Stochastic Target Problems and Applications

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Contents

Part I

Motivation

Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$, W a d-dimensional Brownian motion, $\mathbb{F} = (\mathcal{F}_s, 0 \leq s \leq T)$ the filtration generated by W.

Set of controls: An abstract set $\mathcal{U}$.

Controlled process: A map : $(t, z, \nu) \in [0, T] \times \mathbb{R}^{d+1} \times \mathcal{U}_0 \mapsto Z^{\nu}_{t,z}$ a cadlag $\mathbb{F}$-adapted process satisfying $Z^{\nu}_{t,z}(t) = 0$.

Target : G a Borel subset of $\mathbb{R}^{d+1}$.

Problem: Compute

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ } \mathbf{s.t.} \text{ } Z^{\nu}_{t,z}(T) \in G \; \mathbb{P} - \mathbf{a.s.}\}.$$

1 Application in financial mathematics: super-hedging problems

In “classical” financial market, one can rely on dual formulations that relate prices to singular optimal control problems. What if the notion of “martingale measure” does not apply ?

1.1 Large investor model

Set of controls: $\mathcal{U}$ is the set of $\mathbb{F}$-predictable process with values in $U \subset \mathbb{R}^d$.

Controlled process: $Z^\nu = (X^\nu, Y^\nu) \in \mathbb{R}^d \times \mathbb{R}$ with

$dX^\nu = \mu_X(X^\nu, \nu)dr + \sigma_X(X^\nu, \nu)dW \ , \ dY^\nu = \nu' \mu_X(X^\nu, \nu)dr + \nu' \sigma_X(X^\nu, \nu)dW \ .$

$\Rightarrow X^\nu =$ stocks or factors, $Y^\nu =$ wealth, $\nu =$ number of stocks in the portfolio.

Target: $G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \geq g(x)\}$.

Super-hedging price:

$$\gamma(t, x) := \inf\{y \in \mathbb{R} \ : \ (x, y) \in V(t)\}$$

where

$$V(t) := \{(x, y) \in \mathbb{R}^{d+1} : \ \exists \, \nu \in \mathcal{U} \text{ s.t. } Y^\nu_{t,x,y}(T) \geq g(X^\nu_{t,x}(T)) \ \mathbb{P} - \text{a.s.}\}.$$

1.2 Model with proportional transaction costs

Set of controls: $\mathcal{U}$ is the set of adapted non-decreasing process (component by component) in $\mathbb{R}^2$.

Controlled process:

$$\begin{aligned}
X^1(s) &= x^1 + \int_t^s X^1(r)\mu dr + \int_t^s X^1(r)\sigma dW_r^1 \\
X^{2,\nu}(s) &= x^2 + \int_t^s \frac{X^{2,\nu}(r)}{X^1(r)} dX^1(r) - \int_t^s d\nu_r^1 + \int_t^s d\nu_r^2 \\
Y^\nu(s) &= y + \int_t^s (1-\lambda) d\nu_r^1 - \int_t^s (1+\lambda) d\nu_r^2 \ .
\end{aligned}$$

$\Rightarrow$ X^1 = stock, $X^{2,\nu}$ = value invested in the stock, Y^ν = value invested in cash, ν_t^1 = cumulated amount of stocks sold, ν_t^2 = cumulated amount of stocks bought, $\lambda > 0$ is the proportional transaction cost coefficient.

Controlled process:

$$\begin{aligned}
X^1(s) &= x^1 + \int_t^s X^1(r)\mu dr + \int_t^s X^1(r)\sigma dW_r^1 \\
X^{2,\nu}(s) &= x^2 + \int_t^s \frac{X^{2,\nu}(r)}{X^1(r)} dX^1(r) - \int_t^s d\nu_r^1 + \int_t^s d\nu_r^2 \\
Y^\nu(s) &= y + \int_t^s (1-\lambda) d\nu_r^1 - \int_t^s (1+\lambda) d\nu_r^2 \ .
\end{aligned}$$

Target: $G := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} : y \geq g_c(x^1) \text{ and } x^2 \geq g_d(x^1)\}$.

Super-hedging price:

$$\gamma(t, x^1, x^2) := \inf\{y \in \mathbb{R} \ : \ (x^1, x^2, y) \in V(t)\}$$

where

$$V(t) := \{(x,y) \in \mathbb{R}^3 : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } (Y^\nu_{t,x,y}(T), X^{2,\nu}_{t,x}(T)) \geq (g_c, g_d)(X^{1,\nu}_{t,x}(T)) \ \mathbb{P}\text{−a.s.}\}.$$

1.3 Model with price impact

Set of controls: $\mathcal{U}$ is the set of adapted cadlag piecewise constant non-decreasing process in $\mathbb{R}$.

Controlled process:

$$\begin{aligned} dX^{1,\nu} &= \mu_X(X^{1,\nu})dr + \sigma_X(X^{1,\nu})dW + \beta(X^{1,\nu}, \Delta\nu)\mathbf{1}_{\Delta\nu\neq 0} \\ dX^{2,\nu} &= \Delta\nu \\ dY^{\nu} &= \gamma(X^{1,\nu}, \Delta\nu)\mathbf{1}_{\Delta\nu\neq 0} \ . \end{aligned}$$

$\Rightarrow$ $X^{1,\nu}$ = stock, Y^{ν} = cumulated buying cost (negative), $\Delta\nu$ = number of stocks bought at time t, $\beta(X^{1,\nu}, \Delta\nu)$ = immediate impact factor, $\gamma(X^{1,\nu}, \Delta\nu)$ = buying cost of $\Delta\nu$ shares (negative).

Controlled process:

$$
\begin{aligned}
dX^{1,\nu} &= \mu_X(X^{1,\nu})dr + \sigma_X(X^{1,\nu})dW + \beta(X^{1,\nu}, \Delta\nu)\mathbf{1}_{\Delta\nu\neq 0} \\
dX^{2,\nu} &= \Delta\nu \\
dY^{\nu} &= \gamma(X^{1,\nu}, \Delta\nu)\mathbf{1}_{\Delta\nu\neq 0} \; .
\end{aligned}
$$

Target: $G := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} : y \geq -K \text{ and } x^2 = N\}$.

Super-hedging of a target buying cost

$$\gamma(t, x^1, x^2) := \inf\{y \in \mathbb{R} \;:\; (x^1, x^2, y) \in V(t)\}$$

where

$$V(t) := \{(x,y) \in \mathbb{R}^3 : \; \exists\, \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(T) \geq -K \text{ and } X^{2,\nu}_{t,x}(T) = N \; \mathbb{P}\text{-a.s.}\}.$$

2 Extension to pathwise constraint

Initial problem: Compute

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z} \in \mathcal{O}\}.$$

Pathwise constraint: A family $\{\mathcal{O}(s), s \leq T\}$ of Borel subsets.

Problem with path constraint:

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z} \in \mathcal{O} \text{ on } [t,T]\}.$$

Example: Super-hedging with credit limit

$$V(t,p) := \{(x,y) \in \mathbb{R}^3 : \exists\, \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y}(T) \geq g(X^{\nu}_{t,x}(T)) \text{ and } Y^{\nu}_{t,x,y} \geq -\kappa\}.$$

for

$$\mathcal{O} := \mathbb{R} \times [-\kappa, \infty)\mathbf{1}_{[0,T)} + \{(x,y) \ : \ y \geq g(x) \vee (-\kappa)\}\mathbf{1}_{\{T\}}.$$

Initial problem: Compute

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z} \in \mathcal{O}\}.$$

Pathwise constraint: A family $\{\mathcal{O}(s), s \leq T\}$ of Borel subsets.

Problem with path constraint:

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z} \in \mathcal{O} \text{ on } [t,T]\}.$$

Example: Super-hedging of American options

$$V(t,p) := \{(x,y) \in \mathbb{R}^3 : \exists\, \nu \in \mathcal{U} \text{ s.t. } Y^{\nu}_{t,x,y} \geq g(X^{\nu}_{t,x}) \text{ on } [t,T]\}.$$

for

$$\mathcal{O} := \{(x,y) \;:\; y \geq g(x) \vee (-\kappa)\}.$$

3 Extension to constraints in expectation - controlled loss

Initial problem: Compute

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z} \in \mathcal{O}\}.$$

Relaxed problem: Given a Borel measurable map ℓ and $p \in \mathbb{R}$, compute

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p \text{ and } Z^{\nu}_{t,z} \in \mathcal{O}\}.$$

Remark : Could compute $w(t,z) = \sup_{\nu} \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right]$ and look for z such that $w(t,z) \geq p$. But it is indirect and might lead to additional numerical instability

One can put several constraints in expectation at the same time, e.g. to impose constraints on a terminal P&L distribution (B. and T. N. Vu [9]).

Application to liquidation with a target costs constraint

Set of controls: $\mathcal{U}$ is the set of adapted cadlag piecewise constant non-decreasing process in $\mathbb{R}$.

Controlled process:

$$\begin{aligned} dX^{1,\nu} &= \mu_X(X^{1,\nu})dr + \sigma_X(X^{1,\nu})dW + \beta(X^{1,\nu},\Delta\nu)\mathbf{1}_{\Delta\nu\neq 0} \\ dX^{2,\nu} &= \Delta\nu \\ dY^{\nu} &= \gamma(X^{1,\nu},\Delta\nu)\mathbf{1}_{\Delta\nu\neq 0}\ . \end{aligned}$$

$\Rightarrow X^{1,\nu}$ = stock, Y^{ν} = cumulated buying cost (negative), $\Delta\nu$ = number of stocks bought at time t, $\beta(X^{1,\nu},\Delta\nu)$ = immediate impact factor, $\gamma(X^{1,\nu},\Delta\nu)$ = buying cost of $\Delta\nu$ shares (negative).

Example: Quantile hedging of a target buying cost

$$V(t,p) := \{(x,y) \in \mathbb{R}^3 :\ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,x,y}(T) \geq -K\right] \geq p\ ,\ X^{2,\nu}_{t,x}(T) = N\}.$$

for

$$\ell(x,y) := \mathbf{1}_{y\geq -K} \text{ and } \mathcal{O} := \mathbb{R}^3\mathbf{1}_{[0,T)} + \{(x,y)\ :\ x^2 = N\}\mathbf{1}_{\{T\}}.$$

Application to liquidation with a target costs constraint

Set of controls: $\mathcal{U}$ is the set of adapted cadlag piecewise constant non-decreasing process in $\mathbb{R}$.

Controlled process:

$$
\begin{aligned}
dX^{1,\nu} &= \mu_X(X^{1,\nu})dr + \sigma_X(X^{1,\nu})dW + \beta(X^{1,\nu},\Delta\nu)\mathbf{1}_{\Delta\nu\neq 0} \\
dX^{2,\nu} &= \Delta\nu \\
dY^{\nu} &= \gamma(X^{1,\nu},\Delta\nu)\mathbf{1}_{\Delta\nu\neq 0}\ .
\end{aligned}
$$

$\Rightarrow$ $X^{1,\nu}$ = stock, Y^ν = cumulated buying cost (negative), $\Delta\nu$ = number of stocks bought at time t, $\beta(X^{1,\nu},\Delta\nu)$ = immediate impact factor, $\gamma(X^{1,\nu},\Delta\nu)$ = buying cost of $\Delta\nu$ shares (negative).

Example: Expected loss pricing of a target buying cost

$V(t,p) := \{(x,y)\in\mathbb{R}^3 :\ \exists\,\nu\in\mathcal{U} \text{ s.t. } \mathbb{E}\left[(Y^\nu_{t,x,y}(T)+K)^-\right]\le -p\ ,\ X^{2,\nu}_{t,x}(T)=N\}.$

for

$$\ell(x,y) := -(y+K)^- \text{ and } G := \{(x,y)\ :\ x^2 = N\}.$$

4 Robust criteria - parameter uncertainty

Game version:

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta]}(T))\right] \geq p \text{ for all } \vartheta \in \mathcal{V}\}.$$

Adverse control / Knightian uncertainty: $\vartheta \in \mathcal{V}$ can be interpreted as a control of a parameter by the "nature" (e.g. volatility, correlation, default time, etc...). Can be used to model Knightian uncertainty.

