A $C^{0,1}$-Itô’s Formula for Flows of Semimartingale Distributions

B. Bouchard

CEREMADE, Université Paris Dauphine - PSL

Joint work with Xiaolu Tan and Jixin Wang (Chinese University of Hong Kong)
Motivation

- Replace $C^{1,2}$-regularity by $C^{0,1}$ when applying Itô’s lemma in situations where regularity is difficult to obtain:
  - Path-dependent functionals: see Xiaolu’s talk.
  - McKean-Vlasov optimal control problems.
Motivation

□ Replace $C^{1,2}$-regularity by $C^{0,1}$ when applying Itô’s lemma in situations where regularity is difficult to obtain:
  - Path-dependent functionals: see Xiaolu’s talk.
  - McKean-Vlasov optimal control problems.

□ We know that it is possible for functionals on $[0, T] \times \mathbb{R}^d$ associated to classical Markovian problems.
The classical Markovian situation
Weak Dirichlet processes

- In the Markovian case: works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [4].

### Definitions:

- Let $X$ and $Y$ be two real valued càdlàg processes. The co-quadractic variation $[X, Y]_t$ is defined by

$$[X, Y]_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t (X(s+\varepsilon) \wedge t - X_s) (Y(s+\varepsilon) \wedge t - Y_s) ds,$$

whenever the limit exists in the sense of u.c.p.

- $X$ has finite quadratic variation, if $[X] := [X, X]$ exists and is finite a.s.

- $A$ is orthogonal if $[A, N] = 0$ for any real valued continuous local martingale $N$.

- $X$ is a weak Dirichlet process if $X = X_0 + M + A$, where $M$ is a local martingale and $A$ is orthogonal such that $M_0 = A_0 = 0$. 
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\[ C^{0,1}\text{-Itô's formula} \]

\[ \square \text{Theorem (Gozzi and Russo [4]) : Let } X = X_0 + M + A \text{ be a continuous weak Dirichlet process with finite quadratic variation, } \nu \in C^{0,1}(\[0, T\) \times \mathbb{R}^d). \text{ Then,} \]

\[ \nu(t, X_t) = \nu(0, X) + \int_0^t \partial_x \nu(s, X_s) dM_s + \Gamma_t, \quad t \in [0, T), \]

where \( \Gamma \) is a continuous orthogonal process.
\( C^{0,1}\)-Itô’s formula

Theorem (Gozzi and Russo [4]) : Let \( X = X_0 + M + A \) be a continuous weak Dirichlet process with finite quadratic variation, \( v \in C^{0,1}([0, T) \times \mathbb{R}^d) \). Then,

\[
v(t, X_t) = v(0, X) + \int_0^t \partial_x v(s, X_s) dM_s + \Gamma_t, \quad t \in [0, T),
\]

where \( \Gamma \) is a continuous orthogonal process.

Remark :

- A version is available for processes with jumps, see Bandini and Russo [1].
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  - A version is available for processes with jumps, see Bandini and Russo [1].
  - If \( v(\cdot, X) \) is a martingale, then \( \Gamma \equiv 0 \) (e.g. replication price in finance, value function of an optimal control problem along the optimal path,...)
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\[ \text{□ Theorem (Gozzi and Russo [4]) : Let } X = X_0 + M + A \text{ be a continuous weak Dirichlet process with finite quadratic variation, } \nu \in C^{0,1}(\mathbb{R}^d) \text{. Then,} \]

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- A version is available for processes with jumps, see Bandini and Russo [1].
- If \( \nu(\cdot, X) \) is a martingale, then \( \Gamma \equiv 0 \) (e.g. replication price in finance, value function of an optimal control problem along the optimal path,...)
- Can be extended to path-dependent functionals using the notion of Dupire’s derivatives, see B., Loeper and Tan [2].
$C^1$-Itô’s formula for flows of semimartingale distributions
Consider a continuous semimartingale on a complete probability space $\left( \Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P} \right)$, satisfying the usual conditions,

$$X = X_0 + A + M^X, \quad \text{with} \quad M^X = M + \int_0^\cdot \sigma_s dM_s^\circ.$$

