Some regularity results for a class of PPDE and applications

B. Bouchard

CEREMADE, Université Paris Dauphine - PSL

Based on works with
G. Loeper (Monash Univ. and BNP), X. Tan (Chinese University of Hong Kong) and M. Vallet (Université Paris Dauphine - PSL)
Motivations

- See under which conditions one can apply Itô-Dupire’s formula to value functions associated to path-dependent pricing or optimal control problems.
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- Use $C^{1+\alpha}$-regularity or show $C^2$-regularity using PDEs.
Example #1: second order coupled FBSDE

- B. and Tan [3]: Solve a second order BSDE related to a (perfect) hedging problem under price impact. Find \((X, Y, Z, g, \mathcal{B})\) such that

\[
X_t = x_0 + \int_0^t \sigma_s(X, g_s) dW_s
\]

\[
Y_t = \Phi(X) - \int_t^T F_s(X, g_s) ds - \int_0^t Z_s dX_s \quad \text{and} \quad Z_t = Z_0 + \int_0^t g_s dX_s - \mathcal{B}_t
\]

where \(\Phi, \sigma\) and \(F\) are path-dependent (non-anticipative).
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where \(\Phi, \sigma\) and \(F\) are path-dependent (non-anticipative).

Derive a solution from a dual formulation of the form :

\[
v(t, x) := \sup_{\alpha} \mathbb{E}\left[ \Phi(\bar{X}^{t,x,\alpha}) - \int_t^T G_s(\bar{X}^{t,x,\alpha}, \alpha_s) ds \right], \quad d\bar{X}^{t,x,\alpha} = \alpha dW,
\]

by using Itô’s lemma : \(Y = v(\cdot, X), \ Z = Dv(\cdot, X)\), etc.
Reminder on Dupire’s derivatives

- Notations:
  - $x$ belongs to $C([0, T])$ or $D([0, T])$.
  - $x_{t^\wedge} := (x_{t^\wedge s})_{s \in [0, T]}$.
  - $x \oplus_t y := x + y 1_{[t, T]}$. 

Horizontal derivative:
$$\partial_t v(t, x) = \lim_{\varepsilon \to 0} \frac{v(t + \varepsilon, x_{t^\wedge}) - v(t, x)}{\varepsilon}.$$

Vertical derivative:
$$\nabla_x v(t, x) \cdot y = \lim_{\varepsilon \to 0} \frac{v(t, x \oplus_t \varepsilon y) - v(t, x)}{\varepsilon}.$$
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- **We define** $C^{0,1}$ and $C^{1,2}$ **accordingly.**
Monotonicity and concavity are defined accordingly.
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We say that $v$ is **Dupire-concave** if for $x^1 = x^2$ on $[0, t)$

\[ v(t, \theta x^1 + (1 - \theta)x^2) \geq \theta v(t, x^1) + (1 - \theta)v(t, x^2), \text{ for all } \theta \in [0, 1] \]
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- If $v$ is **Dupire-concave**, one can define its super-differential
  \[ \partial v(t, x) := \{ z : v(t, x \oplus_t y) \leq v(t, x) + z \cdot y, \forall y \}. \]
Let $\mathcal{P} = \{\mathbb{P} \in \mathcal{P}(D([0, T])) : X \text{ is a càdlàg semimartingale under } \mathbb{P} \}$. 

Theorem [B. and Tan [3, 4]]

Assume that $v$ is Dupire-concave and non-increasing in time. Under additional local boundedness and equi-continuity assumptions \[ \cdots \], we have

$v(t, X) = v(0, X) + \int_0^t H_s dX_s - C_{\mathbb{P}t}, t \in [0, T], \mathbb{P}$-a.s.

in which \[
\{C_{\mathbb{P}t} : \mathbb{P} \in \mathbb{P}\}
\]
is a collection of non-decreasing processes and $H_s \in \partial v(s, X_s)$ for all $s \in [0, T], \mathbb{P}$-q.s, where $X_s - t := X_t 1_{t \in [0, s)} + X_s - 1_{t \in [s, T]}$.

$\Rightarrow$ Enough to construct a solution to our second order FBSDE. See also B. and Tan [4] for an application to robust super-hedging with jumps (compare to Nutz 15).
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in which $\{C^\mathbb{P} : \mathbb{P} \in \mathcal{P}\}$ is a collection of non-decreasing processes and $H_s \in \partial v(s, X_s^{-})$ for all $s \in [0, T]$, $\mathcal{P}$-q.s, where

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Example #2: super-hedging under (bounded) volatility uncertainty

Let us consider a payoff function of the form

$$g(X) = g_\circ \left( X_T, \int_0^T X_t dA_t \right), \quad g_\circ \in C^{1+\alpha} (\mathbb{R}).$$
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**Uncertainty** modeled by \( \mathcal{P}_0 : \mathbb{P} \) such that \( \mathbb{P}[X_0 = x_0] = 1 \) and

\[ dX_s = \sigma_s dW_s^{\mathbb{P}}, \quad \sigma_s \in [\underline{\sigma}, \overline{\sigma}], \quad s \in [0, T], \quad \mathbb{P}\text{-a.s.} \]  \hspace{1cm} (1)
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  \[ \quad \text{(1)} \]

- **Dual formulation**:

  \[ v(t, x) := \sup_{\mathbb{P} \in \mathcal{P}(t, x)} \mathbb{E}^\mathbb{P} [g(X)] = \text{robust super-hedging price} \]

where \( \mathcal{P}(t, x) := \{ \mathbb{P} : \mathbb{P}[X_{t \wedge} = x_{t \wedge}] = 1, \quad \text{and (1) holds on } [t, T] \}. \)
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\( \Rightarrow \) **is not** \( C^{1.2} \) **a priori but may be** \( C^{0,1+\alpha} \) **since** \( g \) **is.**
If it is $C^{0,1+\alpha}$, then one can find the hedging strategy (and prove duality) by applying this version of Itô-Dupire’s Lemma.

