overrightarrow{\mathrm{A}}$, $\tilde{\mu} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\tilde{\sigma} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are defined by

$$\overrightarrow{\mathrm{A}}_s := \begin{pmatrix} 1 \\ A_s \end{pmatrix}, \; \tilde{\mu}_s(x) := \mu_s(\mathbf{A}_s x) \text{ and } \tilde{\sigma}_s(x) := \sigma_s(\mathbf{A}_s x), \; (s,x) \in [0,T] \times \mathbb{R}^2. \tag{2.12}$$

The generator $\widetilde{\mathcal{L}}$ of $\widetilde{X}$ is given by

$$\widetilde{\mathcal{L}}\phi(s,x) \; := \; \tilde{\mu}_s(x) \; \overrightarrow{\mathrm{A}}_s \cdot D\phi(s,x) + \frac{1}{2}\tilde{\sigma}_s(x)^2 \; \mathrm{Tr}\left[\overrightarrow{\mathrm{A}}_s(\overrightarrow{\mathrm{A}}_s)^\top D^2\phi(s,x)\right],$$

for smooth functions $\phi : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$.
Assume that the SDE (2.11) has a solution $\widetilde{X}$ which is Markovian and has a smooth transition probability density function $\tilde{f}(s,x;t,y)$, from x at s to y at t, then $(s,x) \longmapsto \tilde{f}(s,x;t,y)$ solves the Kolmogorov backward equation

$$\big(\partial_s + \widetilde{\mathcal{L}}\big)\tilde{f}(s,x;t,y) = 0, \quad \text{for } (s,x) \in [0,t) \times \mathbb{R}^2. \tag{2.13}$$

Notice that, in the above, the operator $\widetilde{\mathcal{L}}$ acts on the first two arguments (s,x) of $\tilde{f}(s,x;t,y)$.

To construct the parametrix, we consider the following process, with volatility frozen at $(r,z) \in [0,T] \times \mathbb{R}^2$,

$$\widetilde{X}^{r,z}_t \; := \; \tilde{\sigma}_r(z) \int_0^t \overrightarrow{\mathrm{A}}_s dW_s, \quad t \in [0,T].$$

The corresponding generator $\widetilde{\mathcal{L}}^{r,z}$ is then given by

$$\widetilde{\mathcal{L}}^{r,z}\phi(s,x) \; := \; \frac{1}{2}\tilde{\sigma}_r(z)^2 \; \mathrm{Tr}\left[\overrightarrow{\mathrm{A}}_s(\overrightarrow{\mathrm{A}}_s)^\top D^2\phi(s,x)\right], \text{ for smooth functions } \phi.$$

We further define $\tilde{f}_{r,z}(s,x;t,y)$ as the corresponding transition probability function from (s,x) to (t,y), for $(s,x,t,y) \in \Theta$. Notice that $\tilde{f}_{r,z}$ is explicitly given and that $y \longmapsto \tilde{f}_{r,z}(s,x;t,y)$ is the density function of the Gaussian random vector $x + \tilde{\sigma}_r(z) \int_s^t \overrightarrow{\mathrm{A}}_r dW_r$. It satisfies

$$\big(\partial_s + \widetilde{\mathcal{L}}^{r,z}\big)\tilde{f}_{r,z}(s,x;t,y) = 0, \quad \text{for } (s,x) \in [0,t) \times \mathbb{R}^2. \tag{2.14}$$

Now, we employ the machinery of the parametrix method (see e.g. [9, Chapter 1] or [8]), taking $\tilde{f}_{t,y}(s,x;t,y)$ as parametrix, and expressing $\tilde{f}(s,x;t,y)$ in the following form:

$$\tilde{f}(s,x;t,y) = \tilde{f}_{t,y}(s,x;t,y) + \int_s^t \int_{\mathbb{R}^2} \tilde{f}_{r,z}(s,x;r,z)\widetilde{\Phi}(r,z,t,y)dzdr, \tag{2.15}$$

for some function $\widetilde{\Phi} : \Theta \longrightarrow \mathbb{R}$. By (2.13) and (2.14), one must have

$$\begin{aligned}
0 &= \big(\partial_s + \widetilde{\mathcal{L}}\big)\tilde{f}(s,x;t,y) \\
&= \big(\partial_s + \widetilde{\mathcal{L}}\big)\tilde{f}_{t,y}(s,x;t,y) + \big(\partial_s + \widetilde{\mathcal{L}}\big)\int_s^t \int_{\mathbb{R}^2} \tilde{f}_{r,z}(s,x;r,z)\widetilde{\Phi}(r,z,t,y)dzdr \\
&= \big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,y}\big)\tilde{f}_{t,y}(s,x;t,y) - \widetilde{\Phi}(s,x;t,y) \\
&\quad + \int_s^t \int_{\mathbb{R}^2} \big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,z}\big)\tilde{f}_{r,z}(s,x;r,z)\widetilde{\Phi}(r,z;t,y)dzdr.
\end{aligned}$$

Therefore, $\widetilde{\Phi}$ must satisfy

$$\widetilde{\Phi}(s,x;t,y) = \big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,y}\big)\tilde{f}_{t,y}(s,x;t,y) + \int_s^t \int_{\mathbb{R}^2} \big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,z}\big)\tilde{f}_{r,z}(s,x;r,z)\widetilde{\Phi}(r,z;t,y)dzdr. \tag{2.16}$$

In view of (2.16), we obtain

$$\widetilde{\Phi}(s,x;t,y) := \sum_{k=0}^{\infty} \widetilde{\Delta}_k(s,x;t,y), \tag{2.17}$$

where $\widetilde{\Delta}_0(s,x;t,y) := \big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,y}\big)\tilde{f}_{t,y}(s,x;t,y)$, and

$$\widetilde{\Delta}_{k+1}(s,x;t,y) := \int_s^t \int_{\mathbb{R}^2} \widetilde{\Delta}_0(s,x;r,z)\widetilde{\Delta}_k(r,z;t,y)dzdr, \quad k \geq 0. \tag{2.18}$$

Notice that $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{L}}^{t,y}$ and $\tilde{f}_{t,y}$ have explicit expressions. The main strategy of the classical parametrix method consists in checking that $\widetilde{\Phi}$ in (2.17) is well-defined and solves the integral equation (2.16). Then, one defines $\tilde{f}$ by (2.15), and check that it provides a solution to (2.13). If $\tilde{f}$ is smooth, one can basically deduce that $\tilde{f}$ is the transition probability density function of $\widetilde{X}$ in (2.11) by using the Feynman-Kac's formula.

The main difficulty here lies in the fact that $\tilde{f}$ is, in general, not smooth enough. For smoothness properties, we will therefore turn back to the initial coordinates (X, I) and define the candidate transition probability function f of the process (X, I) in (2.9) through (2.19)-(2.20) below, and work on it directly.

2.3 Main results

Under some conditions on the constants α and $(\beta)_{i=0,\cdots,4}$ given in Assumptions 2.1 and 2.4, we will show that $\widetilde{\Phi} : \Theta \longrightarrow \mathbb{R}$ is well-defined by (2.17)-(2.18). For $(r,z) \in [0,T] \times \mathbb{R}^2$, we can then define $f_{r,z}$ and Φ by inverting the change of variables in (2.10):

$$f_{r,z}(s,x;t,y) := \tilde{f}_{r,\mathbf{A}_r z}(s,\mathbf{A}_s x;t,\mathbf{A}_t y), \quad \Phi(s,x;t,y) := \widetilde{\Phi}(s,\mathbf{A}_s x;t,\mathbf{A}_t y). \tag{2.19}$$

The corresponding candidate transition density $f : \Theta \longrightarrow \mathbb{R}$ for (X, I) is therefore:

$$f(s,x;t,y) \;:=\; f_{t,y}(s,x;t,y) + \int_s^t \int_{\mathbb{R}^2} f_{r,z}(s,x;r,z)\Phi(r,z;t,y)dzdr, \tag{2.20}$$

for all $(s,x,t,y) \in \Theta$.
For any positive constant $a \in \mathbb{R}_+$ and $0 \le s < t \le T$, let us set

$$\Sigma_{s,t}(a) \;:=\; a\begin{pmatrix} t-s & -\int_s^t (A_r - A_s)dr \\ -\int_s^t (A_r - A_s)dr & \int_s^t (A_r - A_s)^2 dr \end{pmatrix}.$$

For $(r,z) \in [0,T] \times \mathbb{R}^2$, we write $\Sigma_{s,t}(r,z) := \Sigma_{s,t}(\sigma_r^2(z))$ for simplicity. Equivalently,

$$\Sigma_{s,t}(r,z) \;:=\; \sigma_r^2(z)\begin{pmatrix} t-s & -\int_s^t (A_r - A_s)dr \\ -\int_s^t (A_r - A_s)dr & \int_s^t (A_r - A_s)^2 dr \end{pmatrix}. \tag{2.21}$$

Then, it is easy to check that $y \longmapsto f_{r,z}(s,x;t,y)$ is the density function of the Gaussian random vector

$$\Big(x_1 + \sigma_r(z)(W_t - W_s),\ x_2 + \int_s^t \big(x_1 + \sigma_r(z)(W_u - W_s)\big)dA_u\Big)^\top,$$

so that, with $w := \mathrm{w}_{s,t}(x,y)$ as in (2.5),

$$f_{r,z}(s,x;t,y) \;=\; \frac{1}{2\pi\ \det\big(\Sigma_{s,t}(r,z)\big)^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}\big\langle \Sigma_{s,t}^{-1}(r,z)w, w\big\rangle\Big).$$

Let us also define the Gaussian transition probability function $f^\circ : \Theta \longrightarrow \mathbb{R}$ by

$$f^\circ(s,x;t,y) \;:=\; \frac{1}{2\pi\ \det\big(\Sigma_{s,t}(4\bar{\mathfrak{a}})\big)^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}\big\langle \Sigma_{s,t}^{-1}(4\bar{\mathfrak{a}})w, w\big\rangle\Big), \quad (s,x,t,y) \in \Theta. \tag{2.22}$$

As a first main result, we show that f is well-defined under some conditions on the coefficients α and β_0, and then provide some first regularity and bound estimates.

Theorem 2.7. *Let Assumption 2.1.*(i) *and Assumption 2.4.*(i) *hold true.*

(i) *Assume that*

$$\kappa_0 \;:=\; \frac{1-\beta_0}{2} \wedge (\alpha - \beta_0) \;>\; 0. \tag{2.23}$$

Then $\widetilde{\Phi}$ in (2.17)-(2.18) *is well-defined, and so is $f : \Theta \longrightarrow \mathbb{R}$ in* (2.20)*. Moreover, f is continuous on Θ, and there exists a constant $C > 0$ such that*

$$\big|f(s,x;t,y)\big| \;\le\; Cf^\circ(s,x;t,y), \quad \text{for all } (s,x,t,y) \in \Theta. \tag{2.24}$$

(ii) *Assume that* (2.23) *holds and that*

$$\kappa_1 \;:=\; \kappa_0 + \frac{1-\beta_0}{2} \;=\; (1-\beta_0) \wedge \big(\frac{1}{2} + \alpha - \frac{3}{2}\beta_0\big) \;>\; 0. \tag{2.25}$$

Then, the partial derivative $(s,x;t,y) \in \Theta \mapsto \partial_{x_1} f(s,x;t,y)$ exists, is continuous on Θ, and, for some constant $C_{(2.26)} > 0$,

$$\big|\partial_{x_1} f(s,x;t,y)\big| \;\le\; \frac{C_{(2.26)}}{(t-s)^{1-\kappa_1}} f^\circ(s,x;t,y), \ \text{for all } (s,x,t,y) \in \Theta. \tag{2.26}$$

Under further conditions, we can obtain more regularity of f and then check that it is the transition probability function of the Markov process (X, I). To be more precise, let us rephrase this in terms of path-dependent functionals. For $0 \le s < t \le T$, $\mathrm{x} \in D([0,T])$ and $y \in \mathbb{R}^2$, we set

$$(\mathrm{f}, \mathrm{f}^\circ)(s, \mathrm{x}; t, y) \;:=\; (f, f^\circ)\big(s, \mathrm{x}(s), I_s(\mathrm{x}); t, y\big), \text{ with } I_s(\mathrm{x}) := \int_0^s \mathrm{x}(r) dA_r. \tag{2.27}$$

We now fix $\ell : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that, for some constants $C_{\ell,g} > 0$ and $\alpha_\ell > 0$,

$$\big|\ell(t,x)\big| + \big|g(x)\big| \;\le\; C_{\ell,g} \exp\big(C_{\ell,g}|x|\big), \tag{2.28}$$

and

$$\big|\ell(t,x) - \ell(t,x')\big| \;\le\; C_{\ell,g}\big(e^{C_{\ell,g}|x|} + e^{C_{\ell,g}|x'|}\big)\Big(|x_1 - x_1'|^{\frac{2\alpha_\ell}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{2\alpha_\ell}{1+\beta_2'}}\Big), \tag{2.29}$$

for all $t \in [0,T]$ and $x, x' \in \mathbb{R}^2$. In view of the upper-bound estimate of f in (2.24), we can then define

$$\mathrm{v}(s,\mathrm{x}) \;:=\; \int_s^T \int_{\mathbb{R}^2} \ell(t,y) \mathrm{f}(s,\mathrm{x};t,y) dy dt \;+\; \int_{\mathbb{R}^2} g(y) \mathrm{f}(s,\mathrm{x};T,y) dy, \quad (s,\mathrm{x}) \in [0,T) \times D([0,T]). \tag{2.30}$$

Remark 2.8. *By its definition in* (2.27)*, it is straightforward to check that*

$$\partial_{\mathrm{x}} \mathrm{f}(s,\mathrm{x};t,y) \;=\; \partial_{x_1} f\big(s, \mathrm{x}(s), I_s(\mathrm{x}); t, y\big).$$

