# A $C^{0,1}$-Itô's Formula for Flows of Semimartingale Distributions

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Joint work with Xiaolu Tan and Jixin Wang (Chinese University of Hong Kong)

## Motivation

□ Replace $C^{1,2}$-regularity by $C^{0,1}$ when applying Itô's lemma in situations where regularity is difficult to obtain :

- Path-dependent functionals.
- McKean-Vlasov optimal control problems.

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□ We know that it is possible for functionals on $[0, T] \times \mathbb{R}^d$ associated to classical Markovian problems.

# The classical Markovian situation

# Weak Dirichlet processes

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- Let $X$ and $Y$ be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t \ := \ \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

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- $X$ has finite quadratic variation, if $[X] := [X, X]$, exists and is finite a.s.
- $A$ is **orthogonal** if $[A, N] = 0$ for any real valued continuous local martingale $N$.
- $X$ is a **weak Dirichlet process** if $X = X_0 + M + A$, where $M$ is a local martingale and $A$ is orthogonal such that $M_0 = A_0 = 0$.

# $\mathbb{C}^{0,1}$-Itô's formula

□ **Theorem (Gozzi and Russo [4]) :** Let $X = X_0 + M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $v \in C^{0,1}([0,T) \times \mathbb{R}^d)$. Then,

$$v(t, X_t) \ = \ v(0, X) \ + \ \int_0^t \partial_x v(s, X_s) dM_s \ + \ \Gamma_t, \quad t \in [0, T),$$

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- If $v(\cdot, X)$ is a martingale, then $\Gamma \equiv 0$ (e.g. replication price in finance, value function of an optimal control problem along the optimal path,...)
- Can be extended to path-dependent functionals using the notion of Dupire's derivatives, see B., Loeper and Tan [2].

# $C^1$-Itô's formula for flows of semimartingale distributions

# The setting

□ Consider a continuous semimartingale on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfying the usual conditions,

$$X = X_0 + A + M^X, \quad \text{with} \quad M^X = M + \int_0^{\cdot} \sigma_s^\circ dM_s^\circ.$$

Define $\mathcal{G}^\circ = (\mathcal{G}_t^\circ)_{t\geq 0}$, where $\mathcal{G}_t^\circ := \sigma(M_s^\circ,\ 0 \leq s \leq t)$ and

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□ Consider a continuous weak Dirichlet process

$$Y = Y_0 + A^Y + M^Y,$$

with $[Y, Y]_T < \infty$.

# The setting

☐ **Assumption :**

(i) $\sigma^\circ$ is $\mathbb{F}$-progressively measurable, and $\exists$ sequence of stopping times $(\tau_n)_{n\geq 1}$ w.r.t. $\mathcal{G}^\circ$ s.t. $\tau_n \uparrow \infty$ a.s. and

$$\mathbb{E}\Big[[M]_{\tau_n\wedge t} + |A|^2_{\tau_n\wedge t} + \int_0^{\tau_n\wedge t} \big|\sigma^\circ_s\big|^2 d[M^\circ]_s\Big] < +\infty, \text{ for all } t \geq 0 \text{ and } n \geq 1.$$

(ii) $M$ is orthogonal to $N$ (i.e. $[M, N] = 0$), for all $\mathcal{G}^\circ$-martingales $N$.

(iii) $(H)$-hypothesis condition :

$$\mathbb{E}\big[1_D\big|\mathcal{G}^\circ_t\big] = \mathbb{E}\big[1_D\big|\mathcal{G}^\circ\big], \text{ a.s., for all } D \in \mathcal{F}_t, \ \ t \geq 0.$$

Define the $\mathcal{P}(\mathbb{R}^d)$-valued process

$$m_t := \mathcal{L}\,(X_t\ |\mathcal{G}^\circ_t) = \mathcal{L}(X_t\ |\mathcal{G}^\circ), \quad t \geq 0.$$

# Derivative with respect to the measure

- Given $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, let $\delta F/\delta m : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, be s.t.

