Some regularity results for a class of PPDE and applications

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Based on works with
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Motivations

□ See under which conditions one can apply Itô-Dupire's formula to value functions associated to path-dependent pricing or optimal control problems.

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□ Use $C^{1+\alpha}$-regularity or show C^2-regularity using PDEs.

Example #1 : second order coupled FBSDE

□ B. and Tan [3] : Solve a second order BSDE related to a (perfect) hedging problem under price impact. Find $(X, Y, Z, \mathfrak{g}, \mathfrak{B})$ such that

$$X_t = x_0 + \int_0^t \sigma_s(X, \mathfrak{g}_s) dW_s$$

$$Y_t = \Phi(X) - \int_t^T F_s(X, \mathfrak{g}_s) ds - \int_0^t Z_s dX_s \text{ and } Z_t = Z_0 + \int_0^t \mathfrak{g}_s dX_s - \mathfrak{B}_t$$

where Φ, σ and F are path-dependent (non-anticipative).

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where Φ, σ and F are path-dependent (non-anticipative).

□ Derive a solution from a dual formulation of the form :

$$\mathrm{v}(t, \mathrm{x}) := \sup_{\alpha} \mathbb{E}\Big[\Phi\big(\bar{X}^{t,\mathrm{x},\alpha}\big) - \int_t^T G_s\big(\bar{X}^{t,\mathrm{x},\alpha}, \alpha_s\big) ds\Big], \; d\bar{X}^{t,\mathrm{x},\alpha} = \alpha dW,$$

by using Itô's lemma : $Y = \mathrm{v}(\cdot, X)$, $Z = D\mathrm{v}(\cdot, X)$, etc.

Reminder on Dupire's derivatives

- Notations :
  - x belongs to $C([0, T])$ or $D([0, T])$.
  - $\mathrm{x}_{t\wedge} := (\mathrm{x}_{t\wedge s})_{s\in[0,T]}$,
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- **Horizontal derivative** :

$$\partial_t \mathrm{v}(t,\mathrm{x}) = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{v}(t+\varepsilon, \mathrm{x}_{t\wedge}) - \mathrm{v}(t,\mathrm{x})}{\varepsilon}.$$

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- We define $\mathbb{C}^{0,1}$ and $\mathbb{C}^{1,2}$ accordingly.

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$$\mathrm{v}(t, \theta\mathrm{x}^1 + (1-\theta)\mathrm{x}^2) \;\geq\; \theta\mathrm{v}(t, \mathrm{x}^1) + (1-\theta)\mathrm{v}(t, \mathrm{x}^2), \;\text{ for all } \theta \in [0, 1]$$

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□ If v is **Dupire-concave**, one can define its super-differential

$$\partial\mathrm{v}(t,\mathrm{x}) \;:=\; \left\{ z : \mathrm{v}(t, \mathrm{x} \oplus_t y) \leq \mathrm{v}(t,\mathrm{x}) + z \cdot y, \; \forall\, y \right\}.$$

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Theorem [B. and Tan [3, 4]] Assume that v is Dupire-concave and non-increasing in time. Under additional local boundedness and equi-continuity assumptions $[\cdots]$, we have

$$\mathrm{v}(t,X) = \mathrm{v}(0,X) + \int_0^t H_s dX_s - C_t^{\mathbb{P}},\ t \in [0,T],\ \mathbb{P}\text{−a.s. } \forall\, \mathbb{P} \in \mathcal{P},$$

in which $\{C^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}\}$ is a collection of non-decreasing processes and $H_s \in \partial \mathrm{v}(s, X^{s-})$ for all $s \in [0,T]$, $\mathcal{P}$-q.s, where

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⇒ Enough to construct a solution to our second order FBSDE. See also B. and Tan [4] for an application to robust super-hedging with jumps (compare to Nutz 15).