Similarly, let us define, for $(r,z), (t,y) \in [0,T] \times \mathbb{R}^d$,

$$\mathrm{f}_{r,z}(s,\mathrm{x};t,y) \;:=\; f_{r,z}(s, \mathrm{x}(s), I_t(\mathrm{x}); t, y), \;\; (s,\mathrm{x}) \in [0,t) \times D([0,T]).$$

Then, the functional $(s,\mathrm{x}) \longmapsto \mathrm{f}_{r,z}(s,\mathrm{x};t,y)$ is a classical solution to the PPDE

$$\partial_s \mathrm{f}_{r,z}(s,\mathrm{x};t,y) + \frac{1}{2}\sigma_r(z)^2 \partial^2_{\mathrm{xx}} \mathrm{f}_{r,z}(s,\mathrm{x};t,y) = 0, \;\; \textit{for } (s,\mathrm{x}) \in [0,t) \times D([0,T]). \tag{2.31}$$

Theorem 2.9. *Let Assumptions 2.1 and 2.4 hold true. Assume that* (2.23)*,* (2.25)*,* (2.28) *and* (2.29) *hold, and that there exists $\alpha_\Phi \in \mathbb{R}$ such that*

$$0 < \alpha_\Phi < \kappa_0 \wedge \hat{\alpha}_\Phi \wedge \min_{i=1,2} \frac{1+\beta_i'}{2}, \quad \textit{with} \quad \hat{\alpha}_\Phi := \frac{1}{2} - \beta_0 - \frac{\widehat{\Delta\beta}}{2} - \frac{(\beta_0 + 1 - 2\alpha)^+}{2},$$

where

$$\widehat{\Delta\beta} := \max\left\{\beta_0 - \beta_1 \;,\; \beta_3 - \beta_2\right\}, \tag{2.32}$$

and

$$\min\Big(\frac{2\beta_4 + 1 + \beta_1'}{1+\beta_2'}, 1\Big) \min\{\alpha_\Phi, \alpha_\ell, \alpha\} - \beta_0 > 0.$$

(i) *For each $(t,y) \in (0,T] \times \mathbb{R}^2$, the path-dependent functional* $\mathrm{f}(\cdot; t, y)$ *belongs to* $\mathbb{C}^{1,2}([0,t))$.

(ii) $\mathrm{v} \in \mathbb{C}^{1,2}([0,T))$ *and it solves the PPDE* (2.1)*. Moreover, there exists* $C > 0$ *such that, for all* $(s, \mathrm{x}) \in [0,T) \times D([0,T])$,

$$\big|\partial_{\mathrm{x}} \mathrm{v}(s,\mathrm{x})\big| \;\le\; \frac{Ce^{C(|\mathrm{x}_s|+|I_s(\mathrm{x})|)}}{(T-s)^{1-\kappa_1}}, \text{ and } \big|\partial_t \mathrm{v}(s,\mathrm{x})\big| + \big|\partial^2_{\mathrm{x}} \mathrm{v}(s,\mathrm{x})\big| \;\le\; \frac{Ce^{C(|\mathrm{x}_s|+|I_s(\mathrm{x})|)}}{(T-s)^{1+\beta_0}}. \tag{2.33}$$

If, in addition, $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous, then* v *is the unique classical solution to the PPDE* (2.1) *satisfying*

$$\lim_{t \nearrow T} \mathrm{v}(t,\mathrm{x}) = \bar{g}(\mathrm{x}) \text{ and } |\mathrm{v}(s,\mathrm{x})| \le Ce^{C(|\mathrm{x}_s|+|I_s(\mathrm{x})|)} \tag{2.34}$$

for all $(s,\mathrm{x}) \in [0,T] \times C([0,T])$*, for some* $C > 0$.

(iii) *The SDE* (2.9) *has a unique weak solution* X*. Moreover,* (X, I) *is a strong Markov process with transition probability given by* f *and*

$$\mathrm{v}(s,\mathrm{x}) \;=\; \mathbb{E}\Big[\int_s^T \ell(X_t, I_t)dt + g(X_T, I_T)\Big| X_s = \mathrm{x}(s), I_s = I_s(\mathrm{x})\Big],\ (s,\mathrm{x}) \in [0,T] \times D([0,T]). \tag{2.35}$$

Remark 2.10. *To check the conditions on* α *and* β_i*,* $i = 0, \cdots, 4$ *in Theorem 2.9, let us stay in the setting of Example 2.3.*

(i) – (ii) *In these cases,* $\kappa_0 = \frac{1}{2} \wedge \alpha$*,* $\kappa_1 = 1 \wedge (\alpha + \frac{1}{2})$*,* $\hat{\alpha}_\Phi = \frac{1}{2} - [\frac{1}{2} - \alpha]^+ = \alpha \wedge \frac{1}{2}$ *and we can choose* $\alpha_\Phi \in (0, \frac{1}{2} \wedge \alpha)$.
(iii) *In this case,* $\kappa_0 = \frac{1-2(\gamma_1-\gamma_2)}{2} \wedge (\alpha - 2(\gamma_1 - \gamma_2))$ *which requires that* $\gamma_1 - \gamma_2 < \frac{1}{2} \wedge \frac{\alpha}{2}$ *to ensure that* $\kappa_0 > 0$*. Then,* $\kappa_1 > 0$ *and* $\hat{\alpha}_\Phi = \frac{1}{2} - 3(\gamma_1 - \gamma_2) - [\gamma_1 - \gamma_2 + \frac{1}{2} - \alpha]^+$*. If* $2\alpha/(1 + \beta_1') \le 1$*, then* $\alpha \le 1/2$ *and therefore* $\gamma_1 - \gamma_2 + \frac{1}{2} - \alpha \ge 0$*. In this case, we can choose* $\alpha_\Phi \in (0, \alpha - 4(\gamma_1 - \gamma_2))$ *if* $\gamma_1 - \gamma_2 < \alpha/4$*. If* $2\alpha/(1 + \beta_1') > 1$*, then* (μ, σ) *does no depend on its first argument, and the different cases can also be treated explicitly, leading to a suitable* α_Φ *when* $\gamma_1 - \gamma_2$ *is small enough.*

The conditions in Theorem 2.9 ensure that f is smooth enough, so that one can basically apply the Feynman-Kac's formula to justify that it is the transition probability function of a Markov process. It can then be used to prove that the wellposedness (existence and uniqueness) of the SDE (2.9). If one already knows that the SDE (2.9) has a unique a weak solution, then one can rely on Theorem 2.11 below, which requires less technical conditions on A and (μ, σ), to check that f is the corresponding transition probability function. In this case, the path-dependent functional v defined above may only be $\mathbb{C}^{0,1}([0,T))$, but it is enough to deduce that (2.35) holds, and obtain it's Itô-Dupire's decomposition, whenever it satisfies for instance one of the conditions a. or b. of Theorem 2.11 .

Theorem 2.11. *Let Assumption 2.1.(i) and Assumption 2.4.(i) hold true, and assume that the SDE* (2.9) *has a unique weak solution, so that the corresponding process* (X, I) *is a strong Markov process.*

(i) *Assume in addition that* (2.23) *holds true so that* f *is well defined. Then,* f *is the transition probability function of* (X, I)*, and* (2.35) *holds whenever* (2.28) *does.*

(ii) *Assume that* (2.23), (2.25) *and* (2.28) *hold. Then,* $\mathrm{v} \in \mathbb{C}^{0,1}([0,T))$. *Suppose in addition that one of the following holds:*

(a) *there exists* $C_{(2.36)} > 0$ *such that*

$$\big|\mathrm{v}(t,\mathrm{x}) - \mathrm{v}(t,\mathrm{x}')\big| \;\le\; C_{(2.36)} \int_0^t |\mathrm{x}(r) - \mathrm{x}'(r)| d|A|_r, \tag{2.36}$$

for all $t \in [0,T]$, $\mathrm{x}, \mathrm{x}' \in D([0,T])$ *such that* $\mathrm{x}(t) = \mathrm{x}'(t)$, *in which* $|A|$ *denotes the total variation of* A.

(b) A *is monotone and* $0 < \frac{1+\beta_2-\beta_0}{2+4\beta_4} < 1 - \frac{\beta_3-\beta_2+\beta_0}{2}$.

Then,

$$\mathrm{v}(t,X) = \mathrm{v}(0,X) + \int_0^t \partial_{\mathrm{x}}\mathrm{v}(s,X)\bar{\sigma}_s(X)dW_s - \int_0^t \bar{\ell}_s(X)ds, \quad t \in [0,T]. \tag{2.37}$$

Remark 2.12. *When* b/σ *is bounded, and* σ *is Lipschitz in its space variable in the sense that, for some constant* $C_{(2.38)} > 0$,

$$\big|\sigma_s(x) - \sigma_s(x')\big| \;\le\; C_{(2.38)}|x - x'|, \quad s \in [0,T],\ x, x' \in \mathbb{R}^2, \tag{2.38}$$

with $(\sigma_s(0))_{s\le T}$ *bounded, then the SDE* (2.9) *has a unique weak solution.*

Remark 2.13. *To check the conditions in Theorem 2.11.*(ii).(b)*, let us consider the situations of Example 2.3.*

(i) − (ii) *In these cases,* $\frac{1+\beta_2-\beta_0}{2+4\beta_4} = \frac{1}{2}$ *and* $1 - \frac{\beta_3-\beta_2+\beta_0}{2} = 1$, *so that the conditions inTheorem 2.11.*(ii).(b) *hold true.*

(iii) *In this case,* $\frac{1+\beta_2-\beta_0}{2+4\beta_4} = \frac{1}{2} - \frac{\gamma_1-\gamma_2}{1+2\gamma_2}$ *and* $1 - \frac{\beta_3-\beta_2+\beta_0}{2} = 1 - 2(\gamma_1 - \gamma_2)$. *Therefore, the conditions in Theorem 2.11.*(ii).(b) *hold true when* $\gamma_1 - \gamma_2$ *is small enough.*

3 Proofs

This section is devoted to the proof of Theorems 2.7, 2.9 and 2.11.

3.1 A priori estimates

Recall that, with $w := \mathrm{w}_{s,t}(x,y)$ (see (2.5)),

$$f_{r,z}(s,x;t,y) \;:=\; \frac{1}{2\pi\det\left(\Sigma_{s,t}(r,z)\right)^{\frac{1}{2}}}\exp\Big(-\frac{1}{2}\langle\Sigma_{s,t}^{-1}(r,z)w,w\rangle\Big),$$

where

$$\Sigma_{s,t}(r,z) \;:=\; \sigma_r^2(z)\begin{pmatrix} t-s & -\int_s^t(A_r - A_s)dr \\ -\int_s^t(A_r-A_s)dr & \int_s^t(A_r-A_s)^2dr \end{pmatrix}.$$

By direct computation, one has

$$\det\left(\Sigma_{s,t}(r,z)\right) \;=\; \sigma_r^4\big(z\big)(t-s)^2 m_{s,t}, \tag{3.1}$$

and hence

$$\Sigma_{s,t}^{-1}(r,z) \;=\; \sigma_r^{-2}\big(z\big)\begin{pmatrix} \frac{1}{t-s}\frac{\tilde{m}_{s,t}}{m_{s,t}} & \frac{\int_s^t (A_r-A_s)dr}{(t-s)^2 m_{s,t}} \\ \frac{\int_s^t (A_r-A_s)dr}{(t-s)^2 m_{s,t}} & \frac{1}{(t-s)m_{s,t}} \end{pmatrix}. \tag{3.2}$$

The following quantities will play an important role in our analysis. For $i=1,2$, $(r,z)\in\mathbb{R}^2$ and $(s,x,t,y)\in\Theta$, with $w := \mathrm{w}_{s,t}(x,y)$ (see (2.5)), we compute that

$$\partial_{x_i} f_{r,z}(s,x;t,y) = f_{r,z}(s,x;t,y)\Big(-\big(\Sigma_{s,t}^{-1}(r,z)w\big)_i\Big), \tag{3.3}$$

$$\partial^2_{x_1x_i} f_{r,z}(s,x;t,y) = f_{r,z}(s,x;t,y)\Big(\big(\Sigma_{s,t}^{-1}(r,z)w\big)_1\big(\Sigma_{s,t}^{-1}(r,z)w\big)_i - \big(\Sigma_{s,t}^{-1}(r,z)\big)_{1,i}\Big), \tag{3.4}$$

and

$$\begin{aligned}\frac{\partial^3_{x_1x_1x_i} f_{r,z}(s,x;t,y)}{f_{r,z}(s,x;t,y)} \;=\;& 2\big(\Sigma_{s,t}^{-1}(r,z)w\big)_1\big(\Sigma_{s,t}^{-1}(r,z)\big)_{1,i} + \big(\Sigma_{s,t}^{-1}(r,z)w\big)_i\big(\Sigma_{s,t}^{-1}(r,z)\big)_{1,1} \\ &- \big(\Sigma_{s,t}^{-1}(r,z)w\big)_1^2\big(\Sigma_{s,t}^{-1}(r,z)w\big)_i. \end{aligned}\tag{3.5}$$

Let us first provide some estimations in the following lemma.