Define $G^\circ = (G_t^\circ)_{t \geq 0}$, where $G_t^\circ := \sigma(M_s^\circ, 0 \leq s \leq t)$ and

$$\mathbb{E}^\circ[\xi] := \mathbb{E}[\xi|G^\circ]$$
The setting

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Consider a continuous weak Dirichlet process

\[ Y = Y_0 + A^Y + M^Y, \]

with \([Y, Y]_T < \infty\).
The setting

- **Assumption**:
  1. \(\sigma^\circ\) is \(\mathbb{F}\)-progressively measurable, and \(\exists\) sequence of stopping times \((\tau_n)_{n \geq 1}\) w.r.t. \(\mathcal{G}^\circ\) s.t. \(\tau_n \uparrow \infty\) a.s. and

\[
\mathbb{E}\left[[M]_{\tau_n \wedge t} + A^2_{\tau_n \wedge t} + \int_0^{\tau_n \wedge t} |\sigma_s^\circ|^2 d[M^\circ]_s\right] < +\infty, \text{ for all } t \geq 0 \text{ and } n \geq 1.
\]

  2. \(M\) is orthogonal to \(N\) (i.e. \([M, N] = 0\)), for all \(\mathcal{G}^\circ\)-martingales \(N\).

  3. \((H)\)-hypothesis condition:

\[
\mathbb{E}\left[\mathbf{1}_D \mid \mathcal{G}_t^\circ\right] = \mathbb{E}\left[\mathbf{1}_D \mid \mathcal{G}^\circ\right], \text{ a.s., for all } D \in \mathcal{F}_t, \ t \geq 0.
\]

Define the \(\mathcal{P}(\mathbb{R}^d)\)-valued process

\[
m_t := \mathcal{L}(X_t \mid \mathcal{G}_t^\circ) = \mathcal{L}(X_t \mid \mathcal{G}^\circ), \ t \geq 0.
\]
Given $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, let $\delta F / \delta m : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, be s.t.

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta F}{\delta m} (tm' + (1 - t)m, x) [m' - m] (dx) dt$$

and set

$$D_m F(m, x) := \partial_x \frac{\delta F}{\delta m} (m, x).$$
Main result

We consider $F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ such that the following holds.

Assumption:

\[ \forall n \geq 1, T > 0 \text{ and compact } K \subset \mathbb{R}^d, \exists C > 0 \text{ s.t. } E\left[\left(\left. D_m F(r, y, m_n, \lambda, t_s, X_n, \eta, t_s)\right|_{m_n, \lambda, t_s} = (1 - \lambda)m_{\tau n} \wedge s + \lambda m_{\tau n} \wedge t_s, X_n, \eta, t_s = (1 - \eta)x_{\tau n} \wedge s + \eta x_{\tau n} \wedge t_s\right)\right]^2 \leq C, \text{ a.s.} \]

Theorem:

\[ \exists \text{ a continuous orthogonal process } \Gamma \text{ such that } F(t, Y_t, m_t) = F(0, Y_0, m_0) + \int_0^t \partial_y F(s, Y_s, m_s) dM_{Y_s} + \int_0^t E\left[\left. D_m F(s, \cdot, m_s, X_s)\sigma_{\circ s}\right|_{Y_s}\right] dM_{\circ s} + \Gamma_t, t \geq 0. \]
Main result

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- **Assumption**: $\forall \, n \geq 1, \, T > 0$ and compact $K \subset \mathbb{R}^d$, $\exists \, C > 0$ s.t.

  $$
  \mathbb{E}^\circ \left[ \left( D_m F(r, y, m_s^{n,\lambda,t}, X_s^{n,\eta,t}) \right)^2 \right] \leq C, \text{ a.s.,}
  $$

  $\forall \, (r, s, t) \in [0, 2T] \times [0, t] \times [0, T], (\lambda, \eta, y) \in [0, 1]^2 \times K,$

where $m_s^{n,\lambda,t} := (1 - \lambda)m_{\tau_n \wedge s} + \lambda m_{\tau_n \wedge t}$, $X_s^{n,\eta,t} := (1 - \eta)X_{\tau_n \wedge s} + \eta X_{\tau_n \wedge t}$. 
Main result

- We consider $F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times P_2(\mathbb{R}^d))$ such that the following holds

- **Assumption**: $\forall n \geq 1, T > 0$ and compact $K \subset \mathbb{R}^d$, $\exists C > 0$ s.t.

