

## Foreword

This book is dedicated to Master students or young Phd students in applied mathematics who wish to learn mathematical finance. It uses a top-down approach, starting from the fundamentals and going little by little towards applications, while covering a large spectrum of concepts and mathematical tools. This is complemented by various (corrected) exercises in which many examples of practical interest will be provided. However, it is not our aim to detail all practitioners' models. We rather try to keep a general view point, show various classes of techniques and try to adopt as much as possible a rigorous mathematical approach. Very good complementary readings are [8, 24, 43, 63, 66].

We first study the so-called *Fundamental Theorems of Asset Pricing* that are the basis of the theory of pricing and hedging of financial risks. This part is of abstract nature but is the basis of any practical model, and should be understood before going to practical considerations.

Indeed, we explain how the absence of arbitrage opportunities is related to the existence of risk neutral measures, and the central role the latter play in the pricing of derivatives and in the theory of portfolio management. We first consider discrete time models in which the analysis can be carried out in full generality and with rather simple mathematical tools. Continuous time models are studied in a subsequent chapter. In both cases, we shall see that the absence of arbitrage opportunities does not permit, in general, to define a unique possible price for a given financial derivative product, unless the market is complete and transactions are not restricted. This only provides an interval of *viable prices*, i.e. that do not create new arbitrage opportunities. Its upper-bound is the so-called *super-hedging price*: the minimal price at which an option should be sold to ensure that it can be covered without risk. To solve portfolio management issues, we follow the classical approach that consists in modelling preferences by utility functions. We explain how the so-called *dual formulation* approach leads to an explicit characterisation of the optimal portfolios in complete markets and how it can be used in incomplete markets to select viable prices of financial derivatives.

In the second part, we go closer to applications and confine ourselves to Markovian diffusion models, which are the one used in practice. Using the fundamentals of the first part, we provide general tools for the pricing and hedging of risk in practical situations.

We first study complete markets and explain why, in this context, the super-hedging price of a derivative can be characterised as the solution of linear second order parabolic equation, from which a perfect hedging strategy can be inferred as the gradient of the option price with respect to the values of the underlying assets. This is the *delta hedging* strategy. We consider different types of payoffs : plain vanilla options, barrier options and American options. As a first step, we will assume that the pricing functional is smooth, and then appeal to the notion of viscosity solutions in the case where it may be irregular. We also present various techniques allowing one to compute the hedging strategy: tangent process and Malliavin calculus approaches, characterisations by partial differential equations.

We then consider *imperfect markets*: either incomplete or with constraints on the hedging positions. The super-hedging price functional is now related to a non-linear parabolic equation. When the imperfection is due to a constraint on the hedging positions, this equation can be replaced by a linear one after modifying the option payoff, this is the so-called *face-lift*. When the imperfection is due to a risk that cannot be hedged, it cannot be simplified in general but still provides some informations on how the option can be covered without taking any risk. Unfortunately, this typically leads to hedging costs that are much too high for practical purposes, as can be observed in the classical example of stochastic volatility models when the volatility cannot be hedged.

This leads us to the notion of *short-fall price*. Instead of trying to super-hedge a risk, we look for a minimal price such that a given loss criteria can be satisfied by the P&L of a suitable (partial-)hedging strategy. In complete markets, this price can be determined by the same *dual formulation* approach as the one used in the analysis of portfolio management issues. For incomplete markets, we use the recent theory of *stochastic targets* that provides a characterisation of the *short-fall price* as the solution of a non-linear parabolic equation of Hamilton-Jacobi-Bellman type, that can be solved numerically.

The last part of this book goes one step further towards applications: we provide a more detailed description of local and stochastic volatility models, in particular how they can be calibrated. We also discuss the effect of discrete time portfolio rebalancing, the use of semi-static hedging policies and the issues related to the recalibration procedures.