Lemma 3.1. *Let Assumption 2.1.(i) hold. Then, there exists constants $C_{(3.6)}$, $C_{(3.7)}$, $C_{(3.8)}$, $C_{(3.9)} > 0$, such that, for all $(s,x,t,y)\in\Theta$ and $(r,z)\in[0,T]\times\mathbb{R}^2$, with $w := \mathrm{w}_{s,t}(x,y)$ (see (2.5)), we have*

$$\Big|\big(\Sigma_{s,t}^{-1}(r,z)w\big)_1\Big| \;=\; \Big|\frac{\partial_{x_1} f_{r,z}(s,x;t,y)}{f_{r,z}(s,x;t,y)}\Big| \;\le\; \frac{C_{(3.6)}}{(t-s)^{\frac{1+\beta_0}{2}}}\sqrt{\big\langle \Sigma_{s,t}^{-1}(r,z)w, w\big\rangle}, \tag{3.6}$$

$$\Big|\big(\Sigma_{s,t}^{-1}(r,z)w\big)_2\Big| \;\le\; \frac{C_{(3.7)}}{(t-s)^{\frac{1+\beta_3}{2}}}\sqrt{\big\langle \Sigma_{s,t}^{-1}(r,z)w, w\big\rangle}, \tag{3.7}$$

$$\Big|\big(\Sigma_{s,t}^{-1}(r,z)\big)_{1,1}\Big| \;\le\; \frac{C_{(3.8)}}{(t-s)^{1+\beta_0}}, \tag{3.8}$$

$$\Big|\big(\Sigma_{s,t}^{-1}(r,z)\big)_{1,2}\Big| \;\le\; \frac{C_{(3.9)}}{(t-s)^{1+\frac{\beta_0+\beta_3}{2}}}. \tag{3.9}$$

Proof. The bound in (3.8) and (3.9) are immediate consequences of Assumption 2.1.(i) and (3.2), up to appealing to Cauchy-Schwarz's inequality for the latter. By direct computation, one has

$$\begin{aligned}\sigma_t(z)^2\big\langle \Sigma_{s,t}^{-1}(r,z)w, w\big\rangle &= \frac{1}{(t-s)^2 m_{s,t}}\int_s^t \Big((A_r-A_s)^2 w_1^2 + 2(A_r-A_s)w_1w_2 + w_2^2\Big)dr \\ &= \frac{1}{(t-s)^2 m_{s,t}}\int_s^t \big((A_r-A_s)w_1 + w_2\big)^2 dr.\end{aligned}$$

Hence, using Assumption 2.4 and (2.2),

$$\begin{aligned}\underline{\mathfrak{a}}\Big|\big(\Sigma_{s,t}^{-1}(r,z)w\big)_1\Big| &\le \sigma_t(z)^2\Big|\big(\Sigma_{s,t}^{-1}(r,z)w\big)_1\Big| \\ &= \frac{1}{(t-s)^2 m_{s,t}}\Big|\int_s^t \big(A_r - A_s\big)\big((A_r - A_s)w_1 + w_2\big)dr\Big| \\ &\le \sqrt{\frac{\tilde{m}_{s,t}}{(t-s)m_{s,t}}}\sqrt{\bar{\mathfrak{a}}\langle\Sigma_{s,t}^{-1}(r,z)w,w\rangle} \le \frac{\sqrt{\bar{\mathfrak{a}}}C_{(2.2)}}{(t-s)^{\frac{1+\beta_0}{2}}}\sqrt{\langle\Sigma_{s,t}^{-1}(r,z)w,w\rangle}.\end{aligned}$$

Similarly, using the above, Assumption 2.4 and Cauchy-Schwarz inequality, implies that

$$\begin{aligned}\underline{\mathfrak{a}}\Big|\big(\Sigma_{s,t}^{-1}(r,z)w\big)_2\Big| &\le \sigma_t(z)^2\Big|\big(\Sigma_{s,t}^{-1}(r,z)w\big)_2\Big| \\ &= \Big|\frac{1}{(t-s)^2 m_{s,t}}\int_s^t \big((A_r - A_s)w_1 + w_2\big)dr\Big| \\ &\le \frac{\sqrt{\bar{\mathfrak{a}}}C_{(2.3)}}{(t-s)^{\frac{1+\beta_3}{2}}}\sqrt{\langle\Sigma_{s,t}^{-1}(r,z)w,w\rangle}.\end{aligned}$$

□

As usual, an important step consists in providing a suitable upper-bound on the parametrix density. Recall that $y \longmapsto f^\circ(s,x;t,y)$ defined in (2.22) is a Gaussian density function on $\mathbb{R}^2$.

Lemma 3.2. *Let Assumption 2.1.(i) hold. Then, there exists $C_{(3.11)} > 0$ such that, for all $(s,x,t,y) \in \Theta$ and $(r,z) \in [0,T]\times\mathbb{R}^2$, we have*

$$f_{r,z}(s,x;t,y) \;\le\; \varpi(s,x;t,y)\; f^\circ(s,x;t,y), \tag{3.10}$$

in which $\varpi := \varpi^1\varpi^2$ with

$$\begin{cases} \varpi^1(s,x;t,y) \;:=\; C_{(3.11)}\exp\Big(-\frac{1}{C_{(3.11)}}\Big(\frac{|w_1|^2}{(t-s)^{1+\beta_1'}} + \frac{|w_2|^2}{(t-s)^{1+\beta_2'}}\Big)\Big), \\ \varpi^2(s,x;t,y) \;:=\; \exp\Big(-\frac12\langle\Sigma_{s,t}^{-1}(4\bar{\mathfrak{a}})w,w\rangle\Big), \end{cases} \tag{3.11}$$

where $w := \mathrm{w}_{s,t}(x,y)$ as defined in (2.5).

Proof. Let us first observe that $m_{s,t} = \tilde{m}_{s,t} - [(t-s)^{-1}\int_s^t(A_r - A_s)ds]^2$, so that the right-hand side of (2.2) is equivalent to

$$\Big(\frac{1}{t-s}\int_s^t (A_r - A_s)ds\Big)^2 \;\le\; \tilde{m}_{s,t}\Big(1 - \frac{(t-s)^{\beta_0}}{C_{(2.2)}}\Big).$$

Note that, upon changing the value of $C_{(2.2)}$, one can assume that $C_{(2.2)} \ge 2T^{\beta_0}$. Hence, using the inequality $2ab \le a^2 + b^2$ for $a,b \in \mathbb{R}$,

$$\begin{aligned}2\left|\frac{\int_s^t(A_r - A_s)w_1w_2ds}{(t-s)^2 m_{s,t}}\right| &\le 2\left[\tilde{m}_{s,t}\Big(1 - \frac{(t-s)^{\beta_0}}{C_{(2.2)}}\Big)\right]^{\frac12}\frac{w_1w_2}{(t-s)m_{s,t}} \\ &\le \Big(1 - \frac{(t-s)^{\beta_0}}{C_{(2.2)}}\Big)^{\frac12}\Big\{\frac{\tilde{m}_{s,t}}{(t-s)m_{s,t}}|w_1|^2 + \frac{1}{(t-s)m_{s,t}}|w_2|^2\Big\}.\end{aligned}$$

Combining the above with (2.2)-(2.3) and Assumption 2.4 implies that

$$\begin{aligned}\big\langle \Sigma_{s,t}^{-1}(r,z)w,w\big\rangle &\geq \frac{1}{\sigma_r(z)^2}\left(\frac{\tilde{m}_{s,t}}{(t-s)m_{s,t}}|w_1|^2-2\left|\frac{\int_s^t(A_r-A_s)w_1w_2ds}{(t-s)^2m_{s,t}}\right|+\frac{1}{(t-s)m_{s,t}}|w_2|^2\right)\\ &\geq \frac{C}{\bar{\mathfrak{a}}}\left(\frac{|w_1|^2}{(t-s)^{1+\beta_1-\beta_0}}+\frac{|w_2|^2}{(t-s)^{1+\beta_2-\beta_0}}\right),\end{aligned} \tag{3.12}$$

for some $C>0$ that does not depend on (s,x,t,y). The required result then follows from obvious algebra and Assumption 2.4. □

Lemma 3.3. *Let Assumption 2.1.(i) hold. Let us define the transition density function* $f^{\circ,\frac{1}{2}}$ *by, for* $(s,x,t,y)\in\Theta$,

$$f^{\circ,\frac{1}{2}}(s,x;t,y) := \frac{1}{2\pi\ \det\left(\Sigma_{s,t}(8\bar{\mathfrak{a}})\right)^{\frac{1}{2}}}\exp\Big(-\frac{1}{2}\big\langle\Sigma_{s,t}^{-1}(8\bar{\mathfrak{a}})\mathrm{w}_{s,t}(x,y),\mathrm{w}_{s,t}(x,y)\big\rangle\Big). \tag{3.13}$$

Then, there exists $C_{[3.3]}>0$ *such that, for all* $(s,x,t,y)\in\Theta$ *and* $x'\in\mathbb{R}^2$ *satisfying*

$$|x_1-x_1'|^{\frac{1}{1+\beta_1'}}+|x_2-x_2'|^{\frac{1}{1+\beta_2'}} \ \leq\ (t-s)^{1/2}, \tag{3.14}$$

we have

$$f^{\circ}(s,x';t,y)\leq C_{[3.3]}f^{\circ,\frac{1}{2}}(s,x;t,y) \tag{3.15}$$

and

$$\left(\frac{|\mathrm{w}_{s,t}(x',y)_1|^2}{(t-s)^{1+\beta_1'}}+\frac{|\mathrm{w}_{s,t}(x',y)_2|^2}{(t-s)^{1+\beta_2'}}\right)\varpi^1(s,x,t,y)\leq C_{[3.3]}. \tag{3.16}$$

Proof. Set $w:=\mathrm{w}_{s,t}(x,y)$ and $w':=\mathrm{w}_{s,t}(x',y)$. First observe that

$$\left(\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})w,w\rangle\right)^{\frac{1}{2}}\leq\left(\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})(w-w'),(w-w')\rangle\right)^{\frac{1}{2}}+\left(\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})w',w'\rangle\right)^{\frac{1}{2}}.$$

Using that $2ab\leq 2a^2+2b^2$ for $a,b\geq 0$, we deduce that

$$-\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})w',w'\rangle\leq -\frac{1}{2}\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})w,w\rangle+\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})(w-w'),(w-w')\rangle.$$

Now, by the same arguments as in the proof of Lemma 3.2 and Assumption 2.4, we have

$$\big\langle\Sigma_{s,t}^{-1}(\bar{\mathfrak{a}})(w-w'),(w-w')\big\rangle\leq 2\bar{\mathfrak{a}}\left(\frac{|x_1-x_1'|^2}{(t-s)^{1+\beta_1'}}+\frac{|x_2-x_2'|^2}{(t-s)^{1+\beta_2'}}\right)\leq 4\bar{\mathfrak{a}},$$

in which we used (2.5) and our assumption (3.14). This proves (3.15). The assertion (3.16) is proved similarly, upon interchanging the role of x and x'. □

3.2 Wellposedness of $\widetilde{\Phi}$

In this section, we prove that $\widetilde{\Phi}$ in (2.17)-(2.18) is well defined. Recall that f° is defined in (2.22), and let us define

$$\tilde{f}^\circ(s,x;t,y) \;:=\; f^\circ(s,\mathbf{A}_s x;t,\mathbf{A}_t y), \quad (s,x,t,y)\in\Theta.$$

Noticing that $\mathbf{A}=\mathbf{A}^{-1}$, and recalling that

$$f_{r,z}(s,x;t,y) \;:=\; \tilde{f}_{r,\mathbf{A}_r z}(s,\mathbf{A}_s x;t,\mathbf{A}_t y),$$

it is straightforward to check that

$$\partial_{x_1} f_{r,z}(s,x;t,y) \;=\; \overrightarrow{\mathrm{A}}_s\cdot D_x \tilde{f}_{r,\mathbf{A}_r z}(s,\mathbf{A}_s x;t,\mathbf{A}_t y), \tag{3.17}$$

$$\partial^2_{x_1x_1} f_{r,z}(s,x;t,y) \;=\; \mathrm{Tr}\Big[\overrightarrow{\mathrm{A}}_s\big(\overrightarrow{\mathrm{A}}_s\big)^\top D^2_{xx}\tilde{f}_{r,\mathbf{A}_r z}(s,\mathbf{A}_s x;\mathbf{A}_t y)\Big]. \tag{3.18}$$

Lemma 3.4. *Let the conditions of Theorem 2.7.(i) hold. Then, there exist a constant $C_{(3.19)}>0$ such that*

$$\big|\big(\widetilde{\mathcal{L}}-\widetilde{\mathcal{L}}^{t,\tilde{y}}\big)\tilde{f}_{t,\tilde{y}}(s,\tilde{x};t,\tilde{y})\big| \;\le\; \frac{C_{(3.19)}}{(t-s)^{1-\kappa_0}}\tilde{f}^\circ(s,\tilde{x};t,\tilde{y}),\ \textit{for all } (s,\tilde{x},t,\tilde{y})\in\Theta, \tag{3.19}$$

in which κ_0 is defined in (2.23).