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\mathbb{E}^\circ \left[ (D_m F(r, y, m_s^{n,\lambda, t}, X_s^{n,\eta, t}))^2 \right] \leq C, \text{ a.s.,}
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- **Theorem**: $\exists$ a continuous orthogonal process $\Gamma$ such that

\[
F(t, Y_t, m_t) = F(0, Y_0, m_0) + \int_0^t \partial_y F(s, Y_s, m_s) \, dM_s^Y
\]

\[
+ \int_0^t \mathbb{E}^\circ \left[ D_m F(s, \cdot, m_s, X_s) \sigma_s^o \right](Y_s) \, dM_s^o + \Gamma_t, \quad t \geq 0.
\]
Sketch of proof

- We restrict to $F(t, m, y) = F(m)$. 
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- Define

\[
\Gamma_t := F(m_t) - \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s) \sigma_s^\circ \right] dM^\circ_s.
\]
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$$\Gamma_t := F(m_t) - \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s) \sigma_s^\circ \right] dM_s^\circ.$$

We need to show that, for any continuous martingale $N$,

$$\frac{1}{\varepsilon} \int_0^t \left[ \Gamma_{s+\varepsilon} - \Gamma_s \right] (N_{s+\varepsilon} - N_s) ds \to 0,$$
Sketch of proof

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$$\frac{1}{\varepsilon} \int_0^t \left[ \Gamma_{s+\varepsilon} - \Gamma_s \right] (N_{s+\varepsilon} - N_s) ds \to 0,$$

or equivalently

$$I_t^\varepsilon := \frac{1}{\varepsilon} \int_0^t \left[ F(m_{s+\varepsilon}) - F(m_s) \right] (N_{s+\varepsilon} - N_s) ds \to I_t,$$

where

$$I_t := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s) \sigma_s^\circ \right] d[M^\circ, N]_s.$$
By definition of $D_m F$,

$$
\int_0^t \left[ F(m_{s+\epsilon}) - F(m_s) \right] \frac{N_{s+\epsilon} - N_s}{\epsilon} \, ds
$$

$$
= \int_0^t \int_0^1 \int_0^1 \mathbb{E}^0 \left[ D_m F(m_{s+\epsilon}^\lambda, X_{s+\epsilon}^\eta ; X_s) (X_{s+\epsilon} - X_s) \right] \frac{N_{s+\epsilon} - N_s}{\epsilon} \, d\eta d\lambda ds,
$$

where $m_{s+\epsilon}^\lambda := m_s + \lambda (m_{s+\epsilon} - m_s)$ and $X_{s+\epsilon}^\eta := X_s + \eta (X_{s+\epsilon} - X_s)$. 
By definition of $D_mF$,

$$\int_0^t \left[ F(m_{s+\epsilon}) - F(m_s) \right] \frac{N_{s+\epsilon} - N_s}{\epsilon} ds$$

$$= \int_0^t \int_0^1 \int_0^1 \mathbb{E}^0 \left[ D_mF(m_{s}, X_{s}, \eta, \epsilon) (X_{s+\epsilon} - X_s) \right] \frac{N_{s+\epsilon} - N_s}{\epsilon} d\eta d\lambda ds,$$

where $m_{s}^{\lambda, \epsilon} := m_s + \lambda (m_{s+\epsilon} - m_s)$ and $X_{s}^{\eta, \epsilon} := X_s + \eta (X_{s+\epsilon} - X_s)$.

We can show that $\lim_{\epsilon \to 0} I_{t}^{\epsilon} = \lim_{\epsilon \to 0} J_{t}^{\epsilon}$, where

$$J_{t}^{\epsilon} := \int_0^t \mathbb{E}^0 \left[ D_mF(m_{s}, X_{s}) (X_{s+\epsilon} - X_s) \right] \frac{N_{s+\epsilon} - N_s}{\epsilon} ds.$$
We then write
\[ J_t^{\varepsilon} = J_t^{1,\varepsilon} + J_t^{2,\varepsilon} + J_t^{3,\varepsilon}, \]
where
\[ J_t^{1,\varepsilon} := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s) \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \]
\[ J_t^{2,\varepsilon} := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s)(M_{s+\varepsilon} - M_s) \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \]
\[ J_t^{3,\varepsilon} := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s) \int_s^{s+\varepsilon} \sigma_r^\circ dM_r^\circ \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds. \]
We then write

\[ J_\varepsilon = J_{1,\varepsilon} + J_{2,\varepsilon} + J_{3,\varepsilon}, \]

where

\[ J_{1,\varepsilon} := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s) \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \]

\[ J_{2,\varepsilon} := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s)(M_{s+\varepsilon} - M_s) \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \]

\[ J_{3,\varepsilon} := \int_0^t \mathbb{E}^\circ \left[ D_m F(m_s, X_s) \int_s^{s+\varepsilon} \sigma_r^\circ dM_r^\circ \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds. \]

