Poisson process and actuarial sciences
Mid term exam (2015-2016)
March 16, 2016

The quality of the redaction will be taken into account. Be clear and concise in your answers.

Questions. [Answer in maximum 3 lines] (5 points)

1. Give the definition of a standard counting process. [1pt]
2. What it a renewal process ? [1pt]
3. What are the properties of the increments of a mixed Poisson process ? [1pt]
4. What is the Markov property for a Poisson process ? What does it mean ? [1pt]
5. What is the law of the first n jump times of a Poisson process N given that $N_t = n$? [1pt]

In the following problem, most of the questions can be answered independently but results from the previous ones (given in the text) will often be useful. You can therefore skip the questions you can not answer, but read all of them !

Problem.(19 points) The number of vehicles entering a roundabout (*rond point*) is modeled by a Poisson process N of parameter $\lambda > 0$. Namely, N_t is the number of vehicles that have entered the roundabout during the first t minutes. The corresponding sequence of jump times is $(T_i)_{i\geq 1}$.

1. Given that exactly ℓ vehicles entered the roundabout within the first t minutes, give the probability that at least k vehicles have entered the roundabout within the first s minutes, with $\ell \geq k$ and $t > s$. Express the result in terms of s, t, ℓ and k. [1,5pt] Show that it can be interpreted in terms of a binomial distribution. [0,5pt]
2. Because of works on the roundabout, its access will be closed during a time duration of h minutes starting from time t_o. Vehicles will still arrive according to N but will have to queue and wait. We want to ensure that the maximal number of vehicles queuing is strictly less than $K \geq 1$ with probability $p \in (0, 1)$.
   (a) What it the probability that a vehicle arrives exactly at time t_o ? [1pt]
   (b) Set
   $$\tilde{\delta}_i := T_{N_{t_o}+i} - (T_{N_{t_o}+i-1}1_{\{i\geq 2\}} + t_o 1_{\{i=1\}}).$$
   Show that $\tilde{\delta}_1$ is distributed according to an exponential distribution of parameter λ.[1,5pt] Then, show that $(\tilde{\delta}_i)_{i\geq 1}$ is an iid sequence, by induction. [1,5pt]

(c) Set

$$\tilde{T}_n := \sum_{i=1}^{n} \tilde{\delta}_i \text{ and } \tilde{N}_t := \sum_{i\geq 1} 1_{\{\tilde{T}_i \leq t\}}.$$

Justify that $\tilde{N}$ is a Poisson process with parameter λ. [0,5pt]

(d) Provide a condition in terms of element(s) of $(\tilde{T}_n)_{n\geq 1}$ ensuring that the maximal number of vehicles queuing on the time interval $[t_o, t_o + h]$ is strictly less than $K \geq 1$ with probability (at least) $p \in (0,1)$. [0,5pt]

(e) How can we approximate the maximal possible duration h by simply using the quantiles of the Gaussian distribution if K is large ? [1,5pt]

(f) Explain why a good approximation of the number of vehicles queuing is $h\lambda$ if h is large. [1pt]

3. We now assume that the size of the vehicles arriving at the roundabout is given by a sequence of iid random variable $(\xi_i)_{i\geq 1}$. Namely, ξ_i is the size of the vehicles arriving at time T_i. We assume that $(\xi_i)_{i\geq 1}$ is independent of N. We set

$$\tilde{\xi}_i = \xi_{N_{t_o}+i}.$$

(a) Show that $(\tilde{\xi}_i)_{i\geq 1}$ is an iid sequence independent of N_{t_o}, with the same law as $(\xi_i)_{i\geq 1}$.[1,5pt]

(b) Given two bounded functions f and g, and $k, \ell \in \mathbb{N}$, show that

$$\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_k) g(\tilde{\delta}_1, \ldots, \tilde{\delta}_\ell)] = \mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_k)]\mathbb{E}[g(\tilde{\delta}_1, \ldots, \tilde{\delta}_\ell)],$$

[1,5pt] and deduce that $(\tilde{\xi}_i)_{i\geq 1}$ is independent of $\tilde{N}$. [0,5pt]

(c) Set $m := \mathbb{E}[\xi_1]$. The total length of the queue is given by the process

$$\tilde{S}_t = \sum_{i\geq 1} \tilde{\xi}_i 1_{\tilde{T}_i \leq t}.$$

Show that $\tilde{S}$ is a compound Poisson process of parameter λ.[0,5pt] What is the expected queue length after h minutes ?[1pt]

(d) What is the a.s. limit of $\tilde{S}_h/h$ as $h \to \infty$. [0,5pt]

4. We finally discuss approximations in the case $\lambda \to \infty$. From now on, we write $\tilde{N}^\lambda$ and $\tilde{S}^\lambda$ for $\tilde{N}$ and $\tilde{S}$ to insist on the fact that they depend on λ.

