Almost sure hedging under permanent price impact

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Based on joint works with
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Motivation
Option pricing with liquidity impact in the literature (part of)

- Super-hedging/hedging:
  - Cetin, Jarrow and Protter 2004: illiquidity, no impact, pricing à la B&S.
  - Cetin, Soner and Touzi 2009: restrictions on strategies.
  - Bank and Dolinsky 2019.
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  - Cetin, Jarrow and Protter 2004: illiquidity, no impact, pricing à la B&S.
  - Cetin, Soner and Touzi 2009: restrictions on strategies.
  - Bank and Dolinsky 2019.

- Other pricing rules (not replication nor super-replication): Abergel and Loeper 2013, Almgren and Li 2013, Millot and Abergel 2011, Guéant and Pu 2013, Bank, Soner and Voss 2017, ...
Aim of this work

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• Consider a model with price impact and liquidity cost, but in which hedging still makes sense without being degenerate (in any sense).
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- Permanent impact with possible resilience.
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  - Define a continuous time trading dynamics from a discrete time trading rule.
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  • Permanent impact with possible resilience.

□ What we do :
  • Define a continuous time trading dynamics from a discrete time trading rule.
  • Provide a direct argument for the characterization of the hedging policy.
Chapter 1
Impact rule and continuous time trading dynamics
Impact rule and liquidity cost

- **Basic rule**: an order $\delta$ moves the price by

$$X_{t-} \rightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2} \delta^2 f(X_{t-}) = \delta \frac{1}{2} (X_{t-} + X_t) = \int_0^\delta (X_{t-} + \iota f(X_{t-})) \underbrace{d\iota}_{\text{add. quantity}}.$$

1. $X_t$ represents the price of an asset at time $t$.
2. $\delta$ is the size of the order.
3. $f(X)$ is a function that describes the impact of the order size on the price.
4. The integral represents the cumulative cost of the order.
5. The cost includes a term for the average price and a term for the current price.
Impact rule and liquidity cost

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- We just need model the curve around $\delta = 0$. 
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□ We just need model the curve around $\delta = 0$. This should be understood for a “small” order $\delta$ as one can split orders, in continuous time. Would obtain the same with

$$X_{t-} \rightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

and costs

$$\int_0^\delta (X_{t-} + F(X_{t-}, \iota)) d\iota.$$
Impact rule and liquidity cost

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$$X_{t^-} \rightarrow X_t = X_{t^-} + \delta f(X_{t^-}),$$

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and costs

$$\int_0^\delta (X_{t^-} + F(X_{t^-}, \iota)) d\iota$$

if $\partial_\delta F(x, 0) = f(x)$, $\partial_\delta^2 F(x, 0) = f'(x)$, $F(x, 0) = \partial_\delta^2 F(x, 0) = 0$. 
In particular, would lead to the same results if

\[ X_{t^-} \rightarrow X_{t^-} + F(X_{t^-}, \delta) \]

with

\[ F(x, \delta) = \Delta x(x, \delta) := x(x, \delta) - x, \]

and \( x(x, \cdot) \) defined as the solution of

\[ x(x, \cdot) = x + \int_0^\cdot f(x(x, s))ds. \]
In particular, would lead to the same results if

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Interpretation in terms of large order splitting: split \( \delta \) in \( \delta/n \) then

\[ X_{t-} + \frac{\delta}{n} f(X_{t-}) \simeq x(X_{t-}, \frac{\delta}{n}) \sim x(x(X_{t-}, \frac{\delta}{n}), \frac{\delta}{n})) = x(X_{t-}, \frac{2\delta}{n}) \sim \ldots \]
In particular, would lead to the same results if

\[ X_{t-} \longrightarrow X_{t-} + F(X_{t-}, \delta) \]

with

\[ F(x, \delta) = \Delta x(x, \delta) := x(x, \delta) - x, \]

and \( x(x, \cdot) \) defined as the solution of

\[ x(x, \cdot) = x + \int_0^\cdot f(x(x, s))ds. \]

In this case, the cost would be

\[ \int_0^\delta x(X_{t-}, \nu)d\nu. \]
A trading signal is an Itô process of the form

\[ Y = Y_0 + \int_0^t b_s \, ds + \int_0^t a_s \, dW_s. \]
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Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.
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Need to define the dynamics of the wealth and of the asset. As usual, consider discrete trading and pass to the limit.

Trade at times \( t^n_i = iT/n \) the quantity \( \delta^n_{t^n_i} = Y_{t^n_i} - Y_{t^n_{i-1}} \).
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Trade at times \( t^n_i = iT/n \) the quantity \( \delta^n_{t^n_i} = Y_{t^n_i} - Y_{t^n_{i-1}} \).