Then \( J_{1,\varepsilon} \rightarrow 0, J_{2,\varepsilon} \rightarrow 0, \) u.c.p.,
We then write

\[ J_t^\varepsilon = J_t^{1,\varepsilon} + J_t^{2,\varepsilon} + J_t^{3,\varepsilon}, \]

where

\[ J_t^{1,\varepsilon} := \int_0^t \mathbb{E}^o \left[ D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s) \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} \, ds, \]

\[ J_t^{2,\varepsilon} := \int_0^t \mathbb{E}^o \left[ D_m F(m_s, X_s)(M_{s+\varepsilon} - M_s) \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} \, ds, \]

\[ J_t^{3,\varepsilon} := \int_0^t \mathbb{E}^o \left[ D_m F(m_s, X_s) \int_s^{s+\varepsilon} \sigma_r^o \, dM_r^o \right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} \, ds. \]

Then \( J_t^{1,\varepsilon} \rightarrow 0, J_t^{2,\varepsilon} \rightarrow 0, \) u.c.p., and

\[ \lim_{\varepsilon \to 0} J_t^{3,\varepsilon} = \lim_{\varepsilon \to 0} \int_0^t \int_s^{s+\varepsilon} \mathbb{E}^o \left[ D_m F(m_r, X_r)\sigma_r^o \right] dM_r^o \frac{N_{s+\varepsilon} - N_s}{\varepsilon} \, ds \]

\[ = \int_0^t \mathbb{E}^o \left[ D_m F(m_r, X_r)\sigma_r^o \right] d[M^o, N]_r. \]
A verification theorem for a class of McKean-Vlasov optimal control problems
Let $\Omega^0 = \Omega^1 := C([0, T], \mathbb{R}^d)$ with canonical process $X^0$ and $W$, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}_0^0$ and $\mathbb{P}_0^1$. 
A class of McKean-Vlasov optimal control problems

□ Let $\Omega^0 = \Omega^1 := C([0, T], \mathbb{R}^d)$ with canonical process $X^0$ and $W$, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}^0$ and $\mathbb{P}^1_0$.

□ Let $\mathcal{A}^0$ denote the collection of $\mathbb{F}^0$-progressively measurable process $\alpha : [0, T] \times \Omega^0 \longrightarrow A$, bounded.
A class of McKean-Vlasov optimal control problems

Let $\Omega^0 = \Omega^1 := C([0, T], \mathbb{R}^d)$ with canonical process $X^0$ and $W$, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}^0_0$ and $\mathbb{P}^1_0$.

Let $\mathcal{A}^0$ denote the collection of $\mathbb{F}^0$-progressively measurable process $\alpha : [0, T] \times \Omega^0 \rightarrow A$, bounded.

Define

$$\mathcal{P}_W^0(t, x^0) := \left\{ \mathbb{P}^0 \in \mathcal{P}(\Omega^0) : X^0 = x^0_t + \int_t^\cdot \alpha^\mathbb{P}_{t}^0 dr + \int_t^\cdot dW^\mathbb{P}_{t}^0, \mathbb{P}^0\text{-a.s.} \right\}$$

$$\mathbb{P}^0[X^0_{t\wedge .} = x^0_{t\wedge .}] = 1, \text{ where } \alpha^\mathbb{P}_{t}^0 \in \mathcal{A}^0$$

and $W^\mathbb{P}_{t}^0$ is a $(\mathbb{P}^0, \mathbb{F}^0)$-Brownian motion.
A class of McKean-Vlasov optimal control problems

- Let \( \Omega^0 = \Omega^1 := C([0, T], \mathbb{R}^d) \) with canonical process \( X^0 \) and \( W \), canonical filtrations \( F^0 \) and \( F^1 \), and Wiener measures \( P^0 \) and \( P^1 \).