**Theorem [B., Loeper and Tan [2] :** Let $X$ be a semimartingale, $v \in C^{0,1}$ such that $v$ and $\nabla_x v$ are locally uniformly continuous and $[\cdots]$. Then,

$$v(t, X) = v(0, X) + \int_0^t \nabla_x v(s, X) dX_s + \Gamma_t, \quad t \in [0, T],$$

where $\Gamma$ is a continuous orthogonal process, if and only if

$$\frac{1}{\varepsilon} \int_0^t \{v(s+\varepsilon, X) - v(s+\varepsilon, X_s \oplus s+\varepsilon (X_{s+\varepsilon} - X_s))\} \{N_{s+\varepsilon} - N_s\} ds \underset{\varepsilon \downarrow 0}{\longrightarrow} \text{u.c.p.}$$

for all (bounded) continuous martingale $N$.

**Remark :** Compare with Bandini and Russo (17) and Gozzi and Russo (06). Here

$$[X, Y]_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) ds$$
Remark: The above condition holds as soon as for some $L \in \text{BV}$:

$$|v(t, x) - v(t, x')| \leq C \int_0^t |x_s - x'_s| dL_s.$$ 

- Similar result for càdlàg processes (B. and Vallet [5]).
It remains to show that the candidate solution to the PPDE

\[-\partial_t v - \sup_{\sigma \in [\sigma, \overline{\sigma}]} \frac{\sigma^2}{2} \nabla_x^2 v = 0, \ v(T, \cdot) = g\]

is $C^{0,1}$ with locally uniformly continuous vertical Dupire’s derivative.
Approximate viscosity solutions of PPDEs
(A tool for regularity)

\[-\partial_t \varphi(t, x) - F(t, x, \varphi(t, x), \nabla_x \varphi(t, x), \nabla^2_x \varphi(t, x)) = 0, \varphi(T, \cdot) = g\]

B., Loeper and Tan [10].

Related works: Ekren, Touzi and Zhang (16), Ren, Touzi and Zhang (17), Ekren and Zhang (16), Cosso and Russo (19), Jianjun Zhou (21).
Let $\pi = (\pi^n)_n$, with $\pi^n = (t^n_i)_{0 \leq i \leq n}$, be an increasing sequence of time grids. Set

$$\bar{x}^n := \sum_{i=0}^{n-1} x_{t^n_i} 1_{[t^n_i, t^n_{i+1})} + x_{t^n_n} 1\{T\}$$
Definition of solutions by approximation

Let \( \pi = (\pi^n)_n \), with \( \pi^n = (t^n_i)_{0 \leq i \leq n} \), be an increasing sequence of time grids. Set

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We say that a continuous function \( v^n \) is a \( \pi^n \)-viscosity solution of

\[
-\partial_t \varphi(t, x) - F(t, x, \varphi(t, x), \nabla_x \varphi(t, x), \nabla^2_x \varphi(t, x)) = 0 \quad \forall \ t < T
\]

if it is of the form

\[
\sum_{i=0}^{n-1} 1_{[t^n_i, t^n_{i+1})} v^n_i(t, \bar{x}^n_{\wedge t^n_i}, x)
\]
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if it is of the form

$$\sum_{i=0}^{n-1} 1_{[t^n_i, t^n_{i+1})} v^n_i(t, \bar{x}^n_{t^n_i}, x)$$

in which each $v^n_i(\cdot, \bar{x}^n_{t^n_i}, \cdot)$ is a viscosity solution on $\mathbb{R}^d \times [t^n_i, t^n_{i+1})$ of

$$- \partial_t v^n_i(t, \bar{x}^n_{t^n_i}, x) - F(t, \bar{x}^n_{t^n_i}, v^n_i(t, \bar{x}^n_{t^n_i}, x), Dv^n_i(t, \bar{x}^n_{t^n_i}, x), D^2 v^n_i(t, \bar{x}^n_{t^n_i}, x)) = 0$$

$$v^n_i(t_{i+1}^n - , \bar{x}^n_{t^n_i}, x) = v^n_{i+1}(t_{i+1}^n, \bar{x}^n_{t^n_i} 1_{[0, t^n_{i+1})} + 1_{\{t^n_{i+1}\}} x, x)$$
Example: Think about replacing

\[ v(t, x) := \sup_{P \in \mathcal{P}(t, x)} \mathbb{E}_P^P[g(X)] \]

by

\[ v^n(t, \bar{x}^n) := \sup_{P \in \mathcal{P}(t, \bar{x}^n)} \mathbb{E}_P^P[g(\bar{X}_{t_0}^n, \ldots, \bar{X}_{t_n}^n)] \].
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(i) If \( t \in [t^n_i, t^n_{i+1}) \), then \((\bar{X}^n_{t^n_0}, \ldots, \bar{X}^n_{t^n_i})\) is known (and is a parameter for the period \([t, T]\))
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(i) If \( t \in [t^n_{i}, t^n_{i+1}) \), then \( (\bar{X}^n_{t_0^n}, \ldots, \bar{X}^n_{t_i^n}) \) is known (and is a parameter for the period \( [t, T] \))