Proof. For simplicity, we assume that $t-s\le 1$, the case $t-s>1$ being trivially handled. Let us denote

$$x := \mathbf{A}_s\tilde{x} \ \text{ and } y := \mathbf{A}_t\tilde{y}. \tag{3.20}$$

(i) Using (2.12) and (3.17), we first estimate

$$I_1 \;:=\; \tilde{\mu}_s(\tilde{x})\ \overrightarrow{\mathrm{A}}\cdot D_x\tilde{f}_{t,\tilde{y}}(s,\tilde{x};t,\tilde{y}) \;=\; \mu_s(x)\ \partial_{x_1} f_{t,y}(s,x;t,y).$$

Then, by Assumption 2.4, Lemmas 3.1 and 3.2, it follows that

$$|I_1| \;\le\; \frac{\mathfrak{b}C_{(3.6)}}{(t-s)^{\frac{1+\beta_0}{2}}}\sqrt{\langle\Sigma^{-1}_{s,t}(t,y)w,w\rangle}f_{t,y}(s,x;t,y) \;\le\; \frac{\mathfrak{b}C_{(3.6)}C_{(3.22)}}{(t-s)^{1-\frac{1-\beta_0}{2}}}f^\circ(s,x;t,y), \tag{3.21}$$

in which $w:=\mathrm{w}_{s,t}(x,y)$ and, with $w':=\mathrm{w}_{s',t'}(x',y')$,

$$C_{(3.22)} \;:=\; \sup_{(s',x',t',y',z')\in\Theta\times\mathbb{R}^2}\sqrt{\langle\Sigma^{-1}_{s',t'}(t',z')w',w'\rangle}\varpi(s',x';t',y') \;<\; \infty. \tag{3.22}$$

(ii) Using (2.12) and (3.18), we now estimate

$$I_2 \;:=\; \mathrm{Tr}\Big[\big(\tilde{\sigma}^2_t(\tilde{y})-\tilde{\sigma}^2_s(\tilde{x})\big)\overrightarrow{\mathrm{A}}_s(\overrightarrow{\mathrm{A}}_s)^\top D^2_{xx}\tilde{f}_{t,\tilde{y}}(s,\tilde{x};t,\tilde{y})\Big] = \big[\sigma^2_t(y)-\sigma^2_s(x)\big]\partial^2_{x_1x_1}f_y(s,x;t,y).$$

By (3.4) and Lemma 3.1, one obtains

$$|I_2| \;\le\; \big[\sigma^2_t(y)-\sigma^2_s(x)\big]\frac{(C_{(3.6)})^2\vee C_{(3.8)}}{(t-s)^{1+\beta_0}}\Big(\langle\Sigma^{-1}_{s,t}(t,y)w,w\rangle+1\Big)f_y(s,x;t,y).$$

Recalling (2.5), (2.7), (2.6) and Lemma 3.2, it follows that, for some $C > 0$ that does not depend on (s, x, t, y) and $z \in \mathbb{R}^2$,

$$|I_2| \;\le\; C\frac{1}{(t-s)^{1+\beta_0-\alpha}}\Big(1 + \big|\mathrm{w}_{s,t}(x,y)\big|^{\frac{2\alpha}{1+\beta_1'}} + \big|\mathrm{w}_{s,t}(x,y)\big|^{\frac{2\alpha}{1+\beta_2'}}\Big)(\varpi^1 f^\circ)(s,x;t,y),$$

and we conclude by using the definition of ϖ^1 in (3.11). □

Proposition 3.5. *Let the conditions of Theorem 2.7.(i) hold. Then, the sum in* (2.17) *is well-defined and there exists a constant* $C_{(3.23)} > 0$ *such that*

$$\big|\widetilde{\Phi}(s,x;t,y)\big| \;\le\; \frac{C_{(3.23)}}{(t-s)^{1-\kappa_0}}\tilde{f}^\circ(s,x;t,y), \text{ for all } (s,x,t,y) \in \Theta. \tag{3.23}$$

Moreover, $\widetilde{\Phi}$ *is continuous on* Θ *and satisfies*

$$\widetilde{\Phi}(s,x;t,y) = \widetilde{\Delta}_0(s,x;t,y) + \int_s^t \int_{\mathbb{R}^2} \widetilde{\Delta}_0(s,x;r,z)\widetilde{\Phi}(r,z;t,y)dzdr, \text{ for all } (s,x,t,y) \in \Theta. \tag{3.24}$$

Proof. Let us recall that, if well-defined,

$$\widetilde{\Phi}(s,x;t,y) \;:=\; \sum_{k=0}^{\infty} \widetilde{\Delta}_k(s,x;t,y),$$

where $\widetilde{\Delta}_0(s,x;t,y) := \big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,y}\big)\tilde{f}_{t,y}(s,x;t,y)$, and

$$\widetilde{\Delta}_{k+1}(s,x;t,y) := \int_s^t \int_{\mathbb{R}^2} \widetilde{\Delta}_0(s,x;r,z)\widetilde{\Delta}_k(r,z;t,y)dzdr, \quad k \ge 0.$$

We already know from Lemma 3.4 that

$$\big|\big(\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}^{t,y}\big)\tilde{f}_{t,y}(s,x;t,y)\big| \;\le\; \frac{C_{(3.19)}}{(t-s)^{1-\kappa_0}}\tilde{f}^\circ(s,x;t,y),$$

for all $(s,x,t,y) \in \Theta$. By the same induction argument as in [8, proof of Proposition 4.1], together with (3.1) and (2.3), we then deduce that

$$\big|\widetilde{\Delta}_k(s,x;t,y)\big| \;\le\; \frac{M_k}{(t-s)^{1-k\kappa_0}}\tilde{f}^\circ(s,x;t,y) \;\le\; CM_k(t-s)^{k\kappa_0-2-\frac{\beta_3}{2}}, \tag{3.25}$$

in which, $C > 0$ does not depend on (s,x,t,y) and k, and

$$M_k \;:=\; \frac{\{C_{(3.19)}\Gamma(\kappa_0)\}^k}{\Gamma(k\kappa_0)},$$

where Γ denotes the Gamma function. By dominated convergence, each map $\widetilde{\Delta}_k$ is continuous. Then, the well-posedness of $\widetilde{\Phi}$ follows from the fact that the power series $\sum_{k\ge 0} M_k u^k$ has a radius of convergence equal to ∞. Continuity of $\widetilde{\Phi}$ is a consequence of the absolute continuity of the series.

It remains to prove (3.24). Note that, by the above,

$$\widetilde{\Phi}(s,x;t,y) = \widetilde{\Delta}_0(s,x;t,y) + \sum_{k\geq 0} \int_s^t \int_{\mathbb{R}^2} \widetilde{\Delta}_0(s,x;r,z)\widetilde{\Delta}_k(r,z;t,y)dzdr$$

and the family $\{(r,z) \in (s,t) \times \mathbb{R}^2 \mapsto \sum_{k=0}^n \widetilde{\Delta}_0(s,x;r,z)\widetilde{\Delta}_k(r,z;t,y),\ n \geq 1\}$ is uniformly integrable and converges to $\widetilde{\Delta}_0(s,x;\cdot)\widetilde{\Phi}(\cdot;t,y)$. This implies (3.24). □

Recall that

$$\Phi(s,x;t,y) \ := \ \widetilde{\Phi}(s,\mathbf{A}_s x;t,\mathbf{A}_t y).$$

Proposition 3.6. *Let the conditions of Theorem 2.7.(i) hold. Then,* $f : \Theta \longrightarrow \mathbb{R}$ *is well-defined in* (2.20)*. Moreover, it is continuous on* Θ *and, for some* $C_{[3.6]} > 0$*,*

$$\big|f(s,x;t,y)\big| \ \leq \ C_{[3.6]} f^\circ(s,x;t,y), \text{ for all } (s,x,t,y) \in \Theta.$$

Proof. This is an immediate consequence of Proposition 3.5 and Lemma 3.2, recalling that f° is a transition density and observing that $\int_s^t (t-r)^{-1+\kappa_0} dr \leq CT^{\kappa_0}$. □

3.3 C^1-regularity

We now prove that $x = (x_1, x_2) \mapsto f(s,x;t,y)$ is C^1 in its first space variable x_1, with partial derivative dominated by a Gaussian density.

Lemma 3.7. *Let the conditions of Theorem 2.7 hold. Then, there exists* $C_{[3.7]} > 0$ *such that, for all* $(r,z) \in [0,T] \times \mathbb{R}^2$ *and* $(s,x,t,y) \in \Theta$*,*

$$\big|\partial_{x_1} f_{r,z}(s,x;t,y)\big| \ \leq \ \frac{C_{[3.7]}}{(t-s)^{\frac{\beta_0+1}{2}}} f^\circ(s,x;t,y).$$

Moreover, let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a (measurable) function such that* $\int_{\mathbb{R}^2} f^\circ(s,x;t,y)|h(y)|dy < \infty$*, and*

$$V(s,x;t) \ := \ \int_{\mathbb{R}^2} f_{t,y}(s,x;t,y)h(y)dy, \quad (s,x) \in [0,t) \times \mathbb{R}^2,$$

then $(s,x) \in [0,t) \times \mathbb{R}^2 \mapsto V(s,x;t)$ *is continuously differentiable in its first space variable* x_1 *and satisfies*

$$\big|\partial_{x_1} V(s,x;t)\big| \ \leq \ \frac{C_{[3.7]}}{(t-s)^{\frac{\beta_0+1}{2}}} \int_{\mathbb{R}^2} f^\circ(s,x;t,y)\ |h(y)|dy,$$

in which $C_{[3.7]} > 0$ *does not depend on* $(s,x,t) \in [0,T] \times \mathbb{R}^2 \times [0,T]$ *with* $s < t$*.*

Proof. The first inequality follows immediately from Lemmas 3.1 and 3.2, as in the proof of (3.21). The second one then follows by dominated convergence. □

For the following, we recall the defintion of κ_1 in (2.25).

Proposition 3.8. *Let the conditions of Theorem 2.7 hold. Then, for each* $(t,y) \in (0,T] \times \mathbb{R}^2$*, the map* $(s,x) \in [0,t) \times \mathbb{R}^2 \mapsto f(s,x;t,y)$ *is continuously differentiable in its first space variable* x_1*. Moreover, there exists* $C_{[3.8]} > 0$ *such that*

$$|\partial_{x_1} f(s,x;t,y)| \;\le\; \frac{C_{[3.8]}}{(t-s)^{1-\kappa_1}} f^\circ(s,x;t,y), \text{ for all } (s,x;t,y) \in \Theta.$$

Proof. Fix $z \in \mathbb{R}^2$. In view of the estimate in (3.23), together with Lemma 3.7, we can find $C > 0$, that does not depend on $(s,x,t,y) \in \Theta$, such that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big|\partial_{x_1} f_{r,z}(s,x;r,z)\Phi(r,z;t,y)\Big| dz \\
\le\;& C(r-s)^{\frac{-\beta_0-1}{2}} \int_{\mathbb{R}^2} f^\circ(s,x;r,z)\big|\Phi(r,z;t,y)\big| dz \\
\le\;& C(t-r)^{-1+\kappa_0}(r-s)^{\frac{-\beta_0-1}{2}} \int_{\mathbb{R}^2} f^\circ(s,x;r,z) f^\circ(r,z;t,y) dz \\
=\;& C(t-r)^{-1+\kappa_0}(r-s)^{\frac{-\beta_0-1}{2}} f^\circ(s,x;t,y).
\end{aligned}$$

Therefore, by the dominated convergence theorem,

$$\partial_{x_1} \int_s^t \int_{\mathbb{R}^2} f_{r,z}(s,x;r,z)\Phi(r,z,t,y) dz dr$$

is well-defined and continuous, and so is $\partial_{x_1} f(\cdot;t,y)$. The latter is bounded from the above estimates by integrating over r and using the relation between the Euler-Gamma and the Beta functions. □

We conclude this section by a continuity property result on f, which allows one to apply the C^1-Itô's formula in the context of Theorem 2.11.

Proposition 3.9. *Let Assumptions 2.1 and 2.4.(i) hold true. Assume in addition that* (2.23) *holds and that* $\frac{\beta_3-\beta_2+\beta_0}{2} < 1$*, and let us fix* $\alpha' \in \big(0, \frac{1+\beta_2'}{2} \wedge (1 - \frac{\beta_3-\beta_2+\beta_0}{2})\big]$*. Then, for all* $\delta > 0$*, there exists* $C_{[3.9]} > 0$ *such that*

$$\big|f(s,x;t,y) - f(s,x';t,y)\big| \;\le\; C_{[3.9]}|x_2 - x_2'|^{\frac{2\alpha'}{1+\beta_2'}},$$

for all $(s,x,t,y) \in \Theta$ *and* $x' = (x_1', x_2') \in \mathbb{R}^2$ *such that* $t - s \ge \delta$ *and* $x_1 = x_1'$*.*

Proof. Let $I := \big|f_{r,z}(s,x;t,y) - f_{r,z}(s,x';t,y)\big|$ and denote by $C > 0$ a generic constant that can change from line to line but does not depend on (s,x,x',t,y,z). Then, by (3.3), Lemma 3.1 and Lemma 3.2, one can find x_2'' in the interval formed by x_2 and x_2' such that, with $x'' := (x_1, x_2'')$,

$$I \le |x_2 - x_2'|\;|\partial_{x_2} f_{r,z}(s,x'';t,y)| \le C|x_2 - x_2'| \frac{1}{(t-s)^{\frac{1+\beta_3}{2}}} f^\circ(s,x'';t,y).$$

If $(t-s)^{\frac{1+\beta_2'}{2}}/|x_2-x_2'| \ge 1$, then

$$
\begin{aligned}
I &\le C|x_2-x_2'|^{\frac{2\alpha'}{1+\beta_2'}} \frac{1}{(t-s)^{\frac{1+\beta_3}{2}-\frac{1+\beta_2'}{2}+\alpha'}} f^\circ(s,x'';t,y) \\
&= C|x_2-x_2'|^{\frac{2\alpha'}{1+\beta_2'}} \frac{1}{(t-s)^{\alpha'+\frac{\beta_3-\beta_2+\beta_0}{2}}} f^\circ(s,x'';t,y).
\end{aligned}
$$

Otherwise, by (3.10),

$$
I \le |x_2-x_2'|^{\frac{2\alpha'}{1+\beta_2'}} \frac{1}{(t-s)^{\alpha'}} \big(f^\circ(s,x;t,y)+f^\circ(s,x';t,y)\big).
$$

We conclude by using the fact that $\beta_3-\beta_2+\beta_0 \ge 0$ and by appealing to (3.23). □

3.4 C^2-regularity

We now prove that f is C^2 in its first space variable x_1 and that v is a smooth solution of the path-dependent PDE (2.1).