Strategy : $\mathfrak{u} \in \mathfrak{U} : \vartheta \in \mathcal{V} \mapsto \mathfrak{u}[\vartheta] \in \mathcal{U}$ is non-anticipating, ie $\mathfrak{u}[\vartheta^1]_{\cdot\wedge t} = \mathfrak{u}[\vartheta^2]_{\cdot\wedge t}$ on $\vartheta^1_{\cdot\wedge t} = \vartheta^2_{\cdot\wedge t}$.

Remark: When the adverse control is volatility, this relates to G-expectation of Peng [15] and 2BSDEs of Soner, Touzi and Zhang [20]: find $\mathfrak{u}$ such that

$$Y_T^{\mathfrak{u}[\vartheta]} \geq g(X_T^{\vartheta})\ \mathbb{P} - \text{a.s. } \forall\, \vartheta$$

where $X_T^{\vartheta} = x + \int_0^T \vartheta_s dW_s$ and $Y_T^{\mathfrak{u}[\vartheta]} = y + \int_0^T \mathfrak{u}[\vartheta]_s dX_s^{\vartheta}$.

5 Optimal control under risk constraint

General probem

$$w(t,x,p) := \sup_{\nu \in \mathcal{U}(t,x,p)} F(t,x;\nu)$$

where

$$F(t,x;\nu) := \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right]$$

and

$$\mathcal{U}(t,x,p) := \left\{\nu \in \mathcal{U}^t \ : \ G(t,x;\nu) := \mathbb{E}\left[g(X^{\nu}_{t,x}(T))\right] \leq p\right\}.$$

Idea: turn it into a "standard" optimal control problem with state constraint, the domain being given by the stochastic target problem associated to the constraint.

Part II

Geometric dynamic programming and PDE characterization for stochastic target problem in $\mathbb{P}$ – a.s. form

$$V(t) := \{z \in \mathbb{R}^{d+1} : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in G \ \mathbb{P} - \text{a.s.}\}.$$

1 A simple Markovian framework

Notations: Let $\mathbb{F}^t = (\mathcal{F}^t_s, t \leq s \leq T)$ denote the augmented filtration generated by $(W_s - W_t, t \leq s \leq T)$. Let $\mathcal{T}^t$ denote the set of $\mathbb{F}^t$-stopping times and $\mathcal{U}^t$ the set of $\mathbb{F}^t$-predictable elements of $\mathcal{U}$.

Problem:

$$V(t) := \{z \in \mathbb{R}^{d+1} : \; \exists\, \nu \in \mathcal{U}^t \text{ s.t. } Z^\nu_{t,z}(T) \in G \; \mathbb{P} - \text{a.s.}\}.$$

Controls: $\mathcal{U}$ the set of progressively measurable processes with values in a compact set $U \subset \mathbb{R}^\kappa$ (viewed as a subset of $L^2([0,T] \times \Omega)$).

Controlled process : $Z^\nu_{t,z}$ solution of

$$Z(s) = z + \int_t^s \mu(Z(r), \nu_r)dr + \int_t^s \sigma(Z(r), \nu_r)dW_r, \;\; t \leq s \leq T,$$

with μ and σ Lipschitz, uniformly in the control.

2 The GDPP

Problem:

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U}^t \text{ s.t. } Z^{\nu}_{t,z}(T) \in G\ \mathbb{P}-\text{a.s.}\}.$$

Theorem: (Soner and Touzi [17]) **Let $\{\theta^{\nu},\ \nu \in \mathcal{U}^t\}$ be a family of $\mathcal{T}^t$. Then,**

$$V(t) = \overline{V}(t)$$

where

$$\overline{V}(t) := \{\overline{z} \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U}^t \textbf{ s.t. } Z^{\nu}_{t,\overline{z}}(\theta^{\nu}) \in V(\theta^{\nu})\ \mathbb{P}-\textbf{a.s.}\}.$$

2.1 $V(t) \subset \overline{V}(t)$

Fix $z_o \in V(t)$. Then, $\exists\ \nu \in \mathcal{U}^t$ such that

$$\mathbb{P}\left[Z^{\nu}_{t,z}(T) \in G\right] = 1.$$

Then (flow property),

$$\mathbb{P}\left[Z^{\nu}_{\theta^\nu,\xi}(T) \in G \mid \mathcal{F}^t_{\theta^\nu}\right] = 1 \ \ \mathbb{P}-\text{a.s.}$$

where $\xi := Z^{\nu}_{t,z}(\theta^\nu)$. Thus, for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\int \mathbf{1}_{\left\{Z^{\nu_\omega}_{\theta^\nu(\omega),\xi(\omega)}(T)(\omega')\in G\right\}} dP(\omega') = 1$$

where $\nu_\omega : \omega' \in \Omega \mapsto \nu(\omega \mathbf{1}_{[0,\theta^\nu(\omega)]} + (\omega' - \omega'_{\theta^\nu(\omega)})\mathbf{1}_{(\theta^\nu(\omega),T]}) \in \mathcal{U}^{\theta^\nu(\omega)}$.

Thus, $Z^{\nu}_{t,z}(\theta^\nu)(\omega) \in V(\theta^\nu(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Hence,

$$z \in \overline{V}(t) = \{\bar{z} \in \mathbb{R}^{d+1} :\ \exists\ \nu \in \mathcal{U}^t \text{ s.t. } Z^{\nu}_{t,\bar{z}}(\theta^\nu) \in V(\theta^\nu)\ \mathbb{P}-\text{a.s.}\}.$$

2.2 $V(t) \supset \overline{V}(t)$

Let $z \in \overline{V}(t)$ and $\nu \in \mathcal{U}^t$ be such that

$$Z^{\nu}_{t,\overline{z}}(\theta^\nu) \in V(\theta^\nu).$$

Lemma: There exists a Borel measurable map $\phi : (t', z') \in [t,T] \times \mathbb{R}^{d+1} \mapsto \phi(t', z')$ such that

$$\phi(t', z') \in \left\{ \nu' \in \mathcal{U}^{t'} : Z^{\nu'}_{t',z'}(T) \in G \; \mathbb{P} - \text{a.s.} \right\} \; \mu_\nu - a.e.$$

where $\mu_\nu(B) = \mathbb{P}\left[(\theta^\nu, Z^\nu_{t,z}(\theta^\nu)) \in B \right]$. Moreover, there exists $\overline{\nu} \in \mathcal{U}^t$ such that

$$\phi(\theta^\nu, Z^\nu_{t,z}(\theta^\nu)) = \overline{\nu} \;, \; dt \times d\mathbb{P}\text{–a.e.}$$

Take

$$\tilde{\nu} := \nu \mathbf{1}_{[t,\theta^\nu)} + \overline{\nu} \mathbf{1}_{[\theta^\nu, T]}.$$

Then,

$$Z^{\tilde{\nu}}_{t,z}(T) \; = \; Z^{\overline{\nu}}_{\theta^\nu,\xi}(T) = Z^{\phi(\theta^\nu,\xi)}_{\theta^\nu,\xi}(T) \in G \quad \mathbb{P} - \text{a.s.}$$

3 The GDPP in the monotonic case

Problem:

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U}^t \text{ s.t. } Z^{\nu}_{t,z}(T) \in G\ \mathbb{P}-\text{a.s.}\}.$$

Monotonicity assumption:

(i) $Z^{\nu}_{t,x,y} = (X^{\nu}_{t,x}, Y^{\nu}_{t,x,y}) \in \mathbb{R}^d \times \mathbb{R}$

(ii) $(x, y) \in G$ implies $(x, y') \in G$ for $y' \geq y$.

Consequence: $(x, y) \in V(t)$ implies $(x, y') \in V(t)$ for $y' \geq y$

Value function: $\gamma(t,x) := \inf\{y \in \mathbb{R} \ : \ (x,y) \in V(t)\}$.

Theorem : **Let $\{\theta^\nu,\ \nu \in \mathcal{U}^t\}$ be a family of $\mathcal{T}^t$. Then,**

(GDP1) If $y > \gamma(t,x)$, then there exists $\nu \in \mathcal{U}^t$ such that

$$Y^\nu_{t,x,y}(\theta^\nu) \geq \gamma(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\ \mathbb{P} - \textbf{a.s.}$$

(GDP2) If $y < \gamma(t,x)$, then for all $\nu \in \mathcal{U}^t$

$$\mathbb{P}\left[Y^\nu_{t,x,y}(\theta^\nu) > \gamma(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right] < 1.$$

Proof. $y > \gamma(t,x) \Rightarrow (x,y) \in V(t) \Rightarrow y \geq \gamma(t,x)$.

4 Informal PDE derivation

Assume γ smooth and $y = \gamma(t, x)$ implies that there exists $\nu \in \mathcal{U}^t$ such that

$$Y^{\nu}_{t,x,y}(\theta^{\nu}) \geq \gamma(\theta^{\nu}, X^{\nu}_{t,x}(\theta^{\nu}))\ \mathbb{P} - \text{a.s.}$$

For $\theta^{\nu} = t+$:

$$Y^{\nu}_{t,x,y}(t+) \geq \gamma(t+, X^{\nu}_{t,x}(t+))\ \mathbb{P} - \text{a.s.}$$

i.e. (with $\mathcal{L}^{\nu}_X$ the Dynkin operator associated to X^{ν})

$$(\mu_Y(x, y, \nu_t) - \mathcal{L}^{\nu_t}_X \gamma(t, x))\, dt + (\sigma_Y(x, y, \nu_t) - D\gamma(t, x)\sigma_X(x, \nu_t))\, dW_t \geq 0.$$

This implies

$$\sigma_Y(x, y, \nu_t) - D\gamma(t, x)\sigma_X(x, \nu_t) = 0 \text{ and } \mu_Y(x, y, \nu_t) - \mathcal{L}^{\nu_t}_X \gamma(t, x) \geq 0.$$

Hence,

$$\mathcal{H}\gamma(t, x) := \sup_{u \in \mathcal{N}_0\gamma(t,x)} \{\mu_Y(x, \gamma(t, x), u) - \mathcal{L}^u_X \gamma(t, x)\} \geq 0$$

where $\mathcal{N}_0\gamma(t, x) := \{u \in U \ :\ \sigma_Y(x, \gamma(t, x), u) = D\gamma(t, x)'\sigma_X(x, u)\}$.

By optimality

$$\mathcal{H}_0\gamma(t,x) := \sup_{u \in \mathcal{N}_0\gamma(t,x)} \{\mu_Y(x,\gamma(t,x),u) - \mathcal{L}_X^u \gamma(t,x)\} = 0$$

where $\mathcal{N}_0\gamma(t,x) := \{u \in U \;:\; \sigma_Y(x,\gamma(t,x),u) = D\gamma(t,x)'\sigma_X(x,u)\}$.

5 PDE derivation

Technical issue: The operator

$$(t, x, \varphi, \partial_t\varphi, D\varphi, D^2\varphi) \mapsto \mathcal{H}_0\varphi(t, x)$$

may not be continuous.

