Lecture notes on BSDEs
Main existence and stability results

Bruno Bouchard
Université Paris-Dauphine, CEREMADE, and CREST, Paris, France
bouchard@ceremade.dauphine.fr

February 2014 (revised May 2015)
Lectures given at the London School of Economics

Contents

Introduction and general notations

Backward stochastic differential equations (BSDEs) are the non-Markovian (stochastic) counterpart of semi-linear parabolic equations. They have a wide range of applications in economics, and more generally in optimal control. In mathematical finance, the standard hedging theory can be written in terms of BSDEs (possibly reflected or with constraints), but they are also naturally associated to risk measures (g-expectations), utility maximization under constraints, or recursive utilities.
These lectures are an introduction to the theory of BSDEs and to their applications. We will concentrate on various existence and stability results, starting from the classical Lipschitz continuous case up to quadratic BSDEs, and BSDEs with constraints.

Our aim is to present the techniques rather than the results by themselves, so that the reader can enter the subject and further study the references we provide. These notes should be read in the given order, some arguments that are used repeatedly will only be explained the first time they appear.

We shall only consider BSDEs driven by a Brownian motion. Most of the results presented here can be extended to BSDEs driven by a Brownian motion and a jump process, or even by a general rcll martingale.

Very good complementary readings are the lectures notes [10, 29] and the book [31].

We collect here some general notations that will be used all over these notes.

We use the notation $\partial_x f$ to denote the derivative of a function f with respect to its argument x. For second order derivatives, we write $\partial^2_{xx} f$ and $\partial^2_{xy} f$.

The Euclydean norm of $x \in \mathbb{R}^d$ is $|x|$, d is given by the context.

We will always work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a d-dimensional Brownian motion W . We let $\mathbb{F} = (\mathcal{F}_t)_{t \le T}$ denote the augmentation of its raw filtration up to a fixed time horizon T. In general, all identities are taken in the $\mathbb{P}-$a.s. or $dt \times d\mathbb{P}$-a.e. sense, this will be clear from the context.
We shall make use of the following spaces (the dimension of the random variables depends on the

context):

- $\mathcal{P}$: progressively measurable processes.
- $\mathbb{L}^p(\mathcal{F}_t)$: $\mathcal{F}_t$-measurable random variables ξ such that $\|\xi\|_{\mathbb{L}^p} := \mathbb{E}\left[|\xi|^p\right]^{\frac{1}{p}} < \infty$. We write $\mathbb{L}^p$ if $t = T$.
- $\mathbb{S}^p$: ζ in $\mathcal{P}$ with continuous paths such that $\|\zeta\|_{\mathbb{S}^p} < \infty$.
- $\mathbb{S}^p_{\text{rcll}}$: ζ in $\mathcal{P}$ with rcll paths such that $\|\zeta\|_{\mathbb{S}^p} := \mathbb{E}\left[\sup_{[0,T]} |\zeta|^p\right]^{\frac{1}{p}} < \infty$.
- $\mathbb{A}^p$: ζ in $\mathcal{P}$ with non-decreasing rcll paths and such that $\zeta_T \in \mathbb{L}^p$ and $\zeta_0 = 0$.
- $\mathbb{H}^p$: ζ in $\mathcal{P}$ such that $\|\zeta\|_{\mathbb{H}^p} := \mathbb{E}\left[\int_0^T |\zeta_s|^p ds\right]^{\frac{1}{p}} < \infty$.
- $\overline{\mathbb{H}}^p$: ζ in $\mathcal{P}$ such that $\|\zeta\|_{\overline{\mathbb{H}}^p} := \mathbb{E}\left[\left[\int_0^T |\zeta_s|^2 ds\right]^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty$.
- $\mathcal{T}_t$: set of stopping times with values in $[t, T]$. We write $\mathcal{T}$ for $\mathcal{T}_0$.
- $\mathbb{H}^2_{\text{BMO}}$: ζ in $\mathcal{P}$ such that $\|\zeta\|_{\mathbb{H}^2_{\text{BMO}}} := \sup_{\tau \in \mathcal{T}} \|\mathbb{E}\left[\int_\tau^T |\zeta_s|^2 ds \; |\mathcal{F}_\tau\right]^{\frac{1}{2}} \|_{\mathbb{L}^\infty} < \infty$.

Given two processes X and X', we shall always use the notation ΔX for $X - X'$. We apply the same for two functions g and g': $\Delta g = g - g'$.

In all this document, C will denote a generic constant which may change from line to line. Although it will not be said explicitly, it will never depend on quantities that may change in the course of the argument (like a parameter that will be send to ∞ for instance).

Proofs will be given for the one dimensional case although the result is stated in a multivariate framework. This is only to avoid heavy notations.

Chapter 1

Introduction and motivations

1.1 What is a BSDE ?

Given an $\mathbb{R}^d$-valued random variable ξ in $\mathbb{L}^2$ and $g : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$, a solution to the BSDE

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ t \leq T,\ \mathbb{P} - \text{a.s.}, \tag{1.1}$$

is a pair of adapted processes (Y, Z), typically in $\mathbb{S}^2 \times \mathbb{H}^2$, with values in $\mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that (1.1) holds.
It means that the process Y has the dynamics

$$dY_s = -g_s(Y_s, Z_s)ds + Z_s dW_s,$$

but, as opposed to forward SDEs, we prescribe its terminal condition $Y_T = \xi$ rather than its initial condition Y_0.
To fix ideas, let us consider the simple case $g \equiv 0$. Then, a couple $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$ such that (1.1) holds must satisfy

$$Y_t = \mathbb{E}\left[\xi \mid \mathcal{F}_t\right]$$

and the Z component of the solution is uniquely given by the martingale representation theorem

$$\xi = \mathbb{E}\left[\xi\right] + \int_0^T Z_s dW_s, \quad \text{i.e. } \mathbb{E}\left[\xi \mid \mathcal{F}_t\right] = \mathbb{E}\left[\xi\right] + \int_0^t Z_s dW_s.$$

From this particular case, we see that an adapted solution to (1.1) can only be given by a pair: the component Z is here to ensure that the process Y is adapted. Unlike deterministic ODEs, we can not simply revert time as the filtration goes in one direction.

In the rest of this Chapter, we provide various examples of applications. Other examples can be found in the lectures notes [29] and in the book [31].

1.2 Application to the hedging of financial derivatives

Let us first discuss some applications to the pricing and hedging of financial derivatives.[1]

1.2.1 European options

We consider here a financial market with one risky asset S whose evolution is given by

$$dS_t = S_t \mu_t dt + S_t \sigma_t dW_t,$$

in which μ and σ are some predictable and bounded processes. A trader can either invest in the risky asset S or borrow/lend money at an instantaneous risk free interest rate r, which is again bounded and predictable for sake of simplicity. If π_t is the amount of money invested in S at t, and Y is the total wealth of the trader, then $Y - \pi$ is the amount of money that is lend/borrowed, and the dynamics of the wealth process is

$$dY_t = \frac{\pi_t}{S_t} dS_t + r_t(Y_t - \pi_t)dt = \{\pi_t(\mu_t - r_t) + r_t Y_t\} \, dt + \pi_t \sigma_t dW_t.$$

Let us now consider a European option with payoff at time T given by a random variable $\xi \in \mathbb{L}^2$. The aim of a trader who wants to sell this option is to define the minima initial amount of capital Y_0 such that he can cover the payoff ξ. Obviously, if we can find a π such that $Y_T = \xi$, then this minimal amount is Y_0. Otherwise stated we look for a couple (Y, π) such that

$$Y_t = \xi - \int_t^T \{\pi_s(\mu_s - r_s) + r_s Y_s\} \, ds - \int_t^T \pi_s \sigma_s dW_s.$$

If there exists a predictable process λ such that $(\mu - r) = \sigma\lambda$, which is called a risk premium in mathematical finance, then the above reads

$$Y_t = \xi - \int_t^T \{Z_s \lambda_s + r_s Y_s\} \, ds - \int_t^T Z_s dW_s, \tag{1.2}$$

after setting $Z := \pi\sigma$.

Hence, the problem of hedging the option is reduced to finding a solution to a BSDE.

In the above, the solution is explicitly given by $Y_t = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} \xi \,|\mathcal{F}_t]$ in which $\mathbb{Q}$ is the equivalent probability measure such that $W + \int_0^\cdot \lambda_s ds$ is a Brownian motion, the so-called risk neutral measure.

However, the solution is no more explicit if the interest rates for borrowing and lending are different. Let us denote them by r^b and r^l respectively. Then, the dynamics of the wealth is given by

$$Y_t = \xi - \int_t^T \left\{\pi_s \mu_s + r_s^l (Y_s - \pi_s)^+ - r_s^b (Y_s - \pi_s)^-\right\} ds - \int_t^T \pi_s \sigma_s dW_s.$$

[1] This section can be completed by the reading of El Karoui et al. [32, 30].

Assuming that $\sigma > 0$, the corresponding BSDE is

$$Y_t = \xi - \int_t^T \left\{ Z_s \frac{\mu_s}{\sigma_s} + \frac{r_s^l}{\sigma_s}(\sigma_s Y_s - Z_s)^+ - \frac{r_s^b}{\sigma_s}(\sigma_s Y_s - Z_s)^- \right\} ds - \int_t^T Z_s dW_s.$$

1.2.2 Hedging with constraints

Let us now consider the case where the trader wants to confine himself to strategies π satisfying certain bounds: $\pi \in [-m, M]$ $dt \times d\mathbb{P}$-a.e. for given limits $m, M > 0$. Then, he needs to find (Y, Z) satisfying (1.2) and $Z/\sigma \in [-m, M]$ $dt \times d\mathbb{P}$-a.e. In general, this problem does not have a solution and one needs to relax (1.2) into

$$dY_t \leq \{\pi_t(\mu_t - r_t) + r_t Y_t\} dt + \pi_t \sigma_t dW_t \quad \text{with} \quad Y_T = \xi, \tag{1.3}$$

which we write as

$$Y_t = \xi - \int_t^T \{Z_s \lambda_s + r_s Y_s\} ds - \int_t^T Z_s dW_s + A_T - A_t, \tag{1.4}$$

in which A is an adapted non-decreasing process. The A process can be viewed as a consumption process. To ensure to keep $\pi = Z\sigma$ within the prescribed bounds, one needs to start with a higher initial wealth, which might indeed not be used and can therefore be consumed.

Hence, the solution is now a triplet (Y, Z, A). Obviously, uniqueness does not hold in general, as we can always start with an higher Y_0 and compensate with the A process. However, we are interested here in the characterization of the minimal solution, in the sense that Y is minimal among all possible solutions, since the trader wants to minimize the initial capital required for the hedging.

This problem has been widely studied in the mathematical finance literature, and we refer to [5, 14, 20, 22] for an analysis in the context of BSDEs. The corresponding BSDE is usually referred as a BSDE with constraint on the gain process.

1.2.3 American options

An American payoff can be viewed as an adapted process ζ: the amount ζ_t is paid to the holder if he exercises his option at t before the maturity T.
Then, we want to find (Y, Z) solving (1.3) such that $Y \geq \zeta$ on $[0, T]$ $\mathbb{P}-$a.s. Again, we can not expect to have an equality in (1.3) if we look for a minimal solution, which in particular should satisfy $Y_T = \zeta_T$. Then, a solution is again given by a triplet (Y, Z, A), with A adapted and non-decreasing, such that (1.4) holds and $Y \geq \zeta$ on $[0, T]$. This is called a reflected BSDE, as the Y process should be reflected on the lower barrier ζ so has to stay above it at any time. This class of BSDEs has been first introduced in the context of mathematical finance by El Karoui et al. [30].

1.2.4 Hedging according to a loss function

We now consider the problem of finding the minimal initial wealth Y_0 such that there exists π for which

$$\mathbb{E}\left[\ell(Y_T - \xi)\right] \geq m.$$

In the above, m is a threshold, and ℓ is a loss function: a non-decreasing and concave function, which typically strongly penalizes the loss $(Y_T - \xi)^-$. This pricing criteria has be widely studied by Föllmer and Leukert [33, 34]. This leads to a class of BSDEs in which the terminal condition Y_T is no more fixed, but has only to satisfy a certain moment condition. Their properties have been studied by Bouchard, Elie and Réveillac [6].

1.3 Optimal control : the stochastic maximum principle

Let us now turn to an application in optimal control.

We consider here the problem of maximizing an expected gain of the form

$$J(\nu) := \mathbb{E}\left[g(X_T^\nu) + \int_0^T f_t(X_s^\nu, \nu_t)dt\right],$$

in which X^ν is the solution of the one dimensional sde

$$dX_t^\nu = b_t(X_t^\nu, \nu_t)dt + \sigma_t(X_t^\nu, \nu_t)dW_t$$

with ν in the set $\mathcal{U}$ of predictable processes with values in $\mathbb{R}$.
In the above, the random maps f, b and σ are such that $(t, \omega) \mapsto (f_t(\omega, x, u), b_t(\omega, x, u), \sigma_t(\omega, x, u))$ is predictable for any $(x, u) \in \mathbb{R}^2$ (we omit the ω argument in the following). We also assume that they are $dt \times d\mathbb{P}$-a.e. bounded, C^1 in their argument (x, u), and that themselves as well as there first derivatives are Lipschitz. The function g maps $\Omega \times \mathbb{R} \to \mathbb{R}$, $g(0) \in \mathbb{L}^\infty$, and g is a.s. C^1 with bounded first derivative in x.

In the following, we shall show how BSDEs permits to provide necessary and sufficient conditions for optimality. We refer to Peng [50, 51] for further references.

1.3.1 Necessary condition

Let us start with a necessary condition for a control $\hat{\nu}$ to be optimal. The general idea is to used a spike variation of the form $\nu^{\varepsilon,\tau} := \hat{\nu}\mathbf{1}_{[0,\tau)\cup[\tau+\varepsilon,T]} + \nu\mathbf{1}_{[\tau,\tau+\varepsilon)}$ with $\varepsilon \in (0, T-\tau)$ and ν a $\mathcal{F}_\tau$-measurable random variable, $\tau \in \mathcal{T}$.
By optimality of $\hat{\nu}$, we must have

$$J(\hat{\nu}) \geq J(\nu^{\varepsilon,\tau}),$$

and therefore, if $\varepsilon \mapsto J(\nu^{\varepsilon,\tau})$ is smooth,

$$\partial_\varepsilon J(\nu^{\varepsilon,\tau})_{|\varepsilon=0} \leq 0 \,. \tag{1.5}$$

The first problem is therefore to show that this map is smooth. From now on, we write $\hat{X}$ for $X^{\hat{\nu}}$ and $X^{\nu^{\tau,\varepsilon}}$ for $X^{\nu^{\tau,\varepsilon}}$, and we assume that σ does not depend on ν for sake of simplicity, see [51] for the general case.

Under this additional condition, we can first show that $X^{\nu^{\tau,\varepsilon}}$ is smooth with respect to ε.

Proposition 1.1 *Let us consider the process $\hat{Y}^{\tau,\nu}$ defined as the solution of*

$$\begin{aligned} Y_t &= \mathbf{1}_{t\geq\tau}\left(b_\tau(\hat{X}_\tau,\nu) - b_\tau(\hat{X}_\tau,\hat{\nu}_\tau)\right) \\ &+ \int_\tau^t \partial_x b_s\left(\hat{X}_s,\hat{\nu}_s\right) Y_s ds + \int_\tau^t \partial_x\sigma_s\left(\hat{X}_s\right) Y_s dW_s \,. \end{aligned} \tag{1.6}$$

Assume that $\hat{\nu}$ has $\mathbb{P}-a.s.$ right-continuous paths. Then, $\hat{Y}^{\nu,\tau} = \frac{\partial}{\partial\varepsilon} X^{\nu^{\tau,\varepsilon}}{}_{|\varepsilon=0}$ on $[0,T]$ $\mathbb{P}-a.s.$ Moreover,

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} J(\nu^{\varepsilon,\tau})_{|\varepsilon=0} &= \mathbb{E}\left[\partial_x g(\hat{X}_T)\hat{Y}_T^{\nu,\tau} + \int_\tau^T \partial_x f_s(\hat{X}_s,\hat{\nu}_s)\hat{Y}_s^{\nu,\tau} ds\right] \\ &+ \mathbb{E}\left[f_\tau(\hat{X}_\tau,\nu) - f_\tau(\hat{X}_\tau,\hat{\nu}_\tau)\right] \,. \end{aligned} \tag{1.7}$$

The idea of the stochastic maximum principle is to introduce a set of dual variables in order to exploit (1.7). Let us first define the Hamiltonian:

$$\mathcal{H}_t(x,u,p,q) := b_t(x,u)p + \sigma_t(x)q + f_t(x,u).$$

Then, we assume that there exists a couple $(\hat{P},\hat{Q})$ of square integrable adapted processes satisfying the BSDE

$$\hat{P}_t = \partial_x g(\hat{X}_T) + \int_t^T \partial_x \mathcal{H}_s(\hat{X}_s,\hat{\nu}_s,\hat{P}_s,\hat{Q}_s)ds - \int_t^T \hat{Q}_s dW_s \,. \tag{1.8}$$

This equation is called the adjoint equation and $(\hat{P},\hat{Q})$ the adjoint process.

The reason for introducing this process becomes clear once we apply Itô's Lemma to $\hat{P}\hat{Y}^{\tau,\nu}$. Indeed, assuming that the local martingale part of $\hat{P}\hat{Y}^{\tau,\nu}$ is a true martingale, we obtain that $\partial_x g(\hat{X}_T)\hat{Y}_T^{\tau,\nu} = \hat{P}_T\hat{Y}_T^{\tau,\nu}$ is equal in expectation to

$$\begin{aligned} &\hat{P}_\tau(b_\tau(\hat{X}_\tau,\nu) - b_\tau(\hat{X}_\tau,\hat{\nu}_\tau)) - \int_\tau^T \hat{Y}_s^{\tau,\nu}\partial_x\mathcal{H}_s(\hat{X}_s,\hat{\nu}_s,\hat{P}_s,\hat{Q}_s)ds \\ &+ \int_\tau^T \partial_x b_s\left(\hat{X}_s,\hat{\nu}_s\right)\hat{Y}_s^{\tau,\nu}\hat{P}_s ds + \int_\tau^t \partial_x\sigma_s\left(\hat{X}_s\right)\hat{Y}_s^{\tau,\nu}\hat{Q}_s ds, \end{aligned}$$

which, by definition of $\mathcal{H}$, is equal to

$$\hat{P}_\tau(b_\tau(\hat{X}_\tau,\nu_\tau) - b_\tau(\hat{X}_\tau,\hat{\nu})) - \int_\tau^T Y_s^{\tau,\nu}\partial_x f_s\left(\hat{X}_s,\hat{\nu}_s\right) ds \ .$$

It follows that

$$\partial_\varepsilon J(\nu^{\varepsilon,\tau})_{|\varepsilon=0} \quad = \quad \mathbb{E}\left[\mathcal{H}_\tau(\hat{X}_\tau,\nu_\tau,\hat{P}_\tau,\hat{Q}_\tau) - \mathcal{H}_\tau(\hat{X}_\tau,\hat{\nu},\hat{P}_\tau,\hat{Q}_\tau)\right] \ .$$

By arbitrariness of ν, this implies the *necessary condition*

$$\mathcal{H}_\tau(\hat{X}_\tau,\hat{\nu}_\tau,\hat{P}_\tau,\hat{Q}_\tau) = \max_{u\in\mathbb{R}} \mathcal{H}_\tau(\hat{X}_\tau,u,\hat{P}_\tau,\hat{Q}_\tau) \quad \mathbb{P}-\text{a.s.} \tag{1.9}$$

for all $\tau \in \mathcal{T}$.