3.4.1 Potential estimate and Hölder regularity of Φ

Let $0 \le s < t \le T$ and $x \in \mathbb{R}^2$, $h:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be a (measurable) function, we first estimate the second order derivative of the following functional:

$$
V(s,x;t) \ := \int_{\mathbb{R}^2} f_{t,y}(s,x;t,y)h(y)dy.
$$

Let us also denote

$$
E_{s,t}^{-1}(x) \ := \ \begin{pmatrix} 1 & 0 \\ A_t-A_s & 1 \end{pmatrix} x. \tag{3.26}
$$

Lemma 3.10. *Let Assumption 2.1 and Assumption 2.4.(i) hold. Let $h:\mathbb{R}^2 \longrightarrow \mathbb{R}$ and $h_\circ:\mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be such that, for some $\alpha_h>0$ and $C_h>0$,*

$$
\big|h(y)-h(y')\big| \le C_h\Big(|y_1-y_1'|^{\frac{2\alpha_h}{1+\beta_1'}}+|y_2-y_2'|^{\frac{2\alpha_h}{1+\beta_2'}}\Big)\big(h_\circ(y)+h_\circ(y')\big), \ \textit{for all } y,y' \in \mathbb{R}^2,
$$

and

$$
\int_{\mathbb{R}^2} f^\circ(s,x;t,y)h_\circ(y)dy \ < \ \infty, \ \textit{for all } 0 \le s < t \le T, \quad x \in \mathbb{R}^2.
$$

Assume that

$$
\kappa_h \ := \ \min\Big(\frac{2\beta_4+1+\beta_1'}{1+\beta_2'},1\Big)\min\{\alpha_h,\alpha\}-\beta_0 \ > \ 0.
$$

Then, $\partial^2_{x_1x_1}V(s,x;t)$ is well defined and continuous. Moreover

$$
\partial^2_{x_1x_1}V(s,x;t) = \int_{\mathbb{R}^2} \partial^2_{x_1x_1} f_{t,y}(s,x;t,y)h(y)dy,
$$

and there exists $C>0$, that does not depend on $C_h>0$, such that

$$
\big|\partial^2_{x_1x_1}V(s,x;t)\big| \le \frac{CC_h}{(t-s)^{1-\kappa_h}}\left(\big|h\big(E_{s,t}^{-1}(x)\big)\big|+\big|h_\circ\big(E_{s,t}^{-1}(x)\big)\big|+\int_{\mathbb{R}^2} f^\circ(s,x;t,y)h_\circ(y)dy\right),
$$

for all $0 \le s < t \le T$ and $x \in \mathbb{R}^2$.

Proof. For simplicity, we only consider the case $t-s \le 1$. To estimate the second order derivative, we decompose

$$I := \int_{\mathbb{R}^2} \partial^2_{x_1x_1} f_{t,y}(s,x;t,y)h(y)dy$$

into the sum of the three following terms, with $\check{x} := E^{-1}_{s,t}(x)$,

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^2} \partial^2_{x_1x_1} f_{t,y}(s,x;t,y)\big[h(y)-h\big(\check{x}\big)\big]dy,\\
I_2 &:= h\big(\check{x}\big)\int_{\mathbb{R}^2} \big\{\partial^2_{x_1x_1} f_{t,y}(s,x;t,y) - \partial^2_{x_1x_1} f_{t,\check{x}}(s,x;t,y)\big\}\,dy,\\
I_3 &:= h\big(\check{x}\big)\int_{\mathbb{R}^2} \partial^2_{x_1x_1} f_{t,\check{x}}(s,x;t,y)dy.
\end{aligned}$$

All over this proof, $C>0$ denotes a generic constant that may change from line to line but does not depend on C_h, $(s,x;t,y)\in\Theta$ and $z\in\mathbb{R}^2$.

(i) We first estimate I_1. Set $w = \mathrm{w}_{s,t}(x,y)$, recall (2.5). By the Hölder regularity property of h and the inequality $(a+b)^\gamma \le 2^\gamma(a^\gamma+b^\gamma)$ for $a,b\ge 0$ and $\gamma>0$, one has

$$\begin{aligned}
&\Big|\partial^2_{x_1x_1} f_{t,y}(s,x;t,y)\big[h(y)-h\big(\check{x}\big)\big]\Big|\\
&\le C_h\Big|\partial^2_{x_1x_1} f_{t,y}(s,x;t,y)\Big|\Big(|x_1-y_1|^{\frac{2\alpha_h}{1+\beta_1'}} + |y_2-x_2-(A_t-A_s)x_1|^{\frac{2\alpha_h}{1+\beta_2'}}\Big)\big(h_\circ(y)+h_\circ(E^{-1}_{s,t}(x))\big)\\
&\le CC_h\Big|\partial^2_{x_1x_1} f_{t,y}(s,x;t,y)\Big|\Big(|w_1|^{\frac{2\alpha_h}{1+\beta_1'}} + |w_2|^{\frac{2\alpha_h}{1+\beta_2'}} + \big|(A_t-A_s)w_1\big|^{\frac{2\alpha_h}{1+\beta_2'}}\Big)\big(h_\circ(y)+h_\circ(E^{-1}_{s,t}(x))\big).
\end{aligned}$$

Then, arguing as in the proof of Lemma 3.4 and using (2.4), we deduce that

$$\begin{aligned}
\big|I_1\big| \le& CC_h\Big(\frac{1}{(t-s)^{1+\beta_0-\alpha_h}} + \frac{1}{(t-s)^{1+\beta_0-\alpha_h\frac{2\beta_4+1+\beta_1'}{1+\beta_2'}}}\Big)\int_{\mathbb{R}^2} f^\circ(s,x;t,y)\big(h_\circ(y)+h_\circ(E^{-1}_{s,t}(x))\big)dy,\\
\le&\ \frac{CC_h}{(t-s)^{1-\kappa_h}}\Big(\int_{\mathbb{R}^2} f^\circ(s,x;t,y)h_\circ(y)dy + h_\circ\big(E^{-1}_{s,t}(x)\big)\Big).
\end{aligned}$$

(ii) We now consider I_2. By (3.4) and Lemma 3.1,

$$\begin{aligned}
&\big|\partial^2_{x_1x_1} f_{t,y}(s,x;t,y) - \partial^2_{x_1x_1} f_{t,\check{x}}(s,x;t,y)\big|\\
\le&\ \big|f_{t,y}(s,x;t,y) - f_{t,\check{x}}(s,x;t,y)\big|\Big|\big(\Sigma^{-1}_{s,t}(t,y)w\big)_1^2 - \big(\Sigma^{-1}_{s,t}(t,y)\big)_{1,1}\Big|\\
&+ f_{t,\check{x}}(s,x;t,y)\Big|\big(\Sigma^{-1}_{s,t}(t,y)w\big)_1^2 - \big(\Sigma^{-1}_{s,t}(t,\check{x})w\big)_1^2\Big|\\
&+ f_{t,\check{x}}(s,x;t,y)\Big|\big(\Sigma^{-1}_{s,t}(t,y)\big)_{1,1} - \big(\Sigma^{-1}_{s,t}(t,\check{x})\big)_{1,1}\Big|\\
=&\ \big|f_{t,y}(s,x;t,y) - f_{t,\check{x}}(s,x;t,y)\big|\Big|\big(\Sigma^{-1}_{s,t}(t,y)w\big)_1^2 - \big(\Sigma^{-1}_{s,t}(t,y)\big)_{1,1}\Big|\\
&+ f_{t,\check{x}}(s,x;t,y)\,\big|\sigma_t(y)^{-4} - \sigma_t(\check{x})^{-4}\big|\,\Big|\big(\Sigma^{-1}_{s,t}(1)w\big)_1\Big|^2\\
&+ f_{t,\check{x}}(s,x;t,y)\,\big|\sigma_t(y)^{-2} - \sigma_t(\check{x})^{-2}\big|\,\Big|\big(\Sigma^{-1}_{s,t}(1)\big)_{1,1}\Big|,
\end{aligned}$$

in which, by (2.7),

$$
\begin{aligned}
|\sigma_t(y)-\sigma_t(\check{x})| &\le C_{(2.7)}\Big(\big|y_1-x_1\big|^{\frac{2\alpha}{1+\beta_1'}}+\big|y_2-x_1(A_t-A_s)-x_2\big|^{\frac{2\alpha}{1+\beta_2'}}\Big)\\
&\le C\Big(|w_1|^{\frac{2\alpha_g}{1+\beta_1'}}+|w_2|^{\frac{2\alpha_g}{1+\beta_2'}}+\big|(A_t-A_s)w_1\big|^{\frac{2\alpha_g}{1+\beta_2'}}\Big)
\end{aligned}
$$

by the same arguments as in in step 1. Using Lemma 3.12 below, (3.6), (3.8) and (2.6), it follows that

$$
\big|I_2\big| \;\le\; \frac{C|h(\check{x})|}{(t-s)^{1+\beta_0-\alpha\frac{2\beta_4+1+\beta_1'}{1+\beta_2'}}}\int_{\mathbb{R}^2} f^\circ(s,x;t,y)dy \le \frac{C|h\big(E_{s,t}^{-1}(x)\big)|}{(t-s)^{1-\kappa_h}}.
$$

(iii) We finally consider I_3. Notice that $y\mapsto f_{t,\check{x}}(s,x;t,y)$ is a Gaussian density function, so that

$$
\int_{\mathbb{R}^2}\big|\partial^2_{x_1x_1}f_{t,\check{x}}(s,x;t,y)\big|dy \;<\; \infty.
$$

Moreover, by the definition of $f_{t,\check{x}}(s,x;t,y)$, one has

$$
D_y f_{t,\check{x}}(s,x;t,y) = -\begin{pmatrix} 1 & 0\\ -(A_t-A_s) & 1\end{pmatrix}^{\top} D_x f_{t,\check{x}}(s,x;t,y),
$$

so that

$$
\partial_{x_1}f_{t,\check{x}}(s,x;t,y) = -\partial_{y_1}f_{t,\check{x}}(s,x;t,y)-(A_t-A_s)\partial_{y_2}f_{t,\check{x}}(s,x;t,y),
$$

which implies

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_{x_1}f_{t,\check{x}}(s,x;t,y)dy_1dy_2 \;=\; 0,
$$

and therefore $I_3=0$.

Finally, we can apply the Leibniz integral rule to interchange the derivative and the integral, and hence to conclude the proof. □

Remark 3.11. *Let us consider $V(s,x;t)$ as a path-dependent functional:*

$$
\overline{V}(s,\mathrm{x};t) \;:=\; V(s,\mathrm{x}(s),I_s(\mathrm{x});t) \;=\; \int_{\mathbb{R}^2}\mathrm{f}_{t,y}(s,\mathrm{x};t,y)h(y)dy.
$$

In view of Remark 2.8, the above results implies that, in the context of Lemma 3.10, the second order vertical derivative $\partial^2_{\mathrm{xx}}\overline{V}(s,\mathrm{x};t)$ is well defined. Moreover, by (2.31), one has

$$
\partial_s\mathrm{f}_{t,y}(s,\mathrm{x};t,y) \;=\; -\frac{1}{2}\sigma(t,y)^2\partial^2_{\mathrm{xx}}\mathrm{f}_{t,y}(s,\mathrm{x};t,y). \tag{3.27}
$$

Then, by the same technique as in Lemma 3.10, we can deduce that the horizontal derivative $\partial_s\overline{V}(s,\mathrm{x};t)$ is also well defined.

We now provide the following easy estimate which is used in the proof of Lemma 3.10.

Lemma 3.12. *Let Assumption 2.1.(i) and Assumption 2.4.(i) hold. Then, there exists* $C_{[3.12]} > 0$ *such that, for all* $(s,x,t,y) \in \Theta$ *and* $z, z' \in \mathbb{R}^2$,

$$
\begin{aligned}
&\big|f_{t,z}(s,x;t,y) - f_{t,z'}(s,x;t,y)\big| \\
&\le C_{[3.12]} \left(|z_1 - z_1'|^{\frac{2\alpha}{1+\beta_1'}} + |z_2 - z_2'|^{\frac{2\alpha}{1+\beta_2'}} \right) \left[1 + \langle \Sigma_{s,t}^{-1}(1) w, w\rangle\right] (\varpi f^\circ)(s,x;t,y)
\end{aligned}
$$

in which $w := \mathrm{w}_{s,t}(x,y)$.

Proof. Let us write $f_{[a]}$ for $f_{t,z}$ if $\sigma_t^2(z) = a$, and let $\partial_a f_{[a]}$ denote its derivative with respect to this parameter a. Then,

$$
\partial_a f_{[a]}(s,x;t,y) = \left[-\frac{1}{a} + \frac{\sigma_t(0)^2}{2a^2} \langle \Sigma_{s,t}^{-1}(0) w, w\rangle \right] f_{[a]}(s,x;t,y)
$$

in which $w = \mathrm{w}_{s,t}(x,y)$ is as in (2.5). In view of Assumption 2.4 and Lemma 3.2, it follows that

$$
|\partial_a f_{[a]}(s,x;t,y)| \le C \left[1 + \langle \Sigma_{s,t}^{-1}(0) w, w\rangle\right] \varpi(s,x;t,y) f^\circ(s,x;t,y),
$$

for some $C > 0$ that does not depend on a, (s,x,t,y). We conclude by appealing to (2.7). □

In order to apply Lemma 3.10 to (2.15), we need to prove that the function $\Phi(s,x;t,y)$ defined by (2.17) and (2.19) is Hölder in x. Recall the definition of $\widehat{\Delta\beta}$ in (2.32), of κ_0 in (2.23) and of $f^{\circ,\frac{1}{2}}$ in (3.13).

Lemma 3.13. *Let the conditions of Theorem 2.9 hold. Fix* $\alpha_\Phi \in \big(0, \hat\alpha_\Phi \wedge \kappa_0 \wedge \min_{i=1,2} \frac{1+\beta_i'}{2}\big)$. *Then, there exists* $C_{\alpha_\Phi} > 0$ *such that,*

$$
\begin{aligned}
&|\Phi(s,x;t,y) - \Phi(s,x';t,y)| \\
\le\ & C_{\alpha_\Phi} \frac{|x_1 - x_1'|^{\frac{2\alpha_\Phi}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{2\alpha_\Phi}{1+\beta_2'}}}{(t-s)^{1-\eta_\Phi}} \Big(f^{\circ,\frac{1}{2}}(s,x;t,y) + f^{\circ,\frac{1}{2}}(s,x';t,y) \Big),
\end{aligned}
$$

for all $(s,x,t,y) \in \Theta$*, in which*

$$
\eta_\Phi \ :=\ \hat\alpha_\Phi \wedge \kappa_0 - \alpha_\Phi \ >\ 0.
$$

Proof. In all this proof, $C > 0$ denotes a generic constant, whose value can change from line to line, but which does not depend on $(s,x,t,y) \in \Theta$. We set $\Delta_k(s,x;t,y) := \widetilde\Delta_k(s, \mathbf{A}_s x; t, \mathbf{A}_t y)$ and recall that $\Phi(s,x;t,y) := \widetilde\Phi(s, \mathbf{A}_s x; t, \mathbf{A}_t y)$, $(s,x,t,y) \in \Theta$.