We assume that the stock price evolves according to

\[ X = X_{t^n_i} + \int_{t^n_i}^t \mu(X_s) \, ds + \int_{t^n_i}^t \sigma(X_s) \, dW_s \]

between two trades.
The corresponding dynamics are

\[
Y^n_t := \sum_{i=0}^{n-1} Y^n_{t_i} \mathbf{1}_{t_i^* \leq t < t_{i+1}^*} + Y^T \mathbf{1}_{t=T}, \quad \delta^n_{t_i} = Y^n_{t_i} - Y^n_{t_{i-1}}
\]

\[
X^n = X_0 + \int_0^\cdot \mu(X^n_s)ds + \int_0^\cdot \sigma(X^n_s)dW_s + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \delta^n_{t_i} f(X^n_{t_i^n-}),
\]

\[
V^n = V_0 + \int_0^\cdot Y^n_{s-}dX^n_s + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta^n_{t_i})^2 f(X^n_{t_i^n-}),
\]

where

\[
V^n = \text{cash part} + Y^n X^n = \text{“portfolio value”}.
\]
The corresponding dynamics are

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Y^n_t := \sum_{i=0}^{n-1} Y^n_{t_i} 1\{t_i \leq t < t_{i+1}\} + Y^n_T 1\{t = T\}, \quad \delta^n_{t_i} = Y^n_{t_i} - Y^n_{t_{i-1}}
\]

\[
X^n = X_0 + \int_0^\cdot \mu(X^n_s) ds + \int_0^\cdot \sigma(X^n_s) dW_s + \sum_{i=1}^{n} 1_{[t^n_i, \tau]} \delta^n_{t_i} f(X^n_{t_i^-}),
\]

\[
V^n = V_0 + \int_0^\cdot Y^n_{s-} dX^n_s + \sum_{i=1}^{n} 1_{[t_i^n, \tau]} \frac{1}{2} (\delta^n_{t_i})^2 f(X^n_{t_i^-}),
\]

where

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V^n = \text{cash part} + Y^n X^n = \text{"portfolio value".}
\]

**Warning**: The portfolio is \((V^n - Y^n X^n, Y^n)\) whose liquidation will not lead to \(V^n\) in cash!
Passing to the limit $n \to \infty$, it converges in $S_2$ to

\begin{align*}
Y &= Y_0 + \int_0^n b_s \, ds + \int_0^n a_s \, dW_s \\
X &= X_0 + \int_0^n \sigma(X_s) \, dW_s + \int_0^n f(X_s) \, dY_s + \int_0^n (\mu + a_s \sigma f')(X_s) \, ds \\
V &= V_0 + \int_0^n Y_s \, dX_s + \frac{1}{2} \int_0^n a_s^2 f(X_s) \, ds,
\end{align*}

at a speed $\sqrt{n}$. 
More details on the limit... : We have

\[ X^n = X_0 + \int_0^\cdot \mu(X_s^n) ds + \int_0^\cdot \sigma(X_s^n) dW_s + \sum_{i=1}^n 1_{[t_i^n, \tau]} \delta^n_t f(X_{t_i^n -}^n), \]
More details on the limit... : We have

$$X^n = X_0 + \int_0^1 \mu(X^n_s) ds + \int_0^1 \sigma(X^n_s) dW_s + \sum_{i=1}^n 1_{[t^n_i, T]} \delta^n_{t^n_i} f(X^n_{t^n_i^-}),$$

in which

$$\delta^n_{t^n_i+1} f(X^n_{t^n_i+1^-}) = \left( \int_{t^n_i}^{t^n_{i+1}} dY_t \right) f \left( X^n_{t^n_i} + \int_{t^n_i}^{t^n_{i+1}} dX^n_s \right)$$

$$= \int_{t^n_i}^{t^n_{i+1}} f \left( X^n_{t^n_i} + \int_{t^n_i}^t dX^n_{r,c} \right) dY_t$$

$$+ \int_{t^n_i}^{t^n_{i+1}} \langle \int_{t^n_i}^t dY_r, f \left( X^n_{t^n_i} + \int_{t^n_i}^t dX^n_r \right) \rangle_t + \text{neglectable}$$
More details on the limit... : We have

\[ X^n = X_0 + \int_0^\cdot \mu(X^n_s)ds + \int_0^\cdot \sigma(X^n_s)dW_s + \sum_{i=1}^n 1_{[t^n_i, T]} \delta^n_{t^n_i} f(X^n_{t^n_i-}), \]

in which

\[ \delta^n_{t^n_{i+1}} f(X^n_{t^n_{i+1}-}) = (\int_{t^n_i}^{t^n_{i+1}} dY_t) f \left( X^n_{t^n_i} + \int_{t^n_i}^{t^n_{i+1}} dX^n_{t^n_i} \right) \]

\[ = \int_{t^n_i}^{t^n_{i+1}} f \left( X^n_{t^n_i} + \int_{t^n_i}^t dX^n_{t^n_i} \right) dY_t \]