A similar analysis can be carried out when σ does depend on the control ν but it requires a second order expansion in the definition of Y above. See Peng [50, 51].

1.3.2 Sufficient condition

We work within the same framework as above, except that we now allow σ to depend on the control process ν.

We assume here that the maps

$$x \mapsto g(x) \ \text{ and } \ x \mapsto \hat{\mathcal{H}}_t(x,\hat{P}_t,\hat{Q}_t) := \sup_{u\in\mathbb{R}} \mathcal{H}_t(x,u,\hat{P}_t,\hat{Q}_t) \quad \text{are } \mathbb{P}-\text{a.s. concave} \tag{1.10}$$

for almost every $t \in [0,T]$, and that

$$\partial_x \mathcal{H}_\tau(\hat{X}_\tau,\hat{\nu}_\tau,\hat{P}_\tau,\hat{Q}_\tau) = \partial_x \hat{\mathcal{H}}_\tau(\hat{X}_\tau,\hat{P}_\tau,\hat{Q}_\tau) \tag{1.11}$$

for all stopping times τ. Note that the latter corresponds to the envelop principle along the path of $(\hat{X},\hat{P},\hat{Q})$.

Under the above assumptions, the condition

$$\mathcal{H}_\tau(\hat{X}_\tau,\hat{\nu}_\tau,\hat{P}_\tau,\hat{Q}_\tau) = \max_{u\in\mathbb{R}} \mathcal{H}_\tau(\hat{X}_\tau,u,\hat{P}_\tau,\hat{Q}_\tau) \ \ \forall\, \tau \in [0,T] \tag{1.12}$$

is actually a sufficient condition for optimality.

Indeed, we first note that, by concavity of g,

$$\mathbb{E}\left[g(\hat{X}_T) - g(X_T^\nu)\right] \quad \geq \quad \mathbb{E}\left[\partial_x g(\hat{X}_T)(\hat{X}_T - X_T^\nu)\right] = \mathbb{E}\left[\hat{P}_T(\hat{X}_T - X_T^\nu)\right],$$

which, by Itô's Lemma and (1.11), implies

$$\begin{aligned}
\mathbb{E}\left[g(\hat{X}_T) - g(X_T^\nu)\right] &\geq \mathbb{E}\left[\int_0^T \hat{P}_s(b_s(\hat{X}_s, \hat{\nu}_s) - b_s(X_s^\nu, \nu_s))ds\right] \\
&- \mathbb{E}\left[\int_0^T \partial_x \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s)(\hat{X}_s - X_s^\nu)ds\right] \\
&+ \mathbb{E}\left[\int_0^T \left(\sigma_s(\hat{X}_s) - \sigma_s(X_s^\nu)\right)\hat{Q}_s ds\right].
\end{aligned}$$

By definition of $\mathcal{H}$, $\hat{\mathcal{H}}$ and (1.10)-(1.12), this leads to

$$\begin{aligned}
J(\hat{\nu}) - J(\nu) &\geq \mathbb{E}\left[\int_0^T (\mathcal{H}_s(\hat{X}_s, \hat{\nu}_s, \hat{P}_s, \hat{Q}_s) - \mathcal{H}_s(X_s^\nu, \nu_s, \hat{P}_s, \hat{Q}_s))ds\right] \\
&- \mathbb{E}\left[\int_0^T \partial_x \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s)(\hat{X}_s - X_s^\nu)ds\right] \\
&\geq \mathbb{E}\left[\int_0^T \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s) - \hat{\mathcal{H}}_s(X_s^\nu, \hat{P}_s, \hat{Q}_s)ds\right] \\
&- \mathbb{E}\left[\int_0^T \partial_x \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s)(\hat{X}_s - X_s^\nu)ds\right] \\
&\geq 0.
\end{aligned}$$

Remark 1.1 *Let us now assume that μ, σ and f are non-random and assume that there exists a smooth solution φ to the Hamilton-Jacobi-Bellman equation:*

$$0 = \sup_{u \in \mathbb{R}} \left(\frac{\partial}{\partial t}\varphi(t,x) + b_t(x,u)\partial_x \varphi(t,x) + \frac{1}{2}(\sigma_t(x,u))^2 \partial^2_{xx}\varphi(t,x) + f_t(x,u) \right)$$

with terminal condition $\varphi(T, \cdot) = g$. Assume that the sup is attained by some $\hat{u}(t,x)$. Set $p := \partial_x \varphi$ and $q := \partial^2_{xx}\varphi\sigma$. It follows from the envelop theorem, that (p,q) formally solves (take the derivative with respect to x in the above equation)

$$0 = \mathcal{L}^{\hat{u}(t,x)} p(t,x) + \partial_x \hat{\mathcal{H}}_t(x, p(t,x), q(t,x,\hat{u}(t,x)))$$

with the terminal condition $p(T, \cdot) = \partial_x g$. Let now $\hat{X}$ be the controlled process associated to the Markov control $\hat{\nu} = \hat{u}(\cdot, \hat{X}_\cdot)$ (assuming that it is well defined). Then, Itô's Lemma implies that

$$\begin{aligned}
p(t, \hat{X}_t) &= \partial_x g(\hat{X}_T) + \int_t^T \partial_x \mathcal{H}_s(\hat{X}_s, \hat{\nu}_s, p(s, \hat{X}_s), q(s, \hat{X}_s, \hat{\nu}_s))ds \\
&- \int_t^T q(s, \hat{X}_s, \hat{\nu}_s) dW_s.
\end{aligned}$$

Under mild assumptions ensuring that there is only one solution to the above BSDE, this shows that

$$\hat{P}_t = p(t, \hat{X}_t) = \partial_x \varphi(t, \hat{X}_t) \ \text{ and } \ \hat{Q}_t = q(t, \hat{X}_t, \hat{\nu}_t) = \partial^2_{xx}\varphi(t, \hat{X}_t)\sigma_t(\hat{X}_t, \hat{\nu}_t).$$

Otherwise stated, the adjoint process $\hat{P}$ can be seen as the derivative of the value function with respect to the initial condition in space, while $\hat{Q}$ is intimately related to the second derivative.

1.3.3 Examples

Example 1.1 *Let us first consider the problem*

$$\max \mathbb{E}\left[\ln(X_T^\nu)\right]$$

where X^ν is defined as

$$X_t^\nu \;=\; x_0 + \int_0^t X_s^\nu \nu_s \frac{dS_s}{S_s} = x_0 + \int_0^t X_s^\nu \nu_s \mu_s ds + \int_0^t X_s^\nu \nu_s \sigma_s dW_s \tag{1.13}$$

for some $x_0 > 0$ and where

$$S_t = S_0 e^{\int_0^t (\mu_s - \sigma_s^2/2)ds + \int_0^t \sigma_s dW_s}$$

for some bounded predictable processes μ and $\sigma > 0$ with $1/\sigma$ bounded as well.

This corresponds to the problem of maximizing the expected logarithmic utility of the discounted terminal wealth in a one dimensional Black-Scholes type model with random coefficients. Here, ν stands for the proportion of the wealth X^ν which is invested in the risky asset S.

It is equivalent to maximizing $\mathbb{E}\left[X_T^\nu\right]$ with X^ν now defined as

$$X_t^\nu = \int_0^t (\nu_s \mu_s - \nu_s^2 \sigma_s^2/2) ds \quad .$$

The associated Hamiltonian is

$$\mathcal{H}_t(x, u, p, q) = (u\mu_t - (u^2\sigma_t^2/2))p \;.$$

Thus $\hat{\mathcal{H}}_t(x, p, q) = \frac{1}{2}\frac{\mu_t^2}{\sigma_t^2}p$ and the argmax is $\hat{u}(t, x, p, q) := \frac{\mu_t}{\sigma_t^2}$. It follows that the dynamics of the adjoint process $(\hat{P}, \hat{Q})$ is given by

$$\hat{P}_t = 1 - \int_t^T \hat{Q}_s dW_s \;.$$

This implies that $\hat{P} = 1$ and $\hat{Q} = 0$ $dt \times d\mathbb{P}$ a.e. In particular, for $\hat{X} := X^{\hat{\nu}}$ with $\hat{\nu} := \mu/\sigma^2$ the optimality conditions of the previous section are satisfied. This implies that $\hat{\nu}$ is an optimal strategy. Since the optimization problem is clearly strictly concave in ν, this is the only optimal strategy. Observe that the solution is trivial since it only coincides with taking the max inside the expectation and the integral in $\mathbb{E}\left[X_T^\nu\right] = \mathbb{E}\left[\int_0^T (\nu_s\mu_s - \nu_s^2\sigma_s^2/2)ds\right]$.

Example 1.2 *We consider a similar problem as in the previous section except that we now take a general utility function U which is assumed to be C^1, strictly concave and increasing. We also assume that it satisfies the so-called* Inada conditions*: $\partial_x U(\infty) = 0$ and $\partial_x U(0+) = \infty$.*

We want to maximize $\mathbb{E}\left[U(X_T^\nu)\right]$ *where* X^ν *is given by* (1.13). *We write* $\hat{X}$ *for* $X^{\hat{\nu}}$.

In this case, the condition (1.12) *reads*

$$\mathcal{H}_t(\hat{X}_t, \hat{\nu}_t, \hat{P}_t, \hat{Q}_t) \;=\; \sup_{u\in\mathbb{R}} \left(u\;\mu_t \hat{X}_t \hat{P}_t + u\;\sigma_t \hat{X}_t \hat{Q}_t\right)\;.$$

But, it is clear that it can be satisfied only if

$$\hat{Q}_t = -\lambda_t \hat{P}_t \quad \textit{with}\;\; \lambda = \mu/\sigma\;.$$

Thus, by (1.8), $\hat{P}$ *should have the dynamics*

$$\hat{P}_t = \partial_x U(\hat{X}_T) + \int_t^T \lambda_s \hat{P}_s dW_s\;.$$

This implies that we have to find a real $\hat{P}_0 > 0$ *such that*

$$\hat{P}_t = \hat{P}_0 e^{-\frac{1}{2}\int_0^t \lambda_s^2 ds - \int_0^t \lambda_s dW_s}$$

and $\hat{P}_T = \partial_x U(\hat{X}_T)$. *Hence, the optimal control, if it exists, should satisfy*

$$\hat{X}_T \;=\; (\partial_x U)^{-1}\left(\hat{P}_0 e^{-\frac{1}{2}\int_0^T \lambda_s^2 ds + \int_0^T \lambda_s dW_s}\right)\;. \tag{1.14}$$

Now, let $\mathbb{Q}\sim\mathbb{P}$ *be defined by* $d\mathbb{Q} = \hat{P}_T/\hat{P}_0$ *so that* $W^{\mathbb{Q}} = W + \int_0^\cdot \lambda_s ds$ *is a* $\mathbb{Q}$*-Brownian motion, and that* X^ν *is a supermatingale under* $\mathbb{Q}$ *for all* $\nu\in\mathcal{U}$. *If* $\hat{X}$ *is actually a true* $\mathbb{Q}$*-martingale, then we must have*

$$x_0 \;=\; \mathbb{E}^{\mathbb{Q}}\left[(\partial_x U)^{-1}\left(\hat{P}_0 e^{-\frac{1}{2}\int_0^T \lambda_s^2 ds + \int_0^T \lambda_s dW_s}\right)\right]\;. \tag{1.15}$$

Using the Inada conditions imposed above, it is clear that we can find $\hat{P}_0$ *such that the above identity holds. The representation theorem then implies the existence of an admissible control* $\hat{\nu}$ *such that* (1.14) *is satisfied. Since the sufficient conditions of Section 1.3.2 hold, this shows that* $\hat{\nu}$ *is optimal.*

We can also check this by using the concavity of U *which implies*

$$U(X_T^\nu) \;\le\; U(\hat{X}_T) + \partial_x U(\hat{X}_T)\,(X_T^\nu - X_T) = U(\hat{X}_T) + \hat{P}_T\left(X_T^\nu - \hat{X}_T\right)\;.$$

Since, by the above discussion, the last term is non positive in expectation, this shows that the optimal terminal wealth is actually given by (1.14).

1.4 Exponential utility maximization with constraints

We now consider a similar utility maximization problem, but we add constraint on the financial strategy. We restrict to an exponential utility function. Then, the following has been first discussed

in El Karoui and Rouge [55]. See also Hu et al. [38] for more details and for the case of power utility functions, or [37] for general utility functions.
Let $\mathcal{U}$ be predictable processes with values in a compact (for simplicity) subset $A \subset \mathbb{R}$. Given some predictable bounded processes ν and σ, we describe here the wealth associated to a trading strategy $\nu \in \mathcal{U}$ by the dynamics

$$dV_t^\nu = \nu_t(\mu_t dt + \sigma_t dW_s),\ V_0^\nu = 0.$$

We want to compute

$$u_0 := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[U(V_T^\nu)\right] \quad \text{with } U(v) := -e^{-\eta v},\ \ \eta > 0.$$

We use the following approach: find a process Y such that $L^\nu := U(V^\nu - Y)$ satisfies

- L^ν is a super-martingale for all $\nu \in \mathcal{U}$.
- $L^{\hat\nu}$ is a -martingale for one $\hat\nu \in \mathcal{U}$.
- $L_T^\nu = U(V_T^\nu)$ for all $\nu \in \mathcal{U}$.
- L_0^ν does not depend on $\nu \in \mathcal{U}$, we call it L_0.

If such a process exists then

$$\mathbb{E}\left[U(V_T^{\hat\nu})\right] = \mathbb{E}\left[L_T^{\hat\nu}\right] = L_0 \geq \mathbb{E}\left[L_T^\nu\right] = \mathbb{E}\left[U(V_T^\nu)\right],$$

so that $\hat\nu$ is optimal and $u_0 = L_0$.
Let us take Y of the form (1.1). Then,

$$dL_t^\nu \quad = \quad -\eta L_t^\nu\left(\nu_t\mu_t + g_t(Y_t, Z_t) - \frac{\eta}{2}(\nu_t - Z_t\sigma_t)^2\right)dt - \eta(\nu_t\sigma_t - Z_t)L_t^\nu dW_t.$$

Thus, we must have

$$g_t(Y_t, Z_t) = g(Z_t) = \min_{a\in A}\left(\frac{\eta}{2}(a\sigma_t - Z_t)^2 - a\mu\right) = \min_{a\in A}\frac{\eta}{2}\left(\left|a\sigma_t - (Z + \frac{\mu_t}{\eta\sigma_t})\right|^2 - 2Z_t\frac{\mu_t}{\eta\sigma_t} - \left|\frac{\mu_t}{\eta\sigma_t}\right|^2\right).$$

This provides BSDE with a driver which is quadratic in Z. If a solution exists with Z such that $L^{\hat\nu}$ is a true martingale for $\hat\nu \in \mathcal{U}$ defined by

$$g(Z) = \left(\frac{\eta}{2}(\hat\nu\sigma_t - Z_t)^2 - \hat\nu\mu\right),$$

then, $\hat\nu$ is actually the optimal trading strategy. We shall see later, see Theorem 2.5 below, that existence holds and that the corresponding Z belongs to $\mathbb{H}^2_{\mathrm{BMO}}$, which ensures that $L^{\hat\nu}$ is indeed a true martingale, see Kazamaki [39].

Remark 1.2 *For $U(x) = x^\gamma$, we take ν as the proportion of wealth and*

$$L^\nu := e^{\int_0^\cdot \gamma\nu_s dW_s - \frac{1}{2}\int_0^\cdot \gamma|\nu_s|^2 ds} e^{\int_0^\cdot \gamma\nu_s\frac{\mu_s}{\sigma_s}ds + Y}.$$

We obtain by the same arguments as above that the value is $x^\gamma e^{Y_0}$.

1.5 Risk measures representation

Backward stochastic differential equation can also be used to construct risk measures. We briefly discuss this here and refer to Peng [53] for a complete treatment.

Let us first introduce the notion of $\mathcal{F}$-expectation defined by Peng, which is intimately related to the notion of risk measures.

Definition 1.1 *A non-linear $\mathcal{F}$-expectation is an operator $\mathcal{E} : \mathbb{L}^2 \mapsto \mathbb{R}$ such that*

- $X' \geq X$ *implies* $\mathcal{E}[X'] \geq \mathcal{E}[X']$ *with equality if and only if* $X' = X$.
- $\mathcal{E}[c] = c$ *for* $c \in \mathbb{R}$.
- *For each* $X \in \mathbb{L}^2$ *and* $t \leq T$, *there exists* $\eta_t^X \in \mathbb{L}^2(\mathcal{F}_t)$ *such that* $\mathcal{E}[X\mathbf{1}_A] = \mathcal{E}[\eta_t^X \mathbf{1}_A]$ *for all* $A \in \mathcal{F}_t$. *We write* $\mathcal{E}_t[X]$ *for* η_t^X.

Remark 1.3 η_t^X *is uniquely defined. Indeed, if* $\tilde{\eta}_t$ *satisfies the same, we can take* $A = \mathbf{1}_{\eta_t^X > \tilde{\eta}}$ *and deduce from the first item in the above definition that* $\mathcal{E}[\eta_t^X \mathbf{1}_A] > \mathcal{E}[\eta_t \mathbf{1}_A]$ *if* $\mathbb{P}[A] > 0$. *Note that it corresponds to the notion of conditional expectation, in this non-linear framework.*

Let us now consider the solution (Y, Z) of

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s)ds - \int_t^T Z_s dW_s, \ t \leq T,$$

and call the Y component $\mathcal{E}_t^g[\xi]$. We omit t when $t = 0$. We set $g_\mu(y, z) = \mu|z|$.

The following remarkable result shows that not only BSDEs provides non-linear expectations, but that a large class of them (typically the one used for cash invariant risk measures) are actually given by a BSDE.

Theorem 1.1 *Let* $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *be such that* $g(x, y) \in \mathbb{H}^2$ *for all* (x, y), *and* g *is uniformly Lipschitz in* (y, z) $dt \times d\mathbb{P}$*-a.e., then* $\mathcal{E}^g$ *is a non-linear expectation. Conversely, let* $\mathcal{E}$ *be one non-linear* $\mathcal{F}$*-expectation such that for all* $X, X' \in \mathbb{L}^2$

$$\mathcal{E}[X + X'] \leq \mathcal{E}[X] + \mathcal{E}^{g_\mu}[X']$$

and

$$\mathcal{E}_t[X + X'] = \mathcal{E}_t[X] + X' \text{ if } X' \in \mathbb{L}^2(\mathcal{F}_t).$$

Then, there exists a random driver g *which does not depend on* y *such that* $|g(z)| \leq \mu|z|$ *and* $\mathcal{E} = \mathcal{E}^g$.