Relaxation: Set

$$H_\varepsilon(x, y, q, p, A) := \sup_{u \in N_\varepsilon(x,y,p)} \left\{ \mu_Y(x, y, u) - q - \mu_X(x, u)' p - \frac{1}{2} \mathrm{Tr}[\sigma_X \sigma_X'(x, u) A] \right\}$$

with

$$N_\varepsilon(x, y, p) := \{u \in U \ : \ |\sigma_Y(x, y, u) - p' \sigma_X(x, u)| \leq \varepsilon\} \, .$$

Define

$$\begin{aligned} H^*(x, y, q, p, A) \ &:= \limsup_{(\varepsilon,x',y',p',A') \to (0,x,y,p,A)} H_\varepsilon(x', y', q, p', A') \\ H_*(x, y, q, p, A) \ &:= \liminf_{(\varepsilon,x',y',p',A') \to (0,x,y,p,A)} H_\varepsilon(x', y', q, p', A'). \end{aligned}$$

5.1 Super-solution property

Theorem: (Soner and Touzi [18], B., Elie and Touzi [4]) **Assume that** γ **is locally bounded, then its lower-semicontinuous envelope** γ_* **is a viscosity super-solution on** $[0,T) \times \mathbb{R}^d$ **of**

$$\mathcal{H}^* \varphi(t,x) = 0.$$

Proof. For simplicity, we assume $\gamma = \gamma_*$ (the general case is obtained by considering $(t_n, x_n) \to (t_o, x_o)$ such that $\gamma(t_n, x_n) \to \gamma_*(t_o, x_o)$).
Fix (t_o, x_o) that achieves a strict local minimum of $\gamma - \varphi$ such that $(\gamma - \varphi)(t_o, x_o) = 0$. Assume that

$$\mathcal{H}^* \varphi(t_o, x_o) < 0.$$

Then, there exists a neighborhood B, $r > 0$ and $\varepsilon > 0$ s.t.

$$\sup_{u \in N_\varepsilon(x,y,D\varphi(t,x))} \{\mu_Y(x,y,u) - \mathcal{L}^u_X \varphi(t,x)\} \leq 0 \quad (*)$$

for $(t,x) \in B$ and $|y - \varphi(t,x)| \leq r$.

Fix $\iota \in (0, r)$ and let $y_o := \gamma(t_o, x_o) + \iota$. Then, by (GDP1), there exists $\nu \in \mathcal{U}^{t_o}$ such that

$$Y_o(\theta) \geq \gamma(\theta, X_o(\theta)) \geq \varphi(\theta, X_o(\theta)) \quad (**)$$

with $Z_o := Z^{\nu}_{t_o, x_o, y_o}$, and θ the minimum of

$$\theta_B := \inf\{s \geq t_o : (s, X_o) \notin B\}\,,\ \theta_r := \inf\{s \geq t_o : |Y_o(s) - \varphi(s, X_o(s))| \geq r\}.$$

Note that (**) implies that $Y_o(\theta) = \varphi(\theta, X_o(\theta)) + r$ on $\{\theta = \theta_r\}$.
Moreover, after possibly changing $r > 0$, we can assume that

$$\gamma \geq \varphi + r \text{ on } \partial B.$$

Thus

$$Y_o(\theta) \geq \varphi(\theta, X_o(\theta)) + r \quad (***).$$

Set

$$\chi_s := [\mu_Y(Z_o(s), \nu_s) - \mathcal{L}_X^{\nu_s}\varphi(s, Z_o(s))]\,|\delta_s|^{-2}\delta_s \mathbf{1}_A(s)$$

where

$$\delta := \sigma_Y(Z_o, \nu) - D\varphi(\cdot, Z_o)'\sigma_X(X_o, \nu)\ ,\ A := \{|\delta| > \varepsilon\}.$$

One has

$$\sup_{u \in N_\varepsilon(x,y,D\varphi(t,x))} \{\mu_Y(x, y, u) - \mathcal{L}_X^u\varphi(t, x)\} \leq 0 \quad (*)$$

for $(t, x) \in B$ and $|y - \varphi(t, x)| \leq r$, with

$$N_\varepsilon(x, y, D\varphi(t, x)) = \{u \in U : |\sigma_Y(x, y, u) - D\varphi(t, x)'\sigma_X(x, u)| \leq \varepsilon\}.$$

This implies

$$\chi_s \leq 0 \text{ on } A^c(s) = \{|\delta_s| \leq \varepsilon\}$$

Define L by

$$L = 1 - \int_{t_o}^{\cdot} L_s \chi_s dW_s.$$

Then,

$$\begin{aligned} L_\theta(Y_o(\theta) - \varphi(\theta, X_o(\theta))) \;&=\; \iota + \int_{t_o}^{\theta} L_s \overbrace{\left[\mu_Y(Z_o(s), \nu_s) - \mathcal{L}_X^{\nu_s}\varphi(s, X_o(s))\right]}^{\leq 0} \mathbf{1}_{A^c(s)} ds \\ &\;+\; M_\theta - M_{t_o}. \end{aligned}$$

Recalling that

$$Y_o(\theta) \geq \varphi(\theta, X_o(\theta)) + r \quad (***),$$

this implies

$$\iota \geq \mathbb{E}\left[L_\theta(Y_o(\theta) - \varphi(\theta, X_o(\theta)))\right] \geq r.$$

We obtain a contradiction since $\iota < r$.

5.2 Sub-solution property

Definition: Let $\mathcal{C}(t,x)$ be the $C^{1,2}$ functions φ s.t. : $\forall\ \varepsilon > 0$ and B open s.t. $(x, \varphi(t,x), D\varphi(t,x)) \in B$ and $N_0 \neq \emptyset$ on B, and all $\tilde{u} \in N_0(x, \varphi(t,x), D\varphi(t,x))$, $\exists\ B' \subset B$ and a locally Lipschitz map $\hat{u}$ such that $|\hat{u}(x, \varphi(t,x), D\varphi(t,x)) - \tilde{u}| \leq \varepsilon$ and $\hat{u} \in N_0$ on B'.

Remark Assume σ_Y depends only on x and u, σ_X does not depend on u, and that $u \in U \mapsto \sigma_Y(x,u)$ is invertible + regularity, then $\varphi \in \mathcal{C}(t,x)$ if $\sigma_Y^{-1}(x, D\varphi(t,x)'\sigma_X(x)) \in \mathrm{int}(U)$.

Theorem: **Assume that γ is locally bounded, then its upper-semicontinuous envelope γ^* is a viscosity sub-solution on $[0,T) \times \mathbb{R}^d$ of**

$$\mathcal{H}_* \varphi(t,x) \mathbf{1}_{\varphi \in \mathcal{C}(t,x)} = 0$$

Proof. We assume that $\gamma = \gamma^*$ for simplicity. Fix (t_o, x_o) that achieves a strict local maximum of $\gamma - \varphi$ such that $(\gamma - \varphi)(t_o, x_o) = 0$. Assume that $\varphi \in \mathcal{C}(t_o, x_o)$ and

$$\mathcal{H}_* \varphi(t_o, x_o) > 0$$

Then, by definition of $\mathcal{C}$, one can find a Lipschitz continuous map $\hat{u}$, an open ball $B \ni (t_o, x_o)$, and $r > 0$ such that

$$\begin{aligned} \mu_Y(x, y, \hat{u}(t,x,y)) &\geq \mathcal{L}_X^{\hat{u}(t,x,y)} \varphi(t,x) \\ \sigma_Y(x, y, \hat{u}(t,x,y)) &= D\varphi' \sigma_X(t, x, \hat{u}(t,x,y)) \quad (\,\square\,) \end{aligned}$$

for $(t,x) \in B$ and $|y - \varphi(t,x)| \leq r$.

Take $\iota \in (0, r)$, set $y_o := \gamma(t_o, x_o) - \iota$. Let (X_o, Y_o) be associated with the initial conditions (t_o, x_o, y_o) and the Markovian control induced by $\hat{u}$.

Let θ be the minimum of

$\theta_B := \inf\{s \geq t_o : (s, X_o) \notin B\}$, $\theta_r := \inf\{s \geq t_o : |Y_o(s) - \varphi(s, X_o(s))| \geq r\}$.

By ($\square$) and $\iota < r$, we have

$$Y_o(\theta) \geq -\iota + \varphi(\theta, X_o(\theta)) + r\mathbf{1}_{\{\theta=\theta_r\}} \geq \gamma(\theta, X_o(\theta)) + r - \iota$$

where r can be chosen such that $\gamma - \varphi \leq -r$ on ∂B. This implies that $Y_o(\theta) > \gamma(\theta, X_o(\theta))$ since $r > \iota$, while $y_o < \gamma(t_o, x_o)$. This contradicts (GDP2).

5.3 Boundary condition when $G = \{(x,y) : y \geq g(x)\}$

Notations: Set

$$\mathbf{N}(x,y,p) \; := \; \{r \in \mathbb{R} : \; r = |\sigma_Y(x,y,u) - p'\sigma_X(x,u)| \text{ for some } u \in U\} \; ,$$

and

$$\delta \; := \; \operatorname{dist}(0, \mathbf{N}^c) - \operatorname{dist}(0, \mathbf{N}) \; ,$$

Then,

$$0 \in \operatorname{int}(\mathbf{N}(x,y,p)) \text{ iff } \delta(x,y,p) > 0 \; .$$

Remark: Note that γ_* is a super-solution of $\delta^*\varphi \geq 0$.

Example : Assume that $\sigma_Y(x,y,u) = u'\sigma$ and $\sigma_X(x,u) = \sigma$ is invertible. Then, $\delta\varphi := \delta(\cdot, \varphi, D\varphi) \geq 0 \Leftrightarrow D\varphi \in U$, while $\delta\varphi > 0 \Leftrightarrow D\varphi \in \operatorname{int}(U)$.

Theorem: **Assume that γ is locally bounded, then $\gamma_*(T,\cdot)$ and $\gamma^*(T,\cdot)$ are respectively super- and subsolution on $\mathbb{R}^d$ of**

$$\min\{\varphi - g_*, \delta^*\varphi\} \geq 0$$

and

$$\min\{\varphi - g^*, \delta_*\varphi\}\mathbf{1}_{\varphi\in\mathcal{C}(T,\cdot)} \leq 0.$$

Proof. a. Supersolution: Take $(t_n, x_n, \gamma(t_n, x_n)) \to (T, x, \gamma_*(T, x))$ with $t_n < T$ and $y_n := \gamma(t_n, x_n) + n^{-1}$. Then,

$$\gamma_*(T,x) = \lim_n Y^{\nu_n}_{t_n,x_n,y_n}(T) \geq \liminf_n g(X^{\nu_n}_{t_n,x_n}(T)) \geq g_*(x).$$

Moreover, γ_* is a supersolution on $[0,T)\times\mathbb{R}^d$ of $\delta^*\varphi \geq 0$, which propagates at the boundary.

b. Subsolution: If $\delta_*\varphi(x) > 0$ then N_0 is non-empty on a neighborhood of $(x, \varphi(x), D\varphi(x))$. One appeals to the definition of $\mathcal{C}$ and argue as in the interior of the domain.

6 Example: Super-hedging under constraint

Model: Given σ invertible:

$$dX = \mathrm{diag}[X]\sigma dW \text{ and } dY^{\nu} = \nu' dX = \nu' \mathrm{diag}[X]\sigma dW.$$

Simplication: $u \in N_0(x, y, p) \Leftrightarrow p = u \in U$.

Support function of U: Assume that U is closed, convex and contains 0. Set

$$\delta_U(\zeta) := \sup_{u \in U} \zeta' u.$$

Then,

$$p \in U \Leftrightarrow \mathcal{G}(p) := \inf_{|\zeta|=1} (\delta_U(\zeta) - \zeta' p) \geq 0$$

and

$$p \in \mathrm{int} U \Leftrightarrow \mathcal{G}(p) > 0.$$

The PDE and the terminal condition become

$$\min\{-\mathcal{L}_X\varphi\ ,\ \mathcal{G}(D\varphi)\} = 0 \text{ on } [0,T)\times(0,\infty)^d$$

and (for g continuous)

$$\min\{\varphi - g\ ,\ \mathcal{G}(D\varphi)\} = 0 \text{ on } (0,\infty)^d.$$

Compare with, e.g., Cvitanic, Pham and Touzi [11] and Soner and Touzi [19].