(ii) At the boundary \( t = t^n_{i+1} \), the value \( \bar{X}^n_{t_{i+1}} \) is also frozen, and serves as a starting point for \( \bar{X}^n \) on the period \( [t^n_{i+1}, t^n_{i+2}) \).
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(iii) Ends up with a sequence of backward PDEs on: $[t^n_{n-1}, T)$, $[t^n_{n-2}, t^n_{n-1})$, and so on.
We say that $v$ is a $\pi$-approximate-viscosity solution on $D([0, T])$ of

$$- \partial_t v(t, x) - F(t, x, v(t, x), \nabla_x v(t, x), \nabla^2_x v(t, x)) = 0, \ t < T$$

with terminal condition

$$v(T, \cdot) = g$$

if $v^n(t, x, x_t) \to v(t, x)$ for all $(t, x) \in [0, T] \times D([0, T])$ where $(v^n)_n$ is the sequence defined as above with

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v^n(t^n, x, x) = g(\bar{x}^n 1_{[0, t^n]} + 1_{\{t^n\}} x)
\]

Typical examples: Semi-linear PPDEs or HJB equations.

\( \Rightarrow \) In both cases, amounts to replacing \( X \) by \( \bar{X}^n \) in the coefficients and payoff.

But we also want to consider general non-linear parabolic PPDEs.
Existence, comparison, stability

We focus on the case where

\[ F(t, x, r, p, q) = H(t, x, r, p, q) + \rho(t, x)r + b(t, x)p + \frac{1}{2}\sigma^2(t, x)q \]

where all the coefficients are continuous and Lipschitz/uniformly continuous in space \([\cdots]\) + standard assumptions to have comparison and existence of a viscosity solution with linear growth in finite dimension (for the \(F(\cdot, \bar{x}^n_{\vee t_i^n}, \cdot)\)).
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**Theorem**: Let \(g\) be uniformly continuous, then \(\exists\) a unique \(\pi\)-approximate viscosity solution \(v\) on \(D([0, T])\).
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**Theorem:** Let \(g\) be uniformly continuous, then \(\exists\) a unique \(\pi\)-approximate viscosity solution \(v\) on \(D([0, T])\). Moreover,

- It is locally uniformly continuous.
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**Theorem** : Let \(g\) be uniformly continuous, then \(\exists\) a unique \(\pi\)-approximate viscosity solution \(v\) on \(D([0, T])\). Moreover,

- It is locally uniformly continuous.
- If \(\pi'\) is another increasing sequence of time grids and if \(v'\) is the \(\pi'\)-approximate viscosity solution, then \(v' = v\).
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where all the coefficients are continuous and Lipschitz/uniformly continuous in space $\cdots$ + standard assumptions to have comparison and existence of a viscosity solution with linear growth in finite dimension (for the $F(\cdot, x^n_{i,t_0}, \cdot)$).

**Theorem**: Let $g$ be uniformly continuous, then $\exists$ a unique $\pi$-approximate viscosity solution $v$ on $D([0, T])$. Moreover,

- It is locally uniformly continuous.
- If $\pi'$ is another increasing sequence of time grids and if $v'$ is the $\pi'$-approximate viscosity solution, then $v' = v$.

**Proposition**: Comparison and stability holds in the class of solutions.
Existence, comparison, stability

- We focus on the case where

\[ F(t, x, r, p, q) = H(t, x, r, p, q) + \rho(t, x)r + b(t, x)p + \frac{1}{2}\sigma^2(t, x)q \]

where all the coefficients are continuous and Lipschitz/uniformly continuous in space \([\cdots]\) + standard assumptions to have comparison and existence of a viscosity solution with linear growth in finite dimension (for the \(F(\cdot, \bar{x}^n_{\Lambda t^n}, \cdot)\)).

**Theorem**: Let \(g\) be uniformly continuous, then \(\exists\) a unique \(\pi\)-approximate viscosity solution \(v\) on \(D([0, T])\). Moreover,

- It is locally uniformly continuous.
- If \(\pi'\) is another increasing sequence of time grids and if \(v'\) is the \(\pi'\)-approximate viscosity solution, then \(v' = v\).