1.6 Feynman-Kac representation of semi-linear parabolic equations and numerical resolution

Let us conclude with the link between BSDEs and semi-linear partial differential equations.

Consider the solution X of the sde

$$X = x + \int_0^{\cdot} b_s(X_s)ds + \int_0^{\cdot} \sigma_s(X_s)dW_s,$$

in which b and σ are determinist maps that are assumed to be Lipschitz in their space variable. Assume that there exists a solution $v \in C^{1,2}([0,T) \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ to the PDE

$$0 = \mathcal{L}\varphi + g(\cdot, \varphi, \partial_x\varphi\sigma) \text{ on } [0,T) \times \mathbb{R}, \text{ with } v(T,\cdot) = G$$

in which

$$\mathcal{L}\varphi = \partial_t\varphi + b\partial_x\varphi + \frac{1}{2}\sigma^2\partial^2_{xx}\varphi.$$

Then, the couple

$$Y := v(\cdot, X) \ , \ Z := \partial_x v(\cdot, X)\sigma(X)$$

solves

$$Y = G(X_T) + \int_{\cdot}^T g_s(X_s, Y_s, Z_s)ds - \int_{\cdot}^T Z_s dW_s.$$

Indeed, by Itô's Lemma,

$$\begin{aligned} G(X_T) &= v(t, X_t) + \int_t^T \mathcal{L}v(s, X_s)ds + \int_t^T \partial_x v(s, X_s)\sigma_s(X_s)dW_s \\ &= v(t, X_t) - \int_t^T g_s(X_s, v(s, X_s), \partial_x v(s, X_s)\sigma_s(X_s))ds + \int_t^T \partial_x v(s, X_s)\sigma_s(X_s)dW_s. \end{aligned}$$

In particular, if the above BSDE has at most one solution, then solving the BSDE or the PDE is equivalent.

This provides an alternative to the resolution of PDEs by considering backward schemes of the form

$$\begin{aligned} Y^n_{t^n_i} &:= \mathbb{E}\left[Y^n_{t^n_{i+1}} + \frac{T}{n} g_{t^n_i}(X^n_{t^n_i}, Y^n_{t^n_{i+1}}, Z^n_{t^n_i}) \ |\mathcal{F}_{t^n_i}\right], \\ Z^n_{t^n_i} &:= \frac{n}{T}\mathbb{E}_{t^n_i}\left[Y^n_{t^n_{i+1}}(W_{t^n_{i+1}} - W_{t^n_i}) \ |\mathcal{F}_{t^n_i}\right], \end{aligned}$$

in which $Y^n_T = g(X^n_T)$ and X^n is the Euler scheme of X with time step T/n, $t^i_n = iT/n$. When the coefficients are 1/2-Hölder in time and Lipschitz in the other components, this scheme converges at a speed $n^{-\frac{1}{2}}$, see Bouchard and Touzi [8] and Zhang [58]. Obviously this scheme is only theoretic as it requires the computation of conditional expectations. Still, one can use various Monte-Carlo type approaches to turn it into a real numerical scheme, see the references in the survey paper Bouchard and Warin [9] and in the book Gobet [35].

Remark 1.4 *a. A similar representation holds for elliptic equations. In this case, we have to replace T in the BSDE by a random time τ, typically the first exist time of X from a domain, see e.g. the survey paper Pardoux [47].*
b. By considering BSDEs with jumps, one can also provide a representation of systems of parabolic equations. The original idea is due to Pardoux, Pradeilles and Rao [49] and was further discussed in Sow and Pardoux [56]. The corresponding numerical scheme has been studied by Bouchard and Elie [4].

Chapter 2

General existence and comparison results

The aim of this Chapter is to provide various existence and stability results for BSDEs of the form (1.1).

From now on, a driver g will always be a random map $\Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}^d$ such that $(g_t(y,z))_{t\leq T} \in \mathcal{P}$ for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$.

2.1 The case of a Lipschitz driver

We first consider the standard case of a Lipschitz continuous driver.

Assumption 2.1 $g(0) \in \mathbb{H}^2$ *and* g *is uniformly Lipschitz in* (y,z).

The following results are due to Pardoux and Peng [48]. See also [32] for more properties such as differentiability in the Malliavin sense and for application in optimal control and in finance.

We first provide an easy estimate that will be used later on.

Proposition 2.1 *Let Assumption 2.1 hold. Fix* $\xi \in \mathbb{L}^2$. *If* (Y,Z) *satisfies* (1.1) *(assuming it exists) and* $(Y,Z) \in \mathbb{H}^2 \times \mathbb{H}^2$, *then* $Y \in \mathbb{S}^2$.

Proof. We use (1.1), the fact that g has linear growth in (y,z) and the Burkholder-Davis-Gundy inequality to obtain

$$\|Y\|_{\mathbb{S}^2} \leq C\mathbb{E}\left[|\zeta|^2 + \int_0^T [|Y_s|^2 + |Z_s|^2 + |g_s|^2(0)]ds\right].$$

□

Since the construction of a solution will be based on a contraction argument, we also need some a-priori estimates on the stability of solutions with respect to their drivers and terminal conditions. In particular, the following ensures that a BSDE can only have at most one solution.

Proposition 2.2 (Stability) *Let Assumption 2.1 for g and g' holds. Fix ξ and $\xi' \in \mathbb{L}^2$. Let (Y, Z) and (Y', Z') be associated solutions (assuming they exist) in $\mathbb{S}^2 \times \mathbb{H}^2$. Then,*

$$\|\Delta Y\|_{\mathbb{S}^2}^2 + \|\Delta Z\|_{\mathbb{H}^2}^2 \leq C \left(\|\Delta\zeta\|_{\mathbb{L}^2}^2 + \|\Delta g(Y, Z)\|_{\mathbb{H}^2}^2 \right).$$

Proof. We fix $\alpha \in \mathbb{R}$, and apply Itô's Lemma and the Lipschitz continuity of g' to obtain

$$\begin{aligned}
e^{\alpha t}|\Delta Y_t|^2 + \int_t^T e^{\alpha s}|\Delta Z_s|^2 ds &= e^{\alpha T}|\Delta\zeta|^2 + \int_t^T e^{\alpha s}\left(2\Delta Y_s(g_s(Y, Z) - g'_s(Y', Z')) - \alpha|\Delta Y_s|^2\right) ds \\
&\quad -2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s \\
&\leq e^{\alpha T}|\Delta\zeta|^2 + \int_t^T e^{\alpha s}[C|\Delta Y_s|^2 + \frac{1}{2}|\Delta Z_s|^2 + |\Delta g_s|^2(Y_s, Z_s) - \alpha|\Delta Y_s|^2]ds \\
&\quad -2\int_t^T \Delta Y_s \Delta Z_s dW_s.
\end{aligned}$$

The reason for introducing the α is that if we now choose $\alpha = C$ then the ΔY terms cancel in the first integral on the right-hand side:

$$e^{\alpha t}|\Delta Y_t|^2 + \frac{1}{2}\int_t^T e^{\alpha s}|\Delta Z_s|^2 ds \leq e^{\alpha T}|\Delta\zeta|^2 + C\int_t^T |\Delta g_s|^2(Y_s, Z_s)ds - 2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s.$$

Note that $\int_0^{\cdot} e^{\alpha s}\Delta Y_s \Delta Z_s dW_s$ is a uniformly integrable martingale since by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\mathbb{E}\left[\sup_{[0,T]}|\int_0^{\cdot} e^{\alpha s}\Delta Y_s \Delta Z_s dW_s|\right] &\leq C\mathbb{E}\left[(\int_0^T |\Delta Y_s|^2\ |\Delta Z_s|^2 ds)^{\frac{1}{2}}\right] \\
&\leq C\mathbb{E}\left[\sup_{[0,T]}|\Delta Y|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^{\cdot} |\Delta Z_s|^2 ds\right]^{\frac{1}{2}} < \infty.
\end{aligned}$$

Taking expectation in the previous inequality then yields

$$\sup_{t\leq T}\mathbb{E}\left[|\Delta Y_s|^2\right] + \|\Delta Z\|_{\mathbb{H}^2}^2 \leq C\left(\|\Delta\zeta\|_{\mathbb{L}^2}^2 + \|\Delta g(Y, Z)\|_{\mathbb{H}^2}^2\right).$$

We now use the definition of ΔY and the Burkholder-Davis-Gundy inequality to obtain

$$\|\Delta Y\|_{\mathbb{S}^2} \leq C\mathbb{E}\left[|\Delta\zeta|^2 + \int_0^T [|\Delta Y_s|^2 + |\Delta Z_s|^2 + |\Delta g_s|^2(Y_s, Z_s)]ds\right],$$

and the result follows from the previous estimate. □

Remark 2.1 *In the above, we did in fact not use the Lipschitz continuity of g. Existence of a solution associated to g would be enough.*

We are now in position to prove that a solution to (1.1) exists.

Theorem 2.1 (Existence) *Let Assumption 2.1 holds. Then, there exists a unique solution to* (1.1).

Proof. In the case where g does not depend on (y, z) the result follows from the martingale representation theorem. The general case is obtained by a contraction argument.
Let $\mathbb{H}^2_\alpha$ be the set of elements $\zeta \in \mathcal{P}$ such that $(e^{\frac{\alpha}{2}t}\zeta_t)_{t\le T} \in \mathbb{H}^2$, for $\alpha > 0$. Given $(U, V) \in \mathbb{H}_\alpha$ let us define $(Y, Z) := \Phi(U, V)$ as the unique solution of (1.1) for the driver $(t, \omega) \mapsto g_t(\omega, U(\omega), V(\omega))$. Define similarly (Y', Z') from (U', V'). Then,

$$
\begin{aligned}
e^{\alpha t}|\Delta Y_t|^2 + \int_t^T e^{\alpha s}|\Delta Z_s|^2 ds &= \int_t^T e^{\alpha s}\left[2\Delta Y_s(g_s(U,V) - g_s(U',V')) - \alpha|\Delta Y_s|^2\right] ds \\
&\quad -2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s.
\end{aligned}
$$

Since g is Lipschitz,

$$
\Delta Y_s(g_s(U,V) - g_s(U',V')) \le C|\Delta Y_s|\,|(\Delta U, \Delta V)_s)| \le \alpha|\Delta Y_s|^2 + \frac{C}{\alpha}|(\Delta U, \Delta V)_s|^2,
$$

in which we used that $ab \le \eta a^2 + \eta^{-1}b^2$ for all $a, b \in \mathbb{R}$ and $\eta > 0$. Then,

$$
e^{\alpha t}|\Delta Y_t|^2 + \int_t^T e^{\alpha s}|\Delta Z_s|^2 ds \le \frac{C}{\alpha}\int_t^T e^{\alpha s}|(\Delta U, \Delta V)_s|^2 ds - 2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s
$$

and therefore

$$
\|e^{\alpha\cdot}(\Delta Y, \Delta Z)\|_{\mathbb{H}^2} \le \frac{C}{\alpha}\|e^{\alpha\cdot}(\Delta U, \Delta V)\|_{\mathbb{H}^2}.
$$

For α large enough, the map Φ is contracting, and therefore we can find a fix point $(Y, Z) = \Phi(Y, Z)$. This also prove uniqueness in $\mathbb{H}^2_\alpha$. We have $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$ by Proposition 2.1 and uniqueness in $\mathbb{S}^2 \times \mathbb{H}^2$ by Proposition 2.2. $\square$

We now state a comparison result. It is interesting per-se, and it will be of important use for the construction of solutions with more general divers. Also note the technique that we use to prove it, it is a linearization procedure which is part of the standard machinery.

Proposition 2.3 (Comparison) *Let $d = 1$. Let Assumption 2.1 holds for g and assume that existence holds for g'. Assume that $\zeta \le \zeta'$ and $g(Y', Z') \le g'(Z', Y')$ $dt \times d\mathbb{P}$-a.e. Then, $Y_t \le Y'_t$ for all $t \le T$. If moreover $\mathbb{P}[\zeta < \zeta'] > 0$ or $g(Y', Z') < g'(Z', Y')$ on a set of non-zero measure for $dt \times d\mathbb{P}$, then $Y_0 < Y'_0$.*

Proof. Since g is Lipschitz, the following processes are bounded:

$$b := (Y - Y')^{-1}(g(Y,Z) - g(Y',Z))\mathbf{1}_{Y\neq Y'} \text{ and } a := (Z - Z')^{-1}(g(Y',Z) - g(Y',Z'))\mathbf{1}_{Z\neq Z'}.$$

Let then Γ^t be the solution of

$$\Gamma^t = 1 + \int_t^{\cdot} \Gamma_s^t b_s ds + \int_t^{\cdot} \Gamma_s^t a_s dW_s.$$

Since

$$\Delta Y = \Delta\zeta + \int_{\cdot}^T [b_s \Delta Y_s + a_s \Delta Z_s + \Delta g_s(Y_s', Z_s')]ds - \int_{\cdot}^T \Delta Z_s dW_s,$$

we obtain

$$\Delta Y_t = \mathbb{E}\left[\Gamma_T^t \Delta\zeta + \int_t^T \Gamma_s^t \Delta g_s(Y_s', Z_s')ds \;|\mathcal{F}_t\right].$$

□

Remark 2.2 *Note that the same arguments lead to*

$$Y_t = \mathbb{E}\left[\Gamma_T^t \zeta + \int_t^T \Gamma_s^t g_s(0)ds \;|\mathcal{F}_t\right],$$

with

$$b := Y^{-1}(g(Y,Z) - g(0,Z))\mathbf{1}_{Y\neq 0} \text{ and } a := Z^{-1}(g(0,Z) - g(0))\mathbf{1}_{Z\neq 0}.$$

In particular, if $|\xi| + |g(0)| \leq M$ for some real number M, then Y is bounded.

2.2 The monotone case

We now relax the Lipschitz continuity assumption and replace it by a monotonicity condition. The idea is originally due to Darling and Pardoux [23].

Assumption 2.2 (Monotonicity condition) *$g(0) \in \mathbb{H}^2$, g is continuous with linear growth in (y,z), is Lipschitz in z and*

$$(g(y,\cdot) - g(y',\cdot)) \cdot (y - y') \leq \kappa |y - y'|^2,$$

for all $y, y' \in \mathbb{R}^d$.

Note that we can reduce to the case $\kappa = 0$ in Assumption 2.2 by considering $(e^{\kappa t} Y_t, e^{\kappa t} Z_t)_t$ in place of (Y,Z). Thus, the name monotonicity condition.

As in the Lipschitz continuous case, we start with a-priori estimates that will then be used to construct a contraction.

Proposition 2.4 *Let g be such that*

$$yg(y,z) \leq |y|\ |g(0)| + \kappa \left(|y|^2 + |y|\ |z|\right)$$

for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$. Fix $\xi \in \mathbb{L}^2$ and let (Y,Z) be a solution of (1.1)*. Then, there exists $\alpha > 0$ and C_α (both independent on T) such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{\alpha t}|Y_t|^2 + \int_0^T e^{\alpha s}|Z_s|^2 ds\right] \leq C_\alpha \mathbb{E}\left[e^{\alpha T}|\xi|^2 + \left(\int_0^T e^{\frac{\alpha}{2}s}|g_s(0)|ds\right)^2\right].$$

Moreover, there exists $\gamma > 0$, independent on T, such that

$$|Y_t|^2 \leq \mathbb{E}\left[e^{\gamma(T-t)}|\xi|^2 + \int_t^T e^{\gamma(T-t)}|g_s(0)|^2 ds|\mathcal{F}_t\right].$$

Proof. Apply Itô's Lemma to $e^{\alpha t}|Y_t|^2$ to obtain

$$e^{\alpha t}|Y_t|^2 + \int_t^T e^{\alpha s}|Z_s|^2 ds = e^{\alpha T}|\xi|^2 + \int_t^T e^{\alpha s}\left[2Y_s g_s(Y_s,Z_s) - \alpha|Y_s|^2\right] ds - 2\int_t^T e^{\alpha s} Y_s Z_s dW_s \quad (2.1)$$

in which

$$\begin{aligned} 2Y_s g_s(Y_s,Z_s) - \alpha|Y_s|^2 &\leq C|Y_s|^2 + |g_s(0)|^2 + C|Y_s|\ |Z_s| - \alpha|Y_s|^2 \\ &\leq C|Y_s|^2 + |g_s(0)|^2 + \frac{1}{2}|Z_s|^2 - \alpha|Y_s|^2. \end{aligned}$$

Take $\alpha = C$ and use the above to deduce

$$e^{\alpha t}|Y_t|^2 + \frac{1}{2}\int_t^T e^{\alpha s}|Z_s|^2 ds \leq e^{\alpha T}|\xi|^2 + C\int_t^T e^{\alpha s}|g_s(0)|^2 ds - 2\int_t^T e^{\alpha s} Y_s Z_s dW_s.$$

This provides the second assertion.
The first one follows from similar arguments, we first write that

$$\begin{aligned} 2Y_s g_s(Y_s,Z_s) - \alpha|Y_s|^2 &\leq C|Y_s|^2 + 2|Y_s|\ |g_s(0)| + C|Y_s|\ |Z_s| - \alpha|Y_s|^2 \\ &\leq C|Y_s|^2 + 2|Y_s|\ |g_s(0)| + \frac{1}{2}|Z_s|^2 - \alpha|Y_s|^2, \end{aligned}$$

and take $\alpha = C$ to deduce from (2.1) that

$$e^{\alpha t}|Y_t|^2 + \frac{1}{2}\int_t^T e^{\alpha s}|Z_s|^2 ds = e^{\alpha T}|\xi|^2 + \int_t^T e^{\alpha s}\left[2|Y_s|\ |g_s(0)|\right] ds - 2\int_t^T e^{\alpha s} Y_s Z_s dW_s.$$

Then,

$$\frac{1}{2}\mathbb{E}\left[\int_t^T e^{\alpha s}|Z_s|^2 ds\right] \leq \mathbb{E}\left[e^{\alpha T}|\xi|^2 + \int_0^T e^{\alpha s}\left[2|Y_s|\ |g_s(0)|\right] ds\right]$$

and

$$\mathbb{E}\left[\sup_{t\le T} e^{\alpha t}|Y_t|^2+\frac{1}{2}\int_0^T e^{\alpha s}|Z_s|^2 ds\right] \;\le\; \mathbb{E}\left[e^{\alpha T}|\xi|^2+\int_0^T e^{\alpha s}\left[2|Y_s|\;|g_s(0)|\right]ds\right]$$
$$+\mathbb{E}\left[2\sup_{t\le T}|\int_t^T e^{\alpha s}Y_sZ_sdW_s|\right],$$

in which, given $\iota>0$,

$$\begin{aligned}
2\mathbb{E}\left[\sup_{t\le T}\int_t^T e^{\alpha s}Y_sZ_sdW_s\right] &\le 2\mathbb{E}\left[(\int_0^T e^{2\alpha s}|Y_s|^2|Z_s|^2ds)^{\frac{1}{2}}\right]\\
&\le 2\mathbb{E}\left[\sup_s e^{\alpha s}|Y_s|(\int_0^T |Z_s|^2ds)^{\frac{1}{2}}\right]\\
&\le \iota\mathbb{E}\left[\sup_s e^{\alpha s}|Y_s|^2\right]+C_\iota\mathbb{E}\left[\int_0^T |Z_s|^2ds\right]\\
&\le \iota\mathbb{E}\left[\sup_s e^{\alpha s}|Y_s|^2\right]+C_\iota\mathbb{E}\left[e^{\alpha T}|\xi|^2+\int_0^T e^{\alpha s}\left[|Y_s|\;|g_s(0)|\right]ds\right]
\end{aligned}$$

and, given $\eta>0$,

$$2\mathbb{E}\left[\int_0^T e^{\alpha s}\left[|Y_s|\;|g_s(0)|\right]ds\right] \;\le\; \eta\mathbb{E}\left[\sup_s e^{\alpha s}|Y_s|^2\right]+C_\eta\mathbb{E}\left[(\int_0^T |g_s(0)|ds)^2\right].$$

Combining the above leads to

$$\mathbb{E}\left[\sup_{t\le T} e^{\alpha t}|Y_t|^2+\frac{1}{2}\int_t^T e^{\alpha s}|Z_s|^2ds\right] \;\le\; (1+C_\iota)\mathbb{E}\left[e^{\alpha T}|\xi|^2\right]+(\iota+C_\iota\eta+\eta)\mathbb{E}\left[\sup_s e^{\alpha s}|Y_s|^2\right]$$
$$+C_\eta(1+C_\iota)\mathbb{E}\left[(\int_0^T |g_s(0)|ds)^2\right].$$

We conclude by choosing $\iota=1/4$ and $\eta=1/(4C_\iota+4)$. □

Corollary 2.1 (Stability) *Let Assumption 2.2 for g and g' holds. Fix ξ and $\xi'\in\mathbb{L}^2$. Let (Y,Z) and (Y',Z') be associated solutions (assuming they exist). Then, there exists $\alpha>0$ and C_α (both independent on T) such that*

$$\|\Delta Y\|^2_{\mathbb{S}^2}+\|\Delta Z\|^2_{\mathbb{H}^2}\le C_\alpha\mathbb{E}\left[e^{\alpha T}\|\Delta\zeta\|^2_{\mathbb{L}^2}+\left(\int_0^T e^{\frac{\alpha}{2}s}|\Delta g_s|(Y'_s,Z'_s)ds\right)^2\right]$$

Proof. $(\Delta Y,\Delta Z)$ solves the BSDE with driver $(y,z)\mapsto\bar g(y,z)=g(y+Y',z+Z')-g'(Y',Z')$. It satisfies the requirement of Proposition 2.4. □

Theorem 2.2 (Existence) *Let Assumption 2.2 holds. Then, there exists a unique solution to* (1.1).

