No-arbitrage of second kind in countable markets with proportional transaction costs

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Abstract

Motivated by applications to bond markets, we propose a multivariate framework for discrete time financial markets with proportional transaction costs and a countable infinite number of tradable assets. We show that the no-arbitrage of second kind property (NA2 in short), introduced by [17] for finite dimensional markets, allows to provide a closure property for the set of attainable claims in a very natural way, under a suitable efficient friction condition. We also extend to this context the equivalence between NA2 and the existence of multiple (strictly) consistent price systems.

Key words: No-arbitrage, transaction costs, bond market.

Mathematical subject classifications: 91B28, 60G42.

1 Introduction

Motivated by applications to bonds markets, for which it is adknowledged that all possible maturities have to be taken into account, many papers have been devoted to the study of financial models with infinitely many risky assets, see e.g. [1], [4], [5] or [8] and the references therein. To the best of our knowledge, models with proportional transaction costs have not been discussed so far. This paper is a first attempt to treat such situations in a general framework.

As a first step, we restrict to a discrete time setting where a countable infinite number of financial assets is available.

Following the modern literature on financial models with proportional transaction costs, see [12] for a survey, financial strategies are described here by $\mathbb{R}^{\mathbb{N}}$-valued $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $\xi = (\xi_t)_{t\geq 0}$, where $(\mathcal{F}_t)_{t\geq 0}$ is a given filtration that models the flow of available information, and each component ξ_t^i of $\xi_t = (\xi_t^i)_{i\geq 1} \in \mathbb{R}^{\mathbb{N}}$ describes the changes in the position on the financial asset i induced by trading on the market at time t.

When the number of financial assets is finite, say d, one can view each component ξ_t^i as the amount of money invested in the asset i or as a number of units of asset i held in the portfolio.

The main advantage of working in terms of units is that it is numéraire free, see the discussions in [19] and [16]. In such models, the self-financing condition is described by a cone valued process $\hat{K} = (\hat{K}_t)_{t\geq 0}$ which incorporates bid-ask prices. Namely, a financial strategy is said to satisfy the self-financing condition if $\xi_t \in -\hat{K}_t$ $\mathbb{P}-$ a.s. for all $t \geq 0$, where $-\hat{K}_t(\omega) := \{y \in \mathbb{R}^d \ : \ y^i \leq \sum_{i\neq j}(a^{ji} - a^{ij}\pi_t^{ij}(\omega)), \forall\ i \leq d$, for some $a = (a^{ij})_{i,j\geq 1} \in \mathbb{R}^{d\times d}$ with non-negative entries$\}$. In the above formulation, π_t^{ij} stands for the number of units of asset i required in order to buy one unit of asset j at time t. The self-financing condition then just means that the changes ξ_t in the portfolio can be financed (in the large sense) by passing exchange orders $(a^{ij})_{i,j\geq 1}$ on the market, i.e. $a^{ij} \geq 0$ represents the number of units of asset j that are obtained against $a^{ij}\pi_t^{ij}$ units of asset i.

Under the so-called *efficient friction* assumption, namely $\pi_t^{ij}\pi_t^{ji} > 1$ for all i, j and $t \leq T$, and under suitable no arbitrage conditions (e.g. the strict no-arbitrage condition of [15] or the robust no-arbitrage condition of [19], see also [16]), one can show that there exists a martingale $\hat{Z} = (\hat{Z}_t)_{t\leq T}$ such that, for all $t \leq T$, $\hat{Z}_t$ lies in the interior of the (positive) dual cone $\hat{K}_t'$ of $\hat{K}_t$, which turns out to be given by

$$\hat{K}_t'(\omega) = \{z \in \mathbb{R}^d \ : \ 0 \leq z^j \leq z^i\pi_t^{ij}(\omega),\ i, j \leq d\}\ .$$

The martingale $\hat{Z}$ has then the usual interpretation of being associated to a fictitious frictionless market which is cheaper than the original one, i.e. $\hat{Z}_t^j/\hat{Z}_t^i < \pi_t^{ij}$, and such that the classical no-arbitrage condition holds, i.e. $\hat{Z}$ is a martingale. This generalizes to the multivariate setting the seminal result of [11].

The existence of such a martingale can then be extended to the continuous setting, see [10] for a direct approach in a one-dimensional setting and [9] for a multivariate extension based on a discrete time approximation, which in turns allows to prove that the set of attainable claims is closed is some sense, see e.g. Lemma 12 and the proof of Theorem 15 in [3], see also [2] and [7]. Such a property is highly desirable when one is interested by the formulation of a dual representation for the set of super-hedgeable claims, or by existence results in optimal portfolio management, see the above papers and the references therein.

The aim of this paper is to propose a generalized version of the above results to the context of discrete time models with a countable infinite number of assets, with the purpose of providing later a continuous time version.

When the number of assets is countable infinite, the first difficulty comes from the notion of interior associated to the sequence of dual cones $(\hat{K}'_t)_{t\le T}$. Indeed a natural choice would be to define $\hat{K}_t(\omega)$ as a subset of l^1, the set of elements $x = (x^i)_{i\ge 1} \in \mathbb{R}^{\mathbb{N}}$ such that $|x|_{l^1} := \sum_{i\ge 1} |x^i| < \infty$, so as to avoid having an infinite global position in a subset of financial assets, see [21] for a related criticism on frictionless continuous time models. In this case, $\hat{K}'_t$ should be defined in l^∞, the set of elements $x = (x^i)_{i\ge 1} \in \mathbb{R}^{\mathbb{N}}$ such that $|x|_{l^\infty} := \sup_{i\ge 1} |x^i| < \infty$. But, for the topology induced by $|\cdot|_{l^\infty}$, the sets $\hat{K}'_s(\omega)$ have no reason to have a non-empty interior, except under very strong conditions on the bid-ask matrices $(\pi_t^{ij}(\omega))_{i,j}$.

We therefore come back to the original modelisation of [15] in which financial strategies are described through amounts of money invested in the different risky assets. Namely, we assume that the bid-ask matrix $(\pi_t^{ij})_{i,j}$ takes the form $((1+\lambda_t^{ij})S_t^j/S_t^i)_{i,j}$ where S_t^k stands for the price, in some numéraire, of the risky asset k and λ_t^{ij} is a positive coefficient (typically less than 1) interpreted as a proportional transaction cost. The changes ξ_t in the portfolio du to trading at time t, now labeled in amounts of money evaluated in the numéraire, thus take values in the set $-K_t$ where $K_t(\omega) := \{(S_t^i(\omega)y^i)_{i\ge 1},\ y \in \hat{K}_t(\omega)\}$. Viewed as a subset of l^1, $K_t(\omega)$ has a dual cone $K'_t(\omega) \subset l^\infty$ which takes the form

$$K'_t(\omega) := \{z \in l^\infty \ :\ 0 \le z^j \le z^i(1+\lambda_t^{ij}(\omega)),\ i,j \ge 1\}\ ,$$

and whose interior in l^∞ is now non-empty under mild assumptions, e.g. if $\lambda_t^{ij}(\omega) \ge \varepsilon(\omega)$ a.s. for all $i,j \ge 1$ for some random variable ε taking strictly positive values.

This approach, although not numéraire free, allows to bound the global amount invested in the different subsets of assets, by viewing K_t as a subset of l^1, while leaving open the possibility of finding a process Z such that such Z_t lies in the interior of K'_t a.s., i.e. such that $\hat{Z} := ZS$ still satisfies $\hat{Z}_t^j/\hat{Z}_t^i < \pi_t^{ij}$ for all i,j.

We shall see below that, under a suitable no-arbitrage condition, one can actually choose Z in such a way that ZS is a martingale, thus recovering the above interpretation in terms of arbitrage free fictitious market. Moreover, we shall show that the set of terminal wealths induced by financial strategies defined as above is indeed closed in a suitable sense, see Theorem 3.2 and Theorem 3.1. This means that we do not need to consider an additional closure operation in order to build a nice duality theory or to discuss optimal portfolio management problems, as it is the case in frictionless markets (cf. [20] and [21] for a comparison with continuous time settings).

Another difficulty actually comes from the notion of no-arbitrage to be used in such a context. First, we should note that various, a-priori not equivalent, notions of no-arbitrage opportunities can be used in models with proportional transaction costs. We refer to [12] for a complete presentation and only mention one important point: the proofs of the closure properties, of the set of attainable claims, obtained in [15] and [19], under the strict no-arbitrage and the robust no-arbitrage property, heavily rely on the fact that the boundary of the unit ball is closed in $\mathbb{R}^d$ (for the pointwise convergence). This is no more true, for the pointwise convergence, when working in l^1 viewed as a subspace of $\mathbb{R}^{\mathbb{N}}$ with unit ball defined with $|\cdot|_{l^1}$. In particular, it does not seem that they can be reproduced in our infinite dimensional setting.

However, we shall show that the notion of no-arbitrage of second kind (in short NA2), recently introduced by [17] under the label "no-sure profit in liquidation value", is perfectly adapted. It says that the terminal value V_T of a wealth process can not take values a.s. in K_T if the wealth process at time t, V_t, does not already take values a.s. in K_t, for $t \leq T$. Note that $V_t \in K_t$ if and only if $-V_t \in -K_t$. Since $V_t + (-V_t) = 0$, this means that K_t is the set of position holdings at time t that can be turned into a zero position, after possibly throwing away non-negative amounts of financial assets, i.e. K_t is the set of "solvable" positions at time t. Hence, the NA2 condition means that we can not end up with a portfolio which is a.s. solvable if this was not the case before, which is a reasonable condition.

Under this condition, we shall see that a closure property can be proved under the only assumption that K'_t has a.s. a non-empty interior, for all $t \leq T$, which is for instance the case if $\varepsilon \leq \lambda_t^{ij}(\omega) \leq \varepsilon^{-1}$ a.s. for all $i, j \geq 1$ and $t \leq T$, for some $\varepsilon > 0$. We shall also extend to our framework the PCE (Prices Consistently Extendable) property introduced in [17], which we shall call MSCPS (Multiple Strictly Consistent Price System) to follow the terminology of [6].

The rest of the paper is organized as follows. We first conclude this introduction with a list of notations that will be used throughout paper. The model and our key assumptions are presented in Section 2. Our main results are reported in Section 3. The proofs of the closure properties are collected in Section 4, in which we also prove a dual characterization for the set of attainable claims and discuss the so-called B-property. The existence of a Multiple Strictly Consistent Price System is proved in Section 5. The last section discusses elementary properties of cones in infinite dimensional spaces and under which conditions our key assumption, Assumption 2.1 below, holds.

Notations: We identify the set of $\mathbb{R}$-valued maps on $\mathbb{N}$ with the topological vector space (hereafter TVS) $\mathbb{R}^{\mathbb{N}}$, with elements of the form $x = (x^i)_{i\geq 1}$. The set $\mathbb{R}^{\mathbb{N}}$ is endowed with its canonical product topology, also called the topology of pointwise convergence: $(x_n)_{n\geq 1}$ in $\mathbb{R}^{\mathbb{N}}$ converges pointwise to $x \in \mathbb{R}^{\mathbb{N}}$ if $x_n^i \to x^i$ for all $i \geq 1$. We set $\mathbb{M} = \mathbb{R}^{\mathbb{N}^2}$, whose elements are denoted by $a = (a^{ij})_{i,j\geq 1}$, define $\mathbb{M}_+$ as the subset of $\mathbb{M}$ composed by elements with non-negative components, and use the notation $\mathbb{M}_+^1$ (resp. $\mathbb{M}_{f,+}$) to denote the set of elements a in $\mathbb{M}_+$ such that $\sum_{i,j\geq 1} a^{ij} < \infty$ (resp. only a finite number of the a^{ij}'s are not equal to 0).