**Proposition**: Comparison and stability holds in the class of solutions.

**Remark**: We have precise estimates on the approximation error \(|v^n(t, x, x_t) - v(t, x)|\) (depending on the regul. of \(x\)).
Regularity in the fully non-linear case

Recall that

\[- \partial_t v^n_i (t, \bar{x}^n_{t_i}, x) - F(t, \bar{x}^n_{t_i}, v^n_i (t, \bar{x}^n_{t_i}, x), Dv^n_i (t, \bar{x}^n_{t_i}, x), D^2 v^n_i (t, \bar{x}^n_{t_i}, x)) = 0\]

\[v^n_i (t_{i+1}^n - , \bar{x}^n_{t_i}, x) = v^n_{i+1} (t_{i+1}^n, \bar{x}^n_{t_i} 1_{[0,t_{i+1}^n]} + 1_{t_{i+1}^n} x, x)\]
For terminal conditions of the form (can be made more abstract)

\[ g(x) = g_\circ \left( \int_0^T x_t dA_t \right), \]

where \( g_\circ \in C^{1+\alpha}(\mathbb{R}) \) is bounded, and \( A \) is BV with at most finitely many jumps on \([0, T]\).
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Two cases

(a) Either \( \alpha \in (0, 1) \) and \( F(t, x, y, z, \gamma) = F_1(t)y + F_2(t)z + F_3(t, \gamma) \),
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(b) Or \( \alpha = 1 \) and \( F(t, x, y, z, \gamma) = F_1(t, y, \gamma) + F_2(t)z \) with
\( y \in \mathbb{R} \mapsto F_1(t, y, \gamma) \in C^1 \) with bounded and Lipschitz first order derivative, uniformly in \( \gamma \in \mathbb{R} \) and \( t \leq T \).
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In any case \( \gamma \mapsto F(\cdot, \gamma) \) is concave or \( d \leq 2 \).
For terminal conditions of the form (can be made more abstract)

\[ g(x) = g_o \left( \int_0^T x_t dA_t \right), \]

where \( g_o \in C^{1+\alpha}(\mathbb{R}) \) is bounded, and \( A \) is BV with at most finitely many jumps on \([0, T]\).

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In any case \( \gamma \mapsto F(\cdot, \gamma) \) is concave or \( d \leq 2 \).

**Theorem:** \( \nabla_{x,y} \) is well-defined and locally uniformly continuous.

**Remark:** In the semi-linear case \( F = F(t, x, y, z) \), only needs \( C^{1+\alpha} \) (in the Fréchet sense with respect to the path) : just differentiate the corresponding BSDE...
Regularization in the uniformly elliptic case
(Bouchard and Tan [11])

We focus on the linear case (with $d = 1$) and consider

$$-\partial_t v(t, x) - \mu_t(x) \nabla_x v(t, x) - \frac{1}{2} \sigma_t(x)^2 \nabla_x^2 v(t, x) = 0, \quad v(T, \cdot) = g$$

with

$$(\mu_t, \sigma_t)(x) = (\mu_t, \sigma_t)(x_t, \int_0^t x_s dA_s) \text{ and } g(x) = g(x_T, \int_0^T x_s dA_s),$$

in which

$\mu$ and $\sigma^2$ are Hölder (bounded), $\sigma^2 \geq a > 0$ and $g$ measurable (bounded).
Relation with degenerate equations

If $A$ was absolutely continuous, this would amount to looking for regularity for the degenerate PDE

$$-\partial_t \varphi(t, x) - x^1 \dot{A}_t \partial_{x^2} \varphi(t, x) - \mu_t(x) \partial_{x^1} \varphi(t, x) - \frac{1}{2} \sigma_t(x)^2 \partial_{x^1, x^1} \varphi(t, x) = 0$$

in which derivatives are taken in the traditional sense and

$$(t, x) := (t, x_t, \int_0^t x_s \dot{A}_s ds), \varphi(t, x) := v(t, x).$$

Compare with: M. Di Francesco and A. Pascucci (05), V. Konakov, S. Menozzi and S. Molchanov (10).
Relation with degenerate equations

□ If $A$ was absolutely continuous, this would amount to looking for regularity for the degenerate PDE

$$-\partial_t \varphi(t, x) - x^1 A_t \partial_{x^2} \varphi(t, x) - \mu_t(x) \partial_{x^1} \varphi(t, x) - \frac{1}{2} \sigma_t(x)^2 \partial_{x^1, x^1}^2 \varphi(t, x) = 0$$

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⇒ In general $\partial_t \varphi$ and $\partial_{x^2} \varphi$ are not well defined! Even if $\partial_{x^1, x^1}^2 \varphi$ is, due to the regularizing effect of the noise.
Relation with degenerate equations

- If $A$ was absolutely continuous, this would amount to looking for regularity for the degenerate PDE

$$-\partial_t \varphi(t, x) - x^1 \dot{A}_t \partial_{x^2} \varphi(t, x) - \mu_t(x) \partial_{x^1} \varphi(t, x) - \frac{1}{2} \sigma_t(x)^2 \partial_{x^1, x^1}^2 \varphi(t, x) = 0$$

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⇒ In general $\partial_t \varphi$ and $\partial_{x^2} \varphi$ are not well defined! Even if $\partial_{x^1, x^1}^2 \varphi$ is, due to the regularizing effect of the noise.