(a) Let M be a Poisson process with intensity $\gamma > 0$. Given $c > 0$, show that $\bar{M} := (M_{ct})_{t\geq 0}$ is a Poisson process with intensity $c\gamma$. [1pt]

(b) Deduce from the above that $\tilde{S}^\lambda_h/\lambda$ converges a.s. as $\lambda \to \infty$ and identify the limit. [1pt]

(c) Assume that $\sigma := \text{var}[\xi_1]^{\frac{1}{2}} < \infty$. What can we say about $\sqrt{\tilde{N}^\lambda_h}(\tilde{S}^\lambda_h/\tilde{N}^\lambda_h - m)$ as $\lambda \to \infty$? [2pt]

Correction

1. We need to compute $\mathbb{P}[N_s \geq k | N_t = \ell] = \sum_{j=k}^{\ell} \mathbb{P}[N_s = j | N_t = \ell] = \sum_{j=k}^{\ell} \mathbb{P}[N_s = j, N_t - N_s = \ell - j]/\mathbb{P}[N_t = \ell]$. By independence and the stationnarity of the increments of N,

$$\begin{aligned}\mathbb{P}[N_s = j, N_t - N_s = \ell - j] &= [N_s = j]\mathbb{P}[N_t - N_s = \ell - j] \\ &= \frac{[s\lambda]^j}{j!} e^{-\lambda s} \frac{[(t-s)\lambda]^{\ell-j}}{(\ell-j)!} e^{-\lambda(t-s)},\end{aligned}$$

and therefore

$$\begin{aligned}\mathbb{P}[N_s \geq k | N_t = \ell] &= \sum_{j=k}^{\ell} \frac{[s\lambda]^j}{j!} e^{-\lambda s} \frac{[(t-s)\lambda]^{\ell-j}}{(\ell-j)!} e^{-\lambda(t-s)} \frac{\ell!}{[t\lambda]^{\ell}} e^{\lambda t} \\ &= \sum_{j=k}^{\ell} C_{\ell}^{j} \frac{s^j (t-s)^{\ell-j}}{t^{\ell}}.\end{aligned}$$

Since $1 - s/t = (t-s)/t$, this corresponds to a Binomial distribution of parameters $(\ell, s/t)$.

2. (a) We need to compute $\mathbb{P}[\exists\ n \geq 1 : T_n = t_o] = \sum_{n \geq 1} \mathbb{P}[T_n = t_o] = 0$ because each T_n has a density.

   (b) We first compute, for $h \geq 0$,

$$\begin{aligned}\mathbb{P}[\tilde{\delta}_1 \geq h] &= \sum_{k \geq 0} \mathbb{P}[\tilde{\delta}_1 \geq h, N_{t_o} = k] \\ &= \sum_{k \geq 0} \mathbb{P}[\tilde{\delta}_1 \geq h, T_k \leq t_o < T_{k+1}] \\ &= \mathbb{P}[T_1 - t_o \geq h] + \sum_{k \geq 1} \mathbb{P}[T_{k+1} - t_o \geq h, T_k \leq t_o < T_{k+1}] \\ &= \mathbb{P}[T_1 - t_o \geq h] + \sum_{k \geq 1} \mathbb{P}[T_{k+1} - t_o \geq h, T_k \leq t_o],\end{aligned}$$

   because (T_k, T_{k+1}) has a density. Then, since $T_{k+1} - T_k$ is independent of T_k,

$$\begin{aligned}\mathbb{P}[\tilde{\delta}_1 \geq h] &= e^{-\lambda(h+t_o)} + \sum_{k \geq 1} \mathbb{P}[T_{k+1} - T_k \geq h + t_o - T_k, T_k \leq t_o] \\ &= e^{-\lambda(h+t_o)} + \sum_{k \geq 1} \int_0^{t_o} e^{-\lambda(h+t_o-y)} \frac{\lambda^k y^{k-1}}{(k-1)!} e^{-\lambda y} dy \\ &= e^{-\lambda(h+t_o)} + \int_0^{t_o} \lambda e^{-\lambda(h+t_o-y)} dy \\ &= e^{-\lambda h}.\end{aligned}$$