For $p \in [1, \infty)$ (resp. $p = \infty$), we denote by l^p (resp. l^∞) the set of elements $x \in \mathbb{R}^{\mathbb{N}}$ such that $|x|_{l^p} = (\sum_{i \geq 1} |x^i|^p)^{1/p} < \infty$ (resp. $|x|_{l^\infty} = \sup_{i \geq 1} |x^i| < \infty$). For the natural ordering, l^p_+ is the closed cone of positive elements $x \in l^p$, i.e. $x^i \geq 0$ for all i. Given $x, y \in \mathbb{R}^{\mathbb{N}}$, we write xy for $(x^1 y^1, x^2 y^2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, x/y for $(x^1/y^1, x^2/y^2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and $x \cdot y$ for $\sum_{i \geq 1} x^i y^i$ whenever it is well defined. To $j \in \mathbb{N}$, we associate the element e_j of $\mathbb{R}^{\mathbb{N}}$ satisfying $e_j^j = 1$ and $e_j^i = 0$ for $i \neq j$. We shall also use the notation $\mathbf{1} = (1, 1, \ldots)$.

We define c_f as the space of finite real sequences, and c_0 as the closed subspace of elements $x \in l^\infty$ such that $\lim_{i \to \infty} x^i = 0$. In the following, we shall use the notation μ to denote an element of $(0, \infty)^{\mathbb{N}}$ such that $1/\mu \in l^1$. To such a μ, we associate the Banch space $l^1(\mu)$ (resp. the set $l^1_+(\mu)$) of elements $x \in \mathbb{R}^{\mathbb{N}}$ such that $x\mu \in l^1$ (resp. $x\mu \in l^1_+$). The Banach space $c_0(1/\mu)$ is defined accordingly: $x \in c_0(1/\mu)$ iff $x/\mu \in c_0$. Recall that l^1 (resp. $l^1(\mu)$ is the topological dual of c_0 (resp. $c_0(1/\mu)$).

For a normed space $(E, ||\cdot||_E)$, we define the natural distance $d_E(x, y) := ||x - y||_E$, denote by $d_E(x, A)$ (resp. $d_E(B, A)$) the distance between x (resp. the set $B \subset E$) and the set $A \subset E$.

We shall work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ with $\mathbb{T} := \{0, \ldots, T\}$ for some integer $T > 0$. Without loss of generality, we assume that $\mathcal{F}_T = \mathcal{F}$.

Given a real locally convex TVS E, with topological dual E', and a σ-subalgebra $\mathcal{G} \subset \mathcal{F}$, we denote by E_w the linear space E endowed with the weak topology (i.e. the $\sigma(E, E')$ topology), $\mathcal{B}(E_w)$ stands for the corresponding Borel σ-algebra, and we write $L^0(E, \mathcal{G})$ to denote the collection of weakly $\mathcal{G}$-measurable E-valued random variables. A subset B of $\Omega \times E$ is said to be weakly $\mathcal{G}$-measurable if $B \in \mathcal{G} \otimes \mathcal{B}(E_w)$. When $(E, \|\cdot\|_E)$ is a separable Banach space, the elements of $L^0(E, \mathcal{G})$ are indeed strongly measurable (cf. Sect. IV.2 of [22]). For $1 \leq p \leq \infty$, we then use the standard notations $L^p(E, \mathcal{G})$ for the elements $X \in L^0(E, \mathcal{G})$ such that $\mathbb{E}\left[\|X\|_E^p\right] < \infty$ if $1 \leq p < \infty$, and $\|X\|_E$ is essentially bounded if $p = \infty$. In the case of the non-separable space l^∞, the elements $X \in L^0(l^\infty, \mathcal{G})$ still have a $\mathcal{G}$-measurable norm $|X|_{l^\infty}$. We therefore also use the notation $L^p(l^\infty, \mathcal{G})$ as defined above, although this space does not have all the usual "nice properties" of L^p-spaces.

Any inequality between random variables or inclusion between random sets has to be taken in the $\mathbb{P} -$ a.s. sense.

2 Model formulation

2.1 Financial strategies and no-arbitrage of second kind

We consider a financial market in discrete time with proportional transaction costs supporting a countable infinite number of tradable assets. The evolution of the asset prices is described by a $(0,\infty)^{\mathbb{N}}$-valued $\mathbb{F}$-adapted process $S=(S_t)_{t\in\mathbb{T}}$. All over this paper, we shall impose the following technical condition:

$$S_t/S_s \in L^1(l^\infty,\mathcal{F}) \text{ for all } s,t\in\mathbb{T} . \tag{2.1}$$

Similar conditions are satisfied in continuous time models without transaction costs, cf. Theorem 2.2 of [8].

Remark 2.1 Note that one could simply assume that $S_t/S_s \in l^\infty$ $\mathbb{P}-$a.s. for all $s,t\in\mathbb{T}$, which is a natural condition, and replace the original measure $\mathbb{P}$ by $\tilde{\mathbb{P}}$ defined by $d\tilde{\mathbb{P}}/d\mathbb{P} = \exp\left(-\sum_{s,t\in\mathbb{T}}|S_t/S_s|_{l^\infty}\right) / \ \mathbb{E}\left[\exp\left(-\sum_{s,t\in\mathbb{T}}|S_t/S_s|_{l^\infty}\right)\right]$, which is equivalent and for which (2.1) holds.

The transaction costs are modeled as a $\mathbb{M}_+$-valued adapted process $\lambda=(\lambda_t)_{t\le T}$. It means that buying one unit of asset j against units of asset i at time t costs $\pi_t^{ij}:=(S_t^j/S_t^i)(1+\lambda_t^{ij})$ units of asset i.

Throughout the paper, we shall assume that

$$\lambda_t^{ii}=0 \text{ and } (1+\lambda_t^{ij})(1+\lambda_t^{jk})\ge(1+\lambda_t^{ik}) \ \mathbb{P}-\text{a.s. for all } i,j,k\ge 1 \text{ and } t\in\mathbb{T}. \tag{2.2}$$

and that

$$\sup_{t\in\mathbb{T},i,j\ge1}\lambda_t^{ij}<\infty, \ \ \mathbb{P}-\text{a.s.} . \tag{2.3}$$

Note that these conditions have a natural economic interpretation. The first is equivalent to $\pi_t^{ii}=1$ and $\pi_t^{ij}\pi_t^{jk}\ge\pi_t^{ik}$ for all $i,j,k\ge1$ and $t\le T$, compare with [19].
A portfolio strategy is described as a $\mathbb{R}^{\mathbb{N}}$-valued adapted process $\xi=(\xi)_{t\le T}$ satisfying at any time $t\in\mathbb{T}$

$$\xi_t^i \le \sum_{j\ge1}\left(a^{ji}-a^{ij}(1+\lambda_t^{ij})\right) \ \ \forall\, i\ge1\ \mathbb{P}-\text{a.s.} , \ \ \text{for some } a\in L^0(\mathbb{M}_+,\mathcal{F}_t),$$

whenever this makes sense, or equivalently

$$-\xi_t \ge \sum_{i\ne j} a^{ij}\left((1+\lambda_t^{ij})e_i-e_j\right) \ \ \mathbb{P}-\text{a.s.} , \ \ \text{for some } a\in L^0(\mathbb{M}_+,\mathcal{F}_t). \tag{2.4}$$

As explained in the introduction, ξ_t^i should be interpreted as the additional net amount of money transferred at time t to the account invested in asset i after making transactions on the different assets. The quantity a^{ji} should be interpreted as the amount of money transferred to the account i by selling $a^{ji}(1+\lambda_t^{ji})/S_t^j$ units of asset j. The above inequality means that we allow the investor to throw away money from the different accounts.

In order to give a mathematical meaning to the above expressions, let us define the random convex cones $\tilde{K}_t$, $t \in \mathbb{T}$, as the convex cones generated by elements of finite length in l^1_+ and the set of vectors on the r.h.s. of (2.4) obtained by finite sums:

$$\tilde{K}_t(\omega) = \left\{ x \in l^1 : x = \sum_{i \neq j \geq 1} a^{ij} \left((1 + \lambda_t^{ij}(\omega)) e_i - e_j \right) + \sum_{i \geq 1} b^i e_i \ \text{ for some } a \in \mathbb{M}_{f,+},\, b \in c_f \cap l^1_+ \right\},$$

and define the set of admissible strategies as

$$\mathcal{A} := \{ \xi = (\xi_t)_{t \in \mathbb{T}} \ \mathbb{F}\text{-adapted} \ : \ \xi_t \in -K_t \text{ for all } t \in \mathbb{T} \} \ ,$$

where $K_t(\omega)$ denotes the l^1-closure of $\tilde{K}_t(\omega)$ for $t \leq T$.

Remark 2.2 Note that, by construction, $K_t(\omega)$ is a closed convex cone in l^1 of vertex 0 satisfying $l^1_+ \subset K_t(\omega)$ and such that $K_t(\omega) \cap c_f$ is dense in $K_t(\omega)$.

For ease of notations, we also define

$$\mathcal{A}_t^T := \{ \xi \in \mathcal{A} \ : \ \xi_s = 0 \ \text{ for } s < t \},\ t \in \mathbb{T}.$$

To an admissible strategy $\xi \in \mathcal{A}$, we associate the corresponding portfolio process V^ξ corresponding to a zero initial endowment:

$$V_t^\xi := \sum_{s=0}^{t} \xi_s S_t / S_s \ , \ t \in \mathbb{T}. \tag{2.5}$$

The i-th component corresponds to the amount of money invested in the i-th asset at time t. Note that the additional amount of money ξ_s^i invested at time s in the i-th asset corresponds to ξ_s^i / S_s^i units of the i-th asset, whose value at time t is $(\xi_s^i / S_s^i) S_t^i$.
We then define the corresponding sets of terminal portfolio values

$$\mathcal{X}_t^T := \left\{ V_T^\xi \ : \ \xi \in \mathcal{A}_t^T \right\} \ , \ t \in \mathbb{T}.$$

We can now define our condition of no-arbitrage of the second kind, which is similar to the one used in [17] and [6] for finite dimensional markets. It simply says that a trading strategy can not ensure to end up with a solvable position at time T if the position was not already $\mathbb{P}-$a.s. solvent at previous times $t \leq T$.

Condition 2.1 (NA2) *For all* $t \in \mathbb{T}$,

$$\eta \in L^0(l^1, \mathcal{F}_t) \setminus L^0(K_t, \mathcal{F}_t) \ \Rightarrow \ (\eta S_T / S_t + \mathcal{X}_t^T) \cap L^0(K_T, \mathcal{F}_T) = \emptyset.$$

Remark 2.3 For later use, note that it follows from **NA2** that $\mathcal{X}_0^T \cap L^0(K_T, \mathcal{F}_T) = \{0\}$ whenever K_t is $\mathbb{P}-$a.s. proper (i.e. $K_t \cap (-K_t) = \{0\}$) for all $t \leq T$. Indeed, fix a nontrivial $\xi \in \mathcal{A}$ and suppose that $V_T^\xi \in L^0(K_T, \mathcal{F}_T)$. Since $\xi \neq 0$, there is a smallest t^* such that $\xi_{t^*} \neq 0$ (as a random variable). It follows that $V_T^\xi = \xi_{t^*} S_T / S_{t^*} + g$ for some $g \in \mathcal{X}_{t^*+1}^T$. The condition **NA2** then implies that $\xi_{t^*} \in L^0(K_{t^*}, \mathcal{F}_{t^*})$. However $\xi \in \mathcal{A}$, so $\xi_{t^*} \in L^0(-K_{t^*}, \mathcal{F}_{t^*})$. Since $K_{t^*} \cap (-K_{t^*}) = \{0\}$, this leads to a contradiction.

2.2 The efficient friction assumption

In this paper, we shall assume that a version of the so-called *Efficient friction* assumption holds. In finite dimensional settings, it means that $\lambda_t^{ij} + \lambda_t^{ji} > 0$ for all $i \neq j$ and $t \leq T$, or equivalently that K_t is a.s. proper (i.e. $K_t \cap (-K_t) = \{0\}$), or that the positive dual of each K_t has $\mathbb{P}-$ a.s. non-empty interior, for all $t \leq T$, see [15].
In our infinite dimensional setting, the positive dual cone of $K_t(\omega)$ is defined as

$$K_t'(\omega) := \{z \in l^\infty \ : \ z \cdot x \geq 0 \ \text{ for all } \ x \in K_t(\omega)\}\,, \quad t \in \mathbb{T},$$

or, more explicitly,

$$K_t'(\omega) = \left\{z \in l^\infty \ : \ 0 \leq z^j \leq z^i(1 + \lambda_t^{ij}(\omega)),\ i, j \geq 1\right\}, \quad t \in \mathbb{T}\,, \tag{2.6}$$

and the above mentioned condition could naively read

$$\text{essinf}(\lambda_t^{ij}(\omega) + \lambda_t^{ji}(\omega)) > 0, \tag{2.7}$$

where the essinf is taken over $\omega \in \Omega$, $t \in \mathbb{T}$ and $i \neq j$. However, it is not sufficient in order to ensure that K_t' has a.s. a non-empty interior, as shown in Remark 6.1 below.

