Stochastic Control for Optimal Trading:  
State of Art and Perspectives  
(an attempt of)  

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Market Micro-Structure - Confronting View Points - December 2012
Spirit of this talk

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In short: I will essentially only say what can be done by using available tools from optimal control.

If you are interested, the references are given at the end of these slides.
Outline

**Optimal book liquidation**
- The Almgren and Chriss framework for aggressive strategies
- Controlling the intensity of order matching: non-aggressive strategies
- Towards a control of liquidation robots

**Searching for the good market**
- In a nutshell...

**Pricing problems**
- Inverse versus direct approach
- Pricing under uncertainty or with adverse player

**Optimization under risk constraint**
- Coming back to what we know....

**Perspectives**
- Last slide...

- $Q$ stocks to liquidate.
- In discrete time: $t_n := n\nu_N$ with $\nu_N := T/N$.
- Price dynamics with market impact

$$S_{t_n} = S_{t_{n-1}} + \sigma\nu_N^{1/2}\xi_{t_n} - \nu_N g(\Delta q_{t_n}/\nu_N)$$

where

- $\Delta q_{t_n} = q_{t_n} - q_{t_{n-1}}$ is the number of stocks sold between $t_{n-1}$ and $t_n$,
- $(\xi_{t_n})_n$ are iid with mean 0 and variance 1.
- $g$ is the permanent impact function
A simple model for book liquidation

- Terminal liquidation gain with $q_{tN} = Q$

$$V_T^q = QS_0 + \sum_{n=1}^{N} \left( \sigma l_{N}^{\frac{1}{2}} \xi_{tn} - l_{N} g(\Delta q_{tn}) \right) \sum_{k=n+1}^{N} \Delta q_{tk}$$

$$- \sum_{n=1}^{N} \Delta q_{tn} h(\Delta q_{tn} / l_{N})$$

where $h$ stands for the temporary impact.

- The shortfall due to volatility and market impacts is

$$QS_0 - V_T^q = \sum_{n=1}^{N} \left( -\sigma l_{N}^{\frac{1}{2}} \xi_{tn} + l_{N} g(\Delta q_{tn}) \right) \sum_{k=n+1}^{N} \Delta q_{tk}$$

$$+ \sum_{n=1}^{N} \Delta q_{tn} h(\Delta q_{tn} / l_{N})$$
Mean variance criteria and explicit resolution

- If one restricts to deterministic strategies, one can compute explicitly $E[QS_0 - V^q_T]$ and $\text{Var}[QS_0 - V^q_T]$ and try to solve
  \[
  \min_{q: q_t N = Q} (E[QS_0 - V^q_T] + \lambda \text{Var}[QS_0 - V^q_T]).
  \]

- In Almgren and Chriss, this is done for a specific linear setting
  
  \[
  h(\Delta q_{tn} / \iota_N) = \epsilon \text{sign}(\Delta q_{tn}) + \eta \Delta q_{tn} \quad \text{and} \quad g(\Delta q_{tn}) = \gamma \Delta q_{tn}
  \]

- Because of the dynamics and the criteria, adapted (random) strategies would provide the same result.
General comments

- Simple and meaningful optimal strategy computed explicitly.
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- Discrete time setting: How can we compute an optimal time grid? What about being detected by the other participants?
- Market volume? Intraday volatility moves?
Possible extension

Continuous time versions of the same model:

- Forsyth et al. (2011 and 2012), and Gatheral and Schied (2011) study the continuous time setting with prices given by GBM or ABM.
- Again it is explicit: for the same reason than in discrete time.
- Both use proxies of variance (mean quadratic variation) or of VaR (expected cost + proxy of time average VaR).
- LOB shape more taken into account in Alfonsi, Fruth and Schied (2010), and in Obizhaeva and Wang (2012): still enough simple to be explicit.
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One can use general optimal control techniques to study more general settings. This is done via an impulse control approach in e.g. Kharroubi and Pham (2010).