Hence $\tilde{\delta}_1$ has an exponential distribution of parameter λ. Let us assume that $\tilde{\delta}_1, \ldots, \tilde{\delta}_n$ are iid, and show that it is true for $\tilde{\delta}_1, \ldots, \tilde{\delta}_{n+1}$. Fix $(h_i)_{1\le i\le n+1} \subset [0,\infty)^{n+1}$. Then,

$$\begin{aligned}\mathbb{P}[\cap_{i=1}^{n+1}\{\tilde{\delta}_i \ge h_i\}] &= \sum_{k\ge 0} \mathbb{P}[\cap_{i=1}^{n+1}\{\tilde{\delta}_i \ge h_i\} \cap \{N_{t_o} = k\}] \\ &= \sum_{k\ge 0} \mathbb{P}[\{T_{k+1} - t_o \ge h_1\} \cap_{i=2}^{n+1} \{\delta_{k+i} \ge h_i\} \cap \{T_k \le t_o\}].\end{aligned}$$

Since δ_{k+n+1} is independent of $(T_k, T_{k+1}, (\delta_{k+i})_{2\le i\le n})$, we obtain

$$\begin{aligned}\mathbb{P}[\cap_{i=1}^{n+1}\{\tilde{\delta}_i \ge h_i\}] =& \sum_{k\ge 0} \mathbb{P}[\cap_{i=1}^{n+1}\{\tilde{\delta}_i \ge h_i\} \cap \{N_{t_o} = k\}] \\ =& \mathbb{P}[\delta_{k+n+1} \ge h_{n+1}] \\ & \times \sum_{k\ge 0} \mathbb{P}[\{T_{k+1} - t_o \ge h_1\} \cap_{i=2}^{n} \{\delta_{k+i} \ge h_i\} \cap \{T_k \le t_o\}],\end{aligned}$$

which, by our induction hypothesis and the same argument as above, leads to

$$\mathbb{P}[\cap_{i=1}^{n+1}\{\tilde{\delta}_i \ge h_i\}] = e^{-\lambda h_{n+1}} \sum_{k\ge 0} \mathbb{P}[\cap_{i=1}^{n}\{\tilde{\delta}_i \ge h_i\} \cap \{N_{t_o} = k\}] = \prod_{i=1}^{n+1} e^{-\lambda h_i}.$$

This proves the required result.

(c) $\tilde{N}$ is a Poisson process with parameter λ because $(\tilde{\delta}_i)_{i\ge 1}$ is an iid sequence distributed according to an exponential distribution of parameter λ.

(d) The condition is $\mathbb{P}[\tilde{N}_h < K] \ge p$ which is equivalent to $\mathbb{P}[\tilde{T}_K > h] \ge p$.

(e) By the strong law of large numbers, $(\tilde{T}_K - K\lambda^{-1})/(\sqrt{K}/\lambda) \to N(0,1)$ in law as $K \to \infty$, in which $N(0,1)$ is the centered and reduced Gaussian distribution. Let c_p be the solution of $\mathbb{P}[X > c_p] = p$ if $X \sim N(0,1)$. Then, the maximal duration $\hat{h}_p$ is approximated by $c_p = (\hat{h}_p - K\lambda^{-1})/(\sqrt{K}/\lambda)$, i.e. $\hat{h}_p := c_p\sqrt{K}\lambda^{-1} + K\lambda^{-1}$.

(f) One can apply the law of large numbers for N: $N_h/h \to \lambda$ a.s. as $h \to \infty$. This means that N_h is approximately $h\lambda$ for h large.