- Still all can be well-defined if we appeal to the notion of Dupire’s derivative:

$$\partial_t \varphi(t, x) + x^1 \dot{A}_t \partial_{x^2} \varphi(t, x) = \text{horizontal derivative } \partial_t v \text{ of } v!$$
Another way to look at the PPDE is to make the change of variables

\[ (t, x) = (t, x_t, x_t A_t - \int_0^t x_s dA_s) = (t, x_t, \int_0^t A_s dx_s) \]

which leads to

\[ -\partial_t \varphi(t, x) - \mu_t(x)(1, A_t)D\varphi(t, x) - \frac{\sigma_t^2(x)}{2} \text{Tr}[\Sigma_t^A D^2 \varphi(t, x)] = 0 \]

with

\[ \Sigma_t^A := \begin{pmatrix} 1 & A_t \\ A_t & A_t^2 \end{pmatrix}. \]
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with

\[\Sigma_t^A := \begin{pmatrix} 1 & A_t \\ A_t & A_t^2 \end{pmatrix}.\]

⇒ Again \(\partial_t \varphi\) and \(\partial_x^2 \varphi\) are not well defined in general (unless coefficients are smooth). But this opens the door to the use of the parametrix approach.
Given \((t, y)\), look for the transition density
\[
(s, x) \in [0, t] \times \mathbb{R}^2 \mapsto \bar{f}_y(s, x; t, y)
\]
associated to the dynamics with \textit{frozen coefficients} and zero drift
\[
-\partial_t \varphi(s, x) - \frac{\sigma_t^2(y)}{2} \text{Tr}[\sum_s A^2 \varphi(s, x)] = 0.
\]
\[
L_y \varphi(s, x)
\]
Parametrix

□ Given \((t, y)\), look for the transition density

\[(s, x) \in [0, t] \times \mathbb{R}^2 \mapsto \bar{f}_y(s, x; t, y)\]

associated to the dynamics with \textit{frozen coefficients} and zero drift

\[-\partial_t \varphi(s, x) - \frac{\sigma_t^2(y)}{2} \text{Tr}[\Sigma_s^A D^2 \varphi(s, x)] = 0.\]

\[
L_y \varphi(s, x)
\]

It is a Gaussian density

\[
\bar{f}_y(s, x; t, y) = \frac{1}{2\pi|\Sigma_{s,t}(y)|^{\frac{1}{2}}} e^{-\frac{1}{2} (y-x)^\top \Sigma_{s,t}(y)^{-1} (y-x)}
\]

where

\[
\Sigma_{s,t}(y) := \sigma_t(y)^2 \begin{pmatrix} (t-s) & A^{(1)}_{s,t} \\ A^{(1)}_{s,t} & A^{(2)}_{s,t} \end{pmatrix} \text{ with } A^{(p)}_{s,t} := \int_s^t (A_r - A_s)^p \, dr, \; p \in \{1, 2\}.
\]
Let \((s, x) \in [0, t] \times \mathbb{R}^2 \mapsto \bar{f}(s, x; t, y)\) be the density for the original dynamics (without freezing the coefficients and with a drift), assuming it exists and is smooth.
Let \((s, x) \in [0, t] \times \mathbb{R}^2 \mapsto \bar{f}(s, x; t, y)\)

be the density for the original dynamics (without freezing the coefficients and with a drift), assuming it exists and is smooth.

With \(S_{s,t} := [s, t] \times \mathbb{R}^2\) and

\[L\varphi(t, x) := L_x\varphi(t, x) + \mu_t(x)(1, A_t)D\varphi(t, x)\]
Let 

\[(s, x) \in [0, t] \times \mathbb{R}^2 \mapsto \bar{f}(s, x; t, y)\]

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\[L\varphi(t, x) := L_x\varphi(t, x) + \mu_t(x)(1, A_t)D\varphi(t, x) :\]

\[\bar{f}(s, x; t, y) \equiv \bar{f}_y(s, x; t, y)\]

\[= \int_{S_{s,t}} \partial_r \left[ \bar{f}(s, x; r, z)\bar{f}_y(r, z; t, y) \right] \, drdz\]

\[= \int_{S_{s,t}} \left( -L^*\bar{f}(s, x; r, z) \right)\bar{f}_y(r, z; t, y) - \bar{f}(s, x; r, z)L_y\bar{f}_y(r, z; t, y) \, drdz\]

\[= \int_{S_{s,t}} (L - L_y)\bar{f}_y(r, z; t, y)\bar{f}(s, x; r, z) \, drdz.\]
This leads to

$$\tilde{f}(s, x; t, y) = \tilde{f}_y(s, x; t, y) + \int_{S_{s,t}} \tilde{f}_y(s, x; r, z)\tilde{\Phi}(r, z; t, y)\,drdz$$

with $\tilde{\Phi}(r, z; t, y) := \sum_{k \geq 1} \tilde{\Delta}_k(r, z; t, y)$ where

$$\tilde{\Delta}_1(s, x; t, y) := (L - L_y)\tilde{f}_y(s, x; t, y)$$

$$\tilde{\Delta}_{k+1}(s, x; t, y) := \int_{S_{s,t}} (L - L_z)\tilde{f}_z(s, x; r, z)\tilde{\Delta}_k(r, z; t, y)\,drdz.$$
This leads to

\[ \tilde{f}(s, x; t, y) = \tilde{f}_y(s, x; t, y) + \int_{S_{s,t}} \tilde{f}_y(s, x; r, z)\Phi(r, z; t, y) dr dz \]

with \( \Phi(r, z; t, y) := \sum_{k \geq 1} \tilde{\Delta}_k(r, z; t, y) \) where

\[ \tilde{\Delta}_1(s, x; t, y) := (L - L_y)\tilde{f}_y(s, x; t, y) \]
\[ \tilde{\Delta}_{k+1}(s, x; t, y) := \int_{S_{s,t}} (L - L_z)\tilde{f}_z(s, x; r, z)\tilde{\Delta}_k(r, z; t, y) dr dz. \]

⇒ Remains to check that the sum converges...