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(tm' + (1-t)m, x)\,[m' - m]\,(dx)dt$$

and set

$$D_m F(m,x) := \partial_x \frac{\delta F}{\delta m}(m,x).$$

# Main result

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□ **Assumption :** $\forall\ n \geq 1,\ T > 0$ and compact $K \subset \mathbb{R}^d$, $\exists\ C > 0$ s.t.

$$\mathbb{E}^\circ\Big[\big(D_m F(r, y, m_s^{n,\lambda,t}, X_s^{n,\eta,t})\big)^2\Big] \leq C,\ \text{a.s.},$$
$$\forall\ (r,s,t) \in [0,2T] \times [0,t] \times [0,T], (\lambda,\eta,y) \in [0,1]^2 \times K,$$

where $m_s^{n,\lambda,t} := (1-\lambda)m_{\tau_n \wedge s} + \lambda m_{\tau_n \wedge t}$, $X_s^{n,\eta,t} := (1-\eta)X_{\tau_n \wedge s} + \eta X_{\tau_n \wedge t}$.

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□ **Theorem :** $\exists$ a continuous orthogonal process $\Gamma$ such that

$$\begin{aligned} F(t, Y_t, m_t) =& F(0, Y_0, m_0) + \int_0^t \partial_y F(s, Y_s, m_s)\ dM_s^Y \\ & + \int_0^t \mathbb{E}^\circ\big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\big](Y_s) dM_s^\circ + \Gamma_t,\ t \geq 0. \end{aligned}$$

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We need to show that, for any continuous martingale $N$,

$$\frac{1}{\varepsilon}\int_0^t \big[\Gamma_{s+\varepsilon} - \Gamma_s\big](N_{s+\varepsilon} - N_s)ds \;\longrightarrow\; 0,$$

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or equivalently

$$I_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t \big[F(m_{s+\varepsilon}) - F(m_s)\big](N_{s+\varepsilon} - N_s)ds \ \longrightarrow \ I_t,$$

where

$$\begin{aligned} I_t &= \int_0^t \mathbb{E}^\circ\Big[D_m F(m_s, X_s)\sigma_s^\circ\Big] d[M^\circ, N]_s \\ &= \lim_\varepsilon \frac{1}{\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon} \mathbb{E}^\circ\big[D_m F(m_u, X_u)\sigma_u^\circ\big] dM_u^\circ\right)(N_{s+\varepsilon} - N_s)ds \end{aligned}$$

By definition of $D_m F$,

$$\int_0^t \left[F(m_{s+\varepsilon}) - F(m_s)\right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds$$

$$= \int_0^t \int_0^1 \int \frac{\delta F}{\delta m}(m_s^{\lambda,\varepsilon}, x)[m_{s+\varepsilon} - m_s](dx) d\lambda \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds$$

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We can show that $\lim_{\varepsilon\to 0} I_t^\varepsilon = \lim_{\varepsilon \to 0} J_t^\varepsilon$, where

$$J_t^\varepsilon := \int_0^t \mathbb{E}^\circ \Big[ D_m F(m_s, X_s)(X_{s+\varepsilon} - X_s) \Big] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds.$$

We then write

$$J_t^{\varepsilon} = J_t^{1,\varepsilon} + J_t^{2,\varepsilon} + J_t^{3,\varepsilon},$$

where

$$J_t^{1,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s)\big] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds,$$

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$$J_t^{3,\varepsilon} := \int_0^t \mathbb{E}^\circ\Big[D_m F(m_s, X_s) \int_s^{s+\varepsilon} \sigma_r^\circ dM_r^\circ\Big] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds.$$

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Then $J_t^{1,\varepsilon} \longrightarrow 0$, u.c.p., $J_t^{2,\varepsilon} = 0$, and

$$\begin{aligned}\lim_{\varepsilon\to 0} J_t^{3,\varepsilon} \;&=\; \lim_{\varepsilon\to 0}\int_0^t \int_s^{s+\varepsilon} \mathbb{E}^\circ\Big[D_m F(m_r, X_r)\sigma_r^\circ\Big]\,dM_r^\circ\,\frac{N_{s+\varepsilon} - N_s}{\varepsilon}\,ds \\ &= \int_0^t \mathbb{E}^\circ\Big[D_m F(m_r, X_r)\sigma_r^\circ\Big]\,d[M^\circ, N]_r.\end{aligned}$$

# A verification theorem for a class of McKean-Vlasov optimal control problems

# A class of McKean-Vlasov optimal control problems

☐ Let $\Omega^0 = \Omega^1 := \mathcal{C}([0, T], \mathbb{R}^d)$ with canonical process $X^0$ and $W$, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}_0^0$ and $\mathbb{P}_0^1$.