Proof. Uniqueness follows from Corollary 2.1. In the following, we separate the difficulties. Since g is Lipschitz in z, we can first prove that a contraction holds when a solution exists for the BSDE with driver $g(\cdot, V)$ for any $V \in \mathbb{H}^2$. This is a rather direct consequence of Corollary 2.1. Then, we will show that a solution actually exists for $g(\cdot, V)$ by using the monotonicity condition.
Step 1. Let us first assume that, for any $V \in \mathbb{H}^2$, we can find a solution to

$$Y_t = \xi + \int_t^T g_s(Y_s, V_s)ds - \int_t^T Z_s dW_s, \; t \le T. \tag{2.2}$$

Given $(U, V), (U', V')$ let $(Y, Z), (Y', Z')$ be the corresponding solutions and let Φ be the corresponding mapping. Then, it follows from Corollary 2.1 that

$$\|\Delta Y\|^2_{\mathbb{S}^2} + \|\Delta Z\|^2_{\mathbb{H}^2} \le C_\alpha \mathbb{E}\left[\left(\int_0^T e^{\frac{\alpha}{2}s}|g_s(Y'_s, V_s) - g_s(Y'_s, V'_s)|ds\right)^2\right] \le C'_\alpha T\mathbb{E}\left[\int_0^T e^{\alpha s}|V_s - V'_s|^2 ds\right].$$

Hence,

$$\|\Delta Y\|^2_{\mathbb{S}^2} + \|\Delta Z\|^2_{\mathbb{H}^2} \le C'_\alpha T e^{\alpha T}\|\Delta V\|^2_{\mathbb{H}^2}.$$

For $T \le \delta$ small, the map Φ is contracting. For larger values of T, we can glue together the solutions backward: construct a solution on $[T, T-\delta]$, given $Y_{T-\delta}$ construct a solution on $[T-\delta, T-2\delta]$ with terminal condition $Y_{T-\delta}$ at $T-\delta$, and so on. This provides a solution on $[0, T]$.
Step 2. It remains to prove that we can find a solution to (2.2) for any $V \in \mathbb{H}^2$. We now set $h_t(y) = g_t(y, V_t)$ and restrict to the case where $|\xi| + \sup_t |h_t(0)|$ is bounded by a constant M. This will be important as we will modify the driver to reduce to the Lipschitz case and we need to ensure that, in some sense, the modified driver shares the monotonicity property of h (which will come from a bound on Y deduced from the bound on ξ and $h(0)$).
By truncating and mollifying the function h, we can find a sequence of (random) functions h^n which are Lipschitz in y, with values 0 if $|y| \ge n+2$, uniformly bounded on compact sets, and that converges uniformly on compact sets to h. More precisely, we consider a smooth kernel ρ supported by the unit ball, $\theta_n \in C^\infty$ with values in $[0, 1]$ such that $\theta_n(y) = 1$ if $|y| \le n$ and $\theta_n(y) = 0$ if $|y| \ge n+1$. We set $h^n(y) := \int n^d \rho(nu)\theta_n(y-u)h(y-u)du$.
This allows us to come back to the case of a Lipschitz driver. Then, there exists a solution (Y^n, Z^n) to the BSDE associated to h^n. We shall now show that $(Y^n, Z^n)_n$ is Cauchy, which will provide a solution for the original driver g.
Since ξ and $h(0)$ are uniformly bounded, Proposition 2.4 implies that $|Y^n| \le A$, where A is independent of n. This implies that h^n is monotone along the path of Y. More precisely, if $\kappa = 0$ (which we can always assume, see above), then h^n in monotone on the ball center at 0 of radius $n-1$:

$$h^n(y) = \int_{|u|\le 1} n^d \rho(nu)\theta_n(y-u)h(y-u)du = \int_{|u|\le 1} n^d \rho(nu)h(y-u)du \quad \text{for } |y| \le n-1,$$

and therefore, for $|y|, |y'| \leq n-1$,

$$(y-y')(h^n(y)-h^n(y')) = \int_{|u|\leq 1} n^d \rho(nu)(y-u-(y'-u))(h(y-u)-h(y'-u))du \leq 0,$$

since $\kappa = 0$.
In particular, for $n, m \geq A+1$, we have

$$\begin{aligned}(Y^m - Y^n)(h^m(Y^m) - h^n(Y^n)) &= (Y^m - Y^n)(h^m(Y^m) - h^m(Y^n)) + (Y^m - Y^n)(h^m(Y^n) - h^n(Y^n)) \\ &\leq 0 + 2A \sup_{|y|\leq A} |h^m(y) - h^n(y)|.\end{aligned}$$

By arguments already used in the proof of Proposition 2.2, we deduce that

$$\|Y^m - Y^n\|_{\mathbb{S}^2} + \|Z^m - Z^n\|_{\mathbb{H}^2} \leq C\mathbb{E}\left[\int_0^T \sup_{|y|\leq A} |h^m_s(y) - h^n_s(y)| ds\right].$$

Hence the sequence is Cauchy. Let (Y, Z) be the limit. Clearly, for all $\tau \in \mathcal{T}$, $Y^n_\tau \to Y_\tau$ and $\int_\tau^T Z^n_s dW_s \to \int_\tau^T Z_s dW_s$ in $\mathbb{L}^2$. Moreover

$$\int_\tau^T |h^n_s(Y^n_s) - h_s(Y_s)| ds \leq \int_\tau^T \sup_{|y|\leq A} |h^n_s(y) - h_s(y)| ds + \int_\tau^T |h_s(Y^n_s) - h_s(Y_s)| ds.$$

Since h is continuous, convergence holds. Since equality between optional processes on stopping times implies indistinguishability, see e.g. [24, Thm 86, Chap. IV], this shows that (Y, Z) solves (2.2).

Step 3. We reduced above to bounded ξ and $h(0)$ to ensure to be able to make profit of the monotonicity of h. The general case is obtained by a standard truncation argument. Fix $p > 0$ and set $\xi^p := \xi \mathbf{1}_{|\xi|\leq p}$, $h^p := h\mathbf{1}_{|h(0)|\leq p}$. It satisfies the requirements of the previous step. Let Y^p and Y^q be solutions associated to $p \leq q$. By Corollary 2.1,

$$\|\Delta Y\|^2_{\mathbb{S}^2} + \|\Delta Z\|^2_{\mathbb{H}^2} \leq C\mathbb{E}\left[|\xi|^2 \mathbf{1}_{|\xi|>p} + \int_0^T \mathbf{1}_{\{|h_s(0)|>p\}} h^2_s(Y^p_s) ds\right]$$

Since $|Y^p| \leq Cp$ by Proposition 2.4 and $|h(y)| \leq |h(0)| + C|y|$, we have

$$\mathbf{1}_{\{|h_s(0)|>p\}} h^2_s(Y^p_s) \leq C|h_t(0)|^2 \mathbf{1}_{\{|h_s(0)|>p\}},$$

and therefore

$$\|\Delta Y\|^2_{\mathbb{S}^2} + \|\Delta Z\|^2_{\mathbb{H}^2} \leq C\mathbb{E}\left[|\xi|^2 \mathbf{1}_{|\xi|>p} + \int_0^T \mathbf{1}_{\{|h_s(0)|>p\}} h^2_s(0) ds\right]$$

which goes to 0 as $q \geq p \to \infty$. Hence there exists a limit, and it is easy to check (by similar arguments as in the end of Step 2) that it provides a solution. □

2.3 One dimensional case and non-lipschitz coefficients

In the one dimensional case, we can even only assume that g is continuous. In general, we do not have uniqueness, but only the existence of a minimal solution[1] (Y, Z). The following is due to Lepeletier and San Martin [41].

Assumption 2.3 *$g(0) \in \mathbb{H}^2$, g is continuous with linear growth in (y, z).*

Theorem 2.3 (Existence) *Let $d = 1$ and let Assumption 2.3 holds. Then, there exists a minimal solution to* (1.1) *.*

Proof. We proceed by inf-convolution: $g^n(x) := \inf_{x'}(g(x) + n|x - x'|)$ to reduce to the Lipschitz continuous case. Being n-Lipschitz, the driver g^n is associated to a unique solution (Y^n, Z^n). By the comparison result of Proposition 2.3, the sequence $(Y^n)_n$ is non-decreasing and bounded from above by the solution of the BSDE with driver $C(1 + |y| + |z|)$. Hence, it converges $a.s.$ and in $\mathbb{H}^2$. Moreover, it follows from Itô's Lemma and the uniform linear growth property of $(g^n)_n$, which is an immediate consequence of the linear growth property of g, that

$$\begin{aligned} \|Z^n - Z^m\|^2_{\mathbb{H}^2} &\leq C\mathbb{E}\left[\int_0^T (Y^n_s - Y^m_s)(g^n_s - g^m_s)(Y^n_s, Z^n_s)ds\right] \\ &\leq C\mathbb{E}\left[\int_0^T (Y^n_s - Y^m_s)^2 ds\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T (1 + |Y^n_s| + |Z^n_s|)^2 ds\right]^{\frac{1}{2}} \end{aligned}$$

Hence $(Z^n)_n$ is Cauchy and converges to some $Z \in \mathbb{H}^2$.
It remains to prove that $\int_\tau^T g^m_s(Y^m_s, Z^m_s)ds \to \int_\tau^T g_s(Y_s, Z_s)ds$ for all stopping times $\tau \in \mathcal{T}$. But $g^m \to g$ uniformly on compact sets. In particular $g^m(x^m) \to g(x)$ if $x^m \to x$. Since, after possibly passing to a subsequence $(Y^m, Z^m) \to (Y, Z)$ $dt \times d\mathbb{P}$-a.e., we actually obtain that $g^m(Y^m, Z^m) \to g(Y, Z)$ in $\mathbb{H}^2$, by dominated convergence (and up to a subsequence).
We conclude with the minimality of (Y, Z): if (Y', Z') is another solution, then $Y' \geq Y^n$ by Proposition 2.3, but $Y^n \uparrow Y$. □

Remark 2.3 *Note that comparison holds by Proposition 2.3 for the minimal solutions constructed as above.*

Remark 2.4 *Uniqueness can be obtained under stronger conditions. It is for instance the case if there exists a concave increasing function κ such that $\kappa(0) = 0$, $\kappa(x) > 0$ is $x \neq 0$, $\int_{0+}^\infty (x/\kappa(x))dx = \infty$ and*

$$|g(y, z) - g(y', z')|^2 \leq \kappa(|y - y'|^2 + |x - x'|^2),$$

see [45]. See also [44] for different conditions.

[1] i.e. if (Y', Z') is another solution, then $Y \leq Y'$ on $[0, T]$.

2.4 The quadratic case

We restrict here to the one dimensional setting $d = n = 1$.

2.4.1 Existence for bounded terminal values

To prove existence, we use the following idea. Assume that $g(y, z) = \frac{1}{2}|z|^2$ and set $\bar{Y} := e^Y$. Then,

$$d\bar{Y} = \bar{Y}dY + \frac{1}{2}\bar{Y}Z^2dt = \bar{Y}ZdW$$

so that $(e^Y, e^Y Z)$ solves a linear BSDE, which solution is even explicit.
The extension has been first provided by Kobylanski [40]. We start with the case where the driver is bounded in y and the terminal condition is bounded. The extension to a linear growth condition on y will be quite immediate, while working with unbounded terminal condition will require an additional effort.

Assumption 2.4 *ζ is bounded, g is continuous,*

$$|g(y, z)| \leq K_g(1 + |z|^2)$$

for all $(y, z) \in \mathbb{R} \times \mathbb{R}$.

Theorem 2.4 (Existence for bounded terminal values #1) *Let Assumption 2.4 holds. Then, there exists a maximal solution (Y, Z) to* (1.1)*. Moreover, $(Y, Z) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\text{BMO}}$.*

Proof. Saying that $(\bar{Y}, \bar{Z})$ solves (1.1) with terminal condition $\bar{\zeta}$ is equivalent to saying that

$$(Y, Z) = (e^{2K_g\bar{Y}}, 2K_gY\bar{Z})$$

solves

$$Y_t = \zeta + \int_t^T \tilde{g}_s(Y_s, Z_s)ds - \int_t^T Z_sdW_s \tag{2.3}$$

in which $\zeta := e^{2K_g\bar{\zeta}}$ and

$$\tilde{g}(y, z) = 2K_gy\left\{g(\ln y/2K_g, z/(2K_gy)) - z^2/(4K_gy^2)\right\}$$

which satisfies

$$-2K_g^2y - \frac{|z|^2}{y} \leq \tilde{g}(y, z) \leq 2K_g^2y.$$

In the following, we focus on finding a solution to (2.3). The idea of the proof is the following. First, we truncate the driver in y and z so as to recover the case of a continuous driver with linear growth. Knowing that ξ is bounded, we then check that the truncation on y does not operate because solutions are indeed (uniformly) bounded. By comparison, the sequence of Y-components of the solutions will be non-decreasing and therefore convergent. It then remains to prove the convergence of the Z-components.

Step 1. Let θ_p be a smooth $[0,1]$-valued function with value 1 for $|z| \le p$ and 0 for $|z| \ge p+1$. Fix $0 < \beta$ set $\rho(y) = \beta^{-1} \vee y \wedge \beta$ and

$$\bar{g}^p(x,y) := \theta_p(z)\tilde{g}(\rho(y), z) + 2K_g^2\rho(y)(1-\theta_p(z)).$$

Since $\bar{g}^p$ is continuous, existence holds by Theorem 2.3. Let (Y^p, Z^p) be a solution.

Then, we can choose $(\theta_p)_p$ and $(Y^p)_p$ so that the later is non-increasing and bounded. Indeed, if $(\theta_p)_p$ it is non-decreasing, then $\bar{g}^p \downarrow \tilde{g}(\rho, \cdot)$, and we can appeal to Remark 2.3. Moreover, by Proposition 2.3, Y^p is sandwiched by the backward ODE with divers $2K_g^2(y \vee \beta^{-1})$ and $-2K_g^2(y \wedge \beta)$, and terminal condition given by $A > 0$ and A^{-1} such that $A^{-1} \le e^{2K_g\zeta} \le A$. For A large, this coincides with the ode with drivers $2K_g y$ and $-2K_g y$ and the corresponding solutions a, a' can be found such that $\rho(a) = a$ and $\rho(a') = a'$. Hence, we have $Y^p = \rho(Y^p)$ so that Z^p does not depend on ρ as well. The corresponding lower bound is uniformly strictly positive so that $1/Y^p$ is also bounded.

Hence Y^p converges a.s. and in $\mathbb{H}^2$ to some Y as $p \to \infty$, such that Y and $1/Y$ are bounded.

Step 2. We now show that $(Z^p)_p$ is uniformly bounded in $\mathbb{H}^2$. To see this, let us set $\psi(x) := e^{-3cx}$ where $c > 0$ is such that

$$-2c^2Y^p - c|Z^p|^2 \le \tilde{g}(Y^p, Z^p) \le 2c^2Y^p. \tag{2.4}$$

Then,

$$\psi(\zeta) = \psi(Y_0^p) - \int_0^T \left(\psi'(Y_s^p)\bar{g}_s^p(Y_s^p, Z_s^p) - \frac{1}{2}\psi''(Y_s^p)|Z_s^p|^2\right) ds - \int_0^T \psi'(Y_s^p)Z_s^p dW_s.$$

By (2.4) and since $\psi' < 0$, this implies that, for any stopping time τ,

$$\psi(\zeta) \ge \psi(Y_0^p) + \int_\tau^T \left(\psi'(Y_s^p)2c^2Y_s^p + (c\psi'(Y_s^p) + \frac{1}{2}\psi''(Y_s^p))|Z_s^p|^2\right) ds - \int_\tau^T \psi'(Y_s^p)Z_s^p dW_s.$$

We now observe that $c\psi' + \frac{1}{2}\psi'' = 3c^2\psi/2$ and that $\psi(Y^p) \ge \iota$ for some real $\iota > 0$ independent on p. Hence

$$C\mathbb{E}\left[\psi(\zeta) - \psi(Y_0^p) + \int_\tau^T |\psi'(Y_s^p)| 2c^2 Y_s^p |\mathcal{F}_\tau\right] \ge \mathbb{E}\left[\int_\tau^T |Z_s^p|^2 ds |\mathcal{F}_\tau\right].$$

Note that the same is true for $Z^p/(2K_gY^p)$.