7 The GDPP with pathwise constraints

7.1 Problem and GDPP

Problem: Let $\{\mathcal{O}(s), t \leq s \leq T\}$ be a family of Borel sets.

$$V(t) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U}^t \text{ s.t. } Z^{\nu}_{t,z} \in \mathcal{O} \text{ on } [t,T]\ \mathbb{P}-\text{a.s.}\}.$$

Assumption: $t_n \downarrow t$ and $z_n \to z$ with $z_n \in \mathcal{O}(t_n)$ implies $z \in \mathcal{O}(t)$ (upper hemicontinuous from the right in time).

Theorem: (B. and Vu [8]) **Let $\{\theta^\nu,\ \nu \in \mathcal{U}^t\}$ be a family of $\mathcal{T}^t$. Then,**

$$V(t) = \overline{V}(t),$$

where $\overline{V}(t)$ is the set of initial conditions $\bar{z} \in \mathbb{R}^{d+1}$ such that $\exists\ \nu \in \mathcal{U}^t$ satisfying

$$Z^{\nu}_{t,\bar{z}}(\theta^\nu \wedge s) \in \mathcal{O}(s)\mathbf{1}_{s<\theta^\nu} + \mathbf{1}_{s \geq \theta^\nu} V(\theta^\nu)\ \ \forall\, s \in [t,T]\ \ \mathbb{P}-\textbf{a.s.}.$$

7.2 PDE characterization

Domain:

$$D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.$$

In the interior: For $(t, x, \gamma(t, x)) \in \mathrm{int}D$, the characterization for the problem without constraint holds true.

In the domain: Assume that $D \in C^{1,2}$ (or intersection of $C^{1,2}$ domains) and take $\delta \in C^{1,2}$ such that $\delta > 0$ in $\mathrm{int}(D)$, $\delta = 0$ on ∂D and $\delta < 0$ elsewhere.

The state constraint imposes $d\delta(t, Z^{\nu}_{t,z}(t)) \geq 0$ if $(t, z) \in \partial D$.

As above it implies:

$$\mathcal{L}^{\nu_t}_{Z}\delta(t, x, \gamma(t, x)) \geq 0 \text{ and } D\delta(t, x, \gamma(t, x))'\sigma_Z(x, y, \nu_t) = 0$$

when $(t, x, \gamma(t, x)) \in \partial D$.

Define

$$\mathcal{N}_{\varepsilon}^{in}\varphi := \{u \in N_{\varepsilon}(\cdot,\varphi) \ : \ |D\delta(\cdot,\varphi)'\sigma_Z(\cdot,\varphi,u)| \leq \varepsilon\}$$

and

$$\mathcal{H}_{\varepsilon}^{in}\varphi := \sup_{u\in\mathcal{N}_{\varepsilon}^{in}\varphi} \min\left\{\mu_Y(\cdot,\varphi,u) - \mathcal{L}_X^u \ , \ \mathcal{L}_Z^u\delta(\cdot,\varphi)\right\}.$$

The super-solution property is stated as in the unconstrained case on $\overline{D}$ (the fact that the constraint does not appear at the super-solution level is standard, and usually harmless), but for $(t,x,\gamma^*(t,x,)) \in \partial D$ $(t < T)$, the subsolution property reads

$$\mathcal{H}_{*}^{in}\varphi \leq 0.$$

Part III

Stochastic target problem with constraint in expectation

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \; \exists\, \nu \in \mathcal{U}^t \text{ s.t. } \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p\}.$$

1 Problem reduction

Assume that ℓ has quadratic growth.

Let $\mathcal{A}^t$ denote the set of $\mathbb{F}^t$-progressively measurable square integrable processes with values in $\mathbb{R}^d$. Define

$$M^{\alpha}_{t,p} := p + \int_t^{\cdot} \alpha'_s dW_s.$$

Let $\nu \in \mathcal{U}^t$ be such that $\bar{p} := \mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq p$. Then, there exists $\alpha \in \mathcal{A}^t$ such that

$$\ell(Z^{\nu}_{t,z}(T)) = M^{\alpha}_{t,\bar{p}}(T) \geq M^{\alpha}_{t,p}(T).$$

Conversely, let $(\nu, \alpha) \in \mathcal{U}^t \times \mathcal{A}^t$ be such that $\ell(Z^{\nu}_{t,z}(T)) = M^{\alpha}_{t,\bar{p}}(T) \geq M^{\alpha}_{t,p}(T)$. Then,

$$\mathbb{E}\left[\ell(Z^{\nu}_{t,z}(T))\right] \geq \mathbb{E}\left[M^{\alpha}_{t,p}(T)\right] = p.$$

Proposition: (B., Elie and Touzi [4])

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, (\nu,\alpha) \in \mathcal{U}^t \times \mathcal{A}^t \textbf{ s.t. } \ell(Z^{\nu}_{t,z}(T)) \geq M^{\alpha}_{t,p}(T) \ \mathbb{P}-\textbf{a.s.}\}.$$

We are back to the previous part, for an enlarged system.

Proposition (GDP): Fix $\{\theta^{\phi}, \phi \in \mathcal{U}^t \times \mathcal{A}^t\} \subset \mathcal{T}^t$. Then,

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, \phi \in \mathcal{U}^t \times \mathcal{A}^t \textbf{ s.t. } Z^{\phi}_{t,z}(\theta^{\phi}) \in V(\theta^{\phi}, M^{\phi}_{t,p}(\theta^{\phi})) \ \mathbb{P}-\textbf{a.s.}\}.$$

Additional difficulty: α coming from the martingale representation theorem can not be assumed to take values in a compact !

PDE characterization: Remains the same for the enlarged system, except that a boundary layer phenomena may happen at T.

2 Example #1: quantile hedging in the BS model

2.1 Problem

$$\gamma(t,p) := \inf\{y \geq 0 : \ \exists\, \nu \in \mathcal{U}^t \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,y}(T) \geq g(X_{t,x}(T))\right] \geq p\}.$$

where

$$dX = X(\mu dt + \sigma dW) \text{ and } dY^{\nu} = \nu Y^{\nu} dX/X = \nu Y^{\nu}(\mu dt + \sigma dW).$$

with $\sigma > 0$, $U = \mathbb{R}$ and $g \geq 0$ continuous (poly. growth).

2.2 PDE in the domain

$$H_\varepsilon(x,y,r,q,A)$$
$$=$$
$$\sup_{(u,a)\in N_\varepsilon(x,y,q)} \left\{ uy\mu - r - x\mu q_x - \frac{1}{2}\left(x^2\sigma^2 A_{xx} + a^2 A_{pp} + 2x\sigma a A_{xp}\right)\right\}$$

with

$$N_\varepsilon(x,y,q) := \left\{(u,a)\in\mathbb{R}^2 \;:\; |uy\sigma - x\sigma q_x - aq_p| \le \varepsilon\right\}.$$

Thus, for $q_p > 0$,

$$H_\varepsilon(x,y,r,q,A)$$
$$=$$
$$-r + \sup_{a\in\mathbb{R},|\xi|=1} \left\{\frac{\mu}{\sigma}(\varepsilon\xi + aq_p) - \frac{1}{2}\left(x^2\sigma^2 A_{xx} + a^2 A_{pp} + 2x\sigma a A_{xp}\right)\right\}$$

The above is $+\infty$ is $A_{pp} < 0$. For $A_{pp} > 0$,

$$H_*(x, y, r, q, A) = H^*(x, y, r, q, A)$$
$$=$$
$$-r + \sup_{a \in \mathbb{R}} \left\{ \tfrac{\mu}{\sigma} a q_p - \tfrac{1}{2} \left(x^2 \sigma^2 A_{xx} + a^2 A_{pp} + 2x\sigma a A_{xp} \right) \right\}$$
$$=$$
$$-r + \frac{(\frac{\mu}{\sigma} q_p - x\sigma A_{xp})^2}{2A_{pp}^2} - \tfrac{1}{2} x^2 \sigma^2 A_{xx}$$

In particular: γ^* is a subsolution of $D_{pp}\varphi \geq 0$, hence γ^* is convex.

Remark: If γ is smooth and convex then its p-Fenchel transform

$$\tilde{\gamma}(t, x, q) := \sup_{p \in [0,1]} pq - \gamma(t, x, p)$$

solves

$$-\tilde{\gamma}_t - \frac{1}{2}(x^2 \sigma^2 \tilde{\gamma}_{xx} + 2xq\sigma\lambda\tilde{\gamma}_{xq} + q^2 \lambda^2 \tilde{\gamma}_{qq}) = 0$$

with $\lambda := \mu/\sigma$, at points such that $\tilde{\gamma}_q(t, x, q) \in (0, 1)$.

2.3 Boundary conditions

Intuition: The natural boundary condition is

$$\gamma(T, x, p) = g(x)\mathbf{1}_{p>0},$$

which is not continuous. But, since γ^* is convex in p, one should have

$$\gamma^*(T, x, p) \leq p\gamma^*(T, x, 1) + (1-p)\gamma^*(T, x, 0).$$

Moreover, one can show that

$$\gamma^*(\cdot, 1) = \gamma_*(\cdot, 1) = w \text{ and } \gamma^*(\cdot, 0) = \gamma_*(\cdot, 0) = 0,$$

where $w(t, x) = \gamma(t, x, 1)$ is the Black and Scholes hedging price.
Thus

$$\gamma^*(T, x, p) \leq p\gamma^*(T, x, 1) + (1-p)\gamma^*(T, x, 0) = pw(T, x) = pg(x).$$

Conversely, take $(t_n, x_n, p_n, y_n := \gamma(t_n, x_n) + n^{-1}) \to (T, x, p, \gamma_*(T, x, p))$ and ν_n, α_n such that

$$\mathbf{1}_{\{Y_n(T) \geq g(X_n(T))\}} \geq M_n(T) \geq 0,$$

with $(X_n, Y_n, M_n) = (X_{t_n,x_n}, Y^{\nu_n}_{t_n,x_n,y_n}, M^{\alpha_n}_{t_n,p_n})$. Then, (recall that $g \geq 0$)

$$Y_n(T) \geq M_n(T) g(X_n(T)).$$

Letting L_n be the solution of

$$L_n = 1 - \int_{t_n} \lambda L_n(s) dW_s$$

one has

$$L_n(T) Y_n(T) \geq M_n(T) g(x) + M_n(T) \left(L_n(T) g(X_n(T)) - g(x)\right)$$

and $(|M_n(T)| \leq 1)$

$$\gamma_*(T, x, p) \leftarrow y_n \geq pg(x) - \mathbb{E}\left[|L_n(T) g(X_n(T)) - g(x)|\right] \to pg(x).$$

Proposition: $\gamma^*(T, x, p) = \gamma_*(T, x, p) = pg(x)$.