- Let \( A^0 \) denote the collection of \( F^0 \)-progressively measurable process \( \alpha : [0, T] \times \Omega^0 \rightarrow \mathbb{A} \), bounded.

- Define

\[
\mathcal{P}_W^0(t, x^0) := \left\{ P^0 \in \mathcal{P}(\Omega^0) : X^0 = x^0_t + \int_t^\cdot \alpha_{P^0}^r \, dr + \int_t^\cdot dW_r^{P^0}, P^0\text{-a.s.} \right\}
\]

\[P^0[X^0_{t^\wedge} = x^0_{t^\wedge}] = 1, \text{ where } \alpha_{P^0} \in A^0 \]

and \( W^{P^0} \) is a \((P^0, F^0)\)-Brownian motion \}.

and

\[
\mathcal{P}_W(t, x^0) := \left\{ P = P^0 \times P^1_0 : P^0 \in \mathcal{P}_W^0(t, x^0) \right\}.
\]
For $t \in [0, T]$, $m \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}_W(t, x^0)$, we consider the McKean-Vlasov SDE:

$$X_{s, t}^{t,P} = \xi + \int_t^s \sigma_0(r, X_r^{t,P}, \rho_r^{t,m,P})dX_r^0 + \int_t^s \sigma(r, X_r^{t,P}, \rho_r^{t,m,P})dW_r, \ m \times \mathbb{P}\text{-a.s.}$$

with $\rho_r^{t,m,P} := \mathcal{L}^{m \times \mathbb{P}}(X_r^{t,P} | \mathcal{F}_r X^0)$. 

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$$X_{s, t}^t, P = \xi + \int_t^s \sigma_0(r, X_{r, t}^t, P_{r, m}^t, P_{r, m}^t) dX_r^0 + \int_t^s \sigma(r, X_{r, t}^t, P_{r, m}^t, P_{r, m}^t) dW_r, \quad m \times \mathbb{P} \text{-a.s.}$$

with $P_{r, m}^t := \mathcal{L}^{m \times P}(X_{r, t}^t, P | \mathcal{F}_r X^0)$.

Controlled laws of the canonical process $(X^0, W, X, \rho)$ are in

$$\overline{P}_W(t, m) := \left\{ (m \times \mathbb{P}) \circ (X^0, W, X^{t, P}, P_{t, m}^{t, P})^{-1} : \mathbb{P} \in \mathcal{P}_W(t, x^0), x^0 \in \Omega^0 \right\}.$$
The value function of the McKean-Vlasov control problem is:

$$V(t, m) := \sup_{\mathbb{P} \in \mathcal{P}_W(t, m)} J(t, \mathbb{P}), \text{ with } J(t, \mathbb{P}) := \mathbb{E}^\mathbb{P}\left[ \int_t^T L(s, \rho_s, \xi_s^\mathbb{P}) ds + g(\rho_T) \right].$$
The value function of the McKean-Vlasov control problem is:

\[ V(t, m) := \sup_{\bar{P} \in \bar{P}_W(t, m)} J(t, \bar{P}), \text{ with } J(t, \bar{P}) := \mathbb{E}^\bar{P} \left[ \int_t^T L(s, \rho_s, \alpha_s^\bar{P}) \, ds + g(\rho_T) \right]. \]

Define

\[ K := \left\{ \varphi : [0, \, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d : \varphi \text{ is bounded and Borel measurable} \right\}, \]

and \( H : [0, \, T] \times \mathcal{P}_2(\mathbb{R}^d) \times K \to \mathbb{R} \), the Hamiltonian, defined by

\[ H(t, m, p) := \max_{a \in A} h(t, m, p, a), \]

\[ h(t, m, p, a) := L(t, m, a) + a \int (\sigma_0 p)(t, m, y)m(dy). \]
Dual problems

- From now on, we fix the initial law to be $m_0$. 

It is similar in spirit to B. and Dang [4]: stochastic target formulation of the optimal control problem.
Dual problems

- From now on, we fix the initial law to be $m_0$.
- We introduce the dual problems:
  (i) $D_1$ is the infimum over $v_1$ s.t.

$$v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y) \rho_t(dy)dX_t^0 \geq g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha_t^\mathbb{P})dt, \mathbb{P}\text{-a.s.}$$

for all $\mathbb{P} \in \mathcal{P}_W(0, m_0)$, for some $\phi \in \mathcal{K}$. 