Step 3. We can now prove that $(Z^p)_p$ converges. As it is bounded in $\mathbb{H}^2$, it converges weakly, up to a subsequence. It is not difficult to check that

$$\bar{g}^p(Y^p, Z^p) - \bar{g}^p(Y^p, Z^p) \le \lambda(1 + |Z^q|^2) \le \lambda\left(1 + |Z^q - Z^p|^2 + |Z^p - Z|^2 + |Z|^2\right)$$

for some $\lambda > 0$. Set $\psi(x) = e^{4\lambda x} - x/4 - 1$. We can take $\lambda > 1/16$ so that ψ is strictly increasing and $\psi(0) = 0$. Again, we apply Itô's lemma to obtain

$$\mathbb{E}\left[\psi(Y_0^p - Y_0^q) + \int_0^T \{\frac{1}{2}\psi'' - \lambda\psi'\}(Y_s^p - Y_s^q)|Z_s^p - Z_s^q|^2 ds\right]$$
$$\leq \mathbb{E}\left[\int_0^T \psi'(Y_s^p - Y_s^q)\lambda\left(1 + |Z_s^p - Z_s|^2 + |Z_s|^2\right) ds\right].$$

Note that $\frac{1}{2}\psi'' - \lambda\psi' = 4\lambda^2 e^{4\lambda x} + \frac{\lambda}{4} > 0$. Then, being bounded, $\{\frac{1}{2}\psi'' - \lambda\psi'\}^{\frac{1}{2}}(Y^p - Y^q)$ converges strongly in $\mathbb{H}^2$ as $q \to \infty$ to $\{\frac{1}{2}\psi'' - \lambda\psi'\}^{\frac{1}{2}}(Y^p - Y)$, so that $\{\frac{1}{2}\psi'' - \lambda\psi'\}^{\frac{1}{2}}(Y_s^p - Y_s^q)|Z^p - Z^q|$ converges weakly as $q \to \infty$. From this and the fact that $Y^q \geq Y$ and ψ' is non-decreasing, we get

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \{\frac{1}{2}\psi'' - \lambda\psi'\}(Y_s^p - Y_s)|Z_s^p - Z_s|^2 ds\right] &\leq \liminf_{q\to\infty} \mathbb{E}\left[\int_0^T \{\frac{1}{2}\psi'' - \lambda\psi'\}(Y_s^p - Y_s^q)|Z_s^p - Z_s^q|^2 ds\right] \\
&\leq \mathbb{E}\left[\int_0^T \psi'(Y_s^p - Y_s)\lambda\left(1 + |Z_s^p - Z_s|^2 + |Z_s|^2\right) ds\right] \\
&- \mathbb{E}\left[\psi(Y_0^p - Y_0)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \{\frac{1}{2}\psi'' - 2\lambda\psi'\}(Y_s^p - Y_s)|Z_s^p - Z_s|^2 ds\right] &\leq \liminf_{q\to\infty} \mathbb{E}\left[\int_0^T \{\frac{1}{2}\psi'' - \lambda\psi'\}(Y_s^p - Y_s^q)|Z_s^p - Z_s^q|^2 ds\right] \\
&\leq \mathbb{E}\left[\int_0^T \psi'(Y_s^p - Y_s)\lambda\left(1 + |Z_s|^2\right) ds\right] \\
&- \mathbb{E}\left[\psi(Y_0^p - Y_0)\right].
\end{aligned}$$

Since $\{\frac{1}{2}\psi'' - 2\lambda\psi'\} = \lambda/2 > 0$, it holds that $Z^p \to Z$ in $\mathbb{H}^2$.
Step 4. It remains to prove that (Y, Z) is a solution to (2.3). This is done by similar arguments as already used. □

Remark 2.5 (comparison) *The fact that the sequence $(Y^p)_p$ is non-increasing implies that comparison holds for the maximal solutions. One could similarly provide a minimal solution.*

We now consider a more general setting

Assumption 2.5 *ζ is bounded, g is continuous,*

$$|g(y, z)| \leq K_g(1 + |y| + |z|^2).$$

Theorem 2.5 (Existence for bounded terminal value #2) *Let Assumption 2.5 holds. Then, there exists a unique maximal solution (Y, Z) to (1.1). Moreover, $(Y, Z) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\text{BMO}}$.*

Proof. We can truncate the y term in the driver. Then, since ζ is bounded, we can again sandwich the solution is a way that the truncation does not operate. □

2.4.2 Existence for unbounded terminal values

We next turn to the case where ξ is not bounded. We follow Briand and Hu [12].

Assumption 2.6 *g is continuous, there exists $\beta \geq 0$, $\gamma > 0$ $\alpha \geq \beta/\gamma$ such that*

$$|g(y,z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2.$$

There exists $\lambda > \gamma e^{\beta T}$ such that $\mathbb{E}\left[e^{\lambda|\xi|}\right] < \infty$.

Theorem 2.6 (Existence for unbounded terminal value) *Let Assumption 2.6 holds. Then, there exists a solution (Y,Z) to (1.1). Moreover,*

$$-\frac{1}{\gamma}\ln \mathbb{E}\left[\phi_t(-\xi)|\mathcal{F}_t\right] \leq Y_t \leq \frac{1}{\gamma}\ln \mathbb{E}\left[\phi_t(\xi)|\mathcal{F}_t\right],$$

in which ϕ is given by (2.5) below.

Proof. The proof is based on Theorem 2.5 as we shall first truncate the terminal condition. However, we need global estimates on the Y-component of the corresponding solutions, which do not depend on the truncation. To do this, we extend the deterministic bounds obtained in Step 1 of the proof of Theorem 2.4 into (2.6) below. The existence of a solution in the general case will then be obtained thanks to this global control and a localization argument, see Steps 2 and 3. In Step 4, we prove that the Z-component is in $\mathbb{H}^2$.

Step 1. Set $P = e^{\gamma Y}$ and $Q = \gamma e^{\gamma Y} Z$. If $(Y,Z) \in \mathbb{S}^\infty \times \mathbb{H}^2$ solves (1.1) then (P,Q) solves the BSDE with terminal condition $e^{\gamma\xi}$ and driver

$$F(p,q) := \mathbf{1}_{p>0}\left(\gamma p f(\ln p/\gamma, \frac{q}{\gamma p}) - \frac{1}{2}\frac{|q|^2}{p}\right).$$

We set

$$H(p) := \mathbf{1}_{p\geq 1} p(\alpha\gamma + \beta|\ln p|) + \alpha\gamma \mathbf{1}_{p<1} \geq F(p,q)$$

and let $\phi(x)$ be the solution of

$$\phi_t(x) = e^{\gamma x} + \int_t^T H(\phi_s)ds. \tag{2.5}$$

We first prove that any (possible) solution $(Y,Z) \in \mathbb{S}^\infty \times \mathbb{H}^2$ of (1.1) with ξ bounded satisfies

$$-\frac{1}{\gamma}\ln \mathbb{E}\left[\phi_t(-\xi)|\mathcal{F}_t\right] \leq Y_t \leq \frac{1}{\gamma}\ln \mathbb{E}\left[\phi_t(\xi)|\mathcal{F}_t\right]. \tag{2.6}$$

Set

$$\Phi_t := \mathbb{E}\left[\phi_t(\xi)|\mathcal{F}_t\right] = \mathbb{E}\left[e^{\gamma\xi} + \int_t^T H(\phi_s(\xi))ds|\mathcal{F}_s\right].$$

Then, by applying the representation theorem to

$$e^{\gamma\xi}+\int_0^T \mathbb{E}\left[H(\phi_s(\xi))|\mathcal{F}_s\right]ds$$

we can find χ such that

$$\Phi_t=e^{\gamma\xi}+\int_t^T \mathbb{E}\left[H(\phi_s(\xi))|\mathcal{F}_s\right]ds-\int_t^T \chi_s dW_s.$$

On the other hand

$$P_t=e^{\gamma\xi}+\int_t^T F(P_s,Q_s)ds-\int_t^T Q_s dW_s.$$

Since H is convex, we obtain

$$P_t-\Phi_t\leq\int_t^T \left(F(P_s,Q_s)-H(\Phi_s)\right)ds-\int_t^T (Q_s-\chi_s)dW_s.$$

Since H is locally Lipschitz and P,Φ are uniformly controlled in $(0,\infty)$, we can apply comparison based on $F\leq H$. This provides the upper-bound of (2.6). We then consider $(-Y,-Z)$ and work as above to obtain the lower bound.

Step 2. We assume here that $\xi\geq 0$ and set $\xi^n:=\xi\wedge n$. By Theorem 2.5 and Step 1, we can find a solution (Y^n,Z^n) such that

$$-\frac{1}{\gamma}\ln\mathbb{E}\left[\phi_t(-\xi^n)|\mathcal{F}_t\right]\leq Y_t^n\leq\frac{1}{\gamma}\ln\mathbb{E}\left[\phi_t(\xi^n)|\mathcal{F}_t\right]. \tag{2.7}$$

By considering maximal solutions, we must have $Y^n\leq Y^{n+1}$, see Remark 2.5. We define $Y:=\sup_n Y_n$ and observe from (2.7) that it well defined in $\mathbb{L}^1$ and satisfies (2.6).

Set

$$\tau_k:=\inf\{t:\frac{1}{\gamma}\ln\mathbb{E}\left[\phi_0(\xi)|\mathcal{F}_t\right]\geq k\}\wedge T.$$

Since $H\geq 0$, $(\phi_t)_t$ is non-increasing so that $\ln\mathbb{E}\left[\phi_0(\xi)|\mathcal{F}_t\right]\leq k$ implies $\ln\mathbb{E}\left[\phi_t(\xi^n)|\mathcal{F}_t\right]\leq k$. In view of (2.7) this implies that $Y^n_{\cdot\wedge\tau_k}$ is bounded from above.

We can then define $Y^k:=\sup_n Y^n_{\cdot\wedge\tau_k}$ and show, by similar arguments as in the proof of Theorem 2.4, that there exists Z^k which is the limit in $\mathbb{H}^2$ of $Z^n_{\cdot\wedge\tau_k}$ such that

$$Y^k=\sup_n Y^n_{T\wedge\tau_k}+\int_\cdot^{T\wedge\tau_k} g_s(Y_s^k,Z_s^k)ds-\int_\cdot^{T\wedge\tau_k} Z_s^k dW_s.$$

Since $(\tau_k)_k$ is non-decreasing, we have $Y_{\cdot\wedge\tau_k}=Y^k$. By $\tau_k\uparrow T$ and (2.7), we deduce that Y is a continuous process with limit ξ at T. Set $Z_t=Z_t^k$ for $t\leq\tau_k$. Since $Z^{k+1}=Z^k$ on $[0,\tau_k]$, we obtain that

$$Y_{t\wedge\tau_k}=Y_{\tau_k}+\int_{t\wedge\tau_k}^{\tau_k} g_s(Y_s,Z_s)ds-\int_{t\wedge\tau_k}^{\tau_k} Z_s dW_s.$$

It remains to send $k \to \infty$.

Step 3. We now consider the general case. Set $\xi^p := (-p) \vee \xi$. Then, define (Y^p, Z^p). We do the same as a above by taking inf over p.

Step 4. It remains to prove that $Z \in \mathbb{H}^2$. Let $\tau_n := \inf\{t : \int_0^t e^{2\gamma|Y_s|}|Z_s|^2 ds \geq n\}$. Set $\psi(x) := (e^{\gamma x} - 1 - \gamma x)/\gamma^2$. Then, since $x \mapsto \psi(|x|)$ is C^2 and $\psi' \geq 0$ on $\mathbb{R}_+$,

$$\begin{aligned}
\psi(|Y_0|) &= \psi(|Y_{t\wedge\tau_n}|) + \int_0^{t\wedge\tau_n} \left(\psi'(|Y_s|)\text{sign}(Y_s)g_s(Y_s, Z_s) - \frac{1}{2}\psi''(|Y_s|)|Z_s|^2\right) ds \\
&\quad - \int_0^{t\wedge\tau_n} \psi'(|Y_s|)\text{sign}(Y_s)Z_s dW_s \\
&\leq \psi(|Y_{t\wedge\tau_n}|) + \int_0^{t\wedge\tau_n} \psi'(|Y_s|)(\alpha + \beta|Y_s|)ds + \frac{1}{2}\int_0^{t\wedge\tau_n} |Z_s|^2 \left(\gamma\psi'(Y_s) - \psi''(|Y_s|)\right) ds \\
&\quad - \int_0^{t\wedge\tau_n} \psi'(|Y_s|)\text{sign}(Y_s)Z_s dW_s.
\end{aligned}$$

But $\gamma\psi'(Y_s) - \psi''(|Y_s|) = -1$, so that

$$\psi(|Y_0|) + \frac{1}{2}\mathbb{E}\left[\int_0^{t\wedge\tau_n} |Z_s|^2 ds\right] \leq \mathbb{E}\left[\psi(|Y_{t\wedge\tau_n}|) + \int_0^{t\wedge\tau_n} \psi'(|Y_s|)(\alpha + \beta|Y_s|)ds\right].$$

It remains to appeal to (2.6) and our integrability assumptions on ξ. □

2.4.3 General estimates and stability for bounded terminal conditions using Malliavin calculus

The content of this section is due to Briand and Elie [11]. The first idea of this paper is to rely on the fact that the Y-component of a quadratic BSDE with bounded terminal value should be bounded, while the Z-component should be BMO, i.e. belong to $\mathbb{H}^2_{\text{BMO}}$. We make this assertion precise in the next proposition.

Assumption 2.7 *g is deterministic, $g(0)$ and ζ are bounded by K_0,*

$$|g(y, z) - g(y', z')| \leq L_y|y - y'| + K_z(1 + |z| + |z'|)|z - z'|.$$

Proposition 2.5 (Equivalence of the classes of definition) *Let Assumption 2.7 hold.*

(a.) *Assume that $(Y, Z) \in \mathbb{S}^\infty \times \mathbb{H}^2$ solves* (1.1)*, then $Z \in \mathbb{H}^2_{\text{BMO}}$ and there exists $\kappa > 0$ such that*

$$\mathbb{E}\left[\int_\tau^T |Z_s|^2 ds \;|\mathcal{F}_\tau\right] \leq \kappa(1 + \|Y\|_{\mathbb{H}^\infty})e^{\kappa\|Y\|_{\mathbb{H}^\infty}}.$$

(b.) *Assume that $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2_{\text{BMO}}$ solves* (1.1)*, then*

$$\|Y\|_{\mathbb{S}^\infty} \leq e^{L_y T}(1 + T)K_0.$$

(c.) *Any solution in* $\mathbb{S}^\infty \times \mathbb{H}^2$ *or in* $\mathbb{S}^2 \times \mathbb{H}^2_{\mathrm{BMO}}$ *is in* $\mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}}$.

Proof. a. Note that

$$|g(y,z)| \leq K_0 + \frac{K_z}{2} + L_y|y| + \frac{3K_z}{2}|z|^2.$$

Let ψ be defined by

$$\psi(x) = \frac{e^{3K_z|x|} - 1 - 3K_z|x|}{|3K_z|^2},$$

and apply Itô's Lemma to obtain

$$\psi(Y_\tau) \quad \leq \quad \psi(\xi) + \int_\tau^T \left(\frac{3K_z}{2}|\psi'(Y_s)| - \frac{\psi''(Y_s)}{2}\right) Z_s^2 ds + C\int_\tau^T |\psi'(Y_s)|\,(1+|Y_s|)\,ds - \int_\tau^T \psi'(Y_s) Z_s dW_s.$$

Since $\psi'' - 3K_z|\psi'| = 1$ and $\psi \geq 0$, this implies

$$\frac{1}{2}\mathbb{E}\left[\int_\tau^T |Z_s|^2 ds \;|\mathcal{F}_\tau\right] \leq \mathbb{E}\left[\psi(\xi) + \int_\tau^T |\psi'(Y_s)|\left(K_0 + \frac{K_z}{2} + L_y|Y_s|\right) ds \;|\mathcal{F}_\tau\right].$$

b. We can[2] use the linearization procedure of Proposition 2.3 because $Z \in \mathbb{H}^2_{\mathrm{BMO}}$. □

Now that we know that Z should be BMO, we can use the linearization procedure of Proposition 2.3 to obtain comparison of solutions with different terminal conditions.

Proposition 2.6 (Stability in the terminal condition) *Let Assumption 2.7 hold for* (g,ξ) *and* (g,ξ'). *Let* (Y,Z) *and* (Y',Z') *be associated solutions in* $\mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}}$. *Then, there exists an equivalent probability measure* $\bar{\mathbb{P}}$ *and a bounded process* b *such that*

$$Y_\tau - Y'_\tau = \mathbb{E}^{\bar{\mathbb{P}}}\left[e^{\int_\tau^T b_s ds}(\xi - \xi')\;|\mathcal{F}_\tau\right]$$

for all stopping time $\tau \leq T$. *Moreover, there exists* $p_0 > 1$ *such that for all* $p \geq p_0$ *we can find* C_p *which depends only on* p, L_y, K_z *and* K_0 *such that*

$$\|Y - Y'\|_{\mathbb{S}^{2p}} + \|Z - Z'\|_{\overline{\mathbb{H}}^p} \leq C_p\|\xi - \xi'\|_{\mathbb{L}^{2p}}.$$

Proof. Again, we can use the linearization argument of Proposition 2.3 to obtain that there exists an adapted process b, bounded by L_y, and $a \in \mathbb{H}^2_{\mathrm{BMO}}$ such that

$$\Delta Y = \Delta\zeta + \int_\cdot^T b_s\Delta Y_s + a_s\Delta Z_s ds - \Delta Z_s dW_s.$$

The bound on $\|a\|_{\mathbb{H}^2_{\mathrm{BMO}}}$ depends only on K_z. Since $a \in \mathbb{H}^2_{\mathrm{BMO}}$, we can again define an equivalent measure $\bar{\mathbb{P}}$ and a $\bar{\mathbb{P}}$-Brownian motion $\bar{W}$ such that

$$\Delta Y = \Delta\zeta + \int_\cdot^T b_s \Delta Y_s ds - \Delta Z_s d\bar{W}_s.$$

[2]For any $\zeta \in \mathbb{H}^2_{\mathrm{BMO}}$, the Doléans-Dade Exponential $\mathcal{E}(\int_0^\cdot \zeta_s dW_s)$ is a martingale, see Kazamaki [39].