\[ + \int_{t^n_i}^{t^n_{i+1}} d\langle \int_{t^n_i}^\cdot dY_r, f \left( X^n_{t^n_i} + \int_{t^n_i}^\cdot dX^n_{t^n_i} \right) \rangle_t + \text{neglectable} \]

so that

\[ X = X_0 + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (\mu + a_s\sigma f')(X_s)ds. \]
More details on the limit... : We have

\[ V^n = V_0 + \int_0^\cdot Y^n_s \, dX^n_s + \sum_{i=1}^n 1_{[t^n_i, T]} \frac{1}{2} (\delta^n_{t^n_i})^2 f(X^n_{t^n_i-}), \]
More details on the limit... : We have

\[ V^n = V_0 + \int_0^t Y^n_s \, dX^n_s + \sum_{i=1}^n 1_{[t^n_i, T]} \frac{1}{2} (\delta^n_{t^n_i})^2 f(X^n_{t^n_i-}), \]

in which

\[ (\delta^n_{t^n_{i+1}})^2 f(X^n_{t^n_{i+1}-}) = (\int_{t^n_i}^{t^n_{i+1}} dY^n_t)^2 f \left( X^n_{t^n_i} + \int_{t^n_i}^{t^n_{i+1}} dX^n_t, c \right) \]

\[ = \int_{t^n_i}^{t^n_{i+1}} f \left( X^n_{t^n_i} + \int_{t^n_i}^t dX^n_r \right) d\langle Y \rangle_t + \text{neglectable} \]
More details on the limit... : We have

\[ V^n = V_0 + \int_0^T Y^n_s \, dX^n_s + \sum_{i=1}^n 1_{[t^n_i, T]} \frac{1}{2} (\delta^n_{t^n_i})^2 f(X^n_{t^n_i-}), \]

in which

\[ (\delta^n_{t^n_{i+1}})^2 f(X^n_{t^n_{i+1}-}) = (\int_{t^n_i}^{t^n_{i+1}} dY^c_t)^2 f \left( X^n_{t^n_i} + \int_{t^n_i}^{t^n_{i+1}} dX^n_t \right) \]

\[ = \int_{t^n_i}^{t^n_{i+1}} f \left( X^n_{t^n_i} + \int_{t^n_i}^t dX^n_r \right) d\langle Y \rangle_t + \text{neglectable} \]

so that

\[ V = V_0 + \int_0^T Y_s \, dX_s + \frac{1}{2} \int_0^T a_s^2 f(X_s) \, ds. \]
We now consider a trading signal of the form

\[
Y = Y_0 - \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s + \int_0^\cdot \delta \nu(d\delta, ds)
\]

where

\[
\nu(A, B) = \sum_{i \geq 1} 1_{(\delta_i, \tau_i) \in A \times B}
\]

in which \( \tau_i \) is a stopping time and \( \delta_i \) is \( F_{\tau_i} \)-measurable.
Adding jumps and splitting of large orders

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- Approximation: Jump \( \delta_i \) at time \( \tau_i \) is passed on \([\tau_i, \tau_i + \varepsilon]\) at a rate \( \delta_i / \varepsilon \).
Adding jumps and splitting of large orders

- We now consider a trading signal of the form

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Y = Y_0 - \int_0^\cdot b_s \, ds + \int_0^\cdot a_s \, dW_s + \int_0^\cdot \delta \nu(d\delta, ds)
\]

where

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\]

in which \(\tau_i\) is a stopping time and \(\delta_i\) is \(F_{\tau_i}\)-measurable.

- Approximation: Jump \(\delta_i\) at time \(\tau_i\) is passed on \([\tau_i, \tau_i + \varepsilon]\) at a rate \(\delta_i/\varepsilon\). This leads to

\[
Y^\varepsilon = Y_0 - \int_0^\cdot \left( b_s + \sum_{i \geq 1} 1_{[\tau_i, \tau_i + \varepsilon]}(s) \frac{\delta_i}{\varepsilon} \right) ds + \int_0^\cdot a_s \, dW_s.
\]
The limit dynamics when $\varepsilon \to 0$ is

$$
X = X_0 - \int_0^\cdot \sigma(X_s) dW_s + \int_0^\cdot f(X_s) dY_s^c + \int_0^\cdot (\mu + a_s \sigma f')(X_s) ds
$$

$$
+ \int_0^\cdot \int_0^\cdot \Delta x(X_s^-, \delta) \nu(d\delta, ds)
$$

$$
V = V_0 - \int_0^\cdot Y_s dX_s^c + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s) ds
$$

$$
+ \int_0^\cdot \int_0^\cdot (Y_s - \Delta x(X_s^-, \delta) + \mathcal{I}(X_s^-, \delta)) \nu(d\delta, ds)
$$