Example #2 : super-hedging under (bounded) volatility uncertainty

Let us consider a payoff function of the form

$$g(X) \;=\; g_\circ\Big(X_T, \int_0^T X_t dA_t\Big),\; g_\circ \in C^{1+\alpha}(\mathbb{R}).$$

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☐ **Uncertainty** modeled by $\mathcal{P}_0$: $\mathbb{P}$ such that $\mathbb{P}[X_0 = x_0] = 1$ and

$$dX_s = \sigma_s dW_s^{\mathbb{P}}, \ \sigma_s \in [\underline{\sigma}, \overline{\sigma}], \ s \in [0, T], \ \mathbb{P}\text{-a.s.} \tag{1}$$

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□ **Dual formulation :**

$$\mathrm{v}(t, \mathrm{x}) \; := \sup_{\mathbb{P} \in \mathcal{P}(t,\mathrm{x})} \mathbb{E}^{\mathbb{P}}\big[g(X)\big] \; = \; \text{robust super-hedging price}$$

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$\Rightarrow$ **is not $\mathbb{C}^{1,2}$ a priori but may be $\mathbb{C}^{0,1+\alpha}$ since g is.**

If it is $\mathbb{C}^{0,1+\alpha}$, then on can find the hedging strategy (and prove duality) by applying this version of Itô-Dupire's Lemma.

Theorem [B., Loeper and Tan [2]] : Let X be a semimatingale, $\mathrm{v} \in \mathbb{C}^{0,1}$ such that v and $\nabla_{\mathrm{x}}\mathrm{v}$ are locally uniformly continuous and $[\cdots]$. Then,

$$\mathrm{v}(t,X) \;=\; \mathrm{v}(0,X) \;+\; \int_0^t \nabla_{\mathrm{x}}\mathrm{v}(s,X)dX_s \;+\; \Gamma_t, \quad t \in [0,T],$$

where Γ is a continuous orthogonal process, **if and only if**

$$\frac{1}{\varepsilon}\int_0^{\cdot}\{\mathrm{v}(s+\varepsilon,X)-\mathrm{v}(s+\varepsilon,X_{s\wedge}\oplus_{s+\varepsilon}(X_{s+\varepsilon}-X_s))\}\{N_{s+\varepsilon}-N_s\}ds \underset{\varepsilon\downarrow 0}{\longrightarrow}, \text{ u.c.p.}$$

for all (bounded) continuous martingale N.

Remark : Compare with Bandini and Russo (17) and Gozzi and Russo (06). Here

$$[X,Y]_t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}-X_s)(Y_{s+\varepsilon}-Y_s)ds$$

Remark : The above condition holds as soon as for some $L \in \mathrm{BV}$:

$$|\mathrm{v}(t,\mathrm{x}) - \mathrm{v}(t,\mathrm{x}')| \leq C \int_0^t |\mathrm{x}_s - \mathrm{x}'_s| dL_s.$$

□ Similar result for càdlàg processes (B. and Vallet [5]).

□ It remains to show that the candidate solution to the PPDE

$$-\partial_t \mathrm{v} - \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{\sigma^2}{2} \nabla_{\mathrm{x}}^2 \mathrm{v} = 0, \ \mathrm{v}(T, \cdot) = g$$

is $\mathbb{C}^{0,1}$ with locally uniformly continuous vertical Dupire's derivative.

Approximate viscosity solutions of PPDEs
(A tool for regularity)

$$-\partial_t\varphi(t,\mathrm{x}) - F(t,\mathrm{x},\varphi(t,\mathrm{x}),\nabla_{\mathrm{x}}\varphi(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\varphi(t,\mathrm{x})) = 0,\ \varphi(T,\cdot) = g$$

B., Loeper and Tan [10].

Related works : Ekren, Touzi and Zhang (16), Ren, Touzi and Zhang (17), Ekren and Zhang (16), Cosso and Russo (19), Jianjun Zhou (21).

Definition of solutions by approximation

☐ Let $\pi = (\pi^n)_n$, with $\pi^n = (t_i^n)_{0 \le i \le n}$, be an increasing sequence of time grids. Set

$$\bar{\mathrm{x}}^n := \sum_{i=0}^{n-1} \mathrm{x}_{t_i^n} 1_{[t_i^n, t_{i+1}^n)} + \mathrm{x}_{t_n^n} 1_{\{T\}}$$

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☐ We say that a continuous function v^n is a π^n-**viscosity solution** of

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if it is of the form

$$\sum_{i=0}^{n-1} 1_{[t_i^n, t_{i+1}^n)} v_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x)$$

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in which each $v_i^n(\cdot, \bar{\mathrm{x}}^n_{\wedge t_i^n}, \cdot)$ is a viscosity solution on $\mathbb{R}^d \times [t_i^n, t_{i+1}^n)$ of

$$-\partial_t v_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x) - F(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, v_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x), Dv_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x), D^2 v_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x)) = 0$$

$$v_i^n(t_{i+1}^n -, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x) = v_{i+1}^n(t_{i+1}^n, \bar{\mathrm{x}}^n_{\wedge t_i^n} 1_{[0, t_{i+1}^n)} + 1_{\{t_{i+1}^n\}} x, x)$$