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$$\mathcal{P}_W^0(t, \mathrm{x}^0) := \Big\{ \mathbb{P}^0 \in \mathcal{P}(\Omega^0) : X^0 = \mathrm{x}_t^0 + \int_t^{\cdot} \alpha_r^{\mathbb{P}^0} dr + \int_t^{\cdot} dW_r^{\mathbb{P}^0}, \mathbb{P}^0\text{-a.s.}$$

$$\mathbb{P}^0[X_{t\wedge\cdot}^0 = \mathrm{x}_{t\wedge\cdot}^0] = 1, \text{ where } \alpha^{\mathbb{P}^0} \in \mathcal{A}^0$$

$$\text{and } W^{\mathbb{P}^0} \text{ is a } (\mathbb{P}^0, \mathbb{F}^0)\text{-Brownian motion} \Big\}.$$

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and

$$\mathcal{P}_W(t, \mathrm{x}^0) := \{ \mathbb{P} = \mathbb{P}^0 \times \mathbb{P}_0^1 : \mathbb{P}^0 \in \mathcal{P}_W^0(t, \mathrm{x}^0) \}.$$

□ For $t \in [0,T]$, $m \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}_W(t, \mathrm{x}^0)$, we consider the McKean-Vlasov SDE :

$$X_s^{t,\mathbb{P}} = \xi + \int_t^s \sigma_0(r, X_r^{t,\mathbb{P}}, \rho_r^{t,m,\mathbb{P}}) dX_r^0 + \int_t^s \sigma(r, X_r^{t,\mathbb{P}}, \rho_r^{t,m,\mathbb{P}}) dW_r,\ m \times \mathbb{P}\text{-a.s.}$$

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□ Controlled laws of the canonical process $(X^0, W, X, \rho)$ are in

$$\overline{\mathcal{P}}_W(t, m) := \left\{ (m \times \mathbb{P}) \circ \left( X^0, W, X^{t,\mathbb{P}}, \rho^{t,m,\mathbb{P}} \right)^{-1} : \mathbb{P} \in \mathcal{P}_W(t, \mathrm{x}^0), \ \mathrm{x}^0 \in \Omega^0 \right\}.$$

□ The value function of the McKean-Vlasov control problem is :

$$V(t,m) := \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,m)} J\left(t, \overline{\mathbb{P}}\right), \text{ with } J(t,\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T L\big(s,\rho_s,\alpha_s^{\overline{\mathbb{P}}}\big)ds + g(\rho_T)\Big].$$

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$$V(t,m) := \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,m)} J\left(t, \overline{\mathbb{P}}\right), \text{ with } J(t,\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T L\big(s, \rho_s, \alpha_s^{\overline{\mathbb{P}}}\big) ds + g(\rho_T)\Big].$$

□ Define

$$\mathcal{K} := \Big\{\phi : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \ : \ \phi \text{ is bounded and Borel measurable}\Big\},$$

and $H : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{K} \longrightarrow \mathbb{R}$, the Hamiltonian, defined by

$$H(t,m,p) := \max_{a \in \mathrm{A}} h(t,m,p,a),$$
$$h(t,m,p,a) := L(t,m,a) + a \int (\sigma_0 p)(t,m,y) m(dy).$$

# Dual problems

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(i) $D_1$ is the infimum over $v_1$ s.t.

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for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$, for some $\phi \in \mathcal{K}$.

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□ It is similar in spirit to B. and Dang [4] : stochastic target formulation of the optimal control problem.

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$$X^0 = x_0^0 + \int_0^{\cdot} \alpha_r^{\mathbb{P}^{\mathbf{o}}} dr + \int_0^{\cdot} dW_r^{\mathbb{P}^{\mathbf{o}}}$$

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$$v_1 \geq \mathbb{E}^{\overline{\mathbb{P}}}[g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^{\overline{\mathbb{P}}})], \quad \text{for } \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$$

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and therefore

$$D_2 \geq D_1 \geq V(t, m_0).$$

# Duality and verification

☐ **Theorem :** Assume that $V \in C^{0,1}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ and that $D_m V$ is uniformly bounded (or locally as above). Then,

$$V(0, m_0) = D_1 = D_2.$$

If in addition $\exists$ a Borel measurable function $\hat{a} : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathrm{A}$ s.t.

$$H(\cdot, m, D_m V) = h(\cdot, m, D_m V, \hat{a}(\cdot, m)),$$

for all $m \in \mathcal{P}_2(\mathbb{R}^d)$. Then, $\exists\ \widehat{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$ s.t. $\alpha^{\widehat{\mathbb{P}}} = \hat{a}(\cdot, \rho_\cdot)$, $d\widehat{\mathbb{P}} \times dt$ a.e. and $\widehat{\mathbb{P}}$ is optimal for $V(0, m_0)$.