This proves the first identity.
Let $\mathcal{E}^a$ be the Doleans-Dade exponential of $\int_0^\cdot a_s dW_s$. Since the later is BMO, this is a martingale and, since b is bounded by L_y,

$$|\Delta Y_t| \le e^{L_y T}(\mathcal{E}^a_t)^{-1}\mathbb{E}\left[\mathcal{E}^a_T|\Delta\zeta| \;|\mathcal{F}_t\right] \le e^{L_y T}(\mathcal{E}^a_t)^{-1}\mathbb{E}\left[(\mathcal{E}^a_T)^q \;|\mathcal{F}_t\right]^{\frac{1}{q}}\mathbb{E}\left[|\Delta\zeta|^p \;|\mathcal{F}_t\right]^{\frac{1}{p}}$$

in which $1/p+1/q=1$. But, by the reverse Hölder inequality, $\mathbb{E}\left[(\mathcal{E}^a_T)^q \;|\mathcal{F}_t\right]^{\frac{1}{q}} \le C_* \mathcal{E}^a_t$ for $1<q<q^*$ in which q^* depends $\|a\|_{\mathbb{H}^2_{\mathrm{BMO}}}$, see Kazamaki [39, Theorem 3.1]. For the ΔZ term, we again apply Itô's Lemma to ΔY^2 to obtain

$$\int_0^T |\Delta Z_s|^2 ds = |\Delta\zeta|^2 + \int_0^T 2\Delta Y_s(g_s(Y_s,Z_s)-g_s(Y'_s,Z'_s))ds - \int_0^T 2\Delta Y_s\Delta Z_s dW_s$$

in which

$$\Delta Y_s(g_s(Y_s,Z_s)-g_s(Y'_s,Z'_s)) \le (\sup|\Delta Y|^2)(L_y + 2K_z^2(1+|Z_s|^2+|Z'_s|^2)+\frac{1}{2}|\Delta Z|^2.$$

We conclude by using Burkholder-Davis-Gundy inequality, the energy inequality for BMO martingales [3] which implies that

$$\|Z\|^{2p}_{\mathbb{H}^{2p}} + \|Z'\|^{2p}_{\mathbb{H}^{2p}} \le p!(\|Z\|^{2p}_{\mathbb{H}^2_{\mathrm{BMO}}} + \|Z'\|^{2p}_{\mathbb{H}^2_{\mathrm{BMO}}}),$$

together with the bound on Y of Proposition 2.5. □

Remark 2.6 *Extension of the comparison result to different drivers is straightforward.*

We can now look for another proof of existence. The general idea is the following. When $g \equiv 0$, then the Clark-Ocone formula implies that $Z = \mathbb{E}\left[D_\cdot\xi \;|\mathcal{F}_\cdot\right]$ whenever the Malliavin derivative process $D\xi = (D_t\xi)_t$ is well-defined, see Nualart [46]. If $D\xi$ is bounded, then Z is bounded. The same essentially holds for BSDEs. Thus, if $D\xi$ is bounded, then the Z-component of the solution is bounded and everything works as if the driver was uniformly Lipschitz in z. Thanks to Proposition 2.6, the general case can be obtained by approximating any bounded terminal condition by a sequence that is smooth in the Malliavin sense.

Theorem 2.7 (Short existence proof) *Let Assumption 2.7 hold. Then, there exists a unique solution* $(Y,Z) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}}$ *to* (1.1).

Proof. Step 1. If g is C^1 and ξ has a bounded Mallivian derivative, then it follows from El Karoui et al. [32] that for $s \le t$

$$D_s Y_t = D_s\xi + \int_t^T \left(\partial_y g(Y_u,Z_u)D_sY_u + \partial_z g(Y_u,Z_u)D_sZ_u\right)ds - \int_t^T D_sZ_u dW_u,$$

[3]See Kazamaki [39, Section 2.1]

while $D_t Y_t = Z_t$. This readily implies that $|Z| \leq e^{L_y T} \|D\xi\|_{\mathbb{L}^\infty}$.
Step 2. In the case where ξ has a bounded Mallivian derivative but g is general, we can truncate g in the z component to make it Lipschitz without changing the Lipschitz constant in y. Then, by Step 1 and Proposition 2.5, both Y and Z are bounded by a constant which does not depend on the Lipchitz constant in z of the truncated operator. The truncation can then be chosen so that it does not operate.
Step 3. One can approximate ξ by a sequence $(\xi^n)_n$ which has bounded Malliavin derivatives, so that the convergence holds in any $\mathbb{L}^p$, and $\mathbb{P}-$a.s., see Nualart [46]. The sequence can be chosen to be uniformly bounded by K_0. In view of Proposition 2.6, the sequence of solutions is Cauchy in $\mathbb{S}^{2p} \times \overline{\mathbb{H}}^p$ and belongs to $\mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}}$. □

Remark 2.7 *The linearization technique used in Proposition 2.6 allows for comparison. Note that a comparison result is also obtained in Kobylanski [40, Thm 2.6] under very similar conditions, but again the proof is of pde style one and relies on appropriate changes of variables.*

2.4.4 Comparison for concave drivers and general terminal conditions

The content of this section is based on Delbean, Hu and Richou [26, 27]. Their key idea consists in rewriting the solution of a quadratic BSDE in terms of an optimal control problem on parameterized BSDEs. The value of this optimal control problem being unique, there can not be more than one solution to the BSDE.
This requires the following assumptions.

Assumption 2.8 *g is continuous, and there exists $\alpha, \beta \geq 0$ and $\gamma > 0$ such that*

$$|g(y, z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2.$$

Moreover, g is uniformly Lipschitz in y and concave in z.

To define the associated control problem, we introduce the Fenchel transform f of g with respect to z:

$$f(y, q) := \sup_z (g(y, z) - zq). \tag{2.8}$$

Remark 2.8 *We have*

- *If $f(y, q) < \infty$ then $|f(y, q) - f(y', q)| \leq C|y - y'|$.*
- *f is convex in q.*
- *$f(y, q) \geq -\alpha - \beta|y| + \frac{1}{\gamma 2}|q|^2$.*

- $f(y,q) \geq -\alpha + \frac{1}{\gamma 2}|q|^2$ *if* $g(y,\cdot) \geq g(0,\cdot)$.

Theorem 2.8 (Comparison for concave drivers) *Uniqueness holds for solutions* (Y,Z) *of* (1.1) *satisfying :* $\exists\, p > \gamma$ *and* $\varepsilon > 0$ *such that*

$$\mathbb{E}\left[e^{p \sup_{[0,T]} Y^- + \varepsilon \sup_{[0,T]} Y^+}\right] < \infty.$$

Proof. In order to simplify the proof, we assume that $g(y,\cdot) \leq g(0,\cdot)$, see [26] for the additional small time argument required in the general case. Let $\mathcal{A}$ be the of predictable processes q a.s. square integrable such that

$$M := \mathcal{E}(\int_0^{\cdot} q_s dW_s)$$

is a martingale and

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left(|q_s|^2 + |f_s(0,q_s)|\right) ds + |\xi|\right] < \infty$$

where $d\mathbb{Q}/d\mathbb{P} = M_T$.
By Briand et al. [13, Proposition 6.4], we can find (Y^q, Z^q) such that Y^q is of class (D) under $\mathbb{Q}$, $|Z^q|^2$ and $|f(Y^q,q)|$ are a.s. square integrable and

$$Y^q = \xi + \int_{\cdot}^T f_s(Y^q_s, q_s) ds - \int_{\cdot}^T q_s dW_s.$$

We will show that any solution (Y,Z) of (1.1) verifying the requirement of Theorem 2.8 satisfies $Y = \operatorname*{ess\,inf}_{q\in\mathcal{A}} Y^q$.
Step 1. Let us prove that $Y \leq Y^q$. Fix t_0, let

$$\begin{aligned}
\tau_n &:= \inf\{t \geq t_0 : \int_{t_0}^t |Z_s|^2 + |Z^q_s|^2 + |q_s|^2 \geq n\},\\
\eta &:= (Y^q - Y)^{-1}(f(Y^q,Z) - f(Y,Z))\mathbf{1}_{\{Y^q \neq Y\}},
\end{aligned}$$

and note that $|\eta| \leq C$. We apply Itô's formula to $e^{\int_{t_0}^{\cdot} \eta_s ds}\Delta Y$ with $\Delta Y = (Y - Y^q)$, and use (2.8), to get

$$\begin{aligned}
\Delta Y_t &= e^{\int_t^{\tau_n} \eta_r dr}\Delta Y_{\tau_n} + \int_t^{\tau_n} e^{\int_t^s \eta_r dr}\left(g_s(Y_s,Z_s) - f_s(Y^q_s,q_s) - \eta_s \Delta Y_s - q_s Z_s\right) ds - \int_t^{\tau_n} e^{\int_t^s \eta_r dr}\Delta Z_s dW^{\mathbb{Q}}_s\\
&= e^{\int_t^{\tau_n} \eta_r dr}\Delta Y_{\tau_n} + \int_t^{\tau_n} e^{\int_t^s \eta_r dr}\left(g_s(Y_s,Z_s) - q_s Z_s - f_s(Y_s,q_s)\right) ds - \int_t^{\tau_n} e^{\int_t^s \eta_r dr}\Delta Z_s dW^{\mathbb{Q}}_s\\
&\leq e^{\int_t^{\tau_n} \eta_r dr}\Delta Y_{\tau_n} - \int_t^{\tau_n} e^{\int_t^s \eta_r dr}\Delta Z_s dW^{\mathbb{Q}}_s,
\end{aligned}$$

so that

$$\Delta Y_t \leq \mathbb{E}\left[e^{\int_t^{\tau_n} \eta_r dr}\Delta Y_{\tau_n} | \mathcal{F}_t\right].$$

In order to pass to the limit $n \to \infty$, we use the inequality

$$xy \leq e^{px} + \frac{y}{p}(\ln y - \ln p - 1)$$

to obtain

$$\mathbb{E}^{\mathbb{Q}}[\sup_{[0,T]} Y^+] \;\; \leq \;\; \mathbb{E}\left[e^{\varepsilon \sup_{[0,T]} Y^+} + \frac{M_T}{\varepsilon}(\ln M_T - \ln \varepsilon - 1)\right] < \infty,$$

by definition of $\mathcal{A}$ and assumption on Y^+. Since Y^q is of class (D), this implies that

$$\Delta Y_t \leq \limsup_n \mathbb{E}\left[e^{\int_t^{\tau_n} \eta_r dr} \Delta Y_{\tau_n} | \mathcal{F}_t\right] \leq \mathbb{E}\left[e^{\int_t^{T} \eta_r dr} \Delta Y_T | \mathcal{F}_t\right] = 0.$$

Step 2. We now choose $q = \hat{q} \in \partial_z g(Y, Z)$ so that

$$f(Y, \hat{q}) = g(Y, Z) - Z\hat{q}.$$

Then, the computations of Step 1 lead to

$$\Delta Y_t = \mathbb{E}\left[e^{\int_t^{\tau_n} \eta_r dr} \Delta Y_{\tau_n} | \mathcal{F}_t\right].$$

If $q = \hat{q}$ is admissible we can pass to the limit and conclude, since $Y^{\hat{q}}$ will be of class (D) and

$$\mathbb{E}^{\mathbb{Q}}[\sup_{[0,T]} Y^-] \;\; \leq \;\; \mathbb{E}\left[e^{p \sup_{[0,T]} Y^-} + \frac{M_T}{p}(\ln M_T - \ln p - 1)\right] < \infty.$$

To see this, note that, by Remark 2.8 and the assumption $g \geq g(0, \cdot)$,

$$-\alpha + \frac{1}{2\gamma}|\hat{q}|^2 \leq f(Y, \hat{q}) = g(Y, Z) - Z\hat{q}$$

so that

$$\frac{1}{2\gamma}|\hat{q}|^2 \leq \alpha + g(Y, Z) + C|Z|^2 + \frac{1}{4\gamma}|\hat{q}|^2$$

and

$$\frac{1}{4\gamma}|\hat{q}|^2 \leq \alpha + g(Y, Z) + C|Z|^2.$$

Hence, $|\hat{q}|$ is a.s. square integrable. This also implies that

$$\frac{1}{2\gamma}\mathbb{E}\left[M_{\tau_n} \ln M_{\tau_s}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2\gamma}\int_0^{\tau_n} |\hat{q}_s|^2 ds\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_n} (\alpha + g(Y_s, Z_s) - Z_s\hat{q}_s) ds\right] \leq T\alpha - \mathbb{E}^{\mathbb{Q}}[Y_{\tau_n} - Y_0] \leq C,$$

which shows that $(M_{\tau_n})_n$ is uniformly integrable by the de La Valle Poussin Lemma, hence M is a martingale. Moreover, we can pass to the limit:

$$\mathbb{E}\left[M_T \ln M_T\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^{T} |\hat{q}_s|^2 ds\right] \leq \liminf \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^{\tau_n} |\hat{q}_s|^2 ds\right] \leq C.$$

Is it not difficult to check that $f(0, \hat{q})$ satisfies the required integrability condition. □

2.4.5 Additional readings

Lepeltier and San Martin [42] have studied the case of a super-linear growth in y and quadratic growth in z. Tevzadze [57] has provided a construction based on BMO arguments and Picard iterations for systems. Barrieu and El Karoui [1] use an approach based on general properties of quadratic semimartingales which allows one to consider quite abstract settings. Counter-examples for the case of super-quadratic growth in z can be found in Delbean, Hu and Bao [25]. As for numerical schemes, see Chassagneux and Richou [17].

Chapter 3

Monotonic limits and non-linear Doob-Meyer decomposition

In all this chapter $d = 1$. The following material is due to Peng [52]. It provides general stability results for super-solutions of BSDEs as well as important representation results. These are of important use to provide general existence results to BSDEs with constraint, see Section 4.1, or representations to certain optimal control problems, see Section 4.3.

3.1 Monotonic limit

We start with a general stability result under monotonic convergence.
Let us consider the family of processes

$$Y^i = Y_0^i + \int_0^\cdot g_s^i ds - A^i + \int_0^\cdot Z_s^i ds \tag{3.1}$$

where $(g^i, Z^i)_{i\geq 1}$ is bounded in $\mathbb{H}^2$, each $A^i \in \mathbb{A}^2$ and is continuous.

Theorem 3.1 *Assume that $Y_t^i \uparrow Y_t$ for each t, with Y such that $\|Y\|_{\mathbb{S}^2} < \infty$. Assume further that $(g^i, Z^i)_{i\geq 1}$ is bounded in $\mathbb{H}^2$, and that each $A^i \in \mathbb{A}^2$ and is continuous. Then, Y has the form (3.1) where $g \in \mathbb{H}^2$, (g, Z) is the weak limit of $(g^i, Z^i)_i$, $A \in \mathbb{A}^2$. Moreover, $(Z^i)_i$ converges to Z in any $\mathbb{H}^p$ with $p \in [0, 2)$.*

Proof. By weak compactness, up to a subsequence, $(g^i, Z^i)_i$ converges weakly to some (g, Z). Then, for each stopping times τ, we have the weak convergences

$$\int_0^\tau g_s^i ds \to \int_0^\tau g_s ds \quad \text{and} \quad \int_0^\tau Z_s^i dW_s \to \int_0^\tau Z_s dW_s.$$

So A_τ^i weakly converges to

$$A_\tau := -Y_\tau + Y_0 + \int_0^\tau g_s ds + \int_0^\tau Z_s dW_s.$$

Using the fact that Y has the form $Y = B - A$ with B rcll and A non decreasing and a.s. finite, we show that it is rcll (and so is A). Indeed, $Y_{t+} = B_t - A_{t+} \le Y_t$ and on the other hand

$$Y_t \le Y_t^j + \varepsilon_j \le Y_{t+\iota}^j + \varepsilon_j + \varepsilon_j^\iota \le Y_{t+\iota} + \varepsilon_j + \varepsilon_j^\iota,$$

in which $\lim_{j\to\infty} \varepsilon_j \to 0$ and $\lim_{\iota\to 0} \varepsilon_j^\iota = 0$.
We now show that $(Z^i)_{i\ge 1}$ converges strongly in $\mathbb{H}^p$. Fix $\sigma < \tau$ two stopping times, then

$$\begin{aligned}
\mathbb{E}\left[\int_\sigma^\tau |Z_s - Z_s^i|^2 ds\right] &\le \mathbb{E}\left[|Y_\tau - Y_\tau^i|^2 + \sum_{t\in(\sigma,\tau]} |\Delta A_t|^2\right] \\
&\quad +2\mathbb{E}\left[\int_\sigma^\tau (Y_s - Y_s^i)(g_s - g_s^i)ds + \int_\sigma^\tau (Y_s - Y_s^i)dA_s\right] \\
&=: E_i^{\tau,\sigma} + \mathbb{E}\left[\sum_{t\in(\sigma,\tau]} |A_t - A_{t-}|^2\right]
\end{aligned}$$

because A is continuous, $(Y - Y^i)dA^i \ge 0$. The only difficult term to consider is the jumps' term. For this, we use the following which in proved in the Appendix of [52].

Lemma 3.1 *Let $A \in \mathbb{A}^2$, then for any $\delta, \varepsilon > 0$, there exists a finite number of pairs of stopping times $(\sigma_k, \tau_k)_{k\le K}$ such that $0 < \sigma_k \le \tau_k \le T$ and*

- *the sets $(\sigma_k, \tau_k]$ are disjoint.*
- $\sum_{k\le K} \mathbb{E}\left[\tau_k - \sigma_k\right] \ge T - \varepsilon$
- $\sum_{k\le K} \mathbb{E}\left[\sum_{\sigma_k < t \le \tau_k} |A_t - A_{t-}|^2\right] \le \delta\varepsilon/3.$

By using the above, for i large enough,

$$\sum_k \mathbb{E}\left[\int_{\sigma_k}^{\tau_k} |Z_s - Z_s^i|^2 ds\right] \le \sum_k E_i^{\tau_k,\sigma_k} + \delta\varepsilon/3 \le \delta\varepsilon/2$$

From this an the first item in the lemma, we see that (Z^i) converges in measure on the product space $\Omega \times [0, T]$, and therefore in $\mathbb{H}^p$ for $p < 2$, since the sequence is bounded in $\mathbb{H}^2$. $\square$

3.2 Stability of super-solutions of BSDEs

Given $\xi \in \mathbb{L}^2$, we consider the BSDE

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s,\ t \le T \tag{3.2}$$

in which $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$ and A is a bounded variation process. We shall always assume that Assumption 2.1 holds (although it will not be said explicitly).

Definition 3.1 (Super-solution) *Y is called a super-solution of* (1.1) *if A is non-decreasing.*

Before to provide the main stability result of this section, let us discuss two important properties.

Proposition 3.1 *Given $A \in \mathbb{A}^2$, there exists a unique solution to* (3.2) *such that $Y + A$ is continuous.*

Proof. Set $X := Y + A$. Then, (X, Z) solves

$$X_t = \xi + A_T + \int_t^T g_s(X_s - A_s, Z_s)ds - \int_t^T Z_s dW_s.$$

We can then use Theorem 2.1. □

Proposition 3.2 *If Y is a super-solution then the decomposition in* (3.2) *is unique.*

Proof. Let (Z, A) and (Z', A') denote two decomposition. By Itô's Lemma,

$$0 = 0 + \int_0^T |Z_s - Z'_s|^2 ds + \sum_t (A_t - A_{t-} - (A'_t - A'_{t-}))^2.$$

This implies that $Z = Z'$ and $A_0 = A'_0$. Hence,

$$\begin{aligned} Y_t + A_t &= Y_0 + A_0 - \int_0^t g_s(Y_s, Z_s)ds + \int_0^t Z_s dW_s \\ &= Y_0 + A'_0 - \int_0^t g_s(Y_s, Z'_s)ds + \int_0^t Z'_s dW_s \\ &= Y_t + A'_t, \end{aligned}$$

and therefore $A = A'$. □

We conclude this section with a stability result which is a consequence of Theorem 3.1.