□ Example : Think about replacing

$$\mathrm{v}(t,\mathrm{x}) \ := \sup_{\mathbb{P}\in\mathcal{P}(t,\mathrm{x})} \mathbb{E}^{\mathbb{P}}\big[g(X)\big]$$

by

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(i) If $t\in[t^n_i,t^n_{i+1})$, then $(\bar{X}^n_{t^n_0},\cdots,\bar{X}^n_{t^n_i})$ is known (and is a parameter for the period $[t,\ T]$)

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(ii) At the boundary $t=t^n_{i+1}$, the value $\bar{X}^n_{t^n_{i+1}}$ is also frozen, and serves as a starting point for $\bar{X}^n$ on the period $[t^n_{i+1},t^n_{i+2})$.

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(iii) Ends up with a sequence of backward PDEs on : $[t^n_{n-1}, T)$, $[t^n_{n-2}, t^n_{n-1})$, and so on.

□ We say that v is a **π-approximate-viscosity solution** on $D([0, T])$ of

$$-\partial_t \mathrm{v}(t, \mathrm{x}) - F(t, \mathrm{x}, \mathrm{v}(t, \mathrm{x}), \nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x}), \nabla^2_{\mathrm{x}} \mathrm{v}(t, \mathrm{x})) = 0 \ , \ t < T$$

with terminal condition

$$\mathrm{v}(T, \cdot) = g$$

if $v^n(t, \mathrm{x}, \mathrm{x}_t) \to \mathrm{v}(t, \mathrm{x})$ for all $(t, \mathrm{x}) \in [0, T] \times D([0, T])$ where $(v^n)_n$ is the sequence defined as above with

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□ Typical examples : Semi-linear PPDEs or HJB equations.
$\Rightarrow$ In both cases, amounts to replacing X by $\bar{X}^n$ in the coefficients and payoff.

But we also want to consider general non-linear parabolic PPDEs.

Existence, comparison, stability

□ We focus on the case where

$$F(t, \mathrm{x}, r, p, q) = H(t, \mathrm{x}, r, p, q) + \rho(t, \mathrm{x})r + b(t, \mathrm{x})p + \frac{1}{2}\sigma^2(t, \mathrm{x})q$$

where all the coefficients are continuous and Lipschitz/uniformly continuous in space $[\cdots]$ + standard assumptions to have comparison and existence of a viscosity solution with linear growth in finite dimension (for the $F(\cdot, \bar{\mathrm{x}}^n_{\wedge t_i^n}, \cdot)$).

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Proposition : Comparison and stability holds in the class of solutions.

Existence, comparison, stability

□ We focus on the case where

$$F(t, \mathrm{x}, r, p, q) = H(t, \mathrm{x}, r, p, q) + \rho(t, \mathrm{x})r + b(t, \mathrm{x})p + \frac{1}{2}\sigma^2(t, \mathrm{x})q$$

where all the coefficients are continuous and Lipschitz/uniformly continuous in space $[\cdots]$ + standard assumptions to have comparison and existence of a viscosity solution with linear growth in finite dimension (for the $F(\cdot, \bar{\mathrm{x}}^n_{\wedge t_i^n}, \cdot)$).

Theorem : Let g be uniformly continuous, then $\exists$ a unique π-approximate viscosity solution v on $D([0, T])$. Moreover,

- It is locally uniformly continuous.
- If π' is another increasing sequence of time grids and if v' is the π'-approximate viscosity solution, then $\mathrm{v}' = \mathrm{v}$.

Proposition : Comparison and stability holds in the class of solutions.

Remark : We have precise estimates on the approximation error $|v^n(t, \mathrm{x}, \mathrm{x}_t) - \mathrm{v}(t, \mathrm{x})|$ (depending on the regul. of x).