We shall therefore appeal to a generalized version of the *Efficient Friction* (in short **EF**) assumption of [15] which is directly stated in terms of the random cones K_t' in l^∞. Theorem 2.1 below provides a natural condition under which it is satisfied.

Assumption 2.1 (EF) *The $\mathbb{M}_+$-valued adapted process λ, satisfying* (2.2) *and* (2.3)*, has the property that for all $t \in \mathbb{T}$ and $\mathbb{P}-$ a.s. ω the dual cone $K_t'(\omega)$ has an interior point $\theta_t(\omega)$ such that $\theta_t \in L^0(l^\infty, \mathcal{F}_t)$.*

It is easy to find sufficient conditions on the transactions costs λ such that the Efficient Friction Assumption 2.1 is satisfied. The following result is a direct consequence of Proposition 6.1 reported in Section 6 below.

Theorem 2.1 *Assume that*

$$\text{essinf}\, \lambda_t^{ij}(\omega) > 0. \tag{2.8}$$

where the essinf *is taken over $\omega \in \Omega$, $t \in \mathbb{T}$ and $i \neq j$. Then the Efficient Friction Assumption 2.1 is satisfied.*

Remark 2.4

1. If the condition (2.8) is replaced by the weaker one (2.7) used in finite dimensional settings, then Theorem 2.1 is no longer true. See Remark 6.1 for a counter-example.

2. There are λ giving rise to **EF** not covered by Theorem 2.1. One such case is given by λ defined by $\lambda^{ij} = 1$ for all $i \neq j$ except $\lambda^{12} = 0$. In fact for this case Lemma 6.3 gives that $(3/2, 1, 1, \ldots) \in \mathrm{int}(K'_t)$.

3. Under **EF**, $K_t(\omega)$ is $\mathbb{P}-$a.s. normal (see Section 6), for all $t \in \mathbb{T}$. In particular, $K_t(\omega)$ is $\mathbb{P}-$ a.s. a proper cone.

4. Under **EF**, for all $\xi \in L^0(l^\infty, \mathcal{F}_t)$, $d_{l^\infty}(\xi, \partial K'_t)$ is a real $\mathcal{F}_t$-measurable r.v., where $\partial K'_t(\omega)$ is the border of $K'_t(\omega)$ (see Section 6).

5. The choice of the spaces has to be done with some care. For instance, if the λ^{ij}'s are time independent and uniformly bounded by some constant $c > 0$, and if $\tilde{K}$ and K are defined in l^p with $1 < p < \infty$, instead of l^1, then $K' = \{0\}$ and $K = l^p$. In fact, with $p^{-1} + q^{-1} = 1$, $y \in K'$ if and only if $y \in l^q$ and $0 \leq y^j \leq y^i(1 + \lambda^{ij})$ for all $i \neq j \geq 1$. In particular, $\frac{y^j}{1+c} \leq y^i$ for $i \neq j \geq 1$, so that $y \notin l^q$ whenever there exists $j \geq 1$ such that $y^j > 0$. This shows that $K' = \{0\}$, which then implies that $K = l^p$.

3 Main results

In this section, we state our main results. The proofs are collected in the subsequent sections.

From now on, we denote by $L^0_{t,b}$ the subset of random variables $g \in L^0(l^1, \mathcal{F})$ bounded from below in the sense that

$$g + \eta S_T/S_t \in K_T \text{ for some } \eta \in L^0(l^1_+, \mathcal{F}_t) . \tag{3.1}$$

In the following, a subset $B \subset L^0_{t,b}$ is said to be t-bounded from below if there exists $c \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ (called a lower bound) such that any $g \in B$ satisfies (3.1) for some $\eta \in L^0(l^1_+, \mathcal{F}_t)$ such that $|\eta|_{l^1} \leq c$.

Our first main result is a Fatou-type closure property for the sets $\mathcal{X}_t^T$ in the following sense:

Definition 3.1 *Let $(g_n)_{n\geq 1}$ be a sequence in $L^0(l^1, \mathcal{F})$, which converges $\mathbb{P}-$ a.s. pointwise to some $g \in L^0(l^1, \mathcal{F})$ and fix $t \leq T$.*
We say that $(g_n)_{n\geq 1}$ is t-Fatou convergent with limit g if $\{g_n : n \geq 1\}$ is a subset of $L^0_{t,b}$ which is t-bounded from below.
We say that a subset B of $L^0(l^1, \mathcal{F})$ is t-Fatou closed, if, for any sequence $(g_n)_{n\geq 1}$ in B, which t-Fatou converges to some $g \in L^0(l^1, \mathcal{F})$, we have $g \in B$.

Theorem 3.1 *Assume that* **NA2** *and* **EF** *hold. Then $\mathcal{X}_t^T$ is t-Fatou closed, for all $t \in \mathbb{T}$.*

The above Fatou closure property can then be translated in a $*$-weak closure property of the set of terminal portfolio holding labeled in time-t values of the assets, i.e. $S_t\mathcal{X}_t^T/S_T = \{S_tV_T/S_T,\ V_T \in \mathcal{X}_t^T\}$. Recall that μ denotes any element of $\mathbb{R}^{\mathbb{N}}$ such that $1/\mu \in l^1_+$.

Theorem 3.2 *Assume that* **NA2** *and* **EF** *hold. Then,* $(S_t\mathcal{X}_t^T/S_T) \cap L^\infty(l^1(\mu),\ \mathcal{F})$ *is* $\sigma(L^\infty(l^1(\mu),\ \mathcal{F}),\ L^1(c_0(1/\mu),\ \mathcal{F}))$*-closed for all* $t \in \mathbb{T}$*.*

Remark 3.1 Note that we use the spaces $l^1(\mu)$ and $c_0(1/\mu)$, with $\mu \in (0,\infty)^{\mathbb{N}}$ such that $1/\mu \in l^1$, in the above formulation instead of the more natural ones l^1 and c_0. The reason is that bounded sequences $(x_n)_{n\geq 1}$ in $l^1(\mu)$ have components satisfying $|x_n^i| \leq c1/\mu^i$ for some $c > 0$ independent of i and n and where $1/\mu \in l^1_+$. In particular, $x + c/\mu \in l^1_+$. This allows to appeal to the Fatou closure property of Theorem 3.1, see the proof of Theorem 3.2 in Section 4. We shall actually see in Remark 4.1 below that the above closure property can not be true in general if we consider the (more natural) $\sigma(L^\infty(l^1,\ \mathcal{F}),\ L^1(c_0,\ \mathcal{F}))$-topology.

By using standard separation arguments, Theorem 3.2 allows as usual to characterize the set of attainable claims in terms of natural dual processes.
In models with proportional transaction costs, they consist in elements of the sets $\mathcal{M}_t^T(K' \setminus \{0\})$ of $\mathbb{R}^{\mathbb{N}}$-valued $\mathbb{F}$-adapted processes Z on $\mathbb{T}_t := \{t, t+1, \ldots, T\}$ such that $Z_s \in K'_s \setminus \{0\}$, for all $s \in \mathbb{T}_t$, and ZS is a $\mathbb{R}^{\mathbb{N}}$-valued martingale on $\mathbb{T}_t$, $t \leq T$. Following the terminology of [19], elements of the form ZS with $Z \in \mathcal{M}_t^T(K' \setminus \{0\})$ are called consistent price system (on $\mathbb{T}_t$).

Theorem 3.3 *Assume that* **NA2** *and* **EF** *hold. Fix* $t \in \mathbb{T}$*. Then,* $\mathcal{M}_t^T(K' \setminus \{0\}) \neq \emptyset$*. Moreover, for any* $g \in L^0(l^1, \mathcal{F}_T)$ *such that* $g + \eta S_T/S_t \in L^0(l^1_+, \mathcal{F})$ *for some* $\eta \in L^0(l^1_+, \mathcal{F}_t)$*, we have:*

$$g \in \mathcal{X}_t^T \Leftrightarrow \mathbb{E}\left[Z_T \cdot g \mid \mathcal{F}_t\right] \leq 0 \quad \textit{for all} \;\; Z \in \mathcal{M}_t^T(K' \setminus \{0\})\ .$$

We note that the above conditional expectation $\mathbb{E}\left[Z_T \cdot g \mid \mathcal{F}_t\right]$ is well defined as a $\mathbb{R} \cup \{\infty\}$-valued $\mathcal{F}_t$-measurable r.v. In fact $g + \eta S_T/S_t \in L^0(l^1_+, \mathcal{F})$ implies that $Z_T \cdot g \geq -Z_T \cdot (\eta S_T/S_t)$ where $\eta/S_t \in L^0(l^1, \mathcal{F}_t)$ and $Z_TS_T \in L^1(l^\infty, \mathcal{F})$ by definition.
Following arguments used in [17] and [6], one can also prove that the so-called **B** condition holds under **NA2**.

Condition 3.1 (B) *The following holds for all* $t \in \mathbb{T}$ *and* $\xi \in L^0(l^1, \mathcal{F}_t)$*:*

$$Z_t \cdot \xi \geq 0 \;\; \forall\, Z \in \mathcal{M}_t^T(K' \setminus \{0\}) \;\Rightarrow\; \xi \in K_t.$$

Theorem 3.4 **NA2** $\Leftrightarrow$ (***B*** *and* $\mathcal{M}_0^T(K' \setminus \{0\}) \neq \emptyset$*).*

It finally implies the existence of Strictly Consistent Price Systems, i.e. elements of the sets $\mathcal{M}_t^T(\mathrm{int}K')$ of processes $Z \in \mathcal{M}_t^T(K' \setminus \{0\})$ such that $Z_s \in \mathrm{int}K'_s$, for all $s \in \mathbb{T}_t$. The **NA2** condition actually turns out to be equivalent to the existence of a sufficiently big sets of consistent price systems, which is referred to as the Many Consistent Price Systems (**MCPS**) and Many Strictly Consistent Price Systems (**MSCPS**) properties.

Condition 3.2 *We say that the condition* **MCPS** *(resp.* **MSCPS***) holds if:*
For all $t \in \mathbb{T}$ and $\eta \in L^0(\mathrm{int}K'_t, \mathcal{F}_t)$ such that $\eta S_t \in L^1(l^\infty, \mathcal{F}_t)$, there exists $Z \in \mathcal{M}_t^T(K' \backslash \{0\})$ (resp. $Z \in \mathcal{M}_t^T(\mathrm{int}K')$) such that $Z_t = \eta$.

Theorem 3.5 *Assume that* **EF** *holds. Then, the three conditions* **NA2**, **MCPS** *and* **MSCPS** *are equivalent.*

4 Closure properties and duality

We start with the proof of our closure properties which are the main results of this paper.

4.1 Efficient frictions and Fatou closure property

The key idea for proving the closure property of Theorem 3.1 is the following direct consequence of the **EF** Assumption 2.1.