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  for all $\mathbb{P} \in \mathcal{P}_W(0, m_0)$, for some $\phi \in \mathcal{K}$.

  (ii) $D_2$ is the infimum over $v_2$ s.t.

  $v_2 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y) \rho_t(dy) dX^0_t \geq g(\rho_T) + \int_0^T H(t, \rho_t, \phi) dt, \mathbb{P}_0 - \text{a.s.}$

  for some $\phi \in \mathcal{K}$, where $\mathbb{P}_0$ is a probability measure under which the canonical process $X^0$ is a Brownian motion.

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- Since

$$X^0 = x_0^0 + \int_0^\cdot \alpha^P_r dr + \int_0^\cdot dW^P_r$$

the inequality

$$v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y) \rho_t dy dX^0_t \geq g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha^P_t) dt$$

implies

$$v_1 \geq \mathbb{E}^\mathbb{P}[g(\rho_T) + \int_0^T L(t, \rho_t, \alpha^\mathbb{P}_t)], \text{ for } \mathbb{P} \in \mathcal{P}_W(0, m_0)$$
\[ \begin{align*}
\square \text{ We have } D_2 & \geq D_1 \text{ by definition.} \\
\square \text{ Since } \\
X^0 & = x_0^0 + \int_0^\cdot \alpha_r^0 \, dr + \int_0^\cdot dW_r^0 \\
\text{the inequality} \\
v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y) \rho_t(dy) dX_t^0 & \geq g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha_t^\mathbb{P}) \, dt \\
\text{implies} \\
v_1 & \geq \mathbb{E}_\mathbb{P} [g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^\mathbb{P})], \quad \text{for } \mathbb{P} \in \overline{\mathcal{P}}_W(0, m_0) \\
\text{and therefore} \\
D_2 & \geq D_1 \geq V(t, m_0). 
\end{align*} \]
Theorem: Assume that \( V \in C^{0,1}([0, T], \mathcal{P}_2(\mathbb{R}^d)) \) and that \( D_m V \) is uniformly bounded (or locally as above). Then,

\[
V(0, m_0) = D_1 = D_2.
\]

If in addition \( \exists \) a Borel measurable function \( \hat{a} : [0, T) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow A \) s.t.

\[
H(\cdot, m, D_m V) = h(\cdot, m, D_m V, \hat{a}(\cdot, m)),
\]

for all \( m \in \mathcal{P}_2(\mathbb{R}^d) \). Then, \( \exists \hat{\mathbb{P}} \in \overline{\mathbb{P}}_W(0, m_0) \) s.t. \( \alpha^{\hat{\mathbb{P}}} = \hat{a}(\cdot, \rho) \), \( d\hat{\mathbb{P}} \times dt \) a.e. and \( \hat{\mathbb{P}} \) is optimal for \( V(0, m_0) \).
Duality and verification

**Theorem:** Assume that $V \in C^{0,1}([0, T], \mathcal{P}_2(\mathbb{R}^d))$ and that $D_m V$ is uniformly bounded (or locally as above). Then,

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If in addition $\exists$ a Borel measurable function $\hat{a} : [0, T) \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow A$ s.t.

$$H(\cdot, m, D_m V) = h(\cdot, m, D_m V, \hat{a}(\cdot, m)),$$

for all $m \in \mathcal{P}_2(\mathbb{R}^d)$. Then, $\exists \widehat{P} \in \mathcal{P}_W(0, m_0)$ s.t. $\alpha^{\widehat{P}} = \hat{a}(\cdot, \rho.)$, $d\widehat{P} \times dt$ a.e. and $\widehat{P}$ is optimal for $V(0, m_0)$.

**Remark:** If $A$ is compact, existence of $\hat{a}$ holds if $L$ is upper-semicontinuous.
Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S = V(\cdot, \rho) + \int_0^t L(s, \rho_s, \alpha_s) ds$ is a super-martingale under any $\mathbb{P} \in \mathcal{P}_W(0, m_0)$. 