- Less explicit but can solve pdes and find the optimal control out of it.
- More time consuming but:
  - Provides a general idea of the optimal intervention frontiers.
  - What are the important parameters.
  - Can compute abacus / compress the information computed off-line once for all.
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Motivation and main idea

• Motivation :
  ▶ Usually follow an Almgren and Chriss type strategy and then try to optimize the real passage of order to the LOB: only focus on aggressive orders.
  ▶ Passive orders represents most of orders passed by trading algorithms.
  ▶ Take immediately into account the risk of passive orders not being executed in the global strategy.
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- **Main idea**:
  - Propose continuously an ask quote: \( S_t^a = S_t + \delta_t^a \).
  - The number of sold shares evolves according to a jump process \( N^a \) with jumps of size 1.
  - The intensity of \( N^a \) at \( t \) is \( \lambda(\delta_t^a) := \lambda_0 e^{-k\delta_t^a} \).
    - If \( \delta_t^a = 0 \), the time to be executed is exponentially distributed with parameter \( \lambda_0 \).
    - The greater \( \delta_t^a \) is, the longer one has to wait.
Explicit resolution

• In Guéant, Lehalle and Tapia (2010) :
  ▶ The reference price \( S \) is an ABM.
  ▶ The aim is to maximize an exponential utility function.
  ▶ The HJB equation drops down to a simple system of linear ODEs.
  ▶ Given the solution of the system of ODEs, the optimal strategy is explicit.

• In Bayraktar and Ludkovski (2012) :
  ▶ The reference price \( S \) is just a martingale.
  ▶ The aim is to maximize an expectation.
  ▶ The solution is explicit.

Other works are dealing with similar but more complex models : one can only obtain HJB equations that have to be solved numerically or analyzed for small inventory expansion (e.g. Avellaneda and Stoikov 2008, and Guilbaud and Pham 2012).
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Motivation

- **Motivation:**
  - Once a liquidation strategy is determined, orders are usually executed by trading robots.
  - Trading robots are optimized according to a given criteria.
  - The trader chooses the robot according to market conditions.
  - The different slices are executed by different robots, with different sets of parameters.
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  - The trader chooses the robot according to market conditions.
  - The different slices are executed by different robots, with different sets of parameters.
  - How can one optimize this use given a set of parameterized robots?

- Bouchard, Lehalle and Dang (2011) propose to write down the associated optimal control problem.
Main idea

- Optimal control of robots:

\[
\text{d}R_x(t) = \mu(x, \cdot) \, dt + \sigma(x, \cdot) \, dW_t + \beta(x, \cdot) \, dN_x(t)
\]

At (stopping) times \(\tau_k\), launch a robot with parameter \(\xi_{\tau_k}\) for a period \(\delta_k \geq 0\).

At \(\tau_k + \delta_k\) decide to wait a bit or to launch immediately an other robot with parameter \(\xi_{\tau_k + 1}\) for a period \(\delta_k + 1\), and so on...

- Numerical resolution:
  This is a relatively standard impulse control problem.
  Leads to HJB equations which can be solved numerically.
  Can deduce an abacus on how/when/how long to launch robots within a given strategy.

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Main idea

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  - For a value $x$ of the parameter, the robot has a dynamics:
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    dR_t^x = \mu(x, \cdot) \, dt + \sigma(x, \cdot) \, dW_t + \beta(x, \cdot) \, dN_t^x
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  - At (stopping) times $\tau_k$, launch a robot with parameter $\xi_{\tau_k}$ for a period $\delta_k \geq \underline{\delta} > 0$. 

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Searching for the good market: a stochastic algorithm approach

- Works around Laruelle, Lehalle and Pagès: test the different markets to know which one will be the most efficient given the strategy one has in mind.