Corollary 4.1 *Suppose that* ***EF*** *holds. Then, for all $t \in \mathbb{T}$, there exists $\alpha \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ such that*

$$|\xi|_{l^1} \leq \alpha |\eta|_{l^1}, \;\; \forall\, (\xi, \eta) \in L^0(-K_t, \mathcal{F}_t) \times L^0(K_t, \mathcal{F}_t) \;\; \textit{such that} \;\; \xi + \eta \in K_t.$$

Proof According to the **EF** Assumption 2.1 there exists $\theta_t \in L^0(l^\infty, \mathcal{F}_t)$ such that $\theta_t(\omega)$ is an interior point of $K'_t(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Define

$$\alpha(\omega) := 8|\theta_t(\omega)|_{l^\infty} \left(\frac{1}{d_{l^\infty}(\theta_t(\omega), \partial K'_t(\omega))} \right)^2 ,$$

Then $\alpha \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ by 4. of Remark 2.4. We observe that $\xi_t(\omega) \in (K_t(\omega) - \eta_t(\omega)) \cap (\eta_t(\omega) - K_t(\omega))$, according to the hypotheses and the fact that $K_t + K_t = K_t$. Lemma 6.1 and Lemma 6.2, with $C = K_t(\omega)$, $f_0 = \theta_t(\omega)$, $x = \xi_t(\omega)$, $y = \eta_t(\omega)$ and $b = 1/2$, then apply, which proves the corollary with the above defined α. □

As an almost immediate consequence of the above corollary, we can now obtain under **NA2** the following important property of sequential relative compactness of lower bounded subsets (see (3.1)) of

$$\mathcal{X}_{t,b}^T := \mathcal{X}_t^T \cap L_{t,b}^0 \,.$$

Corollary 4.2 *Assume that* ***EF*** *and* ***NA2*** *hold. Fix* $t \in \mathbb{T}$ *and let* $(\xi^n)_{n\geq 1}$ *be a sequence in* $\mathcal{A}_t^T$ *such that* $(V_T^{\xi^n})_{n\geq 1}$ *is a sequence in* $\mathcal{X}_{t,b}^T$ *which is t-bounded from below. Then,*
(i) $(\xi_t^n)_{n\geq 1}$ *is* $\mathbb{P}-$ *a.s. bounded in* l^1*.*
(ii) *There is a sequence* $(n_k)_{k\geq 1}$ *in* $L^0(\mathbb{N}, \mathcal{F}_t)$ *such that* $(\xi_t^{n_k})_{k\geq 1}$ *converges pointwise* $\mathbb{P}-$ *a.s. to some* $\xi_t \in L^0(-K_t, \mathcal{F}_t)$*.*

Proof Let $c \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ be a lower bound for $(V_T^{\xi^n})_{n\geq 1}$ so that $(V_T^{\xi^n}, \eta_n)$ satisfy (3.1) in place of (g, η), for all $n \geq 1$, where the sequence $(\eta_n)_{n\geq 1}$ in $L^0(l^1_+, \mathcal{F}_t)$ satisfies $\sup_{n\geq 1} |\eta_n|_{l^1} \leq c$.
(i). We then have $V_T^{\xi^n} + \eta_n S_T/S_t = (\eta_n + \xi_t^n)S_T/S_t + (V_T^{\xi^n} - \xi_t^n S_T/S_t) \in K_T$ where $V_T^{\xi^n} - \xi_t^n S_T/S_t \in \mathcal{X}_{t+1}^T$, recall (2.5). Hence, **NA2** implies that $\eta_n + \xi_t^n \in K_t$. The claim then follows from Corollary 4.1, $l^1_+ \subset K_t$ and the fact that $\sup_{n\geq 1} |\eta_n|_{l^1} \leq c$, which imply $\sup_{n\geq 1} |\xi_t^n|_{l^1} \leq \alpha c$ for some $\alpha \in L^0(\mathbb{R}_+, \mathcal{F}_t)$.
(ii). It follows in particular from the above claim that $|(\xi_t^n)^i| \leq \alpha c$ for all $n, i \geq 1$. For $i = 1$, we can then construct a $\mathcal{F}_t$-measurable sequence $(n_k^1)_{k\geq 1} \in L^0(\mathbb{N}, \mathcal{F}_t)$ such that $((\xi_t^{n_k^1})^1)_{k\geq 1}$ converges $\mathbb{P}-$a.s. and is also $\mathbb{P}-$a.s. uniformly bounded in l^1, see e.g. [13]. Iterating this procedure on the different components, we obtain after κ steps a sequence $(n_k^\kappa)_{k\geq 1} \in L^0(\mathbb{N}, \mathcal{F}_t)$ such that $((\xi_t^{n_k^\kappa})^i)_{k\geq 1}$ converges $\mathbb{P}-$ a.s. for all $i \leq \kappa$. It follows that the sequence $(\xi_t^{n_k^k})_{k\geq 1}$ converges $\mathbb{P}-$ a.s. pointwise to some $\mathcal{F}_t$-measurable random variable ξ_t with values in $\mathbb{R}^\mathbb{N}$. Since $|\xi_t^n|_{l^1}$ is $\mathbb{P}-$ a.s. uniformly bounded, $\xi_t \in l^1$ $\mathbb{P}-$ a.s. □

We can now conclude the proof of Theorem 3.1 by appealing to an inductive argument.

Proof of Theorem 3.1. If $t = T$, the result is an immediate consequence of Corollary 4.2. We now assume that it holds for some $0 < t+1 \leq T$ and show that this implies that it holds for t as well. Let $(g_n)_{n\geq 1}$ be a sequence in $\mathcal{X}_t^T$ which is t-Fatou convergent with limit $g \in L^0(l^1, \mathcal{F}_T)$. Then, by definition, there exist $c \in L^0(l^1, \mathcal{F}_t)$ and $\eta_n \in L^0(l^1_+, \mathcal{F}_t)$ such that $|\eta_n|_{l^1} \leq c$ and $g_n + \eta_n S_T/S_t \in K_T$ for all $n \geq 1$. Let the sequence $(\xi^n)_{n\geq 1}$ in $\mathcal{A}_t^T$ be such that $V_T^n = g_n$ for all $n \geq 1$, where $V^n = V^{\xi^n}$. It then follows from Corollary 4.2 that we can find a sequence $(n_k)_{k\geq 1}$ in $L^0(\mathbb{N}, \mathcal{F}_t)$ such that $(\xi_t^{n_k})_{k\geq 1}$ is $\mathbb{P}-$ a.s. bounded in l^1 and converges pointwise $\mathbb{P}-$ a.s. to some $\xi_t \in L^0(-K_t, \mathcal{F}_t)$. Clearly, $(\xi^{n_k})_{k\geq 1}$ is a sequence in $\mathcal{A}_t^T$ since $(n_k)_{k\geq 1}$ is $\mathcal{F}_t$-measurable, and $V_T^{n_k} = g_{n_k}$ where the later converges $\mathbb{P}-$ a.s. pointwise to g as $k \to \infty$. Moreover, $g_{n_k} - \xi_t^{n_k} S_T/S_t = V_T^{n_k} - \xi_t^{n_k} S_T/S_t \in \mathcal{X}_{t+1}^T$ and $(g_{n_k} - \xi_t^{n_k} S_T/S_t) + (\eta_{n_k} + \xi_t^{n_k})S_T/S_t \in L^0(K_T, \mathcal{F}_T)$. Since $(\eta_{n_k} + \xi_t^{n_k})_{k\geq 1}$ is $\mathbb{P}-$ a.s. bounded in l^1, and $(g_{n_k} - \xi_t^{n_k} S_T/S_t)_{k\geq 1}$ converges $\mathbb{P}-$ a.s. pointwise to $g - \xi_t S_T/S_t \in \mathcal{X}_{t+1}^T$, the fact that $\mathcal{X}_{t+1}^T$ is $(t+1)$-Fatou closed, this implies that $g - \xi_t S_T/S_t \in \mathcal{X}_{t+1}^T$ and therefore that $g \in \mathcal{X}_t^T$. □

4.2 Weak closure property and the dual representation of attainable claims

We now turn to the proof of Theorem 3.2 which will allow us to deduce the dual representation of Theorem 3.3 by standard separation arguments. It is an easy consequence of Theorem 3.1 once the suitable spaces have been chosen.

Proof of Theorem 3.2. Fix $t \in \mathbb{T}$ and set $F = L^1(c_0(1/\mu), \mathcal{F})$, so that $F' = L^\infty(l^1(\mu), \mathcal{F})$, where we recall that $1/\mu \in l^1_+$. Let B_1 denote the unit ball in F' and define the set $\Theta := (S_t \mathcal{X}_t^T / S_T) \cap B_1$.
By Krein-Šmulian's Theorem, (c.f. Corollary, Ch. IV, Sect. 6.4 of [18]), it suffices to show that Θ is $\sigma(F', F)$-closed. To see this, let $(h_\alpha)_{\alpha \in \mathcal{I}}$ be a net in Θ which converges $\sigma(F', F)$ to some $h \in B_1$. After possibly passing to convex combinations, we can then construct a sequence $(f_n)_{n \geq 1}$ in Θ which convergences $\mathbb{P}-$a.s. pointwise to h. In fact, this follows from Lemma 4.1 below with $E = (L^1(\mathbb{R}, \mathcal{F}))^{\mathbb{N}}$. This implies that the sequence $(f_n S_T/S_t)_{n \geq 1}$ in $\mathcal{X}_t^T$ converges to hS_T/S_t $\mathbb{P}-$a.s. pointwise. Since $f_n \in B_1$, we have $f_n + 1/\mu \in l^1_+$, and therefore $f_n S_T/S_t + (1/\mu) S_T/S_t \in K_T$ $\mathbb{P}-$a.s. This shows that the sequence $(f_n S_T/S_t)_{n \geq 1}$ is t-Fatou convergent with limit $hS_T/S_t \in L^0(l^1, \mathcal{F})$. It thus follows from Theorem 3.1 that $hS_T/S_t \in \mathcal{X}_t^T$ and therefore that $h \in \Theta$. □

To complete the proof of Theorem 3.2, we now state the following technical Lemma which was used in the above arguments.

Lemma 4.1 *Let E and F be locally convex TVS, with topological duals E' and F' and let $\mathfrak{T}(E)$ be the topology of E. Suppose $F' \subset E$, $E' \subset F$ and that E is metrizable. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a net in F', with convex hull J and converging in the $\sigma(F', F)$ topology to x, then there exists a sequence $(y_n)_{n \geq 1}$ in J, which is $\mathfrak{T}(E)$ convergent to x.*

Proof: Since $F' \subset E$ and $E' \subset F$, the topology on F' induced by $\sigma(E, E')$ is weaker than the $\sigma(F', F)$ topology. The net $(x_\alpha)_{\alpha \in \mathcal{I}}$ then also converges in the $\sigma(E, E')$ topology, so $x \in \bar{J}$ the $\sigma(E, E')$-closure of J. Since $\bar{J}$ is also $\mathfrak{T}(E)$-closed (c.f. Corollary 2, Ch II, Sect. 9.2 of [18]) and $(E, \mathfrak{T}(E))$ is metrizable, it now follows that there exists a sequence in J which is $\mathfrak{T}(E)$-convergent to x. □

From now on, we follow the usual ideas based on the Hahn-Banach separation theorem. For ease of notations, we set $\tilde{\mathcal{X}}_0^T = (S_0 \mathcal{X}_0^T / S_T) \cap L^\infty(l^1(\mu), \mathcal{F})$, and let $\tilde{\mathcal{X}}_{s,0}^T$ denote the set of elements of the form $-\alpha e_i S_0^i / S_t^i \chi_{\{S_t^i \geq \varepsilon\}}$ or $\alpha(e_j - (1 + \lambda_t^{ij}) e_i) S_0 / S_t \chi_{\{S_t^j \wedge S_t^i \geq \varepsilon\}}$ for some $t \leq T$, $i, j \geq 1$, $\varepsilon > 0$ and $\alpha \in L^\infty(\mathbb{R}_+, \mathcal{F}_t)$. Note that

$$\tilde{\mathcal{X}}_{s,0}^T \subset \tilde{\mathcal{X}}_0^T \ . \tag{4.1}$$

Proposition 4.1

1. *Suppose that **EF** and **NA2** hold. Then, for all* $\eta \in L^\infty(l^1(\mu), \mathcal{F}) \setminus \tilde{\mathcal{X}}_0^T$*, there exists* $Y \in L^1(c_0(1/\mu), \mathcal{F})$ *such that*

$$\mathbb{E}\left[Y \cdot X\right] \le 0 < \mathbb{E}\left[Y \cdot \eta\right] \quad \text{for all } X \in \tilde{\mathcal{X}}_0^T.$$

2. *Suppose that* $0 \neq Y \in L^1(c_0(1/\mu), \mathcal{F})$ *and that for all* $X \in \tilde{\mathcal{X}}_{s,0}^T$

$$\mathbb{E}\left[Y \cdot X\right] \le 0.$$

Then $Z_t := \mathbb{E}\left[Y \mid \mathcal{F}_t\right] S_0/S_t$ *satisfies* $Z_t S_t = \mathbb{E}\left[S_T Z_T \mid \mathcal{F}_t\right]$ *and* $Z_t \in L^0(K_t', \mathcal{F}_t) \setminus \{0\}$ *for all* $t \in \mathbb{T}$*.*