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□ **Remark :** If $\mathrm{A}$ is compact, existence of $\hat{a}$ holds if $L$ is upper-semicontinuous.

## Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S^{\overline{\mathbb{P}}} := V(\cdot, \rho_\cdot) + \int_0^\cdot L(s, \rho_s, \alpha_s^{\overline{\mathbb{P}}}) ds$ is a super-martingale under any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$.

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$$S^{\overline{\mathbb{P}}} = V(0, m_0) + \int_0^\cdot \int (\sigma_0 D_m V)(t, \rho_t, y)\rho_t(dy)dW_t^{\overline{\mathbb{P}}} - A^{\overline{\mathbb{P}}}$$

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(b) Since $V(T, \rho_T) = g(\rho_T)$ and $A^{\overline{\mathbb{P}}}$ is non-negative,

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Hence, $V(0, m_0) \geq D_2$ by arbitrariness of $\overline{\mathbb{P}}$.

## Proof of the verification argument

Set

$$\ell(t,m) := \int (\sigma_0 D_m V)(t,m,y)m(dy)$$

and note that $(A^{\overline{\mathbb{P}}})_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,\mathbf{m_0})}$ in the decomposition

$$S^{\overline{\mathbb{P}}} = V(0,m_0) + \int_0^{\cdot} \ell(t,\rho_t)dW_t^{\overline{\mathbb{P}}} - A^{\overline{\mathbb{P}}}$$

satisfies

$$\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,\mathbf{m_0})} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] = 0.$$

by classical arguments.

Moreover,

$$\begin{aligned} V(0,m_0) &= g(\rho_T) + \int_0^T h(t,\rho_t,\alpha_t^{\overline{\mathbb{P}}})dt + A_T^{\overline{\mathbb{P}}} - \int_0^T \ell(t,\rho_t)dX_t^0 \\ &= g(\rho_T) + \int_0^T h(t,\rho_t,\alpha_t^{\widehat{\mathbb{P}}})dt + A_T^{\widehat{\mathbb{P}}} - \int_0^T \ell(t,\rho_t)dX_t^0 \\ &\geq g(\rho_T) + \int_0^T h(t,\rho_t,\alpha_t^{\overline{\mathbb{P}}})dt + A_T^{\widehat{\mathbb{P}}} - \int_0^T \ell(t,\rho_t)dX_t^0 \end{aligned}$$

so that $0 \leq A_T^{\widehat{\mathbb{P}}} \leq A_T^{\overline{\mathbb{P}}}$ a.s. for $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$, and

$$0 = \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] \geq \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\widehat{\mathbb{P}}}] = 0.$$

Moreover,

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We indeed have (using the reverse Hölder's inequality)

$$A_T^{\widehat{\mathbb{P}}} = 0, \ \overline{\mathbb{P}} - \text{a.s.} \ \forall \, \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0).$$

Then,

$$V(0, m_0) = \mathbb{E}^{\widehat{\mathbb{P}}}\left[g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^{\widehat{\mathbb{P}}})dt - \int_0^T \ell(t, \rho_t)\alpha_t^{\widehat{\mathbb{P}}}dt\right]$$

$$= \mathbb{E}^{\widehat{\mathbb{P}}}\left[g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^{\widehat{\mathbb{P}}})dt\right].$$

# Example

- Assume that :
  - $\sigma = \sigma_0 \equiv 1$,
  - $\mathrm{A}$ is a convex,
  - $L(t, m, a) = \bar{L}(a)$ is strictly concave.
  - $g(m) = \bar{g}\big(\int \phi(y) m(dy)\big)$ with $\bar{g} : \mathbb{R}^d \to \mathbb{R}$ in $C_b^1$, and $\phi : \mathbb{R} \longrightarrow \mathbb{R}^d$ that is $C_b^1$.

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Then, $V \in C^{0,1}([0, T], \mathcal{P}_2(\mathbb{R}^d))$.

- An optimal control $\widehat{\mathbb{P}}$ exists (and is unique !) and we have

$$D_m V(0, m_0, x) = \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\bar{g}'\Big(\int_{\mathbb{R}^d} \bar{\phi}(y + X_T^0) m_0(dy)\Big) \nabla \bar{\phi}(x + X_T^0)\Big]$$

where

$$\bar{\phi}(y) := \mathbb{E}^{\mathbb{P}_{\mathbf{0}}^{\mathbf{1}}}[\phi(y + W_T)].$$

Thank you !

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