Previous talk....
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An example of un-hedgeable derivative

- Guaranteed VWAP:
  - Offer a protection against execution price.
  - Payoff = \[ \text{Guaranteed mean price} - \text{Effective mean price} \] \[^+\].
  - Usually the Guaranteed mean price is computed as a proportion \( \kappa \) of the mean price observed on the market on the relevant time period.
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  - A perfect hedge is obviously not possible!
  - What is the price of the guarantee?
The inverse problem point of view

- The abstract model:
  - \( \phi \) = liquidation strategy.
  - \( V^\phi_T \) the mean price obtained at \( T \) when following \( \phi \).
  - \( M^\phi_T \) the mean price observed for the period \([0, T]\) (taking into account price impact).
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• Risk minimization problem:
  > If the guarantee is sold at a unit price/share $p$, the terminal loss/unit is $[\kappa M_T - V_T^\phi - p]^+$
  > Fix $\ell$ a loss function (depending on the number of shares to liquidate).
  > One tries to minimize

$$v(p) := \min_{\phi} \mathbb{E}\left[ \ell \left( [\kappa M_T^\phi - V_T^\phi - p]^+ \right) \right].$$
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  - Given a threshold $\gamma$ on the risk, compute the price

$$\hat{p}(\gamma) := \inf \{ p : v(p) \leq \gamma \}.$$
A direct approach

- Assume a Markovian structure: \( X^{t,x,\phi} \) drives the markets (prices, volumes, volatility, etc...), \( M_T^{t,x,\phi} := M(X_T^{t,x,\phi}) \) and \( V^{t,x,p,\phi} \).
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- Define

$$\hat{p}(t,x,\gamma) := \inf \{ p : \exists \phi \text{ s.t. } \mathbb{E} \left[ \ell \left( \left[ \kappa M(X_T^{t,x,\phi}) - Y_T^{t,x,p,\phi} \right]^+ \right) \right] \leq \gamma \}.$$
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- It can be made time consistent (Bouchard, Elie and Touzi 2009): \( \hat{p}(t, x, \gamma) \) is the minimal value of \( p \) s.t. \( \exists (\phi, \alpha) \) for which

\[
\ell \left( \kappa M(X_{T}^{t,x,\phi}) - V_{T}^{t,x,p,\phi} \right)^{+} \leq \Gamma^{t,\gamma,\alpha}_{T} := \gamma + \int_{t}^{T} \alpha_{s} dW_{s}.
\]

\( \Rightarrow \) This is a stochastic target problem in the terminology of Soner and Touzi.
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- PDEs driving the evolution of $\hat{p}$ can be derived directly!
  - Avoids the numerical inversion of the previous approach.
  - Provides a dynamics for $\Gamma^{t,\gamma,\alpha}_T$ which amounts for the evolution of the conditional expected loss.
A direct approach

- It has been investigated within the framework of Guaranteed VWAP pricing by Bouchard and Dang (2010).

- Instead of a loss function, one can put several P&L constraints:

\[ \mathbb{P} \left[ V^{t,x,p,\phi}_T \geq \kappa M(X^{t,x,\phi}_T) - c_i \right] \geq q_i \text{ for } i = 1, \ldots, I, \]

- ... or more generally several constraints (see Bouchard and Vu 2012).
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- The price is now
  \[
  \hat{p}(t, x, \gamma) := \inf \{ p : \exists \phi \text{ s.t. } \mathbb{E} \left[ \ell \left( \kappa M(X_T^{t,x,\phi[\vartheta], \vartheta}) - V_T^{t,x,p,\phi[\vartheta], \vartheta} \right)^+ \right] \leq \gamma \ \forall \ \vartheta \}.
  \]

- Can be made time consistent as in the previous case, PDEs can be derived (see Bouchard, Moreau and Nutz 2012).
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- We want to optimise

$$\max \left\{ \mathbb{E} \left[ U(V_{T}^{t,x,v,\phi}) \right], \ \phi \ \text{s.t.} \ \mathbb{E} \left[ R(V_{T}^{t,x,v,\phi}) \right] \geq \gamma \right\}$$
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  \]
- Example: Use as a proxy of mean-variance problem
  \[
  \min \left\{ \mathbb{E} \left[ (V_{T}^{t,x,v,\phi} - \gamma_{0})^{2} \right], \phi \text{ s.t. } \mathbb{E} \left[ V_{T}^{t,x,v,\phi} \right] \geq \gamma_{0} \right\}
  \]
Reduction to a time consistent optimization problem with path constraint