Formally,

\[ D_x \tilde{f}(s, x; t, y) = D_x \tilde{f}_y(s, x; t, y) + \int_{S_{s,t}} D_x \tilde{f}_y(s, x; r, z)\Phi(r, z; t, y) dr dz \]

in which \((s, x) \mapsto \tilde{f}_y(s, x; t, y)\) is smooth!

⇒ Remains to estimate the derivatives and check that they are integrable...
Existence of a transition density

- It requires structural conditions, which are just enough to obtain the correct estimates. Define

\[ m_{s,t}^{(1)} := \frac{1}{t-s} \int_s^t A_r dr \]

\[ m_{s,t}^{(2)} := \frac{1}{t-s} \int_s^t (A_r - m_{s,t}^{(1)})^2 dr \quad \text{and} \quad m_{s,t}^{(2)} := \frac{1}{t-s} \int_s^t (A_r - A_s)^2 dr. \]
Existence of a transition density

- It requires structural conditions, which are just enough to obtain the correct estimates. Define

\[ m^{(1)}_{s,t} := \frac{1}{t - s} \int_s^t A_r \, dr \]

\[ m^{(2)}_{s,t} := \frac{1}{t - s} \int_s^t (A_r - m^{(1)}_{s,t})^2 \, dr \]

and

\[ m_{s,t} := \frac{1}{t - s} \int_s^t (A_r - A_s)^2 \, dr. \]

- **Assumption:** \( \exists (\beta_i)_{0 \leq i \leq 4} \in \mathbb{R}_{+}^5 \) and \( C > 0 \) s.t., \( \forall 0 \leq s < t \leq T, \)

\[ \frac{1}{C} (t - s)^{-\beta_1} \leq \frac{m^{(2)}_{s,t}}{m^{(2)}_{s,t}} \leq C(t - s)^{-\beta_0}, \]

\[ \frac{1}{C} (t - s)^{1-\beta_2} \leq \frac{1}{m^{(2)}_{s,t}} \leq C(t - s)^{-\beta_3}. \]

\[ |A_t - A_s| \leq C(t - s)^{\beta_4}. \]
Assumption: We have

\[ \beta'_1 := \beta_1 - \beta_0 > -1, \quad \beta'_2 := \beta_2 - \beta_0 > -1, \]

and \( \exists (a, \bar{a}) \in \mathbb{R}^2, b \in \mathbb{R}, C > 0 \) and \( \alpha > 0 \) s.t.

\[ |\mu| \leq b, \quad 0 < a \leq \sigma^2 \leq \bar{a}, \]

\[ |\sigma_s(x) - \sigma_t(y)| \leq C \left( |t - s|^\alpha + |w_{s,t}(x, y)|^{\frac{2\alpha}{1 + \beta'_1}} + |w_{s,t}(x, y)|^{\frac{2\alpha}{1 + \beta'_2}} \right) \]

\[ |\mu_t(x) - \mu_t(y)| \leq C \left( |x_1 - y_1|^\frac{2\alpha}{1 + \beta'_1} + |x_2 - y_2|^\frac{2\alpha}{1 + \beta'_2} \right), \]

with

\[ w_{s,t}(x, y) := x - \begin{pmatrix} 1 & 0 \\ -(A_t - A_s) & 1 \end{pmatrix} y. \]
**Assumption :** We have

\[ \beta'_1 := \beta_1 - \beta_0 > -1, \quad \beta'_2 := \beta_2 - \beta_0 > -1, \]

and \( \exists (a, \bar{a}) \in \mathbb{R}^2, b \in \mathbb{R}, C > 0 \) and \( \alpha > 0 \) s.t.

\[
|\mu| \leq b, \quad 0 < a \leq \sigma^2 \leq \bar{a},
\]

\[
|\sigma_s(x) - \sigma_t(y)| \leq C \left( |t - s|^{\alpha} + |w_{s,t}(x, y)|^{\frac{2\alpha}{1+\beta'_1}} + |w_{s,t}(x, y)|^{\frac{2\alpha}{1+\beta'_2}} \right)
\]

\[
|\mu_t(x) - \mu_t(y)| \leq C \left( |x_1 - y_1|^{\frac{2\alpha}{1+\beta'_1}} + |x_2 - y_2|^{\frac{2\alpha}{1+\beta'_2}} \right),
\]

with

\[
w_{s,t}(x, y) := x - \begin{pmatrix} 1 & 0 \\ -(A_t - A_s) & 1 \end{pmatrix} y.
\]

**Remark :** can also impose standard Hölder continuity conditions on \( \sigma \) (slightly more complex to handle).
Let us now assume that

\[ \kappa_0 := \frac{1 - \beta_0}{2} \land (\alpha - \beta_0) > 0. \]

\[ \square \textbf{Proposition : } \bar{\Phi} \text{ is well-defined as well as } \]

\[ \bar{f}(s, x; t, y) = \bar{f}_y(s, x; t, y) + \int_{S_{s,t}} \bar{f}_y(s, x; r, z)\bar{\Phi}(r, z; t, y)drdz. \]
Let us now assume that

\[ \kappa_0 := \frac{1 - \beta_0}{2} \wedge (\alpha - \beta_0) > 0. \]