Proof In this proof, we use the notations $F := L^1(c_0(1/\mu), \mathcal{F})$ and $F' := L^\infty(l^1(\mu), \mathcal{F})$.
1. The set $\tilde{\mathcal{X}}_0^T$ being convex and $\sigma(F', F)$-closed, by Theorem 3.2, it follows from the Hahn-Banach separation theorem that we can find $Y \in F$ such that

$$\sup_{X \in \tilde{\mathcal{X}}_0^T} \mathbb{E}\left[Y \cdot X\right] < \mathbb{E}\left[Y \cdot \eta\right].$$

Since $\tilde{\mathcal{X}}_0^T$ is a cone that contains 0, we clearly have

$$\sup_{X \in \tilde{\mathcal{X}}_0^T} \mathbb{E}\left[Y \cdot X\right] = 0 < \mathbb{E}\left[Y \cdot \eta\right]. \tag{4.2}$$

2. First note that $\mathbb{E}\left[Y \mid \mathcal{F}_t\right] \in F$, so that Z is well-defined as a $\mathbb{R}^{\mathbb{N}}$-valued process, and that (4.2) implies $Z_T \neq 0$ as a random variable. Moreover, the fact that the left-hand side inequality of the Proposition holds for $-\alpha e_i S_0^i/S_t^i \chi_{\{S_t^i \ge \varepsilon\}}$ and $\alpha(e_j - (1+\lambda_t^{ij})e_i) S_0/S_t \chi_{\{S_t^j \wedge S_t^i \ge \varepsilon\}}$, for all $t \le T$, $i, j \ge 1$, $\varepsilon > 0$ and $\alpha \in L^\infty(\mathbb{R}_+, \mathcal{F}_t)$, implies that $Z_t := \mathbb{E}\left[Y \mid \mathcal{F}_t\right] S_0/S_t = \mathbb{E}\left[S_T Z_T \mid \mathcal{F}_t\right]/S_t$ satisfies $0 \le Z_t^j \le Z_t^i(1 + \lambda_t^{ij})$, $i, j \ge 1$, for all $t \in \mathbb{T}$. Hence, $Z_t \in K_t'$ by (2.6). Finally, $\mathbb{P}\left[Z = Z_T \neq 0\right] > 0$ implies that $\mathbb{P}\left[Z_t \neq 0\right] > 0$ for $t < T$. □

Remark 4.1 Note that the statement of Theorem 3.2 can not be true in general if we consider the weak topology $\sigma(L^\infty(l^1, \mathcal{F}), L^1(c_0, \mathcal{F}))$ on $(S_t \mathcal{X}_t^T / S_T) \cap L^\infty(l^1, \mathcal{F})$ instead of $\sigma(L^\infty(l^1(\mu), \mathcal{F}), L^1(c_0(1/\mu), \mathcal{F}))$ on $(S_t \mathcal{X}_t^T / S_T) \cap L^\infty(l^1(\mu), \mathcal{F})$. Indeed, if $S_0 \mathcal{X}_t^T / S_T \cap L^\infty(l^1, \mathcal{F})$ was closed in $\sigma(L^\infty(l^1, \mathcal{F}), L^1(c_0, \mathcal{F}))$, then the same arguments as in the proof of Proposition 4.1 above would imply the existence of a random variable Z_T such that $Z_T \in K_T' \setminus \{0\}$ and $Z_T S_T / S_0 \in c_0$. Recalling (2.6), this would imply that $0 \le Z_T^j \le (1 + \lambda_T^{ij}) Z_T^i$ for all $i, j \ge 1$ and $Z_T^i S_T^i / S_0^i \to 0$ $\mathbb{P}-$a.s. as $i \to \infty$. Since Z_T^1 is not identically equal to 0, this can not hold, except if $S_T^i / S_0^i \to 0$ as $i \to \infty$ on a set of non-zero measure, which is in contradiction with (2.1). The closure property stated in terms of $\sigma(L^\infty(l^1(\mu), \mathcal{F}), L^1(c_0(1/\mu), \mathcal{F}))$ does obviously not lead to such a contradiction since (2.3) and (2.1) imply that $Z_T S_T / S_0 \in l^\infty$ so that $(Z_T^i S_T^i / S_0^i)/\mu^i \to 0$ $\mathbb{P}-$ a.s. as $i \to \infty$, whenever $1/\mu \in l^1$.

Corollary 4.3 *Suppose that* ***EF*** *and* ***NA2*** *hold. Then,* $\mathcal{M}_t^T(K'\setminus\{0\})\neq\emptyset$ *for all* $t\in\mathbb{T}$.

Proof It follows from **NA2** that $e_1\in L^\infty(l^1(\mu),\mathcal{F})\setminus\tilde{\mathcal{X}}_0^T$. Using Proposition 4.1 and (4.1) then implies that there exists $Y\in L^1(c_0(1/\mu),\mathcal{F})$ such that

$$\mathbb{E}[Y\cdot X]\leq 0<\mathbb{E}[Y\cdot e_1]\quad\text{for all } X\in\tilde{\mathcal{X}}_{s,0}^T\ . \tag{4.3}$$

Let $\mathcal{Y}$ denote the set of random variables $Y\in L^1(c_0(1/\mu),\mathcal{F})$ satisfying the left-hand side of (4.3) for all $X\in\tilde{\mathcal{X}}_{s,0}^T$. We claim that there exists $\tilde{Y}\in\mathcal{Y}$ such that $a:=\sup_{Y\in\mathcal{Y}}\mathbb{P}[Y^1>0]=\mathbb{P}\left[\tilde{Y}^1>0\right]$. To see this, let $(Y_n)_{n\geq 1}$ be a maximizing sequence. It follows from Proposition 4.1 that $\mathbb{E}[Y_n]\in K_0'$ and $Y_n^i\geq 0$ for all $i\geq 1$. Moreover, we can assume that $\mathbb{P}[Y_n^1>0]>0$. We can then choose $(Y_n)_{n\geq 1}$ such that $\mathbb{E}[Y_n^1]=1$. Recalling (2.3)-(2.6), this implies that there exists $c>0$ such that $0\leq\mathbb{E}[Y_n^i]\leq(1+c)\mathbb{E}[Y_n^1]=(1+c)$ for all $i\geq 1$. Using Komlos Lemma, a diagonalization argument and Fatou's Lemma, we can then assume, after possibly passing to convex combinations, that $(Y_n)_{n\geq 1}$ converges $\mathbb{P}-$a.s. pointwise to some $Y\in L^1(\mathbb{R}_+,\mathcal{F})^{\mathbb{N}}$. Set $\tilde{Y}:=\sum_{n\geq 1}2^{-n}Y_n$. It follows from the monotone convergence theorem that it satisfies the left-hand side of (4.3) for all $X\in\tilde{\mathcal{X}}_{s,0}^T$. Moreover, $\mathbb{P}\left[\tilde{Y}^1>0\right]\geq\mathbb{P}[Y_n^1>0]\to a$ so that $\mathbb{P}\left[\tilde{Y}^1>0\right]=a$. We now show that $\mathbb{P}\left[\tilde{Y}^1>0\right]=1$. If not, there exists $A\in\mathcal{F}$ with $\mathbb{P}[A]>0$ such that $\tilde{Y}^1=0$ on A. Since $e_1\chi_A\in L^\infty(l^1(\mu),\mathcal{F})\setminus\tilde{\mathcal{X}}_0^T$, by **NA2**, it follows from Proposition 4.1 that we can find $Y\in L^1(c_0(1/\mu),\mathcal{F})$ such that such that

$$\mathbb{E}[Y\cdot X]\leq 0<\mathbb{E}[Y\cdot e_1\chi_A]\quad\text{for all } X\in\tilde{\mathcal{X}}_0^T\ .$$

By (4.1), $Y+\tilde{Y}\in\mathcal{Y}$ and $\mathbb{P}\left[Y^1+\tilde{Y}^1>0\right]>\mathbb{P}\left[\tilde{Y}^1>0\right]$ since $\mathbb{E}[Y\cdot e_1\chi_A]>0$ implies that $\mathbb{P}[\{Y^1>0\}\cap A]>0$, a contradiction. To conclude the proof it suffices to observe that Z defined by $\tilde{Z}_t:=\mathbb{E}\left[\tilde{Y}\mid\mathcal{F}_t\right]S_0/S_t$ satisfies $\tilde{Z}_tS_t=\mathbb{E}\left[S_T\tilde{Z}_T\mid\mathcal{F}_t\right]$ and $\tilde{Z}_t\in L^0(K_t',\mathcal{F}_t)\setminus\{0\}$ for all $t\in\mathbb{T}$, by Proposition 4.1 again. Moreover, (2.6) and $\mathbb{P}\left[\tilde{Y}^1>0\right]=1$ implies that $\mathbb{P}\left[\tilde{Y}^i>0\right]=1$ for all $i\geq 1$. This shows that $\tilde{Z}_t\in L^0(K_t'\setminus\{0\},\mathcal{F}_t)$ for all $t\in\mathbb{T}$. □

The statement of Theorem 3.3 is then deduced from Proposition 4.1 and the following standard result.

Lemma 4.2 *Fix* $\xi\in\mathcal{A}_t^T$ *and* $Z\in\mathcal{M}_t^T(K'\setminus\{0\})$*, for some* $t\leq T$*. If* $V_T^\xi+\eta S_T/S_t\in K_T$ *for some* $\eta\in L^0(l^1,\mathcal{F}_t)$*, then*

$$Z_s\cdot V_{s-1}^\xi S_s/S_{s-1}\geq Z_s\cdot V_s^\xi\geq\mathbb{E}\left[Z_{(s+1)\wedge T}\cdot V_{(s+1)\wedge T}^\xi\mid\mathcal{F}_s\right]\geq -Z_s\cdot\eta S_s/S_t\ ,\quad\textit{for all } t\leq s\leq T\ ,$$

with the convention $V_{-1}^\xi/S_{-1}=0$.

Proof Note that the left-hand side inequality just follows from the fact that $\xi_s\in -K_s$ while $Z_s\in K_s'$, and the definition of V^ξ in (2.5). We now prove the two other inequalities.

For $s = T$, it follows from the fact that $Z_T \in K'_T$ and $V^\xi_T + \eta S_T/S_t \in K_T$. Assuming that it holds for $t < s+1 \le T$, we have $Z_{s+1} \cdot V^\xi_{s+1} \ge -Z_{s+1} \cdot \eta S_{s+1}/S_t$. On the other hand, the already proved above left-hand side inequality implies $Z_{s+1} \cdot V^\xi_{s+1} \le Z_{s+1} \cdot V^\xi_s S_{s+1}/S_s$. Since, $\mathbb{E}[Z_{s+1}S_{s+1} \mid \mathcal{F}_s] = Z_s S_s$ by definition of $\mathcal{M}^T_t(K' \setminus \{0\})$, this shows that the above property holds for s as well. □

We now turn to the proof of Theorem 3.3. The basic argument is standard, up to additional technical difficulties related to our infinite dimensional setting.

Proof of Theorem 3.3. The fact that $\mathcal{M}^T_t(K' \setminus \{0\}) \neq \emptyset$ for all $t \in \mathbb{T}$ follows from Corollary 4.3. We now fix $g \in L^0_{t,b}$. In view of Lemma 4.2, it is clear that

$$g \in \mathcal{X}^T_t \Rightarrow \mathbb{E}[Z_T \cdot g \mid \mathcal{F}_t] \le 0 \quad \text{for all} \;\; Z \in \mathcal{M}^T_t(K' \setminus \{0\}) \,.$$

It remains to prove the converse implication. We therefore assume that

$$\mathbb{E}[Z_T \cdot g \mid \mathcal{F}_t] \le 0 \text{ for all } Z \in \mathcal{M}^T_t(K' \setminus \{0\}), \tag{4.4}$$

and show that $g \in \mathcal{X}^T_t$.

(i) The case where $S_0 g/S_T \in L^\infty(l^1(\mu), \mathcal{F})$ is handled by very standard arguments based on Proposition 4.1 and Corollary 4.3. We omit the proof.