3. (a) Take two bounded functions f, g and let us compute

$$\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(N_{t_o})] = \mathbb{E}[\mathbb{E}[f(\xi_{N_{t_o+1}}, \ldots, \xi_{N_{t_o+n}})|N_{t_o}]g(N_{t_o})].$$

But, $(\xi_{N_{t_o+1}}, \ldots, \xi_{N_{t_o+n}})$ given N_{t_o} has the same law as $(\xi_1, \ldots, \xi_n)$ because the ξ_i's are iid and independent of N_{t_o}. This implies that

$$\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(N_{t_o})] = \mathbb{E}[\mathbb{E}[f(\xi_1, \ldots, \xi_n)]g(N_{t_o})] = \mathbb{E}[f(\xi_1, \ldots, \xi_n)]\mathbb{E}[g(N_{t_o})].$$

For $g = 1$, this implies that $\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)] = \mathbb{E}[f(\xi_1, \ldots, \xi_n)]$, which, by arbitrariness of f and n, shows that $(\tilde{\xi}_i)_{i\geq 1}$ and $(\xi_i)_{i\geq 1}$ have the same distribution. Combined with the above equalities, this in turn implies that

$$\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(N_{t_o})] = \mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)]\mathbb{E}[g(N_{t_o})],$$

so that $(\tilde{\xi}_i)_{i\geq 1}$ is independent of N_{t_o}, by arbitrariness of f, g and n.

(b) The equality will imply that $(\tilde{\xi}_i)_{i\geq 1}$ and $(\tilde{\delta}_i)_{i\geq 1}$ are independent. Since $\tilde{N}$ depends only on $(\tilde{\delta}_i)_{i\geq 1}$, this will show that it is dependent of $(\tilde{\xi}_i)_{i\geq 1}$. Let us now check the required equality.

$$\begin{aligned}
&\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(\tilde{\delta}_1, \ldots, \tilde{\delta}_n)] \\
&= \mathbb{E}[\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(\tilde{\delta}_1, \ldots, \tilde{\delta}_n)|N_{t_o}]] \\
&= \mathbb{E}[\mathbb{E}[f(\xi_{N_{t_o}+1}, \ldots, \xi_{N_{t_o}+n})g(T_{N_{t_o}+1} - t_o, \delta_{N_{t_o}+2}, \ldots, \delta_{N_{t_o}+n})|N_{t_o}]].
\end{aligned}$$

But, given N_{t_o}, $(T_{N_{t_o}+1} - t_o, \delta_{N_{t_o}+2}, \ldots, \delta_{N_{t_o}+n})$ and $(\xi_{N_{t_o}+1}, \ldots, \xi_{N_{t_o}+n})$ are independent, thus

$$\begin{aligned}
&\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(\tilde{\delta}_1, \ldots, \tilde{\delta}_n)] \\
&= \mathbb{E}[\mathbb{E}[f(\xi_{N_{t_o}+1}, \ldots, \xi_{N_{t_o}+n})|N_{t_o}]\mathbb{E}[g(T_{N_{t_o}+1} - t_o, \delta_{N_{t_o}+2}, \ldots, \delta_{N_{t_o}+n})|N_{t_o}]].
\end{aligned}$$

Since $(\xi_{N_{t_{o+1}}}, \ldots, \xi_{N_{t_{o+n}}})$ given N_{t_o} has the same law as $(\xi_1, \ldots, \xi_n)$ because the ξ_i's are iid and independent of N_{t_o},

$$\begin{aligned}
\mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)g(\tilde{\delta}_1, \ldots, \tilde{\delta}_n)] &= \mathbb{E}[f(\xi_1, \ldots, \xi_n)]\mathbb{E}[g(T_{N_{t_o}+1} - t_o, \ldots, \delta_{N_{t_o}+n})] \\
&= \mathbb{E}[f(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)]\mathbb{E}[g(\tilde{\delta}_1, \ldots, \tilde{\delta}_n)].
\end{aligned}$$

(c) We know from the above that $\tilde{N}$ is a Poisson process of parameter λ and that $(\tilde{\xi}_i)_{i\geq 1}$ is iid and independent of $\tilde{N}$. Hence, by definition, $\tilde{S}$ is a compound Poisson process of parameter λ. We have $\mathbb{E}[\tilde{S}_h] = \mathbb{E}[\mathbb{E}[\tilde{S}_h|\tilde{N}_h]] = \mathbb{E}[\tilde{N}_h\xi_1]$ because $(\tilde{\xi}_i)_{i\geq 1}$ is iid, independent of $\tilde{N}$ and has the same law as $(\xi_i)_{i\geq 1}$. Since $\tilde{N}_h$ is independent of $(\xi_i)_{i\geq 1}$, because so is N, and is a Poisson process of parameter λ, we obtain $\mathbb{E}[\tilde{N}_h]\mathbb{E}[\xi_1]$ which is equal to λhm.