2.4 Explicit resolution

The upper-semicontinuous p-Fenchel transform

$$\tilde{\gamma}^*(t,x,q) := \sup_{p\in[0,1]} pq - \gamma_*(t,x,p)$$

is a viscosity subsolution of

$$-\tilde{\varphi}_t - \frac{1}{2}(x^2\sigma^2\tilde{\varphi}_{xx} + 2xq\sigma\lambda\tilde{\varphi}_{xq} + q^2\lambda^2\tilde{\varphi}_{qq}) = 0 \text{ on } [0,T)\times(0,\infty)^2$$

with $\lambda := \mu/\sigma$, and satisfies

$$\tilde{\gamma}^*(T,x,q) \leq (q-g(x))^+.$$

Hence,

$$\tilde{\gamma}^*(t,x,q) \leq \mathbb{E}^{\mathbb{Q}}\left[(Q_{t,q}(T) - g(X_{t,x}(T)))^+\right]$$

where

$$dQ/Q = \lambda dW^{\mathbb{Q}} \text{ and } d\mathbb{Q} = (1/Q_{0,1}(T))d\mathbb{P}$$

Then,

$$\begin{aligned}\gamma_*(t,x,p) &\geq \sup_{q>0}\left(pq - E^{\mathbb{Q}}\left[(Q_{t,q}(T)-g(X_{t,x}(T)))^+\right]\right)\\ &= p\hat{q} - E^{\mathbb{Q}}\left[(Q_{t,\hat{q}}(T)-g(X_{t,x}(T)))^+\right]\end{aligned}$$

with

$$p = \mathbb{E}^{\mathbb{Q}}\left[Q_{t,1}(T)\mathbf{1}_{\{Q_{t,\hat{q}}(T)\geq g(X_{t,x}(T))\}}\right] = \mathbb{P}[\overbrace{Q_{t,\hat{q}}(T)\geq g(X_{t,x}(T))}^{=:A}].$$

Hence,

$$\gamma_*(t,x,p) \;\geq\; E^{\mathbb{Q}}\left[Q_{t,\hat{q}}(T)\mathbf{1}_A - (Q_{t,\hat{q}}(T)-g(X_{t,x}(T)))\mathbf{1}_A\right] = E^{\mathbb{Q}}\left[g(X_{t,x}(T))\mathbf{1}_A\right].$$

Since $p=\mathbb{P}\left[A\right]$, we conclude

Proposition:

$$\gamma(t,x,p) \;=\; E^{\mathbb{Q}}\left[g(X_{t,x}(T))\mathbf{1}_{A_{t,x}}\right].$$

where

$$A_{t,x} := \{Q_{t,\hat{q}_{t,x}}(T) \geq g(X_{t,x}(T))\}$$

and $\hat{q}_{t,x}$ **such that** $p = \mathbb{P}[A_{t,x}]$.

One retrieves the result of Föllmer and Leukert [12].

3 Example #2: shortfall pricing in models with proportional costs

3.1 Model

Set of controls: $\mathcal{U}$ is the set of adapted continuous and non-decreasing processes (component by component) in $\mathbb{R}^2$.

Controlled process:

$$\begin{aligned}
X^1(s) &= x^1 + \int_t^s X^1(r)\mu dr + \int_t^s X^1(r)\sigma dW_r \\
X^{2,\nu}(s) &= x^2 + \int_t^s \frac{X^{2,\nu}(r)}{X^1(r)} dX^1(r) - \int_t^s d\nu_r^1 + \int_t^s d\nu_r^2 \\
Y^\nu(s) &= y + \int_t^s (1-\lambda) d\nu_r^1 - \int_t^s (1+\lambda) d\nu_r^2 \ .
\end{aligned}$$

Price under shortfall constraint: $\gamma(t,x,p)$ defined as the inf over y such that $\exists\ \nu \in \mathcal{U}^t$ for which

$$Y^{\nu}_{t,y} + l(X^{2,\nu}_{t,x}) \geq -\kappa \text{ and } \mathbb{E}\left[\psi\left(Y^{\nu}_{t,y}(T) + l(X^{2,\nu}_{t,x}(T)) - g(X^{1}_{t,x}(T))\right)\right] \geq p$$

where $l(x^2) = x^2 - \lambda|x^2|$ is the liquidation value of the position in stock, ψ is bounded, non-decreasing.

Reformulation: $\gamma(t,x,p)$ defined as the inf over y such that $\exists\ (\nu,\alpha) \in \mathcal{U}^t \times \mathcal{A}^t$ for which

$$Y^{\nu}_{t,y} + l(X^{2,\nu}_{t,x}) \geq -\kappa \text{ and } \psi\left(Y^{\nu}_{t,y}(T) + l(X^{2,\nu}_{t,x}(T)) - g(X^{1}_{t,x}(T))\right) \geq M^{\alpha}_{t,p}(T)$$

3.2 Informal PDE derivation

Assume γ smooth and $y = \gamma(t,x,p)$ implies that there exists $(\nu,\alpha) \in \mathcal{U}^t$ such that

$$Y^{\nu}_{t,x,y}(\theta) \geq \gamma(\theta, X^{\nu}_{t,x}(\theta), M^{\alpha}_{t,p}(\theta)) \ \mathbb{P} - \text{a.s.}$$

For $\theta = t+$:

$$Y^{\nu}_{t,x,y}(t+) \geq \gamma(t+, X^{\nu}_{t,x}(t+), M^{\alpha}_{t,p}(t+)) \ \mathbb{P} - \text{a.s.}$$

i.e.

$$\begin{aligned} 0 \ &\leq \ -\mathcal{L}^{\alpha_t}_{X,M}\gamma(t,x,p)dt - \left(D_x\gamma(t,x,p)'x\sigma + \alpha_t D_p\gamma(t,x,p)\right) dW_t \\ &+ \ \left((1-\lambda) + D_{x^2}\gamma(t,x,p)\right) d\nu^1_t + \left(-(1+\lambda) - D_{x^2}\gamma(t,x,p)\right) d\nu^2_t \end{aligned}$$

This implies

$$D_x\gamma(t,x,p)'x\sigma + \alpha_t D_p\gamma(t,x,p) = 0 \text{ and } -\mathcal{L}^{\alpha_t}_{X,P}\gamma(t,x,p) \geq 0$$

or

$$\max\left\{(1-\lambda) + D_{x^2}\gamma(t,x,p)\ ,\ -(1+\lambda) - D_{x^2}\gamma(t,x,p)\right\} > 0.$$

3.3 PDE characterization

Set

$$\mathcal{H}\varphi := -\mathcal{L}^{\hat{a}_\varphi}_{X,M}\varphi \ , \ \hat{a}_\varphi := -D_x\varphi' x\sigma / D_p\varphi$$

and

$$\mathcal{G}\varphi := \max\left\{(1-\lambda) - D_{x^2}\varphi \ , \ -(1+\lambda) + D_{x^2}\varphi\right\} .$$

Proposition: γ_* and γ^* are respectively super- and subsolution of

$$\max\left\{\mathcal{H}\varphi \ , \ \mathcal{G}\varphi\right\} \mathbf{1}_{\{D_p\varphi>0\}} \geq 0$$

and

$$\min\left\{\varphi + l(x^2) + \kappa \ , \ \max\{\mathcal{H}\varphi \ , \ \mathcal{G}\varphi\}\right\}\mathbf{1}_{\{D_p\varphi>0\}} \leq 0.$$

3.4 Boundary condition at T

If

$$\psi\left(Y^{\nu}_{t,y}(T)+l(X^{2,\nu}_{t,x}(T))-g(X^{1}_{t,x}(T))\right)\geq M^{\alpha}_{t,p}(T)$$

then

$$y+zx^2\geq\mathbb{E}^{\mathbb{Q}}\left[\psi^{-1}(M^{\alpha}_{t,p}(T))+g(X^{1}_{t,x}(T))\right]\ \forall\ z\in[1-\lambda,1+\lambda].$$

If ψ^{-1} is convex, then

$$y+l(x^2)\geq\psi^{-1}(p)+\mathbb{E}^{\mathbb{Q}}\left[g(X^{1}_{t,x}(T))\right]\to\psi^{-1}(p)+g(x^1)\text{ as }t\to T$$

Conversely, for $y=\psi^{-1}(p+\eta)+g(x^1)$, $\eta>0$, we can find $\varepsilon>0$ such that for all $t\in[T-\varepsilon,T]$

$$\mathbb{E}\left[\psi\left(Y^{0}_{t,y}(T)+l(X^{2,0}_{t,x}(T))-g(X^{1}_{t,x}(T))\right)\right]\geq p.$$

Proposition: For all $p \in \mathrm{int}(\mathrm{Im}(\psi))$,

$$\gamma^*(T,x,p) = \gamma_*(T,x,p) = \max\{\psi^{-1}(p) + g(x^1)\ ,\ -\kappa\} - l(x^2).$$

3.5 Boundary condition at $\partial\mathrm{Im}(\psi)$

Without loss of generality, we can assume that $\mathrm{Im}(\psi) = [0, 1]$.

a- Boundary condition at $p = 0$:

One has: $\gamma(t,x,0) = -\kappa - l(x^2)$.

Step 1. If φ is a test function for γ^* at $(t,x,0)$ then

$$\min\left\{\varphi + \kappa + l(x^2)\ ,\ \max\{\mathcal{H}\varphi\ ,\ \mathcal{G}\varphi\}\right\}\mathbf{1}_{\{D_p\varphi>0\}} \le 0$$

where

$$\mathcal{H}\varphi = -\mathcal{L}_X\varphi - \frac{1}{2}\left(D_x\varphi' x\sigma/D_p\varphi\right)^2 D_{pp}\varphi + \left(D_x\varphi' x\sigma/D_p\varphi\right)\sigma(x^1 D_{x^1p}\varphi + x^2 D_{x^2p}\varphi)$$

Step 2. If ϕ is a test function for $(t,x) \mapsto \gamma^*(t,x,0)$ at $(t_o,x_o,0)$ then one can construct a sequence of test functions φ_n and test points $(t_n,x_n,p_n) \to (t_o,x_o,p_o)$ such that

$$D_p\varphi_n(t_n,x_n,p_n) > 0\ ,\ (D_{pp}\varphi_n/D_p\varphi_n^2, D_{xp}\varphi_n/D_p\varphi_n)(t_n,x_n,p_n) \to 0,$$

and the other derivatives converges to the corresponding derivatives of ϕ at (t_o,x_o). Passing to the limit leads to

$$\min\left\{\phi + \kappa + l(x^2)\ ,\ \max\{-\mathcal{L}_X\phi\ ,\ \mathcal{G}\phi\}\right\} \le 0.$$

b- Boundary condition at $p = 1$: One has: $\gamma(t, x, 1) = w(t, x)$ the super-hedging price of $g(X^1_{t,x}(T))$ starting from x^2. Clearly, $\gamma^*(t, x, 1) \leq w^*(t, x)$.

One can show by similar argument as above that $(t, x) \mapsto \gamma_*(t, x, 1)$ is a supersolution of

$$\min\left\{\phi - w_* \ , \ \max\{-\mathcal{L}_X\phi \ , \ \mathcal{G}\phi\}\right\} = 0$$

3.6 Boundary condition at $\partial \mathrm{Im}(\psi)$ and $t = T$,

Problematic: typically discontinuous... need to regularize the criteria.

Natural boundary conditions: $\gamma(T, \cdot, 1) = \hat{g}$ where $\hat{g} \geq g - l(x^2)$ is the cost of the cheapest buy-and-hold strategy, and $\gamma(T, \cdot, 0) = -\kappa - l(x^2)$.

Discontinuity: $\gamma(T, \cdot, 1-) = \max\{\psi^{-1}(1) + g(x^1)\ ,\ -\kappa\} - l(x^2)$ and $\gamma(T, \cdot, 0+) = \max\{\psi^{-1}(0) + g(x^1)\ ,\ -\kappa\} - l(x^2)$

Regularization: Assume $\psi(r) = (r^- \vee -1) + 1$, $g \geq 0$, $\kappa > 1$. Then, replace $\psi(y + l(x^2) - g(x^1))$ by $\Delta_\varepsilon(x, y)$ such that $\Delta^{-1}(x, \cdot)$ is continuous on $[0, 1]$, $\Delta_\varepsilon^{-1}(x, 1) = \hat{g}$ and $\Delta_\varepsilon^{-1}(x, 0) = -\kappa - \ell$.

Such a technic is applied in B. and Vu [9] for quantile hedging under portfolio constraint (more previsely for a P&L matching version).

4 Multiple constraints

One can similarly handle problems of the form

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U}^t \text{ s.t. } \mathbb{E}\left[\ell^i(Z^{\nu}_{t,z}(T))\right] \geq p^i \;\forall\, i \leq \kappa\}.$$

In this case, the corresponding martingale M is κ dimensional.