(ii) We now turn to the case where $g \in L^0(l^1(\mu), \mathcal{F})$ is such that $g + \eta S_T/S_t \in K_T$ for some $\eta \in L^0(l^1_+(\mu), \mathcal{F}_t)$. We first construct a sequence $(g_n)_{n\ge1}$ defined as $g_n := (g\mathbf{1}_{\{|S_0 g/S_T|_{l^1(\mu)} \le n\}} - \eta(S_T/S_t)\mathbf{1}_{\{|S_0 g/S_T|_{l^1(\mu)} > n\}})\mathbf{1}_{\{|S_0\eta/S_t|_{l^1(\mu)} \le n\}}$. Since (4.4) holds, $g - g_n \in K_T$ on $\{|S_0\eta/S_t|_{l^1(\mu)} \le n\} \in \mathcal{F}_t$ and $Z_T \in K'_T$ for $Z \in \mathcal{M}^T_t(K' \setminus \{0\})$, we have $\mathbb{E}[Z_T \cdot g_n \mid \mathcal{F}_t] \mathbf{1}_{\{|S_0\eta/S_t|_{l^1(\mu)} \le n\}} \le 0$ for all $Z \in \mathcal{M}^T_t(K' \setminus \{0\})$ for all $n \ge 1$. Moreover, $S_0 g_n/S_T \in L^\infty(l^1(\mu), \mathcal{F})$ for $n \ge 1$. It then follows from (i) that the sequence $(g_n)_{n\ge1}$ belongs to $\mathcal{X}^T_t$. Moreover, $g_n + \eta S_T/S_t \in K_T$ for all $n \ge 1$. Hence, $(g_n)_{n\ge1}$ t-Fatou converges to g. Appealing to the t-Fatou closure property of Theorem 3.1 thus implies that $g \in \mathcal{X}^T_t$.

(iii) We then consider the case where $g \in L^0_{t,b}$ and is such that $g^- := ((g^i)^-)_{i\ge1}$ satisfies $-g^- + \eta S_T/S_t \in l^1_+(\mu)$ for some $\eta \in L^0(l^1_+(\mu), \mathcal{F}_t)$. We now define the sequence $(g_n)_{n\ge1}$ by $g^i_n := g^i\mathbf{1}_{\{g^i \le n/(2^i\mu^i)\}}$ for $i \ge 1$. It satisfies the requirement of (ii) above and is t-Fatou convergent to g since $g_n + \eta S_T/S_t \ge -g^- + \eta S_T/S_t \in l^1_+(\mu) \subset K_T$. Moreover, $\mathbb{E}[Z_T \cdot g_n \mid \mathcal{F}_t] \le 0$ for all $Z \in \mathcal{M}^T_t(K' \setminus \{0\})$ since $g^i_n \le g^i$ for all $i \ge 1$ and (4.4) holds. By (ii), this implies that $g_n \in \mathcal{X}^T_t$ for all $n \ge 1$. Since $\mathcal{X}^T_t$ is t-Fatou closed, by Theorem 3.1, this implies that $g \in \mathcal{X}^T_t$.

(iv) We now turn to the case where $g \in L^0(l^1, \mathcal{F})$ and $g + \eta S_T/S_t \in l^1_+$ for some $\eta \in L^0(l^1_+, \mathcal{F}_t)$. Let $\bar{\mathcal{M}}^T_t$ denote the subset of elements $Z \in \mathcal{M}^T_t(K' \setminus \{0\})$ such that $Z^1_t = 1$, fix $\varepsilon > 0$, and note that (4.4) implies that

$$\mathbb{E}[Z_T \cdot (g - \varepsilon e_1 S_T/S_t) \mid \mathcal{F}_t] \le -\varepsilon \;\; \text{for all} \;\; Z \in \bar{\mathcal{M}}^T_t, \tag{4.5}$$

since $Z \in \bar{\mathcal{M}}_t^T$ implies $\mathbb{E}\left[Z_T^1 S_T^1/S_t^1 \mid \mathcal{F}_t\right] = Z_t^1 = 1$. Let g_n be defined by $g_n^i := g^i \mathbf{1}_{\{g^i \geq 0 \text{ or } i<n\}}$, $i \geq 1$. Note that, for all $Z \in \bar{\mathcal{M}}_t^T$,

$$\mathbb{E}\left[Z_T \cdot (g_n - g) \mid \mathcal{F}_t\right] \leq \mathbb{E}\left[\sum_{i \geq n} Z_T^i (g^i)^- \mid \mathcal{F}_t\right] \leq \mathbb{E}\left[\sum_{i \geq n} Z_T^i \eta^i S_T^i / S_t^i \mid \mathcal{F}_t\right] = \sum_{i \geq n} \eta^i Z_t^i \ ,$$

where the second inequality comes from the fact that $g + \eta S_T/S_t \in l^1_+$ implies $(g^i)^- \leq \eta^i S_T^i/S_t^i$ for all $i \geq 1$. Now observe that (2.3) and (2.6) imply that $0 \leq Z_t^i \leq (1 + c_t)$ for all $i \geq 1$ and $Z \in \bar{\mathcal{M}}_t^T$, for some $c_t \in L^0(\mathbb{R}, \mathcal{F}_t)$. It then follows from the above inequalities, (4.5) and the fact that $\eta \in l^1$ that

$$\limsup_{n \to \infty} \operatorname*{ess\,sup}_{Z \in \bar{\mathcal{M}}_t^T} \mathbb{E}\left[Z_T \cdot (g_n - \varepsilon e_1 S_T/S_t) \mid \mathcal{F}_t\right] \leq -\varepsilon \ .$$

We can then find a sequence $(n_\varepsilon)_{\varepsilon > 0}$ in $L^0(\mathbb{N}, \mathcal{F}_t)$ such that $n_\varepsilon \to \infty$ $\mathbb{P}-$a.s. as $\varepsilon \to 0$ and

$$\mathbb{E}\left[Z_T \cdot (g_{n_\varepsilon} - \varepsilon e_1 S_T/S_t) \mid \mathcal{F}_t\right] \leq 0 \quad \text{for all} \quad Z \in \bar{\mathcal{M}}_t^T.$$

Moreover, $g_{n_\varepsilon} - \varepsilon e_1 S_T/S_t$ satisfies the conditions of (iii) above with $\eta_{n_\varepsilon} := (\eta^i \mathbf{1}_{i \leq n_\varepsilon})_{i \geq 1} + \varepsilon e_1$, recall (2.1), and therefore belongs to $\mathcal{X}_t^T$ for all $\varepsilon > 0$. We conclude again by using the fact that $\mathcal{X}_t^T$ is t-Fatou closed, by Theorem 3.1, and that $g_{n_\varepsilon} + \eta S_T/S_t \in l^1_+ \subset K_T$ for all $\varepsilon > 0$. □

We conclude this section with the proof of Theorem 3.4.

Proof of Theorem 3.4. We follow the arguments of [6] which we adapt to our context. Let us first fix an arbitrary $g \in (\xi S_T/S_t + \mathcal{X}_t^T) \cap K_T$. In view of Lemma 4.2 applied with $\eta = \xi$, one has $-Z_t \cdot \xi \leq \mathbb{E}\left[Z_T \cdot g \mid \mathcal{F}_t\right] \leq 0$ for all $Z \in \mathcal{M}_t^T(K' \setminus \{0\})$. It then follows from **B** that $\xi \in K_t$.
We now prove the converse assertion. Let us consider $\xi \in L^0(l^1, \mathcal{F}_t)$ such that $Z_t \cdot \xi \geq 0$ for all $Z \in \mathcal{M}_t^T(K' \setminus \{0\})$. We can then find $\alpha \in L^0(l^1_+, \mathcal{F}_t)$ such that $-\xi + \alpha \in l^1_+$. By definition of $\mathcal{M}_t^T(K' \setminus \{0\})$, we have $0 \leq Z_t \cdot \xi = \mathbb{E}\left[Z_T \cdot \xi S_T/S_t \mid \mathcal{F}_t\right]$ for all $Z \in \mathcal{M}_t^T(K' \setminus \{0\})$. Moreover, $-\xi + \alpha \in l^1_+$ implies $-\xi S_T/S_t + \alpha S_T/S_t \in l^1_+$, according to (2.1). It then follows from Theorem 3.3 applied to $g = -\xi S_T/S_t$ that $-\xi S_T/S_t \in \mathcal{X}_t^T$. Hence, $0 \in \xi S_T/S_t + \mathcal{X}_t^T$, which by **NA2** implies that $\xi \in K_t$. □

5 On the existence of multiple consistent price systems

We split the proof of Theorem 3.5 in three parts. It follows from ideas introduced in [17] and [6] which we adapt to our context.

Theorem 5.1 *Assume that* **EF** *holds. Then,* **NA2** $\Rightarrow$ **MCPS**.

Proof We divide the proof in several points. In this proof, we use the notations $F := L^1(c_0(1/\mu), \mathcal{F})$ and $F' := L^\infty(l^1(\mu), \mathcal{F})$. From now on, we fix $\eta \in L^0(\text{int}K'_t, \mathcal{F})$ such that $\eta S_t \in L^1(l^\infty, \mathcal{F}_t)$. We set $G' = \mathbb{R}_+\eta$, which is the dual cone of $G = \{y \;:\; y \in l^1, y \cdot x \geq 0 \;\forall\, x \in G'\}$. We also set $\Theta := (-L^0(G, \mathcal{F}_t) + \mathcal{X}_t^T S_t/S_T) \cap F'$.

1. We first show that Θ is $\sigma(F', F)$-closed. Let B_1 be the unit ball in F'. Arguing as in the proof of Theorem 3.2, it suffices to show that, for any sequence $(h_n)_{n\geq 1} \subset \Theta \cap B_1$ that converges $\mathbb{P}-$a.s. to some h, we have $h \in \Theta$. Let $(\zeta_n, V_n)_{n\geq 1} \subset -L^0(G, \mathcal{F}_t) \times \mathcal{X}_t^T$ be such that $\zeta_n + V_n S_t/S_T = h_n$ for all $n \geq 1$. Since $h_n \in B_1$, we have $|h_n^i| \leq 1/\mu^i$ for all $i \geq 1$ and therefore $h_n + 1/\mu \in l^1_+$ with $1/\mu \in l^1_+$. It follows that $(\zeta_n + 1/\mu)S_T/S_t + V_n = h_n S_T/S_t + (1/\mu)S_T/S_t \in K_T$, which, by **NA2**, implies that $\zeta_n + 1/\mu \in K_t$. Since $\eta \in L^0(\text{int}K'_t, \mathcal{F}_t)$, we can find $\varepsilon \in L^0((0,1), \mathcal{F}_t)$ such that $\eta_n := \eta - \varepsilon(1_{\zeta_n^i \geq 0} - 1_{\zeta_n^i < 0})_{i\geq 1} \in K'_t$ for all $n \geq 1$. It follows that $0 \leq \eta_n \cdot (\zeta_n + 1/\mu) \leq -\varepsilon|\zeta_n|_{l^1} + \eta \cdot \zeta_n + (\eta + \varepsilon\mathbf{1}) \cdot 1/\mu$. On the other hand, we have $\eta \cdot \zeta_n \leq 0$ by definition of G and G'. This shows that $(|\zeta_n|_{l^1})_{n\geq 1}$ is $\mathbb{P}-$a.s. uniformly bounded. After possibly passing to ($\mathcal{F}_t$-measurable random) subsequences, see the arguments used in the proof of Corollary 4.2, we can then assume that $(\zeta_n)_{n\geq 1}$ converges $\mathbb{P}-$ a.s. in the product topology to some $\zeta \in L^0(l^1, \mathcal{F}_t)$. Moreover, we can find $(\alpha_n)_{n\geq 1} \subset L^0(l^1_+, \mathcal{F}_t)$ satisfying $\operatorname{ess\,sup}_n |\alpha_n|_{l^1} < \infty$ and such that $-\zeta_n + \alpha_n \in l^1_+$ for all $n \geq 1$. The identity $V_n = h_n S_T/S_t - \zeta_n S_T/S_t$ then leads to $V_n + (1/\mu + \alpha_n)S_T/S_t \in K_T$ since $-\zeta_n + \alpha_n \in l^1_+$ and $h_n + 1/\mu \in l^1_+$. We conclude by appealing to Theorem 3.1.