(d) By the law of large numbers for the compound Poisson process, it converges to λm.

4. (a) $M_{ct} \sim \mathcal{P}(ct\lambda)$ because M is a PP(λ). The fact that the increments of $\bar{M}$ are independent and stationary follows from the fact that it holds for M.

(b) Let us set $\bar{N}_t := \tilde{N}^{\lambda}_{t/\lambda}$ so that $\tilde{N}^{\lambda}_t = \bar{N}_{\lambda t}$ and $\tilde{S}^{\lambda}_h = \sum_{i=1}^{\tilde{N}^{\lambda}_h} \tilde{\xi}_i = \sum_{i=1}^{\bar{N}_{\lambda h}} \tilde{\xi}_i$ where $\bar{N}$ is a PP(1), see the above question. Then, $\bar{N}_{\lambda h}/\lambda \to h$ a.s. as $\lambda \to \infty$. Since $n^{-1}\sum_{i=1}^n \tilde{\xi}_i \to m$ a.s. and $\bar{N}_{\lambda h} \to \infty$ a.s., we deduce that

$$\tilde{S}^{\lambda}_h/\lambda = \frac{\bar{N}_{\lambda h}}{\lambda}\frac{1}{\bar{N}_{\lambda h}}\sum_{i=1}^{\bar{N}_{\lambda h}} \tilde{\xi}_i \to hm \text{ a.s. as } \lambda \to \infty.$$

(c) Set $Y^\lambda := \sqrt{\tilde{N}_h^\lambda}(\tilde{S}_h^\lambda/\tilde{N}_h^\lambda - m)$. Let ϕ denote the Fourrier transform of the law of $\tilde{\xi}_1$. We have, for $u \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\big[e^{iuY^\lambda}\big] &= \mathbb{E}\big[\mathbb{E}[e^{iu\sqrt{\tilde{N}_h^\lambda}\left(\frac{\tilde{S}_h^\lambda}{\tilde{N}_h^\lambda}-m\right)}|\tilde{N}_h^\lambda]\big] \\
&= \mathbb{E}\big[\mathbb{E}[e^{iu\frac{\tilde{\xi}_1}{\sqrt{\tilde{N}_h^\lambda}}}|\tilde{N}_h^\lambda]^{\tilde{N}_h^\lambda} e^{-iu\sqrt{\tilde{N}_h^\lambda}m}\big] \\
&= \mathbb{E}\big[\phi(i\frac{u}{\sqrt{\tilde{N}_h^\lambda}})^{\tilde{N}_h^\lambda} e^{-iu\sqrt{\tilde{N}_h^\lambda}m}\big].
\end{aligned}$$

Since $\tilde{N}_h^\lambda \to \infty$ a.s. as $\lambda \to \infty$, a Taylor expansion leads to

$$\begin{aligned}
&\tilde{N}_h^\lambda \ln \phi(i\frac{u}{\sqrt{\tilde{N}_h^\lambda}}) - iu\sqrt{\tilde{N}_h^\lambda}m \\
&= \tilde{N}_h^\lambda \left(i\frac{u}{\sqrt{\tilde{N}_h^\lambda}}\phi'(0) - \frac{u^2}{2\tilde{N}_h^\lambda}(\phi''(0) - (\phi'(0))^2) + o(1/\tilde{N}_h^\lambda) \right) - iu\sqrt{\tilde{N}_h^\lambda}m \\
&\to -\frac{u^2}{2}\sigma^2
\end{aligned}$$

since $m = \phi'(0)$ and $\sigma^2 = \phi''(0) - (\phi'(0))^2$. By dominated convergence, we obtain

$$\mathbb{E}\big[e^{iuY^\lambda}\big] \to e^{-\frac{u^2}{2}\sigma^2} \quad \text{as } \lambda \to \infty.$$

It follows that Y^λ converges in law to $N(0, \sigma^2)$.