See B. and Vu [9] for a P&L matching problem under portfolio constraint:

$$V(t,p) := \{z \in \mathbb{R}^{d+1} : \exists\, \nu \in \mathcal{U}^t \text{ s.t. } \mathbb{P}\left[Y^{\nu}_{t,z}(T) - g(X_{t,x}(T)) \geq -\gamma_i\right] \geq p^i \;\forall\, i \leq \kappa\}$$

with

$$\gamma_1 < \gamma_2 < \cdots < \gamma_\kappa$$

and

$$0 \leq p^1 \leq p^2 \leq \cdots \leq p^\kappa.$$

This leads to multiple boundary conditions on boundary of sets of the form

$$D_{IJ} = \{p \in [0,1]^\kappa : p^i = 0 \text{ if } i \in I,\ p^i = 1 \text{ if } i \in J,\, 0 < p^i < 1 \text{ otherwise}\}$$

One considers π_{IJ} the projection on D_{IJ} and $v_{IJ}(t,x,p) := \gamma(t,x,\pi_{IJ}(p))$. For each v_{IJ}, the PDE is obtained in the domain $[0,T) \times \mathbb{R}^d \times D_{IJ}$ as before and boundary conditions are given on $[0,T) \times \mathbb{R}^d \times \partial D_{IJ}$ in terms of the $v_{I'J'}$ with $I' \supset I$ and $J' \supset J$.

5 Link with standard stochastic control problems

Let us consider:

$$w := \inf_{\nu \in \mathcal{U}} \mathbb{E}\left[f(Z^\nu(T))\right].$$

Let $\mathcal{A}$ denote the set of $\mathbb{P}-$ a.s. square integrable progressively measurable processes such that $M^\alpha_{0,p}$ is a martingale. Then,

$$w = \overline{w} := \inf\{p : \exists\,(\nu, \alpha) \in \mathcal{U} \times \mathcal{A} \text{ s.t. } f(Z^\nu(T)) \leq M^\alpha_{0,p}(T)\}$$

<u>Proof</u>. Given ν, we can find α such that

$$f(Z^\nu(T)) = M^\alpha_{0,p}(T) \text{ for } p := \mathbb{E}\left[f(Z^\nu(T))\right].$$

Thus $p \geq \overline{w}$. For $p \to w$, this leads to $w \geq \overline{w}$. Conversely, for $p > \overline{w}$, $\exists\,(\nu, \alpha) \in \mathcal{U} \times \mathcal{A}$ such that

$$f(Z^\nu(T)) \leq M^\alpha_{0,p}(T)$$

and therefore $w \leq \mathbb{E}\left[f(Z^\nu(T))\right] \leq p$. Hence $\overline{w} \geq w$.

Part IV

Optimal control under stochastic target constraints in controlled loss form

$$w(t,x,p) := \sup_{\nu \in \mathcal{U}(t,x,p)} F(t,x;\nu)$$

where

$$F(t,x;\nu) := \mathbb{E}\left[f(X^{\nu}_{t,x}(T))\right]$$

and

$$\mathcal{U}(t,x,p) := \left\{\nu \in \mathcal{U}^t \ : \ G(t,x;\nu) := \mathbb{E}\left[g(X^{\nu}_{t,x}(T))\right] \le p\right\}.$$

1 Problem reduction

Assumption: $g(X^{\nu}_{t,x}(T)) \in L^2$ for all $\nu \in \mathcal{U}$ and (t,x).

Proposition:

$$\mathcal{U}(t,x,p) = \left\{\nu \in \mathcal{U}^t \ : \ \exists\, \alpha \in \mathcal{A}^t \text{ s.t. } g(X^{\nu}_{t,x}(T)) \le M^{\alpha}_{t,p}(T)\right\}.$$

Corollary: (B., Elie and Imbert [3]) By the (GDPP),

$$w(t,x) = \sup_{(\nu,\alpha)\in\Gamma(t,x,p)} J(t,x;\nu)$$

where

$$\Gamma(t,x,p) := \{(\nu,\alpha) \in \mathcal{U}^t \times \mathcal{A}^t \text{ s.t. } X^{\nu}_{t,x} \in V(\cdot, M^{\alpha}_{t,p}) \ \text{ on } [t,T] \ \mathbb{P}-\text{a.s.}\}$$

and

$$V(t,p) := \{x \in \mathbb{R}^d \ : \ (\nu,\alpha) \in \mathcal{U}^t \times \mathcal{A}^t \text{ s.t. } g(X^{\nu}_{t,x}(T)) \le M^{\alpha}_{t,p}(T)\}.$$

We are back to a "standard" state constraint problems for a domain defined as

$$D := \{(t, x, p) \in [0, T] \times \mathbb{R}^d \times \overline{\mathrm{Im}(g)}^c \ : \ x \in V(t, p)\}$$

which requires to solve the stochastic target problem associated to V first.

Technical issue: V is typically not smooth, and can even be not continuous.

2 Weak dynamic programming principle

"Weak" ? : in terms of test function rather than in term of w itself.

Why ? : to make profit of the regularity of test functions (and measurability).

Proposition: **Let $\varphi_+ \geq w$ be a smooth function. Let $\{\theta^\phi, \phi \in \mathcal{U}^t \times \mathcal{A}^t\}$ be a family of stopping times such that $(X^\phi_{t,x}, M^\phi_{t,p})$ is bounded on $[t, \theta^\phi]$. Then,**

$$w(t,x,p) \leq \sup_{\phi \in \Gamma(t,x,p)} \mathbb{E}\left[\varphi_+(\theta^\phi, X^\phi_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi))\right].$$

In the above, we just use the measurability of φ_+.

Proof. Take $\nu \in \mathcal{U}(t,x,p)$ and let $\alpha \in \mathcal{A}^t$ be such that $g(X^\nu_{t,x}(T)) \le M^\alpha_{t,p}(T)$. We write $\phi = (\nu, \alpha)$.

$$F(t,x,p;\nu) = \mathbb{E}\left[\mathbb{E}\left[f(X^\nu_{\theta^\phi, X^\nu_{t,x}(\theta^\phi)}(T)) \mid \mathcal{F}_{\theta^\phi}\right]\right]$$

with

$$\mathbb{E}\left[f(X^\nu_{\theta^\phi, X^\nu_{t,x}(\theta^\phi)}(T)) \mid \mathcal{F}_{\theta^\phi}\right](\omega) = F(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega); \nu_\omega)$$

where

$$\nu_\omega : \tilde{\omega} \to \nu(\omega \mathbf{1}_{[0,\theta^\phi(\omega)]} + \mathbf{1}_{(\theta^\phi(\omega),T]}(\tilde{w} - \tilde{w}_{\theta^\phi_\omega})).$$

Assuming $\nu_\omega \in \mathcal{U}(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega), M^\alpha_{t,p}(\theta^\phi)(\omega))$, then

$$\begin{aligned} F(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega); \nu_\omega) &\le \gamma(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega), M^\alpha_{t,p}(\theta^\phi)(\omega)) \\ &\le \varphi(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega), M^\alpha_{t,p}(\theta^\phi)(\omega)) \end{aligned}$$

and therefore

$$F(t,x,p;\nu) \le \mathbb{E}\left[\varphi(\theta^\phi, X^\nu_{t,x}(\theta^\phi), M^\alpha_{t,p}(\theta^\phi))\right].$$

Conclude by taking the sup over $\Gamma(t,x,p)$.

It remains to check that $\nu_\omega \in \mathcal{U}(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega), M^\alpha_{t,p}(\theta^\phi)(\omega))$.
But, $g(X^\nu_{t,x}(T)) \leq M^\alpha_{t,p}(T)$ implies

$$G(\theta^\phi(\omega), X^\nu_{t,x}(\theta^\phi)(\omega); \nu_\omega) = \mathbb{E}\left[g(X^\nu_{\theta^\phi, X^\nu_{t,x}(\theta^\phi)}(T)) \mid \mathcal{F}_{\theta^\phi}\right](\omega) \leq M^\alpha_{t,p}(\theta^\phi)(\omega).$$

Assumption: For any $\nu \in \mathcal{U}$, the maps $(t,x) \mapsto F(t,x;\nu)$ and $(t,x) \mapsto -G(t,x;\nu)$ are lower-semicontinuous (on the right in time).

Proposition: Fix $\delta > 0$ and let $\varphi_- \leq w$ be a smooth function. Let $\{\theta^\phi, \phi \in \mathcal{U}^t \times \mathcal{A}^t\}$ be a family of stopping times such that $(X^\phi_{t,x}, M^\phi_{t,p})$ is bounded on $[t, \theta^\phi]$. Then,

$$\sup_{\phi \in \Gamma(t,x,p)} \mathbb{E}\left[\varphi_-(\theta^\phi, X^\phi_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi))\right] \leq w(t,x,p+\delta).$$

Remark: We make two relaxations $w \to \varphi$ and $p \to p + \delta$. We will play on the upper-semicontinuity of φ_- only.

Proof. Fix $\varepsilon > 0$. By the above continuity properties, one can find open radius $r_i > 0$, points (t_i, x_i, p_i) and controls $\nu_i \in \mathcal{U}(t_i, x_i, p_i)$ such that

$$F(\cdot; \nu_i) + \varepsilon \geq F(t_i, x_i; \nu_i) \geq w(t_i, x_i, p_i) - \varepsilon \geq \varphi - 2\varepsilon \quad \text{on } B_i$$

and

$$G(t, x; \nu_i) - \varepsilon \leq G(t_i, x_i; \nu_i) \leq p_i \leq p + \varepsilon \text{ for } (t, x, p) \in B_i$$

where

$$B_i := ([t_i, t_i + r_i) \times B_{r_i}(x_i, p_i)) \cap D$$

and

$$\cup_{i \geq 1} B_i \supset B$$

a compact set which the controlled process does not exit.
It follows that

$$F(t, x; \nu_i) \geq \varphi(t, x) - 3\varepsilon \text{ and } G(t, x; \nu_i) \leq p + 2\varepsilon \quad \text{for } (t, x, p) \in B_i.$$

Since

$$F(t,x;\nu_i) \geq \varphi(t,x,p) - 3\varepsilon \text{ and } G(t,x;\nu_i) \leq p + 2\varepsilon \quad \text{for } (t,x,p) \in B_i.$$

considering $\overline{\nu}$ defined by

$$\overline{\nu} \ = \begin{cases} \nu & \text{on } [t, \theta^\phi] \\ \tilde{\nu} = \nu_i & \text{on } (\theta^\phi, T] \text{ if } (\theta^\phi, X^\nu_{t,x}(\theta^\phi), M^\alpha_{t,p}(\theta^\phi)) \in B_i \end{cases},$$

where $\phi = (\nu, \alpha) \in \Gamma(t,x,p)$, leads to

$$F(\theta^\phi, X^\nu_{t,x}(\theta^\phi); \tilde{\nu}) \geq \varphi(\theta^\phi, X^\nu_{t,x}(\theta^\phi), M^\alpha_{t,p}(\theta^\phi)) - 3\varepsilon$$

and

$$G(\theta^\phi, X^\nu_{t,x}(\theta^\phi); \tilde{\nu}) \leq M^\alpha_{t,p}(\theta^\phi) + 2\varepsilon.$$

Thus,

$$F(t,x;\overline{\nu}) = \mathbb{E}\left[F(\theta^\phi, X^\nu_{t,x}(\theta^\phi); \tilde{\nu})\right] \geq \mathbb{E}\left[\varphi(\theta^\phi, X^\nu_{t,x}(\theta^\phi), M^\alpha_{t,p}(\theta^\phi)) - 3\varepsilon\right]$$

and

$$G(t,x;\overline{\nu}) = \mathbb{E}\left[G(\theta^\phi, X^\nu_{t,x}(\theta^\phi); \tilde{\nu})\right] \leq p + 2\varepsilon.$$

This implies that

$$w(t,x,p+2\varepsilon) \geq F(t,x;\overline{\nu}) \geq \mathbb{E}\left[\varphi(\theta^{\phi}, X_{t,x}^{\nu}(\theta^{\phi}), M_{t,p}^{\alpha}(\theta^{\phi})) - 3\varepsilon\right].$$

Given $\delta > 0$ and $2\varepsilon < \delta$, we have $w(t,x,p+\delta) \geq w(t,x,p+2\varepsilon)$. Remains to take the sup over the control ϕ on the right and send $\varepsilon \to 0$.