Regularity in the fully non-linear case

Recall that

$$-\partial_t v_i^n(t, \bar{x}^n_{\wedge t_i^n}, x) - F(t, \bar{x}^n_{\wedge t_i^n}, v_i^n(t, \bar{x}^n_{\wedge t_i^n}, x), Dv_i^n(t, \bar{x}^n_{\wedge t_i^n}, x), D^2 v_i^n(t, \bar{x}^n_{\wedge t_i^n}, x)) = 0$$

$$v_i^n(t_{i+1}^n -, \bar{x}^n_{\wedge t_i^n}, x) = v_{i+1}^n(t_{i+1}^n, \bar{x}^n_{\wedge t_i^n} 1_{[0, t_{i+1}^n)} + 1_{\{t_{i+1}^n\}} x, x)$$

□ For terminal conditions of the form (can be made more abstract)

$$g(\mathrm{x}) \;=\; g_\circ\Big(\int_0^T \mathrm{x}_t dA_t\Big),$$

where $g_\circ \in C^{1+\alpha}(\mathbb{R})$ is bounded, and A is BV with at most finitely many jumps on $[0, T]$.

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Theorem : $\nabla_\mathrm{x}\mathrm{v}$ is well-defined and locally uniformly continuous.

Remark : In the semi-linear case $F = F(t,\mathrm{x},y,z)$, only needs $C^{1+\alpha}$ (in the Fréchet sense with respect to the path) : just differentiate the corresponding BSDE...

Regularization in the uniformly elliptic case
(Bouchard and Tan [11])

We focus on the linear case (with $d = 1$) and consider

$$-\partial_t \mathrm{v}(t,\mathrm{x}) - \mu_t(\mathrm{x})\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}) - \frac{1}{2}\sigma_t(\mathrm{x})^2\nabla^2_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}) = 0,\ \mathrm{v}(T,\cdot) = g$$

with

$$(\mu_t,\sigma_t)(\mathrm{x}) = (\mu_t,\sigma_t)(\mathrm{x}_t, \int_0^t \mathrm{x}_s dA_s) \text{ and } g(\mathrm{x}) = g(\mathrm{x}_T, \int_0^T \mathrm{x}_s dA_s),$$

in which

μ and σ^2 are Hölder (bounded), $\sigma^2 \geq \underline{\mathfrak{a}} > 0$ and g measurable (bounded).

Relation with degenerate equations

□ If A was absolutely continuous, this would amount to looking for regularity for the degenerate PDE

$$-\partial_t \varphi(t,x) - x^1 \dot{A}_t \partial_{x^2} \varphi(t,x) - \mu_t(x)\partial_{x^1}\varphi(t,x) - \frac{1}{2}\sigma_t(x)^2 \partial^2_{x^1,x^1}\varphi(t,x) = 0$$

in which derivatives are taken in the traditional sense and

$$(t,x) := (t, \mathrm{x}_t, \int_0^t \mathrm{x}_s \dot{A}_s ds)\ ,\ \varphi(t,x) := \mathrm{v}(t,\mathrm{x}).$$

Compare with : M. Di Francesco and A. Pascucci (05), V. Konakov, S. Menozzi and S. Molchanov (10).

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□ Still all can be well-defined if we **appeal to the notion of Dupire's derivative** :

$$\partial_t \varphi(t,x) + x^1 \dot{A}_t \partial_{x^2} \varphi(t,x) = \text{horizontal derivative } \partial_t \mathrm{v} \text{ of } \mathrm{v}\,!$$

Change of variables

□ Another way to look at the PPDE is to make the change of variables

$$(t, x) = (t, \mathrm{x}_t, \mathrm{x}_t A_t - \int_0^t \mathrm{x}_s dA_s) = (t, \mathrm{x}_t, \int_0^t A_s d\mathrm{x}_s)$$

which leads to

$$-\partial_t \varphi(t, x) - \mu_t(x)(1, A_t) D\varphi(t, x) - \frac{\sigma_t^2(x)}{2} \mathrm{Tr}[\Sigma_t^A D^2 \varphi(t, x)] = 0$$

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⇒ Again $\partial_t \varphi$ and $\partial_{x^2} \varphi$ are not well defined in general (unless coefficients are smooth). But this opens the door to the use of the parametrix approach.

Parametrix

☐ Given (t, y), look for the transition density

$$(s, x) \in [0, t] \times \mathbb{R}^2 \mapsto \bar{f}_y(s, x; t, y)$$

associated to the dynamics with *frozen coefficients* and zero drift

$$-\partial_t \varphi(s, x) - \underbrace{\frac{\sigma_t^2(y)}{2} \mathrm{Tr}[\Sigma_s^A D^2 \varphi(s, x)]}_{L_y \varphi(s,x)} = 0 \ .$$