Theorem 3.2 (Monotonic stability) *Let (Y^i, Z^i) solve* (3.2) *for some continuous $A^i \in \mathbb{A}^2$. Assume that $Y^i \uparrow Y$, with $\|Y\|_{\mathbb{S}^2} < \infty$. Then, $Y \in \mathbb{S}^2_{\mathrm{rcll}}$ and Y is a supersolution of* (1.1)*. Moreover, $(Z^i, A^i)_i$ is bounded in $\mathbb{H}^2 \times \mathbb{S}^2$.*

Proof. It suffices to show that $(g^i, Z^i)_i$ is bounded in $\mathbb{H}^2$, in which $-g^i = g(Y^i, Z^i)$, and to apply Theorem 3.1. Let us compute

$$\begin{aligned} \mathbb{E}\left[|A^i_T|^2\right] &= \mathbb{E}\left[\left(Y^i_0 - Y^i_T - \int_0^T g_s(Y^i_s, Z^i_s)ds + \int_0^T Z^i_s dW_s\right)^2\right] \\ &\leq \kappa \mathbb{E}\left[1 + \int_0^T |Z^i_s|^2 ds\right] \end{aligned}$$

for some $\kappa > 0$, while

$$\begin{aligned}\mathbb{E}\left[\int_0^T |Z_s^i|^2 ds\right] &\leq C\mathbb{E}\left[1 + \int_0^T |Y_s^i Z_s^i| ds + \int_0^T Y_s^i dA_s^i\right] \\ &\leq C_\kappa + \frac{1}{4}\mathbb{E}\left[\int_0^T |Z_s^i|^2 ds + \frac{1}{\kappa}|A_T^i|^2\right],\end{aligned}$$

for some $C_\kappa > 0$. We can then deduce from Theorem 3.1 that $(Z^i)_i$ strongly converges in $\mathbb{H}^p$, $p < 2$, to some Z. This implies that $g(Y^i, Z^i)$ converges in $\mathbb{H}^p$ to $g(Y, Z)$, which in turns has to coincide with the weak limit of the g^i's. □

3.3 g-supermartingale : decomposition, monotonic stability and down-crossings

From now on, we write $\mathcal{E}_{\cdot,\tau}[\xi]$ to denote component Y of the solution of the BSDE with driver g and terminal condition $\xi \in \mathbb{L}^2(\mathcal{F}_\tau)$ at the topping time τ. It is the g-expectation operator. In all this section, we assume that Assumption 2.1 holds for g.

Definition 3.2 (g-supermatingale) *Let X be progressively measurable. X is a g-martingale if $X = \mathcal{E}_{\cdot,T}[X_T]$. It is a g-supermatingale (in the strong sense) if for all stopping $\tau \leq T$, we have $\mathbb{E}\left[|X_\tau|^2\right] < \infty$ and $X \geq \mathcal{E}_{\cdot,\tau}[X_\tau]$.*

We now state the Doob-Meyer decomposition for right-continuous g-supermatingale . A version for processes admitting only left-limits can be found in [7].

Theorem 3.3 (Non-linear Doob-Meyer) *Let Y be a right-continuous g-supermatingale such that $\|Y\|_{\mathbb{S}^2} < \infty$. Then, Y is a rcll g-supersolution.*

Proof. We proceed by penalization. Consider the sequence of solutions

$$y_t^i = Y_T + \int_t^T g_s(y_s^i, z_s^i) ds + \int_t^T i(Y_s - y_s^i) ds - \int_t^T z_s^i dW_s. \tag{3.3}$$

We have $Y \geq y^i$. Otherwise, we could find $\delta > 0$ such that $\mathbb{P}\left[\tau > \sigma\right] > 0$ where

$$\sigma := \inf\{t : y_t^i \geq Y_t + \delta\} \wedge T \quad \text{and} \quad \tau := \inf\{t \geq \sigma : y_t^i \leq Y_t\}.$$

By right-continuity, $y_\sigma^i \geq Y_\sigma + \delta$ and $Y_\tau \geq y_\tau^i$ on $\{\sigma < \tau\}$, which contradicts the comparison principle of Proposition 2.3 combined with the fact that $y \geq Y$ on $[\![\sigma, \tau]\!]$: $Y_\sigma \geq \mathcal{E}_{\sigma,\tau}[Y_\tau] \geq \mathcal{E}_{\sigma,\tau}[y_\tau^i] \geq y_\sigma^i$. By comparison again, we then deduce that $(y^i)_i$ is a non-decreasing sequence bounded by Y. Hence, it converges to some $\hat{Y}$. We can then apply Theorem 3.2 to deduce that $\hat{Y}$ is a rcll g-supersolution. But on the other hand Theorem 3.2 also implies that

$$\limsup_{i\to\infty} i^2\mathbb{E}\left[\left(\int_t^T |Y_s - y_s^i|ds\right)^2\right] < \infty,$$

which implies that $\hat{Y} = Y$ (since both are right-continuous). □

Theorem 3.4 (Monotonic stability) *Let $(Y^i)_i$ be a sequence of rcll g-supermatingales that monotonically converges up to Y such that $\|Y\|_{\mathbb{S}^2} < \infty$. Then, Y is a rcll g-supermatingale.*

Proof. Let us consider (3.3) with Y^i in place of Y, and say that (y^i, z^i) is the solution. As above, we have $y^i \le Y^i$, and $(y^i)_i$ is non-decreasing by comparison (since Y^i is non-decreasing). Hence it converges to some $y \le Y$, which is a rcll super-solution, by Theorem 3.2. To see that $y = Y$, let us consider for $i \ge j$ the BSDE

$$y_t^{ij} = Y_T^j + \int_t^T g_s(y_s^{ij}, z_s^{ij})ds + \int_t^T i(Y_s^j - y_s^{ij})ds - \int_t^T z_s^{ij}dW_s.$$

Then, $y^i = y^{ii} \ge y^{ij}$ since $Y^i \ge Y^j$, but $y^{ij} \uparrow Y^j$ so $y \ge Y$. □

We conclude this section with a down-crossing inequality, which is due to Chen and Peng [18]. The version we provide here is an easy extension of their Theorem 6, see [19, Proposition 2.6].
Let $\mu \ge 0$ be the Lipschitz constant of g with respect to its z-variable. We denote by $\mathcal{E}^{\pm\mu}$ the non-linear expectation operator associated to the driver $(y, z) \mapsto \pm\mu|z|$. Given $a < b$ and a finite set $\{0, T\} \subset I \subset [0, T]$, we let $D_b^a(Y, I)$ be the number of down-crossings of the process $(Y_t)_{t\in I}$ from b to a on $[0, T]$.

Theorem 3.5 (Down-crossing) *Let Y be a g-supermartingale such that $\sup_{[0,T]} \|Y\|_{\mathbb{L}^2} < \infty$. Then,*

$$\mathcal{E}_{0,T}^{-\mu}[D_b^a(Y, I)] \le \frac{1}{b-a}\mathcal{E}_{0,T}^{\mu}[Y_0 \wedge b - Y_T \wedge b].$$

Remark 3.1 *As usual, this allows to prove that the paths of a g-supermartingale have a.s. left- and right-limits. As in the linear case, Y admits a rcll modification if and only if $t \mapsto \mathcal{E}_{0,T}[Y_t]$ is right-continuous. See [7, 19] for more details.*

Chapter 4

BSDE with constraints

In this chapter, we discuss various results on BSDE with constraints. All over this chapter, $\xi \in \mathbb{L}^2$, Assumption 2.1 holds, and $d = 1$.

4.1 Minimal supersolution under general constraints

We start from a general existence result which is an easy consequence of Theorem 3.4.

Let ϕ be a non negative adapted random map, Lipschiz in (y, z). We want to find a minimal solution $(Y, Z, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{A}^2$ which solves (3.2) and satisfies

$$\phi(Y, Z) = 0 \;\; dt \times d\mathbb{P} - a.e. \tag{4.1}$$

Theorem 4.1 (Minimal super-solution) *Assume that a solution* $(\hat{Y}, \hat{Z}, \hat{A}) \in \mathbb{S}^2_{\mathrm{rcll}} \times \mathbb{H}^2 \times \mathbb{A}^2$ *to* (3.2)-(4.1) *exists. Then, there exists a minimal one* $(Y, Z, A) \in \mathbb{S}^2_{\mathrm{rcll}} \times \mathbb{H}^2 \times \mathbb{A}^2$*. Moreover,* Y *is the increasing limit of* (Y^i) *defined through*

$$Y^i_t = \xi + \int_t^T g_s(Y^i_s, Z^i_s)ds + \int_t^T i\phi_s(Y^i_s, Z^i_s)ds - \int_t^T Z^i_s dW_s.$$

Proof. By comparison, $Y^i \le Y^{i+1} \le \hat{Y}$. So $Y^i \uparrow$ to some $Y \le \hat{Y}$. By Theorem 3.4, Y is a g-supersolution in $\mathbb{S}^2_{\mathrm{rcll}}$. Moreover, Theorem 3.2 implies that

$$\limsup_i \; i^2 \mathbb{E}\left[\left(\int_t^T \phi_s(Y^i_s, Z^i_s)ds\right)^2\right] < \infty$$

so that, since $(Y^i, Z^i)_i$ converges in $\mathbb{H}^p$, for $p < 2$, the constraint (4.1) is satisfied. □

4.2 Reflected BSDEs

We now turn to the case where the constraint is a lower bound on Y. More precisely, let S be a continuous process in $\mathbb{S}^2$, $\xi \in \mathbb{L}^2$. We look for minimal solution $(Y, Z, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{A}^2$ such that (3.2) holds with

$$Y \geq S \quad \text{on } [0, T]. \tag{4.2}$$

We assume that $S_T \leq \xi$ without loss of generality.

We already know from the previous section that a minimal solution exists, if at least one solution exists. In the present case, we will drop the a-priori existence assumption, and we shall say more on the structure of the minimal solution. In particular, we shall see that the minimality can be characterized by the minimal effort constraint:

$$\int_0^T (Y_s - S_s) dA_s = 0. \tag{4.3}$$

The following is based on El Karoui et al. [30]. We refer to Lepeltier and Xu [43] for the case of a r.c.l.l. barrier.

4.2.1 Existence and minimality

We first show that the constraint (4.3) actually characterizes the minimal solution. This is an easy consequence of Tanaka's formula.

Theorem 4.2 (Minimality) *Assume that* $(Y, Z, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{A}^2$ *solves* (3.2)-(4.2)-(4.3). *Then, it is the minimal solution to* (3.2)-(4.2).

Proof. Let (Y', Z', A') be a solution of (3.2) with the constraint $Y' \geq S$. Then, we can use Tanaka formula to obtain

$$\begin{aligned}
(Y_t - Y'_t)^+ &\leq \int_t^T \mathbf{1}_{Y_s > Y'_s}(g_s(Y_s, Z_s) - g_s(Y'_s, Z'_s))ds - \int_t^T \mathbf{1}_{Y_s > Y'_s}(Z_s - Z'_s)dW_s \\
&\quad + \int_t^T \mathbf{1}_{Y_s > Y'_s} d(A_s - A'_s) \\
&\leq \int_t^T \mathbf{1}_{Y_s > Y'_s}(g_s(Y_s, Z_s) - g_s(Y'_s, Z'_s))ds - \int_t^T \mathbf{1}_{Y_s > Y'_s}(Y_s - Y'_s)(Z_s - Z'_s)dW_s
\end{aligned}$$

since $Y > Y'$ implies $Y > S$. We then linearize the equation to obtain, for some bounded processes a and b, that

$$\begin{aligned}
(Y_t - Y'_t)^+ &\leq \int_t^T \mathbf{1}_{Y_s > Y'_s}(b_s(Y_s - Y'_s) + a_s(Z_s - Z'_s))ds - \int_t^T \mathbf{1}_{Y_s > Y'_s}(Y_s - Y'_s)(Z_s - Z'_s)dW_s \\
&\leq \int_t^T C(Y_s - Y'_s)^+ + a_s(Z_s - Z'_s))ds - \int_t^T \mathbf{1}_{Y_s > Y'_s}(Y_s - Y'_s)(Z_s - Z'_s)dW_s.
\end{aligned}$$

We can the find a probability measure $\bar{\mathbb{P}}$ such that

$$(Y_t - Y'_t)^+ \;\; \leq \;\; C\mathbb{E}^{\bar{\mathbb{P}}}[\int_t^T (Y_s - Y'_s)^+ ds \; |\mathcal{F}_t].$$

We conclude by Gronwall's Lemma. $\square$

We now prove that a solution exists.

Theorem 4.3 (Existence) *There exists a solution* $(Y, Z, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{A}^2$ *to* (3.2)-(4.2)-(4.3).

Proof. We proceed by penalization and consider the BSDEs

$$Y^n = \xi + \int_\cdot^T \left(g_s(Y_s^n, Z_s^n) + n(Y_s^n - S_s)^-\right) ds - \int_\cdot^T Z_s^n dW_s. \tag{4.4}$$

By comparison, $(Y^n)_n$ is non-decreasing. Let us set $A_t^n := \int_0^t n(Y_s^n - S_s)^- ds$.
Step 1. We first prove that

$$\mathbb{E}\left[|Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + |A_T^n|^2\right] \leq C.$$

To see this, let us use Itô's lemma to obtain that, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}\left[|Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds\right] &\leq \mathbb{E}\left[|\xi|^2 + \int_t^T \left(C(1 + |Y_s^n|^2) + \frac{1}{2}|Z_s^n|^2\right) ds\right] \\
&\quad + \mathbb{E}\left[2\int_t^T Y_s^n n(Y_s^n - S_s)^- ds\right] \\
&\leq \mathbb{E}\left[|\xi|^2 + \int_t^T \left(C(1 + |Y_s^n|^2) + \frac{1}{2}|Z_s^n|^2\right) ds\right] \\
&\quad + 2\mathbb{E}\left[\varepsilon^{-1} \sup_{[t,T]} |S^+|^2 + \varepsilon|A_T^n - A_t^n|^2\right]
\end{aligned}$$

in which we used the fact that $y(y - x)^- = 0\mathbf{1}_{y \geq x} + y(x - y)\mathbf{1}_{y \leq x} \leq x(y - x)^-$.
On the other hand, it follows from (4.4) and the linear growth of g that

$$\mathbb{E}\left[|A_T^n - A_t^n|^2\right] \;\; \leq \;\; C\mathbb{E}\left[|\xi|^2 + |Y_t^n|^2 + \int_t^T (1 + |Y_s^n|^2 + |Z_s^n|^2) ds\right].$$

Combining the above for ε small and using Gronwall's lemma leads to the required result.
Step 2. The result of Step 1. implies that $Y^n \uparrow Y \in \mathbb{H}^2$. We now prove that

$$\mathbb{E}\left[\sup_{[0,T]} |(Y^n - S)^-|^2\right] \to 0.$$

Let $(\bar{Y}^n, \bar{Z}^n)$ be the solution associated to the diver $g(Y^n, Z^n) - n(y - S)$. Then, $Y^n \geq \bar{Y}^n$ and

$$\bar{Y}^n_\tau = \mathbb{E}\left[\xi e^{-n(T-\tau)} + \int_\tau^T (g_s(Y^n_s, Z^n_s) + nS_s)e^{-n(s-\tau)}ds|\mathcal{F}_\tau\right].$$

Passing to the limit $n \to \infty$, we obtain $\bar{Y}^n_\tau \to \xi \mathbf{1}_{\tau=T} + S_\tau \mathbf{1}_{\tau<T}$. This implies that $Y_\tau \geq S_\tau$, and so $Y \geq S$ on $[0,T]$ by [24, Thm 86 Chap IV]. In particular, $(Y^n - S)^- \downarrow 0$, which by Dini's theorem implies that its sup over $[0,T]$ also converges to 0. It remains to use the dominated convergence after noticing that $(Y^n - S)^- \leq (Y^1 - S)^-$.

Step 3. We now show that $(Y^n, Z^n, A^n) \to (Y, Z, A)$ in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{S}^2$.

Fix $\Delta Y = Y^n - Y^p$ for $p \geq n$ and defined ΔZ and ΔA similarly. Then,

$$|\Delta Y_t|^2 + \mathbb{E}\left[\int_t^T |\Delta Z_s|^2 ds \;|\mathcal{F}_t\right] \;\leq\; \mathbb{E}\left[\int_t^T \left(C|\Delta Y_s|^2 + \frac{1}{2}|\Delta Z_s|^2\right) ds + 2\int_t^T \Delta Y_s d\Delta A_s \;|\mathcal{F}_t\right].$$

But since $Y^p \geq Y^n$

$$\Delta Y_s d\Delta A_s \leq -(Y^n - Y^p)p(Y^p - S)^- \leq (Y^n - S)^- dA^p$$

so that

$$\frac{1}{2}\mathbb{E}\left[\int_0^T |\Delta Z_s|^2 ds\right] \;\leq\; \mathbb{E}\left[\int_t^T C|\Delta Y_s|^2 ds + 2\sup_{[0,T]}(Y^n - S)^- A^p_T\right].$$

Letting n and then p go to ∞, this shows that Z^n converges to some Z in $\mathbb{H}^2$. The convergence of $(Y^n)_n$ in $\mathbb{S}^2$ is then deduced by applying Itô's Lemma as above and by using the Burkholder-Davis-Gundy inequality. The convergence of $(A^n)_n$ follows.

Step 4. We know from Theorem 3.2 that (Y, Z, A) solves the BSDE. It remains to prove that (4.3) holds. But,

$$\int_0^T (Y^n_s - S_s)dA^n_s = -n\int_0^T |(Y^n_s - S_s)^-|^2 ds \leq 0,$$

while $(A^n)_n$ and $(Y^n - S)_n$ converges in $\mathbb{S}^2$. □

As for non-reflected BSDEs, we can provide a comparison result. Here again, we appeal to Tanaka's formula.

Proposition 4.1 (Comparison) *Let g satisfy Assumption 2.1 holds. Let (Y, Z, A) solve* (3.2)-(4.2)-(4.3). *Given $\xi' \in \mathbb{L}^2$, $S' \in \mathbb{S}^2$, and a map g' such that the corresponding solution (Y', Z', A') satisfies $g' := g'(Y', Z') \in \mathbb{H}^2$. If $\xi \leq \xi'$, $S \leq S'$ and $g(Y', Z') \leq g'(Y', Z')$, then $Y \leq Y'$. If in addition g' satisfies Assumption 2.1 and $S \geq S'$ then $dA \geq dA'$.*

Proof. Set $\Delta X = X - X'$. By Tanaka formula

$$\begin{aligned}(\Delta Y)^+ &\leq (\Delta\xi)^+ + \int_t^T \mathbf{1}_{Y_s > Y'_s}(g_s(Y_s, Z_s) - g'_s)ds \\ &\quad + \int_t^T \mathbf{1}_{Y_s > Y'_s} d\Delta A_s - \int_t^T \mathbf{1}_{Y_s > Y'_s}(Z_s - Z'_s)dW_s.\end{aligned}$$

But $S \leq S' \leq Y'$ implies that $dA = 0$ on $\{Y > Y'\}$. Thus

$$(\Delta Y)^+ \leq (\Delta\xi)^+ + \int_t^T \mathbf{1}_{Y_s > Y'_s}(g_s(Y_s, Z_s) - g'_s)ds - \int_t^T \mathbf{1}_{Y_s > Y'_s}(Z_s - Z'_s)dW_s.$$

It suffices to linearize as in the proof of Proposition 2.3 to conclude on the first assertion of the Proposition.
When g' is Lipschitz, Y' is obtained by the penalization scheme of Theorem 4.3. By comparison on this scheme, we have $Y^{n'} \geq Y^n$ so that $n(Y^n - S)^- \geq n(Y^{n'} - S')^-$. Since, by the proof of Theorem 4.3, the integrals of these two processes converge to A and A', the result follows. □

4.2.2 The case of a smooth barrier

In the case where the barrier is an Itô process, one can even provide a control on the bounded variation process A.