(i) Let us first consider

$$
\begin{aligned}
I \ :=\ & \Delta_0(s,x;t,y) - \Delta_0(s,x';t,y) \\
=\ & \mu_s(x) \partial_{x_1} f_{t,y}(s,x;t,y) + \frac{1}{2}\big(\sigma_s^2(x) - \sigma_t^2(y)\big) \partial^2_{x_1 x_1} f_{t,y}(s,x;t,y) \\
& - \Big(\mu_s(x') \partial_{x_1} f_{t,y}(s,x';t,y) + \frac{1}{2}\big(\sigma_s^2(x') - \sigma_t^2(y)\big) \partial^2_{x_1 x_1} f_{t,y}(s,x';t,y) \Big). \qquad (3.28)
\end{aligned}
$$

(i.1) In the case where

$$
|x_1 - x_1'|^{\frac{1}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{1}{1+\beta_2'}} \ >\ (t-s)^{1/2},
$$

Lemma 3.4 implies that, for $\alpha' \in (0, \kappa_0)$,

$$\begin{aligned}
&\Big|\Delta_0(s,x;t,y) - \Delta_0(s,x';t,y)\Big| \\
\le\ & C_{(3.19)} \frac{1}{(t-s)^{1-\kappa_0}} \big(f^\circ(s,x;t,y) + f^\circ(s,x';t,y)\big) \\
\le\ & C_{(3.19)} \frac{|x_1 - x_1'|^{\frac{2\alpha'}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{2\alpha'}{1+\beta_2'}}}{(t-s)^{1-\kappa_0+\alpha'}} \big(f^\circ(s,x;t,y) + f^\circ(s,x';t,y)\big).
\end{aligned}$$

(i.2) We next consider the case where

$$|x_1 - x_1'|^{\frac{1}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{1}{1+\beta_2'}} \ \le\ (t-s)^{1/2}. \tag{3.29}$$

Let us write

$$I \ :=\ \Delta_0(s,x;t,y) - \Delta_0(s,x';t,y) \ =\ I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 \ &:=\ \big(\mu_s(x) - \mu_s(x')\big)\partial_{x_1} f_{t,y}(s,x;t,y), \\
I_2 \ &:=\ \mu_s(x')\big(\partial_{x_1} f_{t,y}(s,x;t,y) - \partial_{x_1} f_{t,y}(s,x';t,y)\big), \\
I_3 \ &:=\ \frac{1}{2}\big(\sigma_s^2(x) - \sigma_s^2(x')\big)\partial^2_{x_1x_1} f_{t,y}(s,x';t,y),
\end{aligned}$$

and

$$I_4 \ :=\ \frac{1}{2}\big(\sigma_s^2(x) - \sigma_t^2(y)\big)\Big(\partial^2_{x_1x_1} f_{t,y}(s,x;t,y) - \partial^2_{x_1x_1} f_{t,y}(s,x';t,y)\Big).$$

For I_1, we use the Hölder continuity property of μ in (2.8), Lemma 3.1 and Lemma 3.2 to obtain that

$$\big|I_1\big| \ \le\ C\frac{|x_1 - x_1'|^{\frac{2\alpha}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{2\alpha}{1+\beta_2'}}}{(t-s)^{\frac{1+\beta_0}{2}}} f^\circ(s,x;t,y). \tag{3.30}$$

For I_2, let us fix $\rho \in [0,1]$ and $x'' = \rho x + (1-\rho)x'$ so that, using Assumption 2.4,

$$\big|I_2\big| \ \le\ \mathfrak{b}\Big(\big|\partial^2_{x_1x_1} f_{t,y}(s,x'';t,y)\big|\big|x_1 - x_1'\big| + \big|\partial^2_{x_1x_2} f_{t,y}(s,x'';t,y)\big|\big|x_2 - x_2'\big|\Big).$$

Using (3.4), (3.4), Lemma 3.1, Lemma 3.2 and the fact that $\beta_0 \le \beta_3$, it follows that

$$\big|I_2\big| \le C\Big(\frac{|x_1 - x_1'|}{(t-s)^{1+\beta_0}} + \frac{|x_2 - x_2'|}{(t-s)^{1+\frac{\beta_0+\beta_3}{2}}}\Big) f^\circ(s,x'';t,y).$$

Since x'' lies in the interval formed by x and x', Lemma 3.3 and (3.29) imply that

$$\big|I_2\big| \le C\Big(\frac{|x_1 - x_1'|}{(t-s)^{1+\beta_0}} + \frac{|x_2 - x_2'|}{(t-s)^{1+\frac{\beta_0+\beta_3}{2}}}\Big) f^{\circ,\frac{1}{2}}(s,x;t,y). \tag{3.31}$$

Next, using the Hölder property of σ in (2.7), Lemma 3.1 and Lemma 3.2, it follows that

$$\big|I_3\big| \ \le\ \frac{C}{(t-s)^{1+\beta_0}}\Big(|x_1 - x_1'|^{\frac{2\alpha}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{2\alpha}{1+\beta_2'}}\Big) f^\circ(s,x;t,y). \tag{3.32}$$

Finally, I_4 is tackled as I_2. Namely, we can find $\tilde{x}'' = \tilde{\rho}x + (1-\tilde{\rho})x'$ with $\tilde{\rho} \in [0,1]$ such that

$$\begin{aligned} &\left|\partial^2_{x_1x_1} f_{t,y}(s,x;t,y) - \partial^2_{x_1x_1} f_{t,y}(s,x';t,y)\right| \\ \leq\ & \left|\partial^3_{x_1x_1x_1} f_{t,y}(s,\tilde{x}'';t,y)\right|\left|x_1 - x_1'\right| + \left|\partial^3_{x_1x_1x_2} f_{t,y}(s,\tilde{x}'';t,y)\right|\left|x_2 - x_2'\right| \\ \leq\ & C\left(\frac{|x_1-x_1'|}{(t-s)^{\frac{3}{2}(1+\beta_0)}} + \frac{|x_2-x_2'|}{(t-s)^{\frac{3}{2}+\beta_0+\frac{\beta_3}{2}}}\right)(\varpi^1 f^\circ)(s,\tilde{x}'';t,y), \end{aligned}$$

in which we used Lemma 3.1 and Lemma 3.2 again. Next, we appeal to (2.7) to deduce that

$$\left|\sigma_s^2(x) - \sigma_t^2(y)\right| \leq C_{(2.7)}\Big(|t-s|^\alpha + |w_1|^{\frac{2\alpha}{1+\beta_1'}} + |w_2|^{\frac{2\alpha}{1+\beta_2'}}\Big).$$

Using that $\beta_3 \geq \beta_0$, the condition (3.29) together with Lemma 3.3 and the fact that $\tilde{x}''$ lies on the interval formed by x and x' implies that

$$|I_4| \ \leq\ C|t-s|^\alpha\left(\frac{|x_1-x_1'|}{(t-s)^{\frac{3}{2}(1+\beta_0)}} + \frac{|x_2-x_2'|}{(t-s)^{\frac{3}{2}+\beta_0+\frac{\beta_3}{2}}}\right) f^{\circ,\frac{1}{2}}(s,x;t,y). \tag{3.33}$$

Note that there exists $C>0$, that does not depend on (s,x,t,y) such that

$$f^\circ(s,x;t,y) \ \leq\ C f^{\circ,\frac{1}{2}}(s,x;t,y).$$

Thus, combining (3.30)-(3.33) and recalling (3.28) and (3.15) leads to a upper bound for

$$J := \frac{|I|}{C\big(f^{\circ,\frac{1}{2}}(s,x;t,y) + f^{\circ,\frac{1}{2}}(s,x';t,y)\big)}.$$

Namely,

$$\begin{aligned} J \ \leq\ & \frac{|x_1-x_1'|^{\frac{2\alpha}{1+\beta_1'}} + |x_2-x_2'|^{\frac{2\alpha}{1+\beta_2'}}}{(t-s)^{1+\beta_0}} \\ & +\ |x_1-x_1'|\left(\frac{1}{(t-s)^{1+\beta_0}} + \frac{1}{(t-s)^{\frac{3}{2}(1+\beta_0)-\alpha}}\right) \\ & +\ |x_2-x_2'|\left(\frac{1}{(t-s)^{1+\frac{\beta_0+\beta_3}{2}}} + \frac{1}{(t-s)^{\frac{3}{2}+\beta_0+\frac{\beta_3}{2}-\alpha}}\right). \end{aligned}$$

We then use that $(t-s)^{\frac{1+\beta_i'}{2}}/|x_i - x_i'| \geq 1$, for $i=1,2$, to deduce that, for $0 < \alpha' \leq \alpha \wedge \min_{i=1,2} \frac{1+\beta_i'}{2}$,

$$\begin{aligned} J \ \leq\ & \frac{|x_1-x_1'|^{\frac{2\alpha'}{1+\beta_1'}} + |x_2-x_2'|^{\frac{2\alpha'}{1+\beta_2'}}}{(t-s)^{1+\beta_0+\alpha'-\alpha}} \\ & +\ \frac{|x_1-x_1'|^{\frac{2\alpha'}{1+\beta_1'}}}{(t-s)^{(1+\beta_0)\vee(\frac{3}{2}(1+\beta_0)-\alpha)-\frac{1+\beta_1'}{2}+\alpha'}} + \frac{|x_2-x_2'|^{\frac{2\alpha'}{1+\beta_2'}}}{(t-s)^{(1+\frac{\beta_0+\beta_3}{2})\vee(\frac{3}{2}+\beta_0+\frac{\beta_3}{2}-\alpha)-\frac{1+\beta_2'}{2}+\alpha'}}. \end{aligned}$$

Since $\beta_0 \geq \beta_1$ and $\beta_3 \geq \beta_2$,

$$J \leq \frac{|x_1 - x_1'|^{\frac{2\alpha'}{1+\beta_1'}}}{(t-s)^{(\frac{1}{2}+\beta_0+\frac{\beta_0-\beta_1}{2})\vee(1+\frac{3}{2}\beta_0+\frac{\beta_0-\beta_1}{2}-\alpha)+\alpha'}} + \frac{|x_2 - x_2'|^{\frac{2\alpha'}{1+\beta_2'}}}{(t-s)^{(\frac{1}{2}+\beta_0+\frac{\beta_3-\beta_2}{2})\vee(1+\frac{3}{2}\beta_0+\frac{\beta_3-\beta_2}{2}-\alpha)+\alpha'}}.$$

(i.3) We now combine the results of steps (i.1) and (i.2) to deduce that, when $\alpha' = \alpha_\Phi \in (0, \hat{\alpha}_\Phi \wedge \kappa_0)$,

$$|I| \;\leq\; C\frac{|x_1 - x_1'|^{\frac{2\alpha_\Phi}{1+\beta_1'}} + |x_2 - x_2'|^{\frac{2\alpha_\Phi}{1+\beta_2'}}}{(t-s)^{1-\eta_\Phi}} \Big(f^{\circ,\frac{1}{2}}(s,x;t,y) + f^{\circ,\frac{1}{2}}(s,x';t,y)\Big).$$

(ii) To conclude, it remains to use an induction argument as in the end of the proof of Proposition 3.5. □

3.4.2 Smoothness of the transition density and Feynman-Kac's representation

Recall that $\mathrm{f}(s,\mathrm{x};t,y)$ is defined in (2.27).

Proposition 3.14. *Let the conditions of Theorem 2.9 hold. Then, the vertical derivative* $\partial^2_{\mathrm{xx}}\mathrm{f}(s,\mathrm{x};t,y)$ *and horizontal derivative* $\partial_s\mathrm{f}(s,\mathrm{x};t,y)$ *are well-defined for all* $0 \leq s < t \leq T$, $\mathrm{x} \in D([0,T])$ *and* $y \in \mathbb{R}^2$. *Moreover, for all* $(t,y) \in [0,T]\times\mathbb{R}^2$, $\partial^2_{\mathrm{xx}}\mathrm{f}(\cdot;t,y)$ *and* $\partial_s\mathrm{f}(\cdot;t,y)$ *are continuous on* $[0,t)\times C([0,T])$.

Proof. We denote by $C>0$ a generic constant that does not depend on (s,x,t,y). Let us fix $t_0 \in (s,t)$, then by (2.20) and (2.27),

$$\begin{aligned}\mathrm{f}(s,\mathrm{x};t,y) \;&:=\; \mathrm{f}_{t,y}(s,\mathrm{x};t,y) + \int_s^{t_0}\int_{\mathbb{R}^2}\mathrm{f}_{r,z}(s,\mathrm{x};r,z)\Phi(r,z;t,y)dzdr \\ &\quad + \int_{t_0}^{t}\int_{\mathbb{R}^2}\mathrm{f}_{r,z}(s,\mathrm{x};r,z)\Phi(r,z;t,y)dzdr \\ &=:\; \mathrm{f}_{t,y}(s,\mathrm{x};t,y) \;+\; \mathrm{f}_1(s,\mathrm{x};t,y) \;+\; \mathrm{f}_2(s,\mathrm{x};t,y).\end{aligned}$$

First, the existence and continuity of the vertical derivative and horizontal derivative of $\mathrm{f}_{t,y}(s,\mathrm{x};t,y)$ is trivial.