Remark: We do not have to do infinite (countable) pasting as above, but only finite pasting, this leads to an additional ε...

Theorem (Weak DPP): (B. and Nutz [6], B. and Touzi [7])**Let $\varphi_- \leq w \leq \varphi_+$ be smooth functions and fix $\varepsilon > 0$. Let $\{\theta^\phi, \phi \in \mathcal{U}^t \times \mathcal{A}^t\}$ be a family of stopping times such that $(X^\phi_{t,x}, M^\phi_{t,p})$ is bounded on $[t, \theta^\phi]$. Then,**

$$\sup_{\phi \in \Gamma(t,x,p)} \mathbb{E}\left[\varphi_-(\theta^\phi, X^\phi_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi))\right] \leq w(t,x,p+\varepsilon)$$

and

$$w(t,x,p) \leq \sup_{\phi \in \Gamma(t,x,p)} \mathbb{E}\left[\varphi_+(\theta^\phi, X^\phi_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi))\right].$$

Corollary: With the same notations and assumptions:

$$\sup_{\phi \in \Gamma(t,x,p)} \mathbb{E}\left[w_*(\theta^\phi, X^\phi_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi))\right] \leq w(t,x,p+\varepsilon)$$

and

$$w(t,x,p) \leq \sup_{\phi \in \Gamma(t,x,p)} \mathbb{E}\left[w^*(\theta^\phi, X^\phi_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi))\right].$$

Remark: The same arguments allows one to prove a weak GDPP for problems of the form:

$$\overline{\gamma}(t,x,p) := \{y \in \mathbb{R} : \ \exists\, \nu \in \mathcal{U}^t \text{ s.t. } \mathbb{E}\left[\ell(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T))\right] \geq p\}.$$

Proposition (Weak GDP): Fix $\{\theta^\phi, \phi \in \mathcal{U}^t \times \mathcal{A}^t\} \subset \mathcal{T}^t$. Then,

(GDP1) If $y > \overline{\gamma}(t,x,p)$, then there exists $\phi := (\nu, \alpha) \in \mathcal{U}^t \times \mathcal{A}^t$ such that

$$Y^\nu_{t,x,y}(\theta^\phi) \geq \overline{\gamma}(\theta^\phi, X^\nu_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi)).$$

(GDP2) If there exists $\phi := (\nu, \alpha) \in \mathcal{U}^t \times \mathcal{A}^t$ such that

$$Y^\nu_{t,x,y}(\theta^\phi) > \overline{\gamma}(\theta^\phi, X^\nu_{t,x}(\theta^\phi), M^\phi_{t,p}(\theta^\phi)),$$

then $y \geq \overline{\gamma}(t,x,p-\varepsilon)$ for all $\varepsilon > 0$.

3 PDE characterization

In the domain D: In the interior of

$$D = \{(t,x,p) \in [0,T] \times \mathbb{R}^d \times \overline{\mathrm{Im}(g)}^c \;:\; x \in V(t,p)\}$$

the state constraint does not play any role. We have the usual HJB equation, but in terms of (t,x) and p !
Set

$$\mathcal{H}\varphi = - \sup_{(u,a)\in U\times\mathbb{R}^d} \mathcal{L}^{u,a}_{X,M}\varphi$$

Proposition: w_* and w^* are respectively super- and sub-solution on $\overline{D}$ of

$$\mathcal{H}^*\varphi(t,x,p) \geq 0 \;\text{ if } (t,x,p) \in \mathrm{int}(D)$$

and

$$\mathcal{H}_*\varphi(t,x,p) \leq 0 \;\text{ if } (t,x,p) \in \mathrm{int}(D).$$

On the boundary:

To obtain a characterization, we define the value function γ associated to V:

$$\gamma(t,x) := \inf\{p \in \overline{\mathrm{Im}(g)}^c : \exists\ (\nu,\alpha) \in \mathcal{U}^t \times \mathcal{A}^t \text{ s.t. } g(X^\nu_{t,x}(T)) \leq M^\alpha_{t,p}(T)\}.$$

Assuming that γ is smooth, the (GDPP) for γ implies that the only possible controls on ∂D (boundary on $[0,T)$) are such that

$$\alpha_t - \sigma_X(x,\nu_t)^\top D\gamma(t,x) = 0 \text{ and } -\mathcal{L}^{\nu_t}_X \gamma(t,x) \geq 0,$$

where $\mathcal{L}^u_X$ denotes the Dynkin operator of X

Hence, w should satisfy on ∂D

$$\mathcal{H}^v_{\mathrm{in}} w := -\sup_{(u,a)\in\Theta^v_{\mathrm{in}}} \mathcal{L}^{u,a}_{X,M} w = 0$$

with

$$\Theta^v_{\mathrm{in}} = \{(u,a) \in U \times \mathbb{R}^d : a - \sigma_X(\cdot,u)' Dv = 0, -\mathcal{L}^u_X v \geq 0\}.$$

a- Sub-solution part: We define

$\mathcal{W}_*(t,x) = \{\phi \in C^{1,2}([0,T]\times\mathbb{R}^d) \text{ s.t. } (\gamma-\phi) > (\gamma-\phi)(t,x) = 0 \text{ near } (t,x)\}.$

$\mathcal{N}_0^\phi(t,x)$ is again defined as the set of points $(u,a) \in U \times \mathbb{R}^d$ such that

$$a = \sigma_X(x,u)' D\phi(t,x) \ \text{ and } \ -\mathcal{L}_X^u \phi(t,x) \geq 0 \ .$$

Finally, we set

$$H_{in}^\phi \varphi := -\sup_{(u,a)\in\mathcal{N}_0^\phi} \mathcal{L}_{X,M}^{u,a}\varphi$$

Proposition: Assume that γ is continuous on $[0,T) \times \mathbb{R}^d$ and that U is compact. Then, w^* is a subsolution on $\overline{D}$ of

$$\sup_{\phi\in\mathcal{W}_*(t,x)} \mathcal{H}_{in}^\phi \varphi \leq 0 \quad \text{if } \ (t,x,p) \in \partial D.$$

b- Super-solution part: Since γ may not be smooth, we need to use the notion of test functions.

Set

$$\mathcal{W}^*(t,x) = \{\phi \in C^{1,2}([0,T]\times\mathbb{R}^d) \text{ s.t. } (\gamma-\phi) < (\gamma-\phi)(t,x) = 0 \text{ near } (t,x)\}.$$

Then, for $\phi \in \mathcal{W}^*(t,x)$, we define the set $\mathcal{N}_0^{\varphi}(t,x)$ as the set of points $(u,a) \in U \times \mathbb{R}^d$ such that

$$a = \sigma_X(x,u)' D\phi(t,x) \quad \text{and} \quad -\mathcal{L}_X^u \phi(t,x) \geq 0 \ .$$

We let $\mathcal{C}(t,x)$ be defined as above but with respect to $\mathcal{N}_0^{\phi}(t,x)$ in place of the former $N_0(x,\phi(t,x),D\phi(t,x))$.
Finally, we set

$$H^{\phi}\varphi := -\sup_{(u,a)\in\mathcal{N}_0^{\phi}} \mathcal{L}_{X,M}^{u,a}\varphi$$

For $\phi \in \mathcal{W}^*(t,x)$, we define $\mathcal{H}_{in}^{\phi *}$ as the upper semi-relaxed limit as the point, gradient and Hessian converge (all the parameters except ϕ).

Proposition: Assume that γ is continuous on $[0,T) \times \mathbb{R}^d$. Then, w_* is a supersolution on $\overline{D}$ of

$$\inf_{\phi \in \mathcal{W}^*(t,x) \cap \mathcal{C}(t,x)} \mathcal{H}_{in}^{\phi *} \varphi \geq 0 \quad \text{if} \quad (t,x,p) \in \partial D.$$

Remark: We will use the fact that

$$\partial D = \{(t,x,p) : p = \gamma(t,x)\}$$

when γ is continuous.

Proof. Assume $w = w_*$ for simplicity. Let $(t_o, x_o, p_o) \in \partial D$ be a strict minimum point of $w - \varphi$ (equal to 0) on $\overline{D}$. If

$$\inf_{\phi \in \mathcal{W}^*(t,x) \cap \mathcal{C}(t,x)} \mathcal{H}_{in}^{\phi *} \varphi(t_o, x_o, p_o) < 0$$

then, we can find $\phi \in \mathcal{W}^*(t,x)$, a locally Lipschitz map $(\hat{u}, \hat{a})$, $r > 0$ such that

$$-\mathcal{L}_{X,M}^{(\hat{u},\hat{a})} \varphi \leq 0 \ , \ -\mathcal{L}_X^{\hat{u}} \phi \geq 0 \text{ and } \hat{a} = \sigma_X(\cdot, \hat{u})' D\phi \ \text{ on } B := B_r(t,x,p).$$

Let (X_o, M_o) be the process associated to the initial condition (t_o, x_o, p_o) and the Markovian control $(\hat{u}, \hat{a})$. Let θ be the first exit time of $(\cdot, X_o, M_o)$ from B. Then,

$$w(t_o, x_o, p_o) = \varphi(t_o, x_o, p_o) \leq \mathbb{E}\left[\varphi(\theta, X_o(\theta), M_o(\theta))\right] \leq \mathbb{E}\left[w(\theta, X_o(\theta), M_o(\theta))\right] - \iota$$

with $\iota > 0$ (minimun of $w - \varphi$ on ∂B). If the Markovian control associated to $(\hat{u}, \hat{a})$ is admissible, this contradicts the weak DPP.

By (GDP1) for γ, it suffices to show that $M_o(\theta) > \gamma(\theta, X_o(\theta))$. But, by the above system, and the fact that $(t_o, x_o, p_o) \in \partial D$ implies $p_o = \gamma(t_o, x_o) = \phi(t_o, x_o)$,

$$M_o(\theta) \geq \phi(\theta, X_o(\theta)) > \gamma(\theta, X_o(\theta)).$$

Remark: We appealed only to (GDP1) whose proof does not require any measurable selection argument.

Part V

Stochastic target games

1 Problem formulation

Determine the *viability* sets

$$V(t,p) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))\right] \geq p\; \forall\, \vartheta \in \mathcal{V}\}$$

In which:

- $\mathcal{V}$ is a set of admissible adverse controls
- $\mathfrak{U}$ is a set of admissible strategies : $\mathfrak{u} \in \mathfrak{U} : \vartheta \in \mathcal{V} \to \mathfrak{u}[\vartheta] \in \mathcal{U}$.
- $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}$ is an adapted $\mathbb{R}^d$-valued process s.t. $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(t) = z$
- ℓ is a given loss/utility function
- p a threshold.

In finance : $Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,z} = (X^{\mathfrak{u}[\vartheta],\vartheta}_{t,x}, Y^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y})$ where

- $X^{\mathfrak{u}[\vartheta],\vartheta}_{t,x}$ models financial assets or factors with dynamics depending on ϑ
- $Y^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y}$ models a wealth process
- ϑ is the control of the market : parameter uncertainty (e.g. volatility), adverse players, etc...
- $\mathfrak{u}[\vartheta]$ is the financial strategy given the past observations of ϑ.

Flexible enough to embed constraints, transaction costs, market impact, etc...