Hence, $V(0, m_0) \geq \overline{D}_2$ by arbitrariness of $\mathbb{P}$. 
Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S^\mathbb{P} := V(\cdot, \rho.) + \int_0^t L(s, \rho_s, \alpha_s^\mathbb{P}) ds$ is a super-martingale under any $\mathbb{P} \in \mathcal{P}_W(0, m_0)$. Combined with our $C^1$-Itô’s formula, we obtain:

$$S^\mathbb{P} = V(0, m_0) + \int_0^t \int (\sigma_0 D_m V)(t, \rho_t, y) \rho_t(dy) dW_t^\mathbb{P} - A^\mathbb{P}$$

in which $A^\mathbb{P}$ is non-decreasing.
Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S^\overline{P} := V(\cdot, \rho.) + \int_0^\cdot L(s, \rho_s, \alpha_s^\overline{P})ds$ is a super-martingale under any $\overline{P} \in \overline{P}_W(0, m_0)$. Combined with our $C^1$-Itô’s formula, we obtain:

$$S^\overline{P} = V(0, m_0) + \int_0^\cdot \int (\sigma_0 D_{m} V)(t, \rho_t, y)\rho_t(dy) dW_t^\overline{P} - A^\overline{P}$$

in which $A^\overline{P}$ is non-decreasing.

(b) Since $V(T, \rho_T) = g(\rho_T)$ and $A^\overline{P}$ is non-decreasing,

$$V(0, m_0) + \int_0^T \int (\sigma_0 D_{m} V)(t, \rho_t, y)\rho_t(dy) dX_t^0$$

$$\geq g(\rho_T) + \int_0^T h(t, \rho_t, D_{m} V, \alpha_t^\overline{P}) dt.$$ 

Hence, $V(0, m_0) \geq D_2$ by arbitrariness of $\overline{P}$. 
Proof of the verification argument

Set

\[ \ell(t, m) := \int (\sigma_0 D_m V)(t, m, y) m(dy) \]

and note that \((A_P^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_W(0, m_0)}\) in the decomposition

\[ S_P^{\mathbb{P}} = V(0, m_0) + \int_0^T \ell(t, \rho_t, y) dW_t^{\mathbb{P}} - A_P^{\mathbb{P}} \]

satisfies

\[ \inf_{\mathbb{P} \in \mathcal{P}_W(0, m_0)} \mathbb{E}^{\mathbb{P}}[A_T^{\mathbb{P}}] = 0. \]

by classical arguments.
Moreover,

\[ V(0, m_0) = g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_{t, \bar{P}}) dt + A_{T, \bar{P}}^\bar{P} - \int_0^T \ell(t, \rho_t) dX_t^0 \]

\[ = g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_{t, \hat{P}}) dt + A_{T, \hat{P}}^\hat{P} - \int_0^T \ell(t, \rho_t) dX_t^0 \]

\[ \geq g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_{t, \bar{P}}) dt + A_{T, \hat{P}}^\hat{P} - \int_0^T \ell(t, \rho_t) dX_t^0 \]

so that \( 0 \leq A_{T, \hat{P}}^\hat{P} \leq A_{T, \bar{P}}^\bar{P} \) a.s. for \( \bar{P} \in \bar{P}_W(0, m_0) \), and

\[ 0 = \inf_{\bar{P} \in \bar{P}_W(0, m_0)} \mathbb{E}^{\bar{P}} [A_{T, \bar{P}}^\bar{P}] \geq \inf_{\bar{P} \in \bar{P}_W(0, m_0)} \mathbb{E}^{\bar{P}} [A_{T, \hat{P}}^\hat{P}] = 0. \]
Moreover,

\[ V(0, m_0) = g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^\bar{P})dt + A_T^\bar{P} - \int_0^T \ell(t, \rho_t)dX_t^0 \]

\[ = g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^\hat{P})dt + \hat{A}_T - \int_0^T \ell(t, \rho_t)dX_t^0 \]

\[ \geq g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^\bar{P})dt + A_T^\hat{P} - \int_0^T \ell(t, \rho_t)dX_t^0 \]

so that \( 0 \leq \hat{A}_T \leq A_T^\bar{P} \) a.s. for \( \bar{P} \in \bar{P}_W(0, m_0) \), and

\[ 0 = \inf_{\bar{P} \in \bar{P}_W(0, m_0)} E^{\bar{P}} [A_T^\bar{P}] \geq \inf_{\bar{P} \in \bar{P}_W(0, m_0)} E^{\bar{P}} [A_T^\hat{P}] = 0. \]