For $\mathrm{f}_1(s,\mathrm{x};t,y)$, we can use Lemmas 3.10 and 3.13, Proposition 3.5, together with (3.15), to obtain that

$$\int_s^{t_0}\Big|\int_{\mathbb{R}^2}\partial^2_{\mathrm{xx}}\mathrm{f}_{r,z}(s,\mathrm{x};r,z)\Phi(r,z;t,y)dz\Big|dr \;\leq\; C\int_s^{t_0}\frac{I_1(s,\mathrm{x};r;t,y)+I_2(s,\mathrm{x};r;t,y)}{(r-s)^{1-\kappa_\Phi}}dr,$$

where

$$\kappa_\Phi \;:=\; \min\Big(\frac{2\beta_4+1+\beta_1'}{1+\beta_2'},1\Big)\min\{\alpha_\Phi,\alpha\} - \beta_0 \;>\; 0,$$

and, with $x := (\mathrm{x}(s), I_s(\mathrm{x}))$,

$$\begin{aligned}I_1(s,\mathrm{x};r;t,y) \;&:=\; \int_{\mathbb{R}^2} f^{\circ,\frac{1}{2}}(s,x;r,z)\big(f^{\circ,\frac{1}{2}}(r,E^{-1}_{s,r}(x);t,y) + f^{\circ,\frac{1}{2}}(r,z;t,y)\big)dz \\ &=\; f^{\circ,\frac{1}{2}}(r,E^{-1}_{s,r}(x);t,y) + f^{\circ,\frac{1}{2}}(s,x;t,y) \\ I_2(s,\mathrm{x};r;t,y) \;&:=\; f^{\circ,\frac{1}{2}}(r,E^{-1}_{s,r}(x);t,y).\end{aligned}$$

Since $t_0 < t$, we can then easily obtain the existence and continuity of $\partial^2_{\mathrm{xx}}\mathrm{f}_1(\cdot;t,y)$ by dominated convergence. Further, in view of Remark 3.11 and in particular (3.27), we can also deduce the existence and continuity of the horizontal derivative $\partial_s\mathrm{f}_1(\cdot;t,y)$.

For $\mathrm{f}_2(s,\mathrm{x};t,y)$, we notice that

$$\big|\partial_s \mathrm{f}_{r,z}(s,\mathrm{x};r,z)\big| + \big|\partial^2_{\mathrm{xx}}\mathrm{f}_{r,z}(s,\mathrm{x};r,z)\big| \ \le\ C\mathrm{f}^{\circ}(s,\mathrm{x};r,z), \text{ for } r\ge t_0 > s,\ z\in\mathbb{R}^2.$$

Together with the estimate on $\Phi(r,z;t,y)$ in Proposition 3.5, it follows the existence and continuity of the vertical derivative $\partial^2_{\mathrm{xx}}\mathrm{f}_2(\cdot;t,y)$ and the horizontal derivative $\partial_s\mathrm{f}_2(\cdot;t,y)$. □

Recall the growth condition (2.28) on ℓ and g, and the Hölder continuity condition (2.29) on ℓ. Let

$$v(s,x) \ :=\ \int_s^T\int_{\mathbb{R}^2}\ell(t,y)f(s,x;t,y)dydt + \int_{\mathbb{R}^2} g(y)f(s,x;T,y)dy, \quad (s,x)\in[0,T)\times\mathbb{R}^2.$$

Then, with v defined in (2.30), one has, for $x=(\mathrm{x}(s),I_s(\mathrm{x}))$,

$$\mathrm{v}(s,\mathrm{x}) = v(s,x),\ \partial_{\mathrm{x}}\mathrm{v}(s,\mathrm{x}) = \partial_{x_1}v(s,x) \text{ and } \partial^2_{\mathrm{xx}}\mathrm{v}(s,\mathrm{x}) = \partial^2_{x_1x_1}v(s,x).$$

Proposition 3.15. *Let the conditions of Theorem 2.9 hold. Then:*

(i) $\mathrm{v}\in\mathbb{C}^{1,2}([0,T))$ *and the bound estimates in* (2.33) *hold true.*

(ii) *The function* v *is a classical solution to the PPDE* (2.1). *If in addition g is continuous, then* v *is the unique classical solution of* (2.1) *satisfying* (2.34).

Proof. (i) Let us define, for $(r,z)\in[0,T)\times\mathbb{R}^2$,

$$v_\Phi(r,z) \ :=\int_r^T\int_{\mathbb{R}^2}\Phi(r,z;t,y)\ell(t,y)dydt \ + \int_{\mathbb{R}^2}\Phi(r,z;T,y)g(y)dy, \tag{3.34}$$

so that

$$v(s,x) \ = \int_{\mathbb{R}^2} f_{T,y}(s,x;T,y)g(y)dy \ + \int_s^T\int_{\mathbb{R}^2} f_{r,z}(s,x;r,z)\big(v_\Phi(r,z)+\ell(r,z)\big)dzdr. \tag{3.35}$$

Then, it follows from Lemma 3.13 that

$$|v_\Phi(r,z)-v_\Phi(r,z')| \le C_{\alpha_\Phi}\frac{|z_1-z_1'|^{\frac{2\alpha_\Phi}{1+\beta_1'}} + |z_2-z_2'|^{\frac{2\alpha_\Phi}{1+\beta_2'}}}{(T-r)^{1-\eta_\Phi}}\big(v^{\circ,\frac12}(r,z)+v^{\circ,\frac12}(r,z')\big),$$

in which

$$v^{\circ,\frac12}(r,z) \ :=\ \int_r^T\int_{\mathbb{R}^2} f^{\circ,\frac12}(r,z;T,y)\big|\ell(t,y)\big|dydt + \int_{\mathbb{R}^2} f^{\circ,\frac12}(r,z;T,y)\big|g(y)\big|dy.$$

Together with the Hölder continuity condition on ℓ in (2.29), we can then apply Lemma 3.10 to deduce that $\partial^2_{x_1x_1}v(s,x)$ exists and

$$\begin{aligned}&\partial^2_{x_1x_1}v(s,x)\\ &= \int_{\mathbb{R}^2}\partial^2_{x_1x_1}f_{T,y}(s,x;T,y)g(y)dy + \int_s^T\int_{\mathbb{R}^2}\partial^2_{x_1x_1}f_{r,z}(s,x;r,z)\big(v_\Phi(r,z)+\ell(r.z)\big)dzdr.\end{aligned}$$

Then, using (3.4) and Lemma 3.1, we deduce that, for some constant $C > 0$,

$$\int_{\mathbb{R}^2} \Big| \partial^2_{x_1 x_1} f_{T,y}(s,x;T,y) g(y) \Big| dy \;\le\; \frac{C}{(T-s)^{1+\beta_0}} \int_{\mathbb{R}^2} f^\circ(s,x;T,y) |g(y)| dy \;\le\; \frac{C e^{C|x|}}{(T-s)^{1+\beta_0}}.$$

By Lemma 3.10, one can choose $C > 0$ such that,

$$\begin{aligned} & \Big| \int_s^T \int_{\mathbb{R}^2} \partial^2_{x_1 x_1} f_{r,z}(s,x;r,z) \ell(r.z) dz dr \Big| \\ \le\; & \int_s^T \frac{C}{(r-s)^{1-\kappa_\ell}} \Big(\big|\ell(E^{-1}_{s,r}(x))\big| + C e^{|E^{-1}_{s,r}(x)|} + \int_{\mathbb{R}^2} f^\circ(s,x;r,z) e^{C|z|} dz \Big) dr \;\le\; C e^{C|x|}, \end{aligned}$$

in which

$$\kappa_\ell \;:=\; \min\Big(\frac{2\beta_4 + 1 + \beta_1'}{1 + \beta_2'}, 1 \Big) \min\{\alpha_\ell, \alpha\} - \beta_0 \;>\; 0,$$

and

$$\Big| \int_s^T \int_{\mathbb{R}^2} \partial^2_{x_1 x_1} f_{r,z}(s,x;r,z) v_\Phi(r.z) dz dr \Big| \le \int_s^T \frac{C}{(r-s)^{1-\kappa_\Phi} (T-r)^{1-\eta_\Phi}} e^{C|x|} dr \;\le\; C e^{C|x|}.$$

This proves the bound estimate on $\partial^2_{x_1 x_1} v(s,x)$ (or equivalently $\partial^2_{\mathrm{xx}} \mathrm{v}(s,\mathrm{x})$) in (2.33). In view of (3.27), one can obtain the same bound on $\partial_s \mathrm{v}(s,\mathrm{x})$ in (2.33). Finally, $\partial_{\mathrm{x}} \mathrm{v}(s,\mathrm{x})$ is estimated by appealing to Proposition 3.8 and (2.28). The bound on the right-hand side of (2.34) is proved similarly.

(ii) Recall that

$$\mathrm{f}(s,\mathrm{x};t,y) \;=\; \mathrm{f}_{t,y}(s,\mathrm{x};t,y) \;+ \int_s^t \int_{\mathbb{R}^2} \mathrm{f}_{r,z}(s,\mathrm{x};r,z) \Phi(r,z,t,y) dz dr,$$

and that $(s,\mathrm{x}) \in [0,t) \times D([0,T]) \mapsto \mathrm{f}_{t,y}(s,\mathrm{x};t,y)$ solves

$$\mathrm{L}_{t,y} \mathrm{f}_{t,y}(\cdot;t,y) = 0 \text{ on } [s,t) \times C([0,T]), \tag{3.36}$$

where

$$\mathrm{L}_{t,y} \;:=\; \partial_s + \frac{1}{2} \sigma_t(y)^2 \partial^2_{\mathrm{xx}}.$$

Let

$$\mathrm{L}\phi(s,\mathrm{x}) \;:=\; \partial_s \phi(s,\mathrm{x}) + \mu_s(\mathrm{x}) \partial_{\mathrm{x}} \phi(s,\mathrm{x}) + \frac{1}{2} \sigma^2_s(\mathrm{x}) \partial^2_{\mathrm{x}} \phi(s,\mathrm{x}),$$

for $\phi \in \mathbb{C}^{1,2}([0,T))$. Recalling the definition of v_Φ in (3.34) and using (3.35), we obtain that, with $x := (\mathrm{x}(s), I_s(\mathrm{x}))$,

$$\begin{aligned} \mathrm{L}\mathrm{v}(s,\mathrm{x}) = & \int_{\mathbb{R}^2} \mathrm{L}\mathrm{f}_{T,y}(s,\mathrm{x};T,y) g(y) dy - v_\Phi(s,x) - \ell(s,x) \\ & + \int_s^T \int_{\mathbb{R}^2} \mathrm{L}\mathrm{f}_{r,z}(s,\mathrm{x};r,z) \big(v_\Phi(r,z) + \ell(r,z) \big) dz dr. \end{aligned} \tag{3.37}$$

At the same time, as a consequence of (3.24) and (3.17)-(3.18), we observe that

$$\begin{aligned} \Phi(s,x;T,y) = \; & (\mathrm{L} - \mathrm{L}_{t,y}) \mathrm{f}_{t,y}(s,\mathrm{x};t,y) \\ & + \int_s^t \int_{\mathbb{R}^2} (\mathrm{L} - \mathrm{L}_{r,z}) \mathrm{f}_{r,z}(s,\mathrm{x};r,z) \Phi(r,z;t,y) dz dr. \end{aligned}$$

Hence, recalling Lemma 3.4 and Proposition 3.5, it follows by (3.34) that

$$
\begin{aligned}
v_\Phi(s,x) = & \int_{\mathbb{R}^2} (\mathrm{L}-\mathrm{L}_{T,y})\mathrm{f}_{T,y}(s,\mathrm{x};T,y)g(y)dy \\
& + \int_s^T \int_{\mathbb{R}^2} (\mathrm{L}-\mathrm{L}_{r,z})\mathrm{f}_{r,z}(s,\mathrm{x};r,z)\big(v_\Phi(r,z)+\ell(r,z)\big)dzdr.
\end{aligned}
$$

We then use (3.36) to obtain

$$
v_\Phi(s,x) \ = \int_{\mathbb{R}^2} \mathrm{L}\mathrm{f}_{T,y}(s,\mathrm{x};T,y)g(y)dy + \int_s^T \int_{\mathbb{R}^2} \mathrm{L}\mathrm{f}_{r,z}(s,\mathrm{x};r,z)\big(v_\Phi(r,z)+\ell(r,z)\big)dzdr.
$$

It follows then by (3.37) that v is a classical solution to the PPDE (2.1).

(iii) We now prove that $\lim_{s\nearrow T} \mathrm{v}(s,\mathrm{x}) = g(\mathrm{x}_T, I_T(\mathrm{x}))$, or equivalently $\lim_{s\nearrow T} v(s,x) = g(x)$, whenever g is continuous. In view of the estimates in (3.10) and (3.23), and Proposition 3.6, one has

$$
\begin{aligned}
\lim_{s\nearrow T} v(s,x) &= \lim_{s\nearrow T} \int_{\mathbb{R}^2} f(s,x;T,y)g(y)dy \ = \ \lim_{s\nearrow T} \int_{\mathbb{R}^2} f_{T,y}(s,x;T,y)g(y)dy \\
&= \lim_{M\to\infty} \lim_{s\nearrow T} \int_{D^M_{s,T}} f_{T,y}(s,x;T,y)g(y)dy \ = \lim_{M\to\infty} \lim_{s\nearrow T} \int_{D^M_{s,T}} f_{T,x}(s,x;T,y)g(y)dy \\
&= \lim_{s\nearrow T} \int_{\mathbb{R}^2} f_{T,x}(s,x;T,y)g(y)dy \ = \ g(x),
\end{aligned}
$$

in which

$$
D^M_{s,T} := \Big[x_1 - M\sqrt{T-s},\ x_1 + M\sqrt{T-s}\Big] \times \Big[x_2 - M\sqrt{(T-s)\tilde{m}_{s,t}},\ x_2 + M\sqrt{(T-s)\tilde{m}_{s,t}}\Big],
$$

so that third and fifth equalities are true since both $f_{T,y}(s,x;T,y)$ and $f_{T,x}(s,x;T,y)$ are dominated by $Cf^\circ(s,x;T,y)$ in which the covariance matrix in f° is given by $\Sigma_{s,T}(4\bar{\mathfrak{a}})$, and the fourth equality follows by the fact that, for every fixed $M > 0$,

$$
\lim_{s\nearrow T} \sup_{y\in D^M_{s,T}} \left| \frac{f_{T,y}(s,x;T,y)}{f_{T,x}(s,x;T,y)} - 1 \right| = 0.
$$

(iv) The fact that v is the unique solution of (2.1) satisfying (2.34) holds true follows easily by a verification argument based on Itô-Dupire's formula, see [3], whenever g is continuous.