□ **Proposition** : \( \Phi \) is well-defined as well as

\[ \bar{f}(s, x; t, y) = \bar{f}_y(s, x; t, y) + \int_{S_{s, t}} \bar{f}_y(s, x; r, z)\Phi(r, z; t, y)drdz. \]

□ This is however not enough for \( \bar{f} \) to be even \( C^1 \) in \( x \)...
To obtain more regularity, we need to come back to the original variables (and think in terms of Dupire’s derivatives):

\[
(t, x_t, \int_0^t x_s dA_s) = \left( t, \Gamma_t(x_t, x_tA_t - \int_0^t x_s dA_s) \right)
\]

with

\[
\Gamma_t = \begin{pmatrix} 1 & 0 \\ A_t & -1 \end{pmatrix}.
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\]

with

\[
\Gamma_t = \begin{pmatrix} 1 & 0 \\ A_t & -1 \end{pmatrix}.
\]

The corresponding density is

\[
f(s, x; t, y) = \bar{f}(s, \Gamma_s x; t, \Gamma_t y),
\]

which we write as

\[
f(s, x; t, y) := f(s, (x_s, \int_0^s x_r dA_r); t, y).
\]
\( \mathcal{C}^{0,1} \)-regularity in the sense of Dupire

We now also assume that

\[
(1 - \beta_0) \wedge \left( \frac{1}{2} + \alpha - \frac{3}{2} \beta_0 \right) > 0.
\]
$C^{0,1}$-regularity in the sense of Dupire

We now also assume that

$$(1 - \beta_0) \land (\frac{1}{2} + \alpha - \frac{3}{2} \beta_0) > 0.$$ 

□ **Theorem**: $(s, x) \in [0, t) \times C([0, t]) \mapsto f(s, x; t, y)$ is $C^{0,1}$.
$\mathcal{C}^{0,1}$-regularity in the sense of Dupire

We now also assume that

$$(1 - \beta_0) \wedge \left(\frac{1}{2} + \alpha - \frac{3}{2} \beta_0\right) > 0.$$ 

□ **Theorem** : $(s, x) \in [0, t) \times C([0, t]) \mapsto f(s, x; t, y)$ is $\mathcal{C}^{0,1}$ and, for all (bounded) $g : x \mapsto g(x_T, \int_0^T x_s dA_s)$, the map

$$(s, x) \in [0, T] \times C([0, T]) \mapsto v(s, x) := \int_{\mathbb{R}^2} f(s, x; T, y)g(y)dy$$

is $\mathcal{C}^{0,1}([0, T])$. 
$C^{0,1}$-regularity in the sense of Dupire

We now also assume that

$$(1 - \beta_0) \wedge \left( \frac{1}{2} + \alpha - \frac{3}{2} \beta_0 \right) > 0.$$ 

□ **Theorem:** $(s, x) \in [0, t) \times C([0, t]) \mapsto f(s, x; t, y)$ is $C^{0,1}$ and, for all (bounded) $g : x \mapsto g(x_T, \int_0^T x_s dA_s)$, the map

$$(s, x) \in [0, T] \times C([0, T]) \mapsto v(s, x) := \int_{\mathbb{R}^2} f(s, x; T, y) g(y) dy$$

is $C^{0,1}([0, T])$. If moreover

$$X_t = X_0 + \int_0^t \mu_s(X_s, I_s) ds + \int_0^t \sigma_t(X_s, I_s) dW_s, \quad l_t = \int_0^t X_s dA_s,$$

admits a unique (strong Markov) weak solution, then $f$ is the transition density of $(X, I)$
\(C^{0,1}\)-regularity in the sense of Dupire

We now also assume that

\[
(1 - \beta_0) \wedge \left( \frac{1}{2} + \alpha - \frac{3}{2} \beta_0 \right) > 0.
\]

\(\square\) Theorem : \((s, x) \in [0, t) \times C([0, t]) \mapsto f(s, x; t, y)\) is \(C^{0,1}\) and, for all (bounded) \(g : x \mapsto g(x_T, \int_0^T x_s dA_s)\), the map

\[
(s, x) \in [0, T] \times C([0, T]) \mapsto v(s, x) := \int_{\mathbb{R}^2} f(s, x; T, y)g(y)dy
\]

is \(C^{0,1}([0, T])\). If moreover

\[
X_t = X_0 + \int_0^t \mu_s(X_s, I_s)ds + \int_0^t \sigma_t(X_s, I_s)dW_s, \quad I_t = \int_0^t X_s dA_s,
\]

admits a unique (strong Markov) weak solution, then \(f\) is the transition density of \((X, I)\) and (under additional technical conditions)

\[
v(t, X) = v(0, X) + \int_0^t \nabla_x v(s, X) \sigma_s(X) dW_s.
\]
$\mathbb{C}^{1,2}$-regularity in the sense of Dupire

We finally also assume that $\exists \alpha_\Phi > 0$ s.t.

$$0 < \alpha_\Phi < \kappa_0 \wedge \hat{\alpha}_\Phi \wedge \min_{i=1,2} \frac{1 + \beta_i'}{2}, \text{ with } \hat{\alpha}_\Phi := \frac{1}{2} - \beta_0 - \frac{\Delta \beta}{2} - \frac{(\beta_0 + 1 - 2\alpha)^+}{2},$$

where $\Delta \beta := \max \{\beta_0 - \beta_1, \beta_3 - \beta_2\}$, and

$$\min \left( \frac{2\beta_4 + 1 + \beta_1'}{1 + \beta'_2}, 1 \right) \min \{\alpha_\Phi, \alpha\} - \beta_0 > 0.$$
$C^{1,2}$-regularity in the sense of Dupire