Proposition 4.2 *Assume that*

$$dS_t = \mu_t dt + \sigma_t dW_t$$

for some a.s. square integrable and predictable processes μ and σ. Let $(Y, Z, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{S}^2$ be a solution of (3.2)-(4.2)-(4.3)*. Then, $Z = V$ $d\mathbb{P} \times dt$ on $\{Y = S\}$. Moreover, there exists a predictable process α with values in $[0,1]$ such that*

$$dA = \alpha \mathbf{1}_{Y=S} \left[g(Y, Z) + \mu\right]^- dt.$$

Proof. We use Tanaka formula and (4.3) to obtain

$$d(Y - S)^+ = -\mathbf{1}_{Y>S}[g(Y, Z) + \mu]dt + \mathbf{1}_{Y>S}(Z - \sigma)dW + dL,$$

in which L is the local time of $Y - S$ at 0. But $(Y - S)^+ = (Y - S)$, and therefore

$$-\mathbf{1}_{Y>S}[g(Y, Z) + \mu]dt + \mathbf{1}_{Y>S}(Z - \sigma)dW + dL = -[g(Y, Z) + \mu]dt + (Z - \sigma)dW - dA,$$

so that $Z = \sigma$ on $\{Y = S\}$ and

$$dA \leq -\mathbf{1}_{Y=S}\left[g(Y, Z) + \mu\right]dt.$$

□

4.2.3 Link with optimal stopping problems

We conclude this section with a (not surprising) Snell-envelop style characterization.

Proposition 4.3 *Let* $(Y, Z, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{A}^2$ *be a solution of* (3.2)-(4.2)-(4.3). *Then,*

$$Y_t = \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\xi \mathbf{1}_{\tau=T} + S_\tau \mathbf{1}_{\tau<T} + \int_t^\tau g_s(Y_s, Z_s)ds \mid \mathcal{F}_t\right],$$

in which $\mathcal{T}_t$ *denotes the set of stopping times with values in* $[t, T]$.

Proof. One inequality is trivial. On the other hand, let τ_n be the first time s after t such that $Y_s \le S_s + \frac{1}{n}$ or $s = T$. On $[t, \tau_n]$, we have $dA = 0$. Hence,

$$Y_t = \mathbb{E}\left[Y_{\tau_n} + \int_t^{\tau_n} g_s(Y_s, Z_s)ds \mid \mathcal{F}_t\right].$$

Note that $\tau_n \uparrow \hat{\tau} \in \mathcal{T}_t$. Passing to the limit in the above leads to

$$Y_t = \mathbb{E}\left[\xi \mathbf{1}_{\hat{\tau}=T} + S_{\hat{\tau}} \mathbf{1}_{\hat{\tau}<T} + \int_t^{\hat{\tau}} g_s(Y_s, Z_s)ds \mid \mathcal{F}_t\right].$$

□

4.2.4 Further readings

Many works have been done on reflected BSDEs. Let us a quote a few of them in which the interested reader will be able to find further references.
Reflected BSDEs with very irregular lower bounds have been studied by Pend and Xu [54]. They also considered double reflections, i.e. from above and below. Applications of doubly reflected BSDEs to Dynkin games can be found in Hamadène and Zhang [36]. For multivariate BSDEs with reflections, we refer to Chassagneux, Elie and Kharroubi [15], they have applications in optimal switching problems. Obviously, one can also consider reflected BSDEs with quadratic drivers, see e.g. Bayraktar and Yao [2]. As for numerical schemes, see Bouchard and Chassagneux [3], and Chassagneux, Elie and Kharroubi [16].

4.3 Constraints on the gain process

In this section, we provide a dual representation for BSDEs with constraint on the Z-component. They have been introduced in Cvitanic, Karatzas and Soner [21]. An extension of their existence result and new regularity properties have been obtained recently by Bouchard, Elie and Moreau [5]. For sake of simplicity, we consider here the situation where g is deterministic and the constraint is of the form

$$Z \in \mathcal{O}\ dt \times d\mathbb{P} \tag{4.5}$$

with $\mathcal{O}$ a convex closed set which contains 0.
To provide our characterization of the minimal solution, let us define the support function

$$\delta : u \in \mathbb{R}^d \mapsto \sup\{u \cdot z,\ z \in \mathcal{O}\}.$$

Let us also define $\mathcal{U}$ as the class of bounded progressively measurable processes with values in the domain of $\mathcal{O}$ $dt \times d\mathbb{P}$-a.e. Given $\nu \in \mathcal{U}$, we let $\mathbb{P}^\nu$ be the probability measure of density given by the Doléans-Dade exponential of $\int_0^\cdot \nu_s dW_s$, and denote by $W^\nu := W - \int_0^\cdot \nu_s ds$ the corresponding $\mathbb{P}^\nu$-Brownian motion. Then, given $X \in \mathbb{L}^\infty(\mathcal{F}_\tau)$, $\tau \in \mathcal{T}$, we define $\mathcal{E}^\nu_{\cdot,\tau}[X]$ as the Y-component of the solution (Y, Z) of the BSDE

$$Y = X + \int_\cdot^\tau (g_s(Y_s, Z_s) - \delta(\nu_s))\, ds - \int_\cdot^\tau Z_s dW_s^\nu.$$

Theorem 4.4 (Minimality) *Assume that there exists an adapted r.c.l.l. process $\hat{Y}$ such that*[1]

$$\hat{Y}_\tau = \operatorname{esssup}\left\{\mathcal{E}^\nu_{\tau,T}[\xi],\ \nu \in \mathcal{U},\ \nu \mathbf{1}_{[\![0,\tau]\!]} = 0\right\},\ \forall\, \tau \in \mathcal{T},$$

and $\|\hat{Y}\|_{\mathbb{S}^2} < \infty$. Then, the minimal super-solution (Y, Z) of (1.1)-(4.5) satisfies $Y = \hat{Y}$.

Proof. Let (Y, Z) be a super-solution. Then, for $\tau \le \sigma \le T$,

$$\begin{aligned} Y_\tau &\ge Y_\sigma + \int_\tau^\sigma g_s(Y_s, Z_s) ds - \int_\tau^\sigma Z_s dW_s \\ &= Y_\sigma + \int_\cdot^T (g_s(Y_s, Z_s) - \nu_s Z_s) ds - \int_\tau^\sigma Z_s dW_s^\nu \\ &\ge Y_\sigma + \int_\tau^\sigma (g_s(Y_s, Z_s) - \delta(\nu_s)) ds - \int_\tau^\sigma Z_s dW_s^\nu \end{aligned}$$

since $\nu Z \le \delta(\nu)$ if $Z \in \mathcal{O}$. This implies that Y is a super-solution of the BSDE with driver $g(\cdot) - \delta(\nu)$ under $\mathbb{P}^\nu$, and therefore a $\mathcal{E}^\nu$-super-martingale. Hence, $Y_\tau \ge \mathcal{E}^\nu_{\tau,T}[\xi]$ for all $\nu \in \mathcal{U}$.
Conversely, it is not difficult to deduce from the definition of $\hat{Y}$ that it satisfies a dynamic programming principle[2]

$$\hat{Y}_\tau = \operatorname{esssup}\left\{\mathcal{E}^\nu_{\tau,\sigma}[\hat{Y}_\sigma],\ \nu \in \mathcal{U}\right\},\ \forall\, \tau \le \sigma \in \mathcal{T}.$$

Then, we can use the decomposition of Theorem 3.3 to deduce that, for each $\nu \in \mathcal{U}$, we can find $(\hat{Z}^\nu, \hat{A}^\nu) \in \mathbb{H}^2(\mathbb{P}^\nu) \times \mathbb{A}^2(\mathbb{P}^\nu)$ such that

$$\hat{Y}_\tau = \xi + \int_\tau^T (g_s(\hat{Y}_s, \hat{Z}_s^\nu) - \delta(\nu_s)) ds + \hat{A}_T^\nu - \hat{A}_\tau^\nu - \int_\tau^T \hat{Z}_s^\nu dW_s^\nu.$$

[1]See [7] for a recent more general statement.
[2]See e.g. [5].

Identifying the quadratic variation terms implies that $\hat{Z}^\nu = \hat{Z}^0 =: \hat{Z}$. Thus,

$$e := \int_0^T (\nu_s \hat{Z}_s - \delta(\nu_s))ds \leq \int_0^T (\nu_s \hat{Z}_s - \delta(\nu_s))ds + \hat{A}_T^\nu - \hat{A}_0^\nu = \hat{A}_T^0 - \hat{A}_0^0.$$

If $\hat{Z}$ does not take values in $\mathcal{O}$ $dt \times d\mathbb{P}$-a.e., then a measurable selection argument[3] allows us to construct $\hat{\nu} \in \mathcal{U}$ such that

$$e := \int_0^T (\hat{\nu}_s \hat{Z}_s - \delta(\hat{\nu}_s))ds$$

satisfies $e \geq 0$ and $\mathbb{P}[e > 0] > 0$. Since $(\lambda \hat{\nu}_s \hat{Z}_s - \delta(\lambda \hat{\nu}_s)) = \lambda e$ for $\lambda > 0$, this contradicts the above upper-bound: $\lambda e \leq \hat{A}_T^0 - \hat{A}_0^0$ for all $\lambda \geq 0$. □

Remark 4.1 *We refer to [5, 21] for sufficient conditions under which the r.c.l.l. process $\hat{Y}$ exists. When g is linear, this can be proved by using the aggregation arguments contained in El Karoui [28].*

[3] See [5].

Bibliography

[1] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic bsdes. *The Annals of Probability*, 41(3B):1831–1863, 2013.

[2] E. Bayraktar and S. Yao. Quadratic reflected bsdes with unbounded obstacles. *Stochastic processes and their applications*, 122(4):1155–1203, 2012.

[3] B. Bouchard and J.-F. Chassagneux. Discrete-time approximation for continuously and discretely reflected bsdes. *Stochastic Processes and their Applications*, 118(12):2269–2293, 2008.

[4] B. Bouchard and R. Elie. Discrete-time approximation of decoupled forward–backward sde with jumps. *Stochastic Processes and their Applications*, 118(1):53–75, 2008.

[5] B. Bouchard, R. Elie, and L. Moreau. Regularity of bsdes with a convex constraint on the gains-process. *arXiv preprint arXiv:1409.5369*, 2014.

[6] B. Bouchard, R. Elie, and A. Réveillac. Bsdes with weak terminal condition. *To appear in The Annals of Probability*, 2012.

[7] B. Bouchard, D. Possamai, and X. Tan. A general doob-meyer-mertens decomposition for g-supermartingale systems. *arXiv preprint arXiv:1505.00597*, 2015.

[8] B. Bouchard and N. Touzi. Discrete-time approximation and monte-carlo simulation of backward stochastic differential equations. *Stochastic Processes and their applications*, 111(2):175–206, 2004.

[9] B. Bouchard and X. Warin. Monte-carlo valuation of american options: facts and new algorithms to improve existing methods. In *Numerical methods in finance*, pages 215–255. Springer, 2012.

[10] P. Briand. *Equations Différentielles Stochastiques Rétrogrades*. Lectures notes of University of Savoie.

[11] P. Briand and R. Elie. A simple constructive approach to quadratic bsdes with or without delay. *Stochastic processes and their applications*, 123(8):2921–2939, 2013.

[12] P. Briand and Y. Hu. Bsde with quadratic growth and unbounded terminal value. *Probability Theory and Related Fields*, 136(4):604–618, 2006.

[13] Ph Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica. L p solutions of backward stochastic differential equations. *Stochastic Processes and their Applications*, 108(1):109–129, 2003.

[14] M. Broadie, J. Cvitanić, and H.M. Soner. Optimal replication of contingent claims under portfolio constraints. *Review of Financial Studies*, 11(1):59–79, 1998.

[15] J.-F. Chassagneux, R. Elie, and I. Kharroubi. A note on existence and uniqueness for solutions of multidimensional reflected bsdes. *Electronic Communications in Probability*, 16:120–128, 2011.

[16] J.-F. Chassagneux, R. Elie, and I. Kharroubi. Discrete-time approximation of multidimensional bsdes with oblique reflections. *The Annals of Applied Probability*, 22(3):971–1007, 2012.

[17] J.-F. Chassagneux and A. Richou. Numerical simulation of quadratic bsdes. *arXiv preprint arXiv:1307.5741*, 2013.

[18] Z. Chen and S. Peng. A general downcrossing inequality for g-martingales. *Statistics & probability letters*, 46(2):169–175, 2000.

[19] F. Coquet, Y. Hu, J. Mémin, and S. Peng. Filtration-consistent nonlinear expectations and related g-expectations. *Probability Theory and Related Fields*, 123(1):1–27, 2002.

[20] J. Cvitanić and I. Karatzas. Hedging contingent claims with constrained portfolios. *The Annals of Applied Probability*, 3(3):652–681, 1993.

[21] J. Cvitanic, I. Karatzas, and H. M. Soner. Backward stochastic differential equations with constraints on the gains-process. *Annals of Probability*, pages 1522–1551, 1998.

[22] J. Cvitanić, I. Karatzas, and H.M. Soner. Backward stochastic differential equations with constraints on the gains-process. *Ann. Probab.*, 26(4):1522–1551, 1998.

[23] R. W.R. Darling and E. Pardoux. Backwards sde with random terminal time and applications to semilinear elliptic pde. *The Annals of Probability*, 25(3):1135–1159, 1997.

[24] C. Delacherie and P.A. Meyer. Probabilities and potential. *Math. Studies*, (29), 1978.

[25] F. Delbaen, Y. Hu, and X. Bao. Backward sdes with superquadratic growth. *Probability Theory and Related Fields*, 150(1-2):145–192, 2011.

[26] F. Delbaen, Y. Hu, and A. Richou. On the uniqueness of solutions to quadratic bsdes with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat*, 47(2):559–574, 2011.

[27] F. Delbaen, Y. Hu, and A. Richou. On the uniqueness of solutions to quadratic bsdes with convex generators and unbounded terminal conditions: the critical case. *arXiv preprint arXiv:1303.4859*, 2013.

[28] N. El Karoui. Les aspects probabilites du contrôle stochastique. *Ecole d'été de probabilités de Saint-Flour VIII, 1979, Lecture Notes in Mathematics*, 876, 1981.

[29] N. El Karoui, S. Hamadène, and A. Matoussi. Backward stochastic differential equations and applications. *Indifference pricing: theory and applications*, pages 267–320, 2008.

[30] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M.-C. Quenez. Reflected solutions of backward sde's, and related obstacle problems for pde's. *the Annals of Probability*, 25(2):702–737, 1997.

[31] N. El Karoui and L. Mazliak. *Backward stochastic differential equations*, volume 364. CRC Press, 1997.

[32] N. El Karoui, S. Peng, and M.C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.

[33] H. Föllmer and P. Leukert. Quantile hedging. *Finance and Stochastics*, 3(3):251–273, 1999.

[34] H. Föllmer and P. Leukert. Efficient hedging: cost versus shortfall risk. *Finance and Stochastics*, 4(2):117–146, 2000.

[35] E. Gobet. *Méthodes de Monte-Carlo et processus stochastiques: du linéaire au non-linéaire*. Editions de l'Ecole Polytechnique, 2013.

[36] S. Hamadene and J. Zhang. The continuous time nonzero-sum dynkin game problem and application in game options. *SIAM Journal on Control and Optimization*, 48(5):3659–3669, 2010.

[37] U. Horst, Y. Hu, P. Imkeller, A. Réveillac, and J. Zhang. Forward–backward systems for expected utility maximization. *Stochastic Processes and their Applications*, 124(5):1813–1848, 2014.

[38] Y. Hu, P. Imkeller, and M. Muller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15:1691–1712, 2005.

[39] N. Kazamaki. *Continuous exponential martingales and BMO*. Springer, 1994.

[40] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability*, pages 558–602, 2000.

[41] J.-P. Lepeltier and J. San Martin. Backward stochastic differential equations with continuous coefficient. *Statistics & Probability Letters*, 32(4):425–430, 1997.

[42] J.-P. Lepeltier and J. San Martin. Existence for bsde with superlinear–quadratic coefficient. *Stochastics: An International Journal of Probability and Stochastic Processes*, 63(3-4):227–240, 1998.

[43] J-P Lepeltier and M. Xu. Penalization method for reflected backward stochastic differential equations with one rcll barrier. *Statistics & probability letters*, 75(1):58–66, 2005.

[44] J. Liu and J. Ren. Comparison theorem for solutions of backward stochastic differential equations with continuous coefficient. *Statistics & probability letters*, 56(1):93–100, 2002.

[45] X. Mao. Adapted solutions of backward stochastic differential equations with non-lipschitz coefficients. *Stochastic Processes and their Applications*, 58(2):281–292, 1995.

[46] D. Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[47] E. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic pdes of second order. In *Stochastic Analysis and Related Topics VI*, pages 79–127. Springer, 1998.

[48] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[49] E. Pardoux, F. Pradeilles, and Z. Rao. Probabilistic interpretation of a system of semi-linear parabolic partial differential equations. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 33(4):467–490, 1997.

[50] S. Peng. A general stochastic maximum principle for optimal control problems. *Siam J. of Control and Optimization*, 28(4):966–979, 1990.

[51] S. Peng. Backward stochastic differential equations and applications to optimal control. *Appl Math Optim*, 27:125–144, 1993.

[52] S. Peng. Monotonic limit theorem of bsde and nonlinear decomposition theorem of doob–meyers type. *Probability theory and related fields*, 113(4):473–499, 1999.

[53] S. Peng. Backward stochastic differential equation, nonlinear expectation and their applications. In *Proceedings of the International Congress of Mathematicians*, volume 1, pages 393–432, 2011.

[54] S. Peng and M. Xu. The smallest g-supermartingale and reflected bsde with single and double l2 obstacles. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 41(3):605–630, 2005.

[55] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[56] A.B. Sow and E. Pardoux. Probabilistic interpretation of a system of quasilinear parabolic pdes. *Stochastics and Stochastic Reports*, 76(5):429–477, 2004.

[57] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic processes and their Applications*, 118(3):503–515, 2008.

[58] J. Zhang. A numerical scheme for bsdes. *The Annals of Applied Probability*, 14(1):459–488, 2004.