2. We now show that $\Theta \cap L^0(\mathbb{R}_+^{\mathbb{N}}, \mathcal{F}) = \{0\}$. Fix $(\zeta, V) \in (-L^0(G, \mathcal{F}_t) \times \mathcal{X}_t^T)$ such that $\zeta + VS_t/S_T \in \Theta \cap L^0(\mathbb{R}_+^{\mathbb{N}}, \mathcal{F})$. Then $\zeta S_T/S_t + V \in L^0(l^1_+, \mathcal{F})$, so that $\zeta \in K_t$ by **NA2**. Since $\eta \in \text{int}K'_t$, this implies that $\eta \cdot \zeta > 0$ on $\{\zeta \neq 0\}$. On the other hand, the definition of G and G' leads to $\eta \cdot \zeta \leq 0$. This shows that $\zeta = 0$. An induction argument, based on **NA2** and the fact that $K_s \cap (-K_s) = 0$ for all $s \in \mathbb{T}$, then implies that $V = 0$.

3. We can now conclude the proof. By the Hahn-Banach separation theorem, the fact that Θ is a convex $\sigma(F', F)$-closed cone, that $\Theta \cap L^0(\mathbb{R}_+^{\mathbb{N}}, \mathcal{F}) = \{0\}$ and a standard exhaustion argument, we can find $Y \in F$ such that $\mathbb{E}[Y \cdot h] \leq 0$ for all $h \in \Theta$, and $Y^i > 0$ for all $i \geq 1$. Defining the process Z by $Z_s := \mathbb{E}[YS_t \mid \mathcal{F}_s]/S_s$ for $t \leq s \leq T$, we obtain $Z^i > 0$ for all $i \geq 1$. Using the fact that $-L^0(G, \mathcal{F}_t) \cap F' \subset \Theta$, we also obtain that $Z_t \in G'$. From the fact that $\mathcal{X}_t^T S_t/S_T \cap F' \subset \Theta$, we then deduce, as in the proof of Proposition 4.1, that $Z_s \in K'_s$, for $t \leq s \leq T$. Since $Z_t \in G'$, we can find a non-negative $\mathcal{F}_t$-measurable α such that $Z_t = \alpha\eta$. Since $Z_t \neq 0$, it follows that $\alpha > 0$ $\mathbb{P}-$ a.s. Thus, $(Z_s/\alpha)_{t\leq s\leq T}$ satisfies the required result. □

Lemma 5.1 *Assume that* **EF** *holds. Then,* **MCPS** $\Leftrightarrow$ **MSCPS**.

Proof As in [6], we use a finite recursion from time T to time 0 to prove that **MCPS** $\Rightarrow$ **MSCPS**. Let **MSCPS**(t) be the statement in **MSCPS** for $t \leq T$ given. Suppose that

MCPS is true. Then **MSCPS**(T) is trivially satisfied.
We now suppose that **MSCPS**$(s+1)$ is true for some $0 \le s < T$. Then, there exists an element $\tilde{X} \in \mathcal{M}_{s+1}^T(\mathrm{int}K')$. Since $\tilde{X}_{s+1}S_{s+1} \in L^1(l^\infty, \mathcal{F})$, we can define $\tilde{X}_s := \mathbb{E}\left[\tilde{X}_{s+1}S_{s+1} \mid \mathcal{F}_s\right]/S_s$ and $X_t := \tilde{X}_t/(1+|\tilde{X}_s|_{l^\infty})$ for $s \le t \le T$. Then $0 < |X_s|_{l^\infty} < 1$ and X restricted to the interval $(s,T]$ belongs to $\mathcal{M}_{s+1}^T(\mathrm{int}K')$.
Fix $\eta \in L^0(\mathrm{int}K'_s, \mathcal{F}_s)$, let d be its distance to the border of K'_s and set $\alpha = (1 \wedge d)/2$. It follows from formula (6.2) of Lemma 6.3 below that α is $\mathcal{F}_s$-measurable. Since $|X_s|_\infty < 1$, we have

$$\eta - \alpha X_s \in L^0(\mathrm{int}K'_s, \mathcal{F}_s). \tag{5.1}$$

Let us now choose η such that $\eta S_s \in L^1(l^\infty, \mathcal{F}_s)$. Then $\eta S_s - \alpha X_s S_s \in L^1(l^\infty, \mathcal{F}_s)$, and **MCPS** implies that there exists $Y \in \mathcal{M}_s^T(K' \setminus \{0\})$ such that $Y_s = \eta - \alpha X_s$. In view of (5.1), $Y_s \in L^0(\mathrm{int}K'_s, \mathcal{F}_s)$.
For $s \le t \le T$, define $Z_t = Y_t + \alpha X_t$. Then $Z_s = \eta \in L^0(\mathrm{int}K'_s, \mathcal{F}_s)$. Since, for $s+1 \le t \le T$, $Y_t \in L^0(K'_t \setminus \{0\}, \mathcal{F}_t)$ and $X_t \in L^0(\mathrm{int}K'_t, \mathcal{F}_t)$, and since $\alpha > 0$, it follows that $Z_t \in L^0(\mathrm{int}K'_t, \mathcal{F}_t)$ for such t. Hence $Z \in \mathcal{M}_s^T(\mathrm{int}K')$, so **MSCPS**$(s)$ is true. □

Proof of Theorem 3.5. In view of the above results, it remains to show that **MCPS** ⇒ **NA2**. Fix $\xi \in L^0(l^1, \mathcal{F}_t) \setminus L^0(K_t, \mathcal{F}_t)$ such that $(\xi S_T/S_t + \mathcal{X}_t^T) \subset L^0(K_T, \mathcal{F}_T)$. Without loss of generality, we can assume that $\xi \in L^\infty(l^1, \mathcal{F}_t)$, since otherwise we could replace ξ by $\xi/|\xi|_{l^1}$ and use the fact that $\mathcal{X}_t^T/|\xi|_{l^1} = \mathcal{X}_t^T$, recall that K is a cone valued process. It then follows from Lemma 4.2 that $0 \ge -Z_t \cdot \xi$ for all $Z \in \mathcal{M}_t^T(K' \setminus \{0\})$. By definition of **MCPS**, this implies that $\eta \cdot \xi \ge 0$ for all $\eta \in L^\infty(\mathrm{int}K'_t, \mathcal{F}_t)$. This shows that $\xi \in K_t$ $\mathbb{P}-$a.s. □

6 Elementary properties of K and K'

For ease of notations, we restrict in this section to the case where λ is deterministic and constant in time. We therefore omit the time index in λ, K and K'. We set $\Lambda := (1+\lambda)$.

In this section, by a cone is meant a convex cone C of vertex $0 \in C$, and $(E, \|\cdot\|_E)$ denotes a Banach space with canonical bilinear form $\langle \cdot\,, \cdot \rangle$. We recall that a cone C in E, is said to be normal if there exists $k \ge 1$ such that

$$\|x\|_E \le k\|x+y\|_E \quad \forall\, x, y \in C. \tag{6.1}$$

We start with two easy abstract Lemmas which emphasizes the role plaid by the fact that the sets $K_t(\omega)$ are normal cones with dual cones $K'_t(\omega)$ having non-empty interior, under **EF**.

The purpose of the first result is also to have an explicit expression of the constant k, used to establish measurability properties of the random cones K_t and K'_t.

Lemma 6.1 *Let C be a cone in the Banach space E and suppose that the dual cone C' has an interior point f_0. Then C is a normal cone and one can choose $k = 4||f_0||_{E'}/d_{E'}(f_0, \partial C')$ in* (6.1).

Proof Let $d = d_{E'}(f_0, \partial C')$ and let $\bar{B}(a, r)$ denote the closed ball in E' of radius $r > 0$ centered at a. For $x \in E$, we define

$$p(x) = \sup\{|\langle f\,,\,x\rangle| \;:\; f \in \bar{B}(f_0, d)\}.$$

Substitution of $f = f_0 + dg$, $g \in \bar{B}(0,1)$ into this definition and the fact that $d \leq ||f_0||_{E'}$ give that $p(x) \leq ||f_0||_{E'}||x||_E + d||x||_E \leq 2||f_0||_{E'}||x||_E$. On the other hand, we have

$$||x||_E = \sup\{|\langle g\,,\,x\rangle| \;:\; g \in \bar{B}(0,1)\},$$

which for $g = (f - f_0)/d \in \bar{B}(0,1)$ with $f \in \bar{B}(f_0, d)$ similarly provides

$$||x||_E \leq \sup\{\frac{1}{d}|\langle f\,,\,x\rangle| + \frac{1}{d}|\langle f_0\,,\,x\rangle| \;:\; f \in \bar{B}(f_0, d)\} \leq \frac{2}{d}p(x).$$

Hence $p(\cdot)$ and $||\cdot||_E$ are equivalent norms: For all $x \in E$

$$\frac{d}{2}||x||_E \leq p(x) \leq 2||f_0||_{E'}||x||_E.$$

For $x, y \in C$, it follows directly from the definition of p and the fact that $\bar{B}(f_0, d) \subset C'$ that $p(x + y) \geq p(x)$. Then by the equivalence of the norms, for all $x, y \in C$,

$$||x||_E \leq \frac{2}{d}p(x) \leq \frac{2}{d}p(x + y) \leq \frac{4}{d}||f_0||_{E'}||x + y||_E,$$

which concludes the proof by comparing with (6.1). □

Lemma 6.2 *Let C be a cone in the Banach space E and suppose that f_0 is an interior point of the dual cone C'. Then, there exists $a > 0$ such that for all $x, y \in E$*

$$x \in (C - y) \cap (y - C) \;\Rightarrow\; \|x\|_E \leq a\langle f_0\,,\,y\rangle\;.$$

Moreover (since C is a normal cone), for any $k \geq 1$ satisfying (6.1) and any $b \in (0, 1)$, one can choose

$$a = k/(b\; d_{E'}(f_0, \partial C')).$$

Proof One observes that $x \in (C - y) \cap (y - C)$ iff $z_+ := x + y \in C$ and $z_- := y - x \in C$. Since C is normal according to Lemma 6.1, it follows that, for $\epsilon = \pm$,

$$\|z_\epsilon\|_E \leq k\|z_+ + z_-\|_E = 2k\|y\|_E.$$

Then

$$\|x\|_E = \frac{1}{2}\|z_+ - z_-\|_E \le \frac{1}{2}(\|z_+\|_E + \|z_-\|_E) \le 2k\|y\|_E.$$

Since f_0 is an interior point of C', there exists $r > 0$, such that $f_0 - rg \in C'$ for all $g \in E'$ such that $\|g\|_{E'} \le 1$. For $r > 0$ sufficiently small, we thus have

$$\|y\|_E = \sup_{\|g\|_{E'}\le 1} |\langle g\,,\, y\rangle| = \sup_{\|g\|_{E'}\le 1} \langle g\,,\, y\rangle = \sup_{g\in A_y} \langle g\,,\, y\rangle = \frac{1}{r}\sup_{g\in A_y}(\langle f_0\,,\, y\rangle + \langle rg - f_0\,,\, y\rangle) \le \frac{1}{r}\langle f_0\,,\, y\rangle,$$

where A_y denotes the set of elements $g \in E'$ satisfying $\|g\|_{E'} \le 1$ and $\langle g\,,\, y\rangle \ge 0$, and the last inequality follows from $f_0 - rg \in C'$ while $y \in C$. This shows that the inequality of the lemma is satisfied with $a = 2k/r$. One can choose $r = b\, d_{E'}(f_0, \partial C'))$ with $b \in (0, 1)$, which gives the stated choice of a. □

Lemma 6.3 *Assume that there exists some $c > 0$ such that $\lambda^{ii} = 0$ and $0 \le \lambda^{ij} \le c$ for all $i \ne j \ge 1$. Then, u is an interior point of K' (in l^∞) if and only if $\delta_u > 0$ where*

$$\delta_u := \inf_{i\ne j}\left(u^i\Lambda^{ij} - u^j\right) .$$

Moreover, suppose that the interior of K' is non-empty. Then $u \in \partial K'$ iff $\delta_u = 0$, $u \in l^\infty\backslash K'$ iff $\delta_u < 0$ and the distance between a point $u \in l^\infty$ and the border $\partial K'$ is

$$d_{l^\infty}(u, \partial K') = \left|\inf_{i\ne j}\frac{1}{1+\Lambda^{ij}}\left(u^i\Lambda^{ij} - u^j\right)\right|. \tag{6.2}$$

Proof By definition, $u \in \text{int}K'$ iff $\exists\, r > 0$ such that $u + \bar{B}(0, r) \subset K'$, where $\bar{B}(0, r)$ denotes the closed ball in l^∞ centered at 0 and with radius r. Equivalently, $z = u + |u|_{l^\infty} r'\epsilon$ satisfies (2.6) for all $\epsilon \in \bar{B}(0, 1)$, where $r' = r/|u|_{l^\infty}$ and $u \ne 0$. For given $i \ne j$, choosing $\epsilon = -e_i + e_j$ leads to

$$r'|u|_{l^\infty}(1 + \Lambda^{ij}) \le u^i\Lambda^{ij} - u^j. \tag{6.3}$$

In particular, $\delta_u \ge r'|u|_{l^\infty} > 0$ if $u \in \text{int}K'$. Conversely, if $\delta_u > 0$, then we can find $r' > 0$ such that (6.3) holds. This implies that

$$u^j + |u|_{l^\infty} r' \le (u^i - |u|_{l^\infty} r')\Lambda^{ij},\ i, j \ge 1\ ,$$

so that $u + |u|_{l^\infty} r'\epsilon \in K'$ for all $\epsilon \in \bar{B}(0, 1)$, i.e. $u \in \text{int}K'$.