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It is a Gaussian density

$$\bar{f}_y(s, x; t, y) = \frac{1}{2\pi |\Sigma_{s,t}(y)|^{\frac{1}{2}}} e^{-\frac{1}{2}(y-x)^\top \Sigma_{s,t}(y)^{-1}(y-x)}$$

where

$$\Sigma_{s,t}(y) := \sigma_t(y)^2 \begin{pmatrix} (t-s) & A_{s,t}^{(1)} \\ A_{s,t}^{(1)} & A_{s,t}^{(2)} \end{pmatrix} \text{ with } A_{s,t}^{(p)} := \int_s^t (A_r - A_s)^p dr, \ p \in \{1, 2\}.$$

Link with the original density

☐ Let

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With $S_{s,t} := [s,t] \times \mathbb{R}^2$ and
$L\varphi(t,x) := L_x\varphi(t,x) + \mu_t(x)(1, A_t)D\varphi(t,x)$:

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$$\begin{aligned}
&\bar{f}(s,x;t,y) - \bar{f}_y(s,x;t,y) \\
&= \int_{S_{s,t}} \partial_r \left[\bar{f}(s,x;r,z)\bar{f}_y(r,z;t,y)\right] drdz \\
&= \int_{S_{s,t}} (-L^*\bar{f}(s,x;r,z))\bar{f}_y(r,z;t,y) - \bar{f}(s,x;r,z)L_y\bar{f}_y(r,z;t,y)drdz \\
&= \int_{S_{s,t}} (L - L_y)\bar{f}_y(r,z;t,y)\bar{f}(s,x;r,z)drdz.
\end{aligned}$$

□ This leads to

$$\bar{f}(s,x;t,y) = \bar{f}_y(s,x;t,y) + \int_{S_{s,t}} \bar{f}_y(s,x;r,z)\bar{\Phi}(r,z;t,y)drdz$$

with $\bar{\Phi}(r,z;t,y) := \sum_{k\geq 1} \bar{\Delta}_k(r,z;t,y)$ where

$$\bar{\Delta}_1(s,x;t,y) := (L - L_y)\bar{f}_y(s,x;t,y)$$

$$\bar{\Delta}_{k+1}(s,x;t,y) := \int_{S_{s,t}} (L - L_z)\bar{f}_z(s,x;r,z)\bar{\Delta}_k(r,z;t,y)drdz.$$

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□ Formally,

$$D_x\bar{f}(s,x;t,y) = D_x\bar{f}_y(s,x;t,y) + \int_{S_{s,t}} D_x\bar{f}_y(s,x;r,z)\bar{\Phi}(r,z;t,y)drdz$$

in which $(s,x) \mapsto \bar{f}_y(s,x;t,y)$ is smooth !

⇒ Remains to estimate the derivatives and check that they are integrable...

Existence of a transition density

□ It requires structural conditions, which are just enough to obtain the correct estimates. Define

$$m_{s,t}^{(1)} := \frac{1}{t-s}\int_s^t A_r\,dr$$

$$m_{s,t}^{(2)} := \frac{1}{t-s}\int_s^t (A_r - m_{s,t}^{(1)})^2 dr \quad \text{and} \quad m_{s,t}^{(2)} := \frac{1}{t-s}\int_s^t (A_r - A_s)^2 dr.$$

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□ **Assumption :** $\exists\ (\beta_i)_{0\le i\le 4} \in \mathbb{R}^5_+$ and $C > 0$ s.t., $\forall\ 0 \le s < t \le T$,

$$\frac{1}{C}(t-s)^{-\beta_1} \le \frac{m^{(2)}_{s,t}}{m^{(2)}_{s,t}} \le C(t-s)^{-\beta_0},$$

$$\frac{1}{C}(t-s)^{-\beta_2} \le \frac{1}{m^{(2)}_{s,t}} \le C(t-s)^{-\beta_3}.$$

$$|A_t - A_s| \le C(t-s)^{\beta_4}.$$

☐ **Assumption :** We have

$$\beta_1' := \beta_1 - \beta_0 > -1, \qquad \beta_2' := \beta_2 - \beta_0 > -1,$$

and $\exists\ (\underline{\mathfrak{a}}, \bar{\mathfrak{a}}) \in \mathbb{R}^2$, $\mathfrak{b} \in \mathbb{R}$, $C > 0$ and $\alpha > 0$ s.t.