# Contents

## Part A. Fundamental theorems

<b>1</b>	<b>Discrete time models</b> .....	5
1	Financial assets and portfolio strategies .....	5
2	No-arbitrage and martingale measures .....	8
2.1	Definition of the absence of arbitrage opportunities .....	8
2.2	First Fundamental Theorem .....	8
2.3	Proof of the first fundamental theorem .....	9
3	European option pricing .....	14
3.1	Various notions of prices .....	14
3.2	Dual description of the payoffs that can be super-hedged ...	15
3.3	Dual formulation for the super-hedging price and the set of viable prices .....	18
3.4	Characterisation of the hedging strategy of a replicable payoff .....	19
4	Complete markets .....	20
4.1	Definition and characterisation .....	20
4.2	Martingale representation theorem and European option hedging .....	22
5	American options .....	23
5.1	Supermartingale and <i>Snell envelope</i> .....	23
5.2	Super-replication price and $\mathcal{M}(\tilde{S})$ -Snell envelope .....	25
5.3	Super-replication price and optimal stopping .....	26
5.4	Viable prices .....	29
5.5	Rational exercise strategy .....	30
6	Models with portfolio constraints .....	31
7	Problems .....	32
<b>2</b>	<b>Continuous time models</b> .....	53
1	Financial asset and portfolio strategies .....	53
1.1	Financial assets .....	53

1.2	Portfolio strategies	55
2	Absence of arbitrage and martingale measures	56
2.1	Necessary condition	56
2.2	Sufficient condition	57
2.3	Necessary and sufficient condition	60
3	Pricing by super-hedging	61
4	Complete markets	62
4.1	Characterisation	62
4.2	The case of an invertible volatility	64
4.3	Hedging and Malliavin calculus	65
4.4	American options: hedging and exercise strategy	69
5	Portfolio constraints	71
5.1	Dual formulation of the super-hedging price	72
5.2	An auxiliary family of unconstrained problems	73
5.3	Study of the dual problem	74
6	Problems	76
<b>3</b>	<b>Optimal management and price selection</b>	<b>101</b>
1	Optimal management	101
1.1	Duality in complete markets	102
1.2	Extension to incomplete markets	103
1.3	Indifference price	105
2	Loss function hedging	106
2.1	Quantile hedging	106
2.2	Loss function hedging	109
2.3	Comments	112
3	Problems	112
<b>Part B. Markovian models and PDE approach</b>		
<b>4</b>	<b>Delta hedging in complete market</b>	<b>119</b>
1	Markovian models	119
2	Vanilla options	121
2.1	Regular case: Feynman-Kac formula and <i>delta-hedging</i>	122
2.2	Non-smooth case: price characterisation using viscosity solutions	126
2.3	Tangent process, Malliavin derivatives and delta-hedging	130
3	Barrier options	132
3.1	Pricing equation with Dirichlet boundary condition	133
3.2	Delta-hedging, exploding behaviour and regularisation techniques	135
4	American options	137
4.1	Dynamic Programming Principle	137
4.2	Associated quasi-variational inequalities	140
4.3	Delta-hedging in the smooth case	141

5	Problems	142
<b>5</b>	<b>Super-replication and its practical limits</b>	155
1	Hedging with Portfolio constraints	156
1.1	Framework	156
1.2	Pricing equation	158
1.3	Equivalence property: hedging a modified payoff without constraint	163
2	Application to incomplete markets	166
2.1	Non-hedgeable Volatility: The Black-Scholes-Barenblatt Equation	167
2.2	Non-hedgeable Volatility: the unbounded case, <i>buy-and-hold</i> strategy	169
3	Problems	170
<b>6</b>	<b>Hedging under loss constraints</b>	179
1	Super-replication : a direct approach	179
1.1	Framework	179
1.2	Dynamic programming principle	181
1.3	Pricing equation	182
1.4	Terminal condition of the pricing equation	187
2	Hedging under loss control	190
2.1	Problem Reduction	193
2.2	Pricing equation	194
2.3	Time boundary Condition	197
3	Comments	198
4	Problems	199
<b>Part C. Practical implementation in local and stochastic volatility models</b>		
<b>7</b>	<b>Local volatility models</b>	211
1	Black and Scholes model and implicit volatility	211
2	Local volatility surface	212
2.1	Dupire's approach	212
2.2	Calibration of the volatility curve on a finite number of calls	215
2.3	The recalibration issue	218
3	Impact of the Gamma on the hedging	218
3.1	Impact of a volatility misspecification	218
3.2	Impact of discrete rebalancing	219
4	Example: the CEV model	222
5	Problems	224

<b>8</b>	<b>Stochastic volatility models</b> .....	237
1	Hedging with liquid options .....	237
2	Static and semi-static strategies .....	238
2.1	Decomposition of payoff on a basis of calls and puts .....	239
2.2	Application to variance swaps .....	240
3	Example: the Heston's model .....	242
3.1	The model .....	242
3.2	Fourier's transform computation .....	243
3.3	FFT techniques for the calibration on call prices .....	243
3.4	The recalibration issue .....	246
4	Problems .....	246
	References .....	257