- Set

\[ \varpi(t, x, \gamma) := \inf \left\{ v : \exists \phi \text{ s.t. } \mathbb{E}\left[ R(V_{T}^{t,x,v,\phi}) \right] \geq \gamma \right\} \]
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- It falls within the previous framework related to pricing issues.
- \( \varpi \) can be computed by solving a PDE.
Reduction to a time consistent optimization problem with path constraint

- Set
  \[ \varpi(t, x, \gamma) := \inf \left\{ v : \exists \phi \text{ s.t. } \mathbb{E}\left[ R(V_{T}^{t, x, v, \phi}) \right] \geq \gamma \right\} \]

  - It falls within the previous framework related to pricing issues.
  - \(\varpi\) can be computed by solving a PDE.

- Problem reduction as a state constrained problem
  \[
  \max \left\{ \mathbb{E}\left[ U(V_{T}^{t, x, v, \phi}) \right], \phi \text{ s.t. } \mathbb{E}\left[ R(V_{T}^{t, x, v, \phi}) \right] \geq \gamma \right\} = 
  \max \left\{ \mathbb{E}\left[ U(V_{T}^{t, x, v, \phi}) \right], (\phi, \alpha) \text{ s.t. } V_{s}^{t, x, v, \phi} \geq \varpi(s, X_{s}^{t, x, \phi}, \Gamma_{s}^{t, \gamma, \alpha}) \forall s \in [t, T] \right\}
  \]

  \[ \Rightarrow \text{back to a standard optimization problem with state constraints :} \]
Reduction to a time consistent optimization problem with path constraint

- Set
  \[ \varpi(t,x,\gamma) := \inf \left\{ v : \exists \phi \text{ s.t. } \mathbb{E} \left[ R(V^t_{T,x,v,\phi}) \right] \geq \gamma \right\} \]

  - It falls within the previous framework related to pricing issues.
  - \( \varpi \) can be computed by solving a PDE.

- Problem reduction as a state constrained problem
  \[
  \max \left\{ \mathbb{E} \left[ U(V^t_{T,x,v,\phi}) \right] , \phi \text{ s.t. } \mathbb{E} \left[ R(V^t_{T,x,v,\phi}) \right] \geq \gamma \right\} = \\
  \max \left\{ \mathbb{E} \left[ U(V^t_{T,x,v,\phi}) \right] , (\phi, \alpha) \text{ s.t. } V^t_{s,x,v,\phi} \geq \varpi(s, X^t_{s,x,\phi}, \Gamma_{s,\gamma,\alpha}) \forall s \in [t, T] \right\}
  \]

  \( \Rightarrow \) back to a standard optimization problem with state constraints:
  - Domain given by \( \varpi \) which can be pre-computed.
  - Leads to standard HJB equations with constraint: numerical resolution in general.

- First suggested in Bouchard, Elie and Imbert (2009), and then further developed by Bouchard and Nutz (2012).
Outline

Optimal book liquidation
  The Almgren and Chriss framework for aggressive strategies
  Controlling the intensity of order matching : non-aggressive strategies
  Towards a control of liquidation robots

Searching for the good market
  In a nutshell..

Pricing problems
  Inverse versus direct approach
  Pricing under uncertainty or with adverse player

Optimization under risk constraint
  Coming back to what we know....

Perspectives
  Last slide...
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- A lot has been done to build up relevant simple models allowing for explicit solutions.
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- Can we do more without relying on numerical procedures?

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- Do more on uncertainty or models with aggressive players?

- Optimal control techniques require to postulate some a-priori laws/distributions: more model free strategies? (cf previous talk of G. Pagès and the next talk on Reinforcement Learning by Yuriy Nevmyvaka).
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