$$|\mu| \le \mathfrak{b}, \quad 0 < \underline{\mathfrak{a}} \le \sigma^2 \le \bar{\mathfrak{a}},$$

$$\left|\sigma_s(x) - \sigma_t(y)\right| \ \le\ C\Big(|t-s|^{\alpha} + \left|\mathrm{w}_{s,t}(x,y)\right|^{\frac{2\alpha}{1+\beta_1'}} + \left|\mathrm{w}_{s,t}(x,y)\right|^{\frac{2\alpha}{1+\beta_2'}}\Big)$$

$$\left|\mu_t(x) - \mu_t(y)\right| \le C\Big(\left|x_1 - y_1\right|^{\frac{2\alpha}{1+\beta_1'}} + \left|x_2 - y_2\right|^{\frac{2\alpha}{1+\beta_2'}}\Big),$$

with

$$\mathrm{w}_{s,t}(x,y) := x - \begin{pmatrix} 1 & 0 \\ -(A_t - A_s) & 1 \end{pmatrix} y.$$

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□ **Remark :** can also impose standard Hölder continuity conditions on σ (slightly more complex to handle).

Let us now assume that

$$\kappa_0 \ := \ \frac{1-\beta_0}{2} \wedge (\alpha - \beta_0) \ > \ 0.$$

□ **Proposition :** $\bar{\Phi}$ is well-defined as well as

$$\bar{f}(s,x;t,y) = \bar{f}_y(s,x;t,y) + \int_{S_{s,t}} \bar{f}_y(s,x;r,z)\bar{\Phi}(r,z;t,y)drdz.$$

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□ This is however not enough for $\bar{f}$ to be even C^1 in x...

Back to the original variables

□ To obtain more regularity, we need to come back to the original variables (and think in terms of Dupire's derivatives) :

$$(t, \mathrm{x}_t, \int_0^t \mathrm{x}_s dA_s) = \left(t, \Gamma_t(\mathrm{x}_t, \mathrm{x}_t A_t - \int_0^t \mathrm{x}_s dA_s) \right)$$

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□ The corresponding density is

$$f(s, x; t, y) = \bar{f}(s, \Gamma_s x; t, \Gamma_t y),$$

which we write as

$$\mathrm{f}(s, \mathrm{x}; t, y) := f(s, (\mathrm{x}_s, \int_0^s \mathrm{x}_r dA_r); t, y).$$

$\mathbb{C}^{0,1}$-regularity in the sense of Dupire

We now also assume that

$$(1 - \beta_0) \wedge \left(\frac{1}{2} + \alpha - \frac{3}{2}\beta_0\right) \ > \ 0.$$

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$$(s, \mathrm{x}) \in [0, T] \times C([0, T]) \mapsto \mathrm{v}(s, \mathrm{x}) := \int_{\mathbb{R}^2} \mathrm{f}(s, \mathrm{x}; T, y) g(y) dy$$

is $\mathbb{C}^{0,1}([0, T))$.

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is $\mathbb{C}^{0,1}([0,T))$. If moreover

$$X_t = X_0+\int_0^t \mu_s(X_s,I_s)ds+\int_0^t \sigma_t(X_s,I_s)dW_s, \quad I_t=\int_0^t X_s dA_s,$$

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$$\mathrm{v}(t,X)=\mathrm{v}(0,X)+\int_0^t \nabla_{\mathrm{x}}\mathrm{v}(s,X)\sigma_s(X)dW_s.$$

$\mathbb{C}^{1,2}$-regularity in the sense of Dupire

We finally also assume that $\exists \alpha_\Phi > 0$ s.t.

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$$-\partial_t \mathrm{v}(t, \mathrm{x}) - \mu_t(\mathrm{x})\nabla_\mathrm{x} \mathrm{v}(t, \mathrm{x}) - \frac{1}{2}\sigma_t(\mathrm{x})^2 \nabla_\mathrm{x}^2 \mathrm{v}(t, \mathrm{x}) = 0, \ \mathrm{v}(T, \cdot) = g.$$

Toy examples

(i) $A_t = \int_0^t \rho(s)ds$, with $\varepsilon \leq \rho \leq 1/\varepsilon$ a.e. for some $\varepsilon > 0$.
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(iii) $\exists\ 1 \geq \gamma_1 \geq \gamma_2 > 0$ and $C_1, C_2 > 0$ s.t.

$$C_1|t-s|^{\gamma_1} \leq A_t - A_s \leq C_2|t-s|^{\gamma_2}.$$

$\Rightarrow$ all assumptions are satisfied for $\gamma_1 - \gamma_2 > 0$ small enough (with respect to α). Typically, $\gamma_1 = 1$ in this case.

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Thank you !