In the sequel of the proof, suppose that $\text{int}K'$ is non-empty. According to (2.6), $u \in K'$ iff $\delta_u \ge 0$ and we have proved that $u \in \text{int}K'$ iff $\delta_u > 0$. So it follows that $u \in l^\infty \setminus K'$ iff $\delta_u < 0$ and that $u \in \partial K'$ iff $\delta_u = 0$.

It remains to prove (6.2). Let d denote the right-hand side of (6.2). Suppose first that $\delta_u > 0$. For all $\delta > 0$ we can choose $i \ne j$ such that $\frac{1}{1+\Lambda^{ij}}\left(u^i\Lambda^{ij} - u^j\right) < d + \delta$. Then,

$\delta_{u+(d+\delta)(-e_i+e_j)} < 0$, so $u + (d+\delta)(-e_i + e_j) \notin K'$. This shows that $d_{l^\infty}(u, \partial K') \le d$. Conversely, for all $\epsilon \in \bar{B}(0,1)$ $\delta_{u+d\epsilon} \ge 0$, so $u + d\epsilon \in K'$. Hence, $d \le d_{l^\infty}(u, \partial K')$ which proves (6.2), when $\delta_u > 0$. Proceeding similarly, we obtain for the case $\delta_u < 0$ that $\delta_{u+d\epsilon} \le 0$ for all $\epsilon \in \bar{B}(0,1)$, and that for all $\delta > 0$ there exists $i \ne j$ such that $\delta_{u+(d+\delta)(e_i-e_j)} > 0$. To conclude the proof we note that (6.2) gives $d_{l^\infty}(u, \partial K') = 0$, when $\delta_u = 0$. □

Proposition 6.1 *Assume that there exists some $c > 0$ such that $\lambda^{ii} = 0$ and $0 \le \lambda^{ij} \le c$ for all $i \ne j \ge 1$. Then, the following assertions*

1. *$\exists\, \varepsilon > 0$ such that $\lambda^{ij} \ge \varepsilon \;\forall\, i \ne j$,*
2. ***1** is an interior point of K',*
3. *K is a normal cone,*
4. *K' has the generating property, i.e. $l^\infty = K' - K'$,*
5. *$\exists\, \varepsilon > 0$ such that $\lambda^{ij} + \lambda^{ji} \ge \varepsilon \;\forall\, i \ne j$,*

satisfy: 1. ⇔ 2. ⇒ 3. ⇔ 4. ⇒ 5.

Proof The equivalence of 1. and 2. is a direct consequence of Lemma 6.3. The equivalence between 3. and 4. is standard, cf. Chap. V, Sect.3.5 of [18].
In the rest of the proof, we shall use the following notations:

$$f_{ij} := \Lambda^{ij} e_i - e_j \;\text{ for } i \ne j \ge 1\;,\; x := \sum_{i \ne j} a^{ij} f_{ij} \quad \text{and} \quad y := \sum_{i \ne j} b^{ij} f_{ij}$$

where $a, b \in \mathbb{M}_{f,+}$ will be given by the context.
We now prove that 1. implies 3. Since $x = \sum_{i \ne j} (\Lambda^{ij} a^{ij} - a^{ji}) e_i$ and $|f_{ij}|_{l^1} = \Lambda^{ij} + 1$, we have

$$\sum_{i \ne j} (\Lambda^{ij} - 1) a^{ij} \le |x|_{l^1} \le \sum_{i \ne j} (\Lambda^{ij} + 1) a^{ij} \le (2 + c) \sum_{i \ne j} a^{ij}.$$

Then, according to the above inequality,

$$\varepsilon \sum_{i \ne j} a^{ij} \le |x|_{l^1} \le (2 + c) \sum_{i \ne j} a^{ij}.$$

Similarly,

$$\varepsilon \sum_{i \ne j} (a^{ij} + b^{ij}) \le |x + y|_{l^1}.$$

Combining the above inequalities leads to

$$|x|_{l^1} \le (2 + c) \sum_{i \ne j} a^{ij} \le (2 + c) \sum_{i \ne j} (a^{ij} + b^{ij}) \le \frac{2 + c}{\varepsilon} |x + y|_{l^1}.$$

It then follows that

$$|x|_{l^1} \le \frac{2+c}{\varepsilon}|x+y|_{l^1},$$

for all $x, y \in K$, which proves that K is normal.

It remains to prove that 3. implies 5. Let us assume that the condition 3. is satisfied. Let x and y be defined as above with $a, b \in \mathbb{M}_{f,+}$ such that $b^{ij} = a^{ji}$ for all $i, j \ge 1$, and set $d^{ij} := a^{ij} + b^{ij} = a^{ij} + a^{ji}$, so that $d^{ij} = d^{ji}$, and $x + y = \sum_{i \ne j} d^{ij}(\Lambda^{ij} - 1)e_i$. Then,

$$|x+y|_{l^1} = \sum_{i \ne j} d^{ij}(\Lambda^{ij} - 1) = \frac{1}{2}\sum_{i \ne j} d^{ij}(\lambda^{ij} + \lambda^{ji}) = \sum_{i \ne j} a^{ij}(\lambda^{ij} + \lambda^{ji}) \; .$$

Since K is normal, there is $k \ge 1$, independent on x and y, such that $|x|_{l^1} \le k|x+y|_{l^1}$, which, combined with the previous inequality, implies

$$|x|_{l^1} \le k \sum_{i \ne j} a^{ij}(\lambda^{ij} + \lambda^{ji}).$$

Considering the case where $x = f_{mn}$ for some $m \ne n$, then leads to $2 + \lambda^{mn} \le k(\lambda^{mn} + \lambda^{nm})$. It follows that $\lambda^{mn} + \lambda^{nm} \ge 2/k$, which, by arbitrariness of (m, n), proves that 5. is satisfied.

□

Remark 6.1 Assertion 5. of Proposition 6.1 does not imply that K is normal (assertion 3), or equivalently that K' has the generating property 4. Since $\mathrm{int}K' \ne \emptyset$ implies that K' has the generating property, this shows that 5. does no imply that $\mathrm{int}K' \ne \emptyset$. An example is given by the case where $\lambda^{ij} = 1$ for $i < j$ and $\lambda^{ij} = 0$ for $i \ge j$.

Indeed, assume that λ satisfies the above condition, let $x \in l^\infty$ be defined by $x = (1, 0, 1, 0, \ldots)$ and suppose that it can be written as $x = y_1 - y_2$, for some $y_1, y_2 \in K'$. First note that the definition of λ implies that

$$0 \le y^j \le y^i \le 2y^j \text{ for } j < i \text{ whenever } y \in K'. \tag{6.4}$$

In view of the left-hand side of (6.4) and the identity $x = y_1 - y_2$, we should then have $y_1^{2n-1} = a^{2n-1} + n$, $y_1^{2n} = a^{2n} + n$, $y_2^{2n-1} = a^{2n-1} + n - 1$ and $y_2^{2n} = a^{2n} + n$ for $n \ge 1$, where $(a^n)_{n \ge 1}$ is an increasing non-negative sequence. On the other hand, the right-hand side of (6.4) implies that $0 \le y^i \le 2y^1$ for $i > 1$. This leads to a contradiction, therefore showing $x \notin K' - K'$, i.e. that the generating property is not satisfied.

References

[1] Bjork T., G. Di Masi, Y. Kabanov and W. Runggaldier. Towards a general theory of bond markets. *Finance and Stochastics*, 1, 141-174, 1997.

[2] Bouchard B. and J.-F. Chassagneux. Representation of continuous linear forms on the set of ladlag processes and the pricing of American claims under proportional costs. *Electronic Journal of Probability*, 14, 612-632, 2009.

[3] Campi L. and W. Schachermayer. A Super-Replication Theorem in Kabanov's Model of Transaction Costs. *Finance and Stochastics*, 10(4), 579-596, 2006.

[4] De Donno M. and M. Pratelli. Stochastic integration with respect to a sequence of semimartingales. *Séminaire de Probabilités XXXIX*. 2005

[5] De Donno M. and M. Pratelli. A theory of stochastic integration for bond markets. *The Annals of Applied Probability*, 15(4), 2773-2791, 2005.

[6] Denis E. and Y. Kabanov: Consistent Price Systems and Arbitrage Opportunities of the Second Kind in Models with Transaction Costs. To appear in *Finance and Stochastics*.

[7] De Vallière D., E. Denis and Y. Kabanov. Hedging of American options under transaction costs. *Finance and Stochastics*, 13(1), 105-119, 2009.

[8] Ekeland I. and E. Taflin. A theory of bond portfoliod. *The Annals of Applied Probability*, 15, 1260-1305, 2005.

[9] Grépat J. and Y. Kabanov. Small transaction costs, absence of arbitrage and consistent price systems. Preprint, 2010.

[10] Guasoni P., M. Rásonyi and W. Schachermayer. The Fundamental Theorem of Asset Pricing for Continuous Processes under Small Transaction Costs. To appear in *Annals of Finance*, 2008.

[11] Jouini E. and H. Kallal. Martingales, arbitrage and equilibrium in securities markets with transaction costs. *Journal of Economic Theory*, 66(1), 178-197, 1995.

[12] Kabanov Y. and M. Safarian. *Markets with Transaction Costs, Mathematical Theory*. Springer Finance, Springer-Verlag, 2009.

[13] Kabanov Y. and C. Stricker. A teachers' note on no-arbitrage criteria. *Séminaire de Probabilités XXXV*, Lect. Notes Math. 1755, Springer, 149-152, 2001.

[14] Kabanov Y. and C. Stricker. The Harrison-Pliska arbitrage pricing theorem under transaction costs. *J. Math. Econ.*, 35(2), 185-196, 2001.

[15] Kabanov Y., C. Stricker and M. Rásonyi. No arbitrage criteria for financial markets with efficient friction. *Finance and Stochastics*, 6(3), 371-382, 2002.

[16] Kabanov Y., C. Stricker and M. Rásonyi. On the closedness of sums of convex cones in L^0 and the robust no-arbitrage property. *Finance and Stochastics*, 7(3), 403-411, 2003.

[17] Rásonyi M. Arbitrage under transaction costs revisited. In *Optimality and Risk - Modern Trends in Mathematical Finance*, ed.: F. Delbaen, M. Rásonyi, Ch. Stricker. Springer, 2009.

[18] Schaefer H.H. *Topological Vector Spaces.* 2nd ed. Springer, 1999.

[19] Schachermayer W. The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time. *Mathematical Finance*, 14(1), 19-48, 2004.

[20] Taflin, E.: *Bond Market Completeness and Attainable Contingent Claims*, Fin. Stoch. **9**, 429–452 (2005). Preprint http://arxiv.org/abs/math.OC/0402364

[21] Taflin E. Generalized bond portfolios: Pitfalls and counter examples. To appear in *Annals of Applied Probability*, 2009.

[22] Yosida K. *Functional Analysis*. Grundlehren der mathematischen Wissenschaften, Band 123, 4th. ed., Spriner-Verlag, 1974.