Examples

$$V(t,p) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,z}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

Almost sure constraint:

$$V(t) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,z}(T) \in \mathcal{O}\ \mathbb{P} - \text{a.s.} \;\forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z\in\mathcal{O}}$, $p = 1$.

$\Rightarrow$ Super-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

Compare with Peng (G-expectations) and Soner, Touzi and Zhang (2BSDE).

Examples

Constraint in probability:

$$V(t,p) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{P}\left[Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T) \in \mathcal{O}\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z\in\mathcal{O}}$, $p \in (0,1)$.

$\Rightarrow$ Quantile-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

Examples

$$V(t,p) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

Expected loss control for $\ell(z) = -[y - g(x)]^-$

Can impose several constraint : B. and Thanh Nam (discrete P&L constraints).

Give sense to problems that would be degenerate under $\mathbb{P}-$ a.s. constraints : B. and Dang (guaranteed VWAP pricing).

2 How to get the Geometric dynamic programming principle...

The original problem:

$$V(t,p) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))\right] \geq p \;\forall\, \vartheta \in \mathcal{V}\}$$

Key point: Submartingale property: for $\mathfrak{u}$ fixed

$$S_s^{\mathfrak{u},\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

defines a family of submartingales parameterized by ϑ.

Doob-Meyer decomposition: $\exists$ a family of martingales $\{M^{\mathfrak{u},\vartheta},\vartheta\}$ such that $S^{\mathfrak{u},\vartheta} \geq M^{\mathfrak{u},\vartheta}$ with $M^{\mathfrak{u},\vartheta}(t) = p$.

GDP for target games: $z \in V(t,p)$ if and only if there exists $\mathfrak{u} \in \mathfrak{U}$ and a family of martingales $\{M^{\vartheta},\vartheta\}$ with $M^{\vartheta}(t) = p$ such that

$$\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T)) \geq M^{\vartheta}(T)\ \mathbb{P}-\text{a.s.}\ \forall\, \vartheta \in \mathcal{V}.$$

3 Markovian framework

Strategies: $\mathfrak{U}$ is the set of maps $\mathfrak{u} : \mathcal{V} \to \mathcal{U}$ such that $\{\vartheta_1 =_{(0,t]} \vartheta_2\} \subset \{\mathfrak{u}[\vartheta_1] =_{(0,t]} \mathfrak{u}[\vartheta_2]\}$ for all $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $t \leq T$.

State processes: $Z_{t,z}^{\mathfrak{u},\vartheta} = (X_{t,x}^{\mathfrak{u},\vartheta}, Y_{t,x,y}^{\mathfrak{u},\vartheta})$ is the strong solution of

$$Z(s) = z + \int_t^s \mu(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)dr + \int_t^s \sigma(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)dW_r.$$

(Lipschitz coefficients + controls valued in bounded sets)

Controlled martingales: $\{M_{t,p}^{\alpha}, \alpha \in \mathcal{A}\}$ with

$$M_{t,p}^{\alpha} := p + \int_t^{\cdot} \alpha_s dW_s$$

Martingale strategies: $\mathfrak{A}$ the set of maps $\mathfrak{a}[\cdot]$: $\mathcal{V} \mapsto \mathcal{A}$ such that $\{\vartheta_1 =_{(0,t]} \vartheta_2\} \subset \{\mathfrak{a}[\vartheta_1] =_{(0,t]} \mathfrak{a}[\vartheta_2]\}$ for $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $t \leq T$.

Target

Given a continuous map (with poly growth) $\ell : (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mapsto \ell(x, y) \in \mathbb{R}$, non-decreasing in its y-variable, define

$$I(t, z, \mathfrak{u}, \vartheta) := \mathbb{E}\left[\ell\left(Z^{\mathfrak{u},\vartheta}_{t,z}(T)\right) | \mathcal{F}_t\right] \text{ and } J(t, z, \mathfrak{u}) := \operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} I(t, z, \mathfrak{u}, \vartheta).$$

and

$$\gamma(t, x) := \inf \left\{y \in \mathbb{R} :\ \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } J(t, x, y, \mathfrak{u}) \geq p\, \mathbb{P} - \text{a.s.}\right\}.$$

For later use:

$$K(t, z) := \operatorname*{esssup}_{\mathfrak{u} \in \mathfrak{U}} J(t, z, \mathfrak{u})$$

which can be shown to be deterministic, see [10].

4 The GDP

Theorem

(GDP1) If $y > \gamma(t,x,p+\varepsilon)$ with $\varepsilon > 0$, then $\exists\, \mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^\vartheta, \vartheta \in \mathcal{V}\} \subset \mathcal{A}$ s.t.

$$Y^{\mathfrak{u},\vartheta}_{t,x,y}(\tau) \geq \gamma_*\left(\tau, X^{\mathfrak{u},\vartheta}_{t,x}(\tau), M^{\alpha^\vartheta}_{t,p}(\tau)\right) \quad \mathbb{P}\text{-a.s.}$$

for all $\vartheta \in \mathcal{V}$, $\tau \in \mathcal{T}_t$.

(GDP2) Fix a bounded open set $O \ni (t,x,y,p)$, $(\mathfrak{u},\mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and let τ^ϑ denote the first exit time of $(\cdot, X^{\mathfrak{u},\vartheta}_{t,x}, Y^{\mathfrak{u},\vartheta}_{t,x,y}, M^{\mathfrak{a}[\vartheta]}_{t,p})$, $\vartheta \in \mathcal{V}$. Assume that there exists $\eta > 0$ and a continuous function $\varphi \geq \gamma$ such that

$$Y^{\mathfrak{u},\vartheta}_{t,x,y}(\tau^\vartheta) \geq \varphi\left(\tau, X^{\mathfrak{u},\vartheta}_{t,x}(\tau^\vartheta), M^{\mathfrak{a}[\vartheta]}_{t,p}(\tau^\vartheta)\right) + \eta \;\mathbb{P}-\text{a.s. for all } \vartheta \in \mathcal{V}.$$

Then, $y \geq \gamma(t,x,p-\varepsilon)$ for all $\varepsilon > 0$.

Sketch of proof for GDP1

Assume

$$S_s^{\mathfrak{u},\vartheta} := \operatorname*{ess\,inf}_{\tilde{\vartheta}\in\mathcal{V}_s} \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta\oplus_s\tilde{\vartheta}],\vartheta\oplus_s\tilde{\vartheta}}(T))|\mathcal{F}_s\right]$$

is such that $S_t^{\mathfrak{u},\vartheta} \geq p$.

It admits a càlàg decomposition (up to a modification),+ Doob-Meyer-type decomposition : $S^{\mathfrak{u},\vartheta} \geq M^{\mathfrak{u},\vartheta}$ a càdlàg martingale.

$$K(\tau, Z_{t,z}^{\mathfrak{u},\vartheta}(\tau)) \geq M^{\mathfrak{u},\vartheta}(\tau)$$

which leads to

$$Z_{t,z}^{\mathfrak{u},\vartheta}(\tau) \in V\left(\tau, M^{\mathfrak{u},\vartheta}(\tau) - \varepsilon\right) \quad \mathbb{P}-\text{a.s.}$$

Ok for stopping times τ with values in a countable set. Pass to the limit.

Sketch of proof for GDP2

τ_n^ϑ an approximation of τ^ϑ on a sequence of finite grids π_n.

If

$$Z_{t,z}^{\mathfrak{u},\vartheta}\left(\tau^\vartheta\right) \in \mathring{V}_\iota\left(\tau^\vartheta, M^\vartheta\left(\tau^\vartheta\right)\right) \quad \mathbb{P}-\text{a.s.}$$

then

$$Z_{t,z}^{\mathfrak{u},\vartheta}\left(\tau_n^\vartheta\right) \in V\left(\tau_n^\vartheta, M^\vartheta\left(\tau_n^\vartheta\right)\right) \text{ on } E_n^\vartheta \text{ for all } \vartheta \in \mathcal{V}$$

with $\mathbb{P}\left[E_n^\vartheta\right] \to 1$ as $n \to \infty$ (uniformly in ϑ).

Hence,

$$K(\tau_n^\vartheta, Z_{t,z}^{\mathfrak{u},\vartheta}(\tau_n^\vartheta)) \geq M^\vartheta(\tau_n^\vartheta).$$

Regularity + covering: there exists $\mathfrak{u}_\varepsilon \in \mathfrak{U}$ such that

$$\mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}_\varepsilon,\vartheta}(T))|\mathcal{F}_{\tau_n^\vartheta}\right] \geq M^\vartheta(\tau_n^\vartheta) - \varepsilon.$$

which implies

$$\mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}_\varepsilon,\vartheta}(T))|\mathcal{F}_t\right] \geq p - \varepsilon.$$

References

[1] B. Bouchard (2002), Stochastic Target with Mixed diffusion processes, Stochastic Processes and their Applications, 101, 273-302.

[2] B. Bouchard and N. M. Dang (2010), Generalized stochastic target problems for pricing and partial hedging under loss constraints - Application in optimal book liquidation, preprint.

[3] B. Bouchard, R. Elie, and C. Imbert (2010), Optimal Control under Stochastic Target Constraints. SIAM Journal on Control and Optimization, 48(5), 3501-3531.

[4] B. Bouchard, R. Elie and N. Touzi (2009), Stochastic Target problems with controlled loss, SIAM Journal on Control and Optimization, 48 (5), 3123-3150.

[5] B. Bouchard, L. Moreau and M. Nutz (2012), Stochastic Target Games with Controlled Loss, preprint.

[6] B. Bouchard and M. Nutz (2011), Weak Dynamic Programming for Generalized State Constraints, preprint.

[7] B. Bouchard and N. Touzi (2011), Weak Dynamic Programming Principle for Viscosity Solutions, SIAM Journal on Control and Optimization, 49 (3).

[8] B. Bouchard and T. N. Vu (2010), The American version of the geometric dynamic programming principle: Application to the pricing of american options under constraints, Applied Mathematics and Optimization, 61(2), 235–265.

[9] B. Bouchard and T. N. Vu (2011), A stochastic target approach for P&L matching problems. Preprint Ceremade, University Paris-Dauphine.

[10] R. Buckdahn, Y. Hu, and J. Li (2011), Stochastic representation for solutions of Isaacs' type integral-partial differential equations. *Stochastic Process. Appl.*, 121(12):2715–2750.

[11] J. Cvitanic, H. Pham and N. Touzi (1999), Super-replication in stochastic volatility models with portfolio constraints, Journal of Applied Probability 36, 523-545.

[12] H. Föllmer and P. Leukert (1999), Quantile Hedging, Finance and Stochastics, 3(3), 251–273.

[13] H. Föllmer and P. Leukert (2000), Efficient hedging : cost versus shortfall risk, Finance and Stochastics, 4, 117–146.

[14] L. Moreau (2010), Stochastic Target Problems with Controlled Loss in jump diffusion models. To appear in SIAM Journal on Control and Optimization.

[15] S. Peng (2007), G-expectation, G-Brownian motion and related stochastic calculus of Itô type. In Stochastic Analysis and Applications, volume 2 of Abel Symp., pages 541–567, Springer, Berlin.

[16] R.T. Rockafellar (1970). *Convex Analysis*. Princeton University Press, Princeton, NJ.

[17] H.M. Soner and N. Touzi (2002), Dynamic programming for stochastic target problems and geometric flows, Journal of European Mathematical Society, 4, 201-236.

[18] H.M. Soner and N. Touzi (2002), Stochastic target problems, dynamic programming and visosity solutions, SIAM Journal on Control and Optimization, 41, 404-424.

[19] H.M. Soner and N. Touzi (2004), The problem of super-replication under constraints, Paris-Princeton Lectures in Mathematical Finance, Lecture Notes in Mathematics, Springer-Verlag.

[20] M. Soner, N. Touzi and J. Zhang, Wellposedness of second order backward SDEs, Probability Theory and Related Fields, 153, 149Ð190.