We finally also assume that $\exists \alpha_\Phi > 0$ s.t.

$$0 < \alpha_\Phi < \kappa_0 \wedge \hat{\alpha}_\Phi \wedge \min_{i=1,2} \frac{1 + \beta_i'}{2}, \text{ with } \hat{\alpha}_\Phi := \frac{1}{2} - \beta_0 - \frac{\hat{\Delta}\beta}{2} - \frac{(\beta_0 + 1 - 2\alpha)^+}{2},$$

where $\hat{\Delta}\beta := \max \{\beta_0 - \beta_1, \beta_3 - \beta_2\}$, and

$$\min \left( \frac{2\beta_4 + 1 + \beta_1'}{1 + \beta_2'}, 1 \right) \min \{\alpha_\Phi, \alpha\} - \beta_0 > 0.$$

**Theorem**: The SDE

$$X_t = X_0 + \int_0^t \mu_s(X_s, I_s) ds + \int_0^t \sigma_t(X_s, I_s) dW_s, \quad I_t = \int_0^t X_s dA_s$$

admits a unique weak solution that is a strong Markov process.
$\mathbb{C}^{1,2}$-regularity in the sense of Dupire

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**Theorem:** The SDE

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admits a unique weak solution that is a strong Markov process. $f$ is its transition density and $(s, x) \in [0, t) \times \mathcal{C}([0, t]) \mapsto f(s, x; t, y)$ is $\mathbb{C}^{1,2}$. 
C^{1,2}\text{-regularity in the sense of Dupire}

We finally also assume that \( \exists \alpha_\Phi > 0 \) s.t.

\[
0 < \alpha_\Phi < \kappa_0 \land \hat{\alpha}_\Phi \land \min_{i=1,2} \frac{1 + \beta'_i}{2}, \quad \text{with} \quad \hat{\alpha}_\Phi := \frac{1}{2} - \beta_0 - \frac{\hat{\Delta} \beta}{2} - \frac{(\beta_0 + 1 - 2\alpha)^+}{2},
\]

where \( \hat{\Delta} \beta := \max \{ \beta_0 - \beta_1, \beta_3 - \beta_2 \} \), and

\[
\min \left( \frac{2\beta_4 + 1 + \beta'_1}{1 + \beta'_2}, 1 \right) \min \{ \alpha_\Phi, \alpha \} - \beta_0 > 0.
\]

\[ \square \textbf{Theorem} : \text{The SDE} \]

\[
X_t = X_0 + \int_0^t \mu_s(X_s, I_s) ds + \int_0^t \sigma_t(X_s, I_s) dW_s, \quad I_t = \int_0^t X_s dA_s
\]

admits a unique weak solution that is a strong Markov process. \( \tilde{f} \) is its transition density and \((s, x) \in [0, t) \times C([0, t]) \mapsto \tilde{f}(s, x; t, y) \) is \( \mathbb{C}^{1,2} \).

\( \tilde{v} \) is \( \mathbb{C}^{1,2}([0, T]) \) and solves the PPDE

\[
-\partial_t \tilde{v}(t, x) - \mu_t(x) \nabla_x \tilde{v}(t, x) - \frac{1}{2} \sigma_t(x)^2 \nabla_x^2 \tilde{v}(t, x) = 0, \quad \tilde{v}(T, \cdot) = g.
\]
Toy examples

(i) $A_t = \int_0^t \rho(s) ds$, with $\varepsilon \leq \rho \leq 1/\varepsilon$ a.e. for some $\varepsilon > 0$. 
⇒ all assumptions are satisfied for $\alpha > 0$. 
Toy examples

(i) $A_t = \int_0^t \rho(s) ds$, with $\varepsilon \leq \rho \leq 1/\varepsilon$ a.e. for some $\varepsilon > 0.$
⇒ all assumptions are satisfied for $\alpha > 0.$

(ii) $A_t = t^\gamma$ for some $\gamma \in (0, 1).$
⇒ all assumptions are satisfied for $\alpha > 0.$
Toy examples

(i) $A_t = \int_0^t \rho(s)ds$, with $\varepsilon \leq \rho \leq 1/\varepsilon$ a.e. for some $\varepsilon > 0$. 
$\Rightarrow$ all assumptions are satisfied for $\alpha > 0$.

(ii) $A_t = t^\gamma$ for some $\gamma \in (0, 1)$. 
$\Rightarrow$ all assumptions are satisfied for $\alpha > 0$.

(iii) $\exists 1 \geq \gamma_1 \geq \gamma_2 > 0$ and $C_1, C_2 > 0$ s.t.

$$C_1 |t - s|^{\gamma_1} \leq A_t - A_s \leq C_2 |t - s|^{\gamma_2}.$$ 

$\Rightarrow$ all assumptions are satisfied for $\gamma_1 - \gamma_2 > 0$ small enough (with respect to $\alpha$). Typically, $\gamma_1 = 1$ in this case.
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Thank you!