We indeed have (using the reverse Hölder’s inequality)

\[ A_T^\bar{P} = 0, \ \bar{P} - \text{a.s. } \forall \bar{P} \in \bar{P}_W(0, m_0). \]
Then,

\[ V(0, m_0) = \mathbb{E}^\hat{\mathbb{P}} \left[ g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^\hat{\mathbb{P}}) dt - \int_0^T \ell(t, \rho_t) \alpha_t^\hat{\mathbb{P}} dt \right] = \mathbb{E}^\hat{\mathbb{P}} \left[ g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^\hat{\mathbb{P}}) dt \right]. \]
Proof of the verification argument

Set

\[ \ell(t, m) := \int (\sigma_0 D_m V)(t, m, y) m(dy) \]

and note that \((A^\mathbb{P})_{\mathbb{P} \in \mathbb{P}_W(0, m_0)}\) in the decomposition

\[ S^\mathbb{P} = V(0, m_0) + \int_0^t \ell(t, \rho_t, y) dW_t^\mathbb{P} - A^\mathbb{P} \]

satisfies

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by classical arguments.
Example

- Assume that:
  - \( \sigma = \sigma_0 \equiv 1 \),
  - \( A \) is a convex,
  - \( L(t, m, a) = \bar{L}(a) \) is strictly concave.
  - \( g(m) = \bar{g}(\int \phi(y)m(dy)) \) with \( \bar{g} : \mathbb{R}^d \to \mathbb{R} \) convex and \( C^1_b \), and \( \phi : \mathbb{R} \to \mathbb{R}^d \) that is \( C^1_b \).
Example

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Then, $V \in C^{0,1}([0, T], \mathcal{P}_2(\mathbb{R}^d))$. 
Example

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Then, $V \in C^{0,1}([0, T], P_2(\mathbb{R}^d))$.

An optimal control $\hat{P}$ exists (is unique) and we have

$$D_m V(0, m_0, x) = E^{\hat{P}} \left[ \bar{g}' \left( \int_{\mathbb{R}^d} \bar{\phi}(y + X_T^0)m_0(dy) \right) \nabla \bar{\phi}(x + X_T^0) \right]$$

where

$$\bar{\phi}(y) := E^{\hat{P}_0}[\phi(y + W_T)].$$
Thank you!
Elena Bandini and Francesco Russo.
Weak Dirichlet processes with jumps.

Bruno Bouchard, Grégoire Loeper and Xiaolu Tan.
A $C^{0,1}$-functional Itô’s formula and its applications in mathematical finance.

Bruno Bouchard, Grégoire Loeper and Xiaolu Tan.
Approximate viscosity solutions of path-dependent PDEs and Dupire’s vertical differentiability.

Bruno Bouchard and Ngoc Minh Dang.
Optimal control versus stochastic target problems : an equivalence result.

Bruno Bouchard and Xiaolu Tan.
Understanding the dual formulation for the hedging of path-dependent options with price impact.

Bruno Bouchard and Xiaolu Tan.
A quasi-sure optional decomposition and super-hedging result on the Skorokhod space.

Bruno Bouchard and Maximilien Vallet.
Itô-Dupire’s formula for $C^{0,1}$-functionals of càdlàg weak Dirichlet processes.
Jean-François Chassagneux, Dan Crisan, and François Delarue.
A probabilistic approach to classical solutions of the master equation for large population equilibria.
*American Mathematical Society*, 280(1379), 2022.

Rama Cont and David-Antoine Fournié.
Functional Itô calculus and stochastic integral representation of martingales.

Bruno Dupire.
Functional Itô calculus.

Fausto Gozzi and Francesco Russo.
Weak dirichlet processes with a stochastic control perspective.

Xin Guo, Huyen Pham and Xiaoli Wei.
Itô’s formula for flows of measures on semimartingales.

Yuri F. Saporito.
The functional Meyer–Tanaka formula.
*Stochastics and Dynamics*, 18(04) :1850030, 2018.

Mehdi Talbi, Nizar Touzi and Jianfeng Zhang.
Dynamic programming equation for the mean field optimal stopping problem.