Foreword

This book is dedicated to Master students or young Phd students in applied mathematics who wish to learn mathematical finance. It uses a top-down approach, starting from the fundamentals and going little by little towards applications, while covering a large spectrum of concepts and mathematical tools. This is complemented by various (corrected) exercises in which many examples of practical interest will be provided. However, it is not our aim to detail all practitioners' models. We rather try to keep a general view point, show various classes of techniques and try to adopt as much as possible a rigorous mathematical approach. Very good complementary readings are [8, 24, 43, 63, 66].

We first study the so-called *Fundamental Theorems of Asset Pricing* that are the basis of the theory of pricing and hedging of financial risks. This part is of abstract nature but is the basis of any practical model, and should be understood before going to practical considerations.
Indeed, we explain how the absence of arbitrage opportunities is related to the existence of risk neutral measures, and the central role the latter play in the pricing of derivatives and in the theory of portfolio management. We first consider discrete time models in which the analysis can be carried out in full generality and with rather simple mathematical tools. Continuous time models are studied in a subsequent chapter. In both cases, we shall see that the absence of arbitrage opportunities does not permit, in general, to define a unique possible price for a given financial derivative product, unless the market is complete and transactions are not restricted. This only provides an interval of *viable prices*, i.e. that do not create new arbitrage opportunities. Its upper-bound is the so-called *super-hedging price*: the minimal price at which an option should be sold to ensure that it can be covered without risk. To solve portfolio management issues, we follow the classical approach that consists in modelling preferences by utility functions. We explain how the so-called *dual formulation* approach leads to an explicit characterisation of the optimal portfolios in complete markets and how it can be used in incomplete markets to select viable prices of financial derivatives.

In the second part, we go closer to applications and confine ourselves to Markovian diffusion models, which are the one used in practice. Using the fundamentals of the first part, we provide general tools for the pricing and hedging of risk in practical situations.
We first study complete markets and explain why, in this context, the super-hedging price of a derivative can be characterised as the solution of linear second order parabolic equation, from which a perfect hedging strategy can be inferred as the gradient of the option price with respect to the values of the underlying assets. This is the *delta hedging* strategy. We consider different types of payoffs : plain vanilla options, barrier options and American options. As a first step, we will assume that the pricing functional is smooth, and then appeal to the notion of viscosity solutions in the case where it may be irregular. We also present various techniques allowing one to compute the hedging strategy: tangent process and Malliavin calculus approaches, characterisations by partial differential equations.
We then consider *imperfect markets*: either incomplete or with constraints on the hedging positions. The super-hedging price functional is now related to a non-linear parabolic equation. When the imperfection is due to a constraint on the hedging positions, this equation can be replaced by a linear one after modifying the option payoff, this is the so-called *face-lift*. When the imperfection is due to a risk that cannot be hedged, it cannot be simplified in general but still provides some informations on how the option can be covered without taking any risk. Unfortunately, this typically leads to hedging costs that are much too high for practical purposes, as can be observed in the classical example of stochastic volatility models when the volatility cannot be hedged.
This leads us to the notion of *short-fall price*. Instead of trying to super-hedge a risk, we look for a minimal price such that a given loss criteria can be satisfied by the P&L of a suitable (partial-)hedging strategy. In complete markets, this price can be determined by the same *dual formulation* approach as the one used in the analysis of portfolio management issues. For incomplete markets, we use the recent theory of *stochastic targets* that provides a characterisation of the *short-fall price* as the solution of a non-linear parabolic equation of Hamilton-Jacobi-Bellman type, that can be solved numerically.

The last part of this book goes one step further towards applications: we provide a more detailed description of local and stochastic volatility models, in particular how they can be calibrated. We also discuss the effect of discrete time portfolio rebalancing, the use of semi-static hedging policies and the issues related to the recalibration procedures.

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