□

3.5 Proofs of Theorems 2.7, 2.9 and 2.11

Proof of Theorem 2.7. (i) First, the well-posedness of $\widetilde{\Phi}$ in (2.17)-(2.18) is proved in Proposition 3.5. Further, the well-posedness of f in (2.20) as well as its continuity and growth property is proved in Proposition 3.6.

(ii) Under further conditions, the existence of $\partial_{x_1} f(s,x;t,y)$ as well as its continuity and growth property is proved in Proposition 3.8. □

Proof of Theorem 2.9. (i) The fact that $\mathrm{f}(\cdot; t, y) \in \mathbb{C}^{1,2}([0,t))$ is proved in Proposition 3.14.

(ii) The fact that v provides a classical solution to the PPDE, as well as the estimation on the derivatives are proved in Proposition 3.15.

(iii) We now use the PPDE results in Item (ii) to study the path-dependent SDE (2.9). To study the weak solution of the SDE (2.9), we consider the martingale problem on the canonical space $C([0,T])$ of all $\mathbb{R}^2$-valued continuous paths on $[0,T]$. By abuse of notation, we denote by $(X_t, I_t)_{t\in[0,T]}$ the canonical process, which generates the canonical filtration $\mathbb{F}$. Then, given an initial condition $(t,x) \in [0,T] \times \mathbb{R}^2$, a solution to the corresponding martingale problem is a probability measure $\mathbb{P}$ on $C([0,T])$ such that $\mathbb{P}[(X_s, I_s) = x = (x_1, x_2),\ s \in [0,t]] = 1$, $\mathbb{P}[I_s = x_2 + \int_t^s X_r dA_r,\ s \in [t,T]] = 1$ and the process

$$\varphi(X_s) - \int_t^s \Big(\bar{\mu}_r(X) D\varphi(X_r) + \frac{1}{2}\bar{\sigma}_r(X) D^2\varphi(X_r)\Big) dr,\ s \in [t,T],$$

is a $(\mathbb{P},\mathbb{F})$-martingale for all bounded smooth functions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$. Let us denote, for all $(t,x) \in [0,T] \times \mathbb{R}^2$,

$$\mathcal{P}(t,x) \ := \ \big\{\mathbb{P} \ : \mathbb{P} \text{ is solution to the martingale problem with initial condition } (t,x)\big\}.$$

Notice that $\bar{\mu}$ and $\bar{\sigma}$ are both bounded continuous, it is then classical to know that $\mathcal{P}(t,x)$ is a nonempty compact set (see e.g. Stroock and Varadhan [14, Chapter VI]).

We next apply the classical Markovian selection technique (see e.g. [14, Chapter 12.2]) to construct a weak solution to the SDE (2.9) such that $(X_t, I_t)_{t\in[0,T]}$ is a strong Markov process. Let $(\phi_n)_{n\geq 1}$ be a sequence of bounded continuous functions from $[0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that it is a measure determining sequence in the sense that the sequence

$$\Big\{\mathbb{E}^{\mathbb{P}}\Big[\int_0^T \phi_n(t, X_t, I_t) dt\Big]\Big\}_{n\geq 1}$$

can determinate the probability measure $\mathbb{P}$ on $C([0,T])$. For each $(t,x) \in [0,T] \times \mathbb{R}^2$, let $\mathcal{P}_0^+(t,x) := \mathcal{P}(t,x)$, and then define, for each $n \geq 0$,

$$\mathcal{P}_{n+1}^+(t,x) = \Big\{\mathbb{P} \in \mathcal{P}_n^+(t,x) \ : \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \phi_n(t, X_t, I_t) dt\Big] = \max_{\mathbb{P}' \in \mathcal{P}_n^+(t,x)} \mathbb{E}^{\mathbb{P}'}\Big[\int_0^T \phi_n(t, X_t, I_t) dt\Big]\Big\}.$$

It is easy to see that each $\mathcal{P}_n^+(t,x)$ is a non-empty compact set, so that $\mathcal{P}^+(t,x) := \cap_{n\geq 1}\mathcal{P}_n^+(t,x)$ is also non-empty compact, as the sequence is non-increasing. Moreover, since any two probability measures in $\mathcal{P}^+(t,x)$ has the same value by evaluating w.r.t. any ϕ_n, this implies that $\mathcal{P}^+(t,x)$ contains exactly one probability measure denoted by $\mathbb{P}_{t,x}^+$. By the dynamic programming principle for the optimal control problem in the definition of $\mathcal{P}_{n+1}^+$, it follows that $(X, I, (\mathbb{P}_{t,x}^+)_{(t,x)\in[0,T]\times\mathbb{R}^2})$ provides a Markov process solution to SDE (2.9) such that (X, I) is a strong Markov process.

At the same time, one can apply the above Markovian selection argument to construct another Markov process $(X, I, (\mathbb{P}_{t,x}^-)_{(t,x)\in[0,T]\times\mathbb{R}^2}$ by replacing "max" by "min" in the definition of $\mathcal{P}_{n+1}^+(t,x)$. If the class of all martingale solutions $\mathcal{P}(t,x)$ is not unique, then $\mathbb{P}_{t,x}^+ \neq \mathbb{P}_{t,x}^-$ as $(\phi_n)_{n\geq 1}$ is measure determining.

At the same time, by the results in Item (ii) and the Feynman-Kac's formula in the case $g \equiv 0$, one has

$$\mathbb{E}^{\mathbb{P}^+_{s,x}}\Big[\int_s^T \ell_t(X_t, I_t)dt\Big] = \mathbb{E}^{\mathbb{P}^-_{s,x}}\Big[\int_s^T \ell_t(X_t, I_t)dt\Big] = \int_s^T \int_{\mathbb{R}^2} f(s,x;t,y)\ell_t(y)dydt.$$

Since ℓ could be an arbitrary bounded continuous function, this implies that $\mathbb{P}^+_{s,x} = \mathbb{P}^-_{s,x}$ for all $(s,x) \in [0,T] \times \mathbb{R}^2$. Therefore, for all initial condition (t,x), there exists a unique solution to the martingale problem, i.e. a unique weak solution. Moreover the (unique) solution process (X,I) is a strong Markov process, and the transition probability function is given by f. □

Proof of Theorem 2.11. When the SDE (2.9) admits weak uniqueness, the above Markovian selection argument shows that the only solution (X,I) is a strong Markov process.

(i) Let $W^\perp$ be a Brownian motion independent of W, $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive constants such that $\varepsilon_n \longrightarrow 0$. For each $n > 1$, let us define $\widetilde{X}^n = (\widetilde{X}^{n,1}, \widetilde{X}^{n,2})$ as the unique (Markovian) solution to the SDE

$$d\widetilde{X}^{n,1}_t = \tilde{\mu}_t(\widetilde{X}^n_t)dt + \tilde{\sigma}_t(\widetilde{X}^n_t)dW_t, \quad d\widetilde{X}^{n,2}_t = \tilde{\mu}_t(\widetilde{X}^n_t)A_t dt + \tilde{\sigma}_t(\widetilde{X}^n_t)A_t dW_t + \varepsilon_n dW^\perp_t.$$

By stability of weak solutions of SDEs, it is clear that, by using the same initial condition for the above SDE as that in (2.11) for $\widetilde{X}$, one has $\widetilde{X}^n \longrightarrow \widetilde{X}$ weakly.

At the same time, it follows from e.g. [8] that, for each $t \in (s,T]$, $\widetilde{X}^n_t$ has a density $\tilde{f}^n(s,x;t,\cdot)$ whenever $\widetilde{X}^n_s = x$. Moreover, $\tilde{f}^n$ can be defined in the form

$$\tilde{f}^n(s,x;t,y) = \tilde{f}^n_{t,y}(s,x;t,y) + \int_s^t \int_{\mathbb{R}^2} \tilde{f}^n_{r,z}(s,x;r,z)\widetilde{\Phi}^n(r,z;t,y)dzdr$$

in which $\tilde{f}^n_{t,y}$ is defined as $\tilde{f}_{t,y}$ but with

$$\Sigma^n_{s,t}(r,z) = \sigma^2_r(z)\begin{pmatrix} t-s & -\int_s^t (A_u - A_s)du \\ -\int_s^t (A_u - A_s)du & \int_s^t \big[(A_u - A_s)^2 + \varepsilon_n^2 \sigma_r(z)^{-2}\big]du \end{pmatrix}$$

in place of $\Sigma_{s,t}(r,z)$, and

$$\widetilde{\Phi}^n(s,x;t,y) \ := \ \sum_{k=0}^{\infty} \widetilde{\Delta}^n_k(s,x;t,y),$$

where $\widetilde{\Delta}^n_0(s,x;t,y) := \big(\widetilde{\mathcal{L}}^n_s - \widetilde{\mathcal{L}}^{n,t,y}_s\big)\tilde{f}^n_{t,y}(s,x;t,y)$,

$$\widetilde{\Delta}^n_{k+1}(s,x;t,y) := \int_s^t \int_{\mathbb{R}^2} \widetilde{\Delta}^n_0(s,x;r,z)\widetilde{\Delta}^n_k(r,z;t,y)dzdr, \quad k \geq 0.$$

In the above, $\widetilde{\mathcal{L}}^n$ is the generator of $\widetilde{X}^n$ and $\widetilde{\mathcal{L}}^{n,t,y}$ is defined from $\widetilde{\mathcal{L}}^n$ as $\widetilde{\mathcal{L}}^{t,y}$ is defined from $\widetilde{\mathcal{L}}$ by freezing σ to $\sigma_t(y)$ and erasing the drift term. Then, we define $f^n_{r,z}$ from $\tilde{f}^n_\cdot$ as $f_{r,z}$ is defined from $\tilde{f}_\cdot$ in Section 2.3.

It is straightforward to check that the estimates in (3.25) hold for $(\widetilde{\Delta}^n_k)_{k\geq 0}$ in place of $(\widetilde{\Delta}_k)_{k\geq 0}$, uniformly in $n>0$. Then, an induction argument combined with the fact that $\tilde{f}^n_{t,y}(s,x;t,y)\to \tilde{f}_{t,y}(s,x;t,y)$ as $n\to\infty$, for all $(s,x,t,y)\in\Theta$, implies that $f^n(s,x;t,y):=\tilde{f}^n(s,\mathbf{A}_s x;t,\mathbf{A}_t y)$ converges to $f(s,x;t,y)$ as $n\longrightarrow\infty$, for all $(s,x,t,y)\in\Theta$. By the weak convergence of the sequence of processes $(\widetilde{X}^n)_{n\geq 1}$ to X, this shows that f is the transition probability function of (X,I).

(ii) As shown in Theorem 2.7, one has $\mathrm{v}\in\mathbb{C}^{0,1}([0,T))$ and the vertical derivative $\partial_{\mathrm{x}}\mathrm{v}$ is locally bounded. Let (X,I) be the solution of SDE (2.9), then by Feynman-Kac's formula, the process

$$\mathrm{v}(t,X)+\int_0^t \bar{\ell}(s,X)ds,\ t\in[0,T],\ \text{is a local martingale.}$$

One can further apply the C^1-Itô formula for path-dependent functionals in [1] to prove (2.37). Indeed, when (2.36) holds true, one can directly apply [1, Proposition 2.11 and Theorem 2.5].

Otherwise, when A is monotone and $0<\frac{1+\beta_2-\beta_0}{2+4\beta_4}<1-\frac{\beta_3-\beta_2+\beta_0}{2}$, we can fix $\alpha'\in(\frac{1+\beta_2-\beta_0}{2+4\beta_4},1-\frac{\beta_3-\beta_2+\beta_0}{2})$, and, by Proposition 3.9, there exists a constant $C>0$ such that, for all $\varepsilon>0$,

$$\begin{aligned}&\mathbb{E}\Big[\Big|\mathrm{v}\big(s+\varepsilon,X\big)-\mathrm{v}\big(s+\varepsilon,X_{s\wedge\cdot}\oplus_{s+\varepsilon}(X_{s+\varepsilon}-X_s)\big)\Big|^2\Big]\\ \leq\ & C\mathbb{E}\Big[\Big(\sup_{s\leq t\leq s+\varepsilon}\big|X_t-X_s\big|\,\varepsilon^{\beta_4}\Big)^{\frac{4\alpha'}{1+\beta_2'}}\Big]\ \leq\ C\varepsilon^{\frac{\alpha'(2+4\beta_4)}{1+\beta_2'}},\end{aligned}$$

where $\big(X_{s\wedge\cdot}\oplus_{s+\varepsilon}(X_{s+\varepsilon}-X_s)\big)_t:=\mathbf{1}_{[0,s+\varepsilon)}(t)X_{s\wedge t}+\mathbf{1}_{[s+\varepsilon,T]}(t)X_{s+\varepsilon}$ for all $t\in[0,T]$. Since $\frac{\alpha'(2+4\beta_4)}{1+\beta_2'}>1$, it follows that

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\mathbb{E}\Big[\Big|\mathrm{v}\big(s+\varepsilon,X\big)-\mathrm{v}\big(X_{s\wedge\cdot}\oplus_{s+\varepsilon}(X_{s+\varepsilon}-X_s)\big)\Big|^2\Big]\ =\ 0.$$

Finally, we can apply [1, Proposition 2.6 and Theorem 2.5] to deduce (2.37). □

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