in which $Y^c$ is the continuous part of $Y$, and

$$
x(x, \delta) = x + \int_0^\delta f(x(x, s)) ds \quad , \quad \Delta x(x, \delta) := x(x, \delta) - x
$$

$$
\mathcal{I}(x, \delta) := \int_0^\delta sf(x(x, s)) ds.
$$
Adding resilience

\[ X = X_0 + \int_0^t \sigma(X_s) dW_t + R \]

\[ R = R_0 + \int_0^t f(X_t) dY_t + \int_0^t \left( a_t (f' \sigma)(X_t) - \rho R_t \right) dt \]

\[ Y = y + \int_0^t a_t dW_t + \int_0^t b_t dt \]

\[ V = V_0 + \int_0^t Y_t dX_t + \int_0^t \frac{1}{2} a_t^2 f_t(X_t) dt. \]

Zero cost immediate round trips

- A jump of size $\delta$ moves the stock price to

$$x(x, \delta) = x + \int_0^\delta f(x(x, s)) ds,$$

but

$$x(x, \delta) = x - \delta.$$

Similarly, the impact on the portfolio value is

$$y \Delta x(x, \delta) + I(x, \delta) = (y + \delta) \Delta x(x(x, \delta), -\delta) + I(x(x, \delta), -\delta) = -[y \Delta x(x, \delta) + I(x, \delta)].$$

**Warning:** be careful with barrier-like options!
Zero cost immediate round trips

A jump of size $\delta$ moves the stock price to

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Zero cost immediate round trips

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Zero cost immediate round trips

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$$y \Delta x(x, \delta) + \mathcal{I}(x, \delta)$$

but

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□ There is no hidden cost: this is why perfect hedging will be possible!!
Zero cost immediate round trips

- A jump of size $\delta$ moves the stock price to

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but

$$x(x(x, \delta), -\delta) = x.$$  

Similarly, the impact on the portfolio value is

$$y \Delta x(x, \delta) + \mathcal{J}(x, \delta)$$

but

$$(y + \delta) \Delta x(x(x, \delta), -\delta) + \mathcal{J}(x(x, \delta), -\delta) = -[y \Delta x(x, \delta) + \mathcal{J}(x, \delta)].$$

- **Warning**: be careful with barrier-like options!
Other possible specifications

- Multiplicative formulation

\[ X = X^\circ \ell(Y) \]

Other possible specifications

☐ Multiplicative formulation
\[ X = X^\circ \ell(Y) \]


☐ Immediate partial resilience
Chapter 2 - Hedging of un-covered options
Fix a claim $g = (g_0, g_1)$ with
- $g_0 = \text{cash part}$
- $g_1 = \# \text{ of stocks to deliver.}$
Super-hedging problem

- Fix a claim $g = (g_0, g_1)$ with
  - $g_0 =$ cash part
  - $g_1 =$ # of stocks to deliver.

- Super-hedging price = minimal initial cash so that
  \[ V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T). \]
Super-hedging problem

- Fix a claim \( g = (g_0, g_1) \) with
  - \( g_0 = \) cash part
  - \( g_1 = \# \) of stocks to deliver.

- Super-hedging price = minimal initial cash so that

\[
V_T - Y_T X_T \geq g_0(X_T) \quad \text{and} \quad Y_T = g_1(X_T).
\]

\( \Rightarrow \) Match perfectly the number of stocks and be above the cash requirement.
Super-hedging price

\[ w(0, X_0-) \] is the inf over \( V_0- \) such that one super-hedges for some \((a, b, \nu)\), starting from \( Y_0- = 0 \).
Super-hedging price

\( w(0, X_{0^-}) \) is the inf over \( V_{0^-} \) such that one super-hedges for some \((a, b, \nu)\), starting from \( Y_{0^-} = 0 \).

\[
w(0, X_{0^-}) := \inf \{ V_{0^-} : \exists (a, b, \nu) \text{ s.t. } V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T) \}.
\]
Super-hedging price

- $w(0, X_{0^-})$ is the inf over $V_{0^-}$ such that one super-hedges for some $(a, b, \nu)$, starting from $Y_{0^-} = 0$.

- $\hat{w}(0, X_{0^-}, Y_{0^-})$ is the inf over $V_{0^-}$ such that one super-hedges for some $(a, b, \nu)$, starting from $Y_{0^-} \in \mathbb{R}$.
Super-hedging price

\( w(0, X_{0-}) \) is the inf over \( V_{0-} \) such that one super-hedges for some \((a, b, \nu)\), starting from \( Y_{0-} = 0 \).

\( \hat{w}(0, X_{0-}, Y_{0-}) \) is the inf over \( V_{0-} \) such that one super-hedges for some \((a, b, \nu)\), starting from \( Y_{0-} \in \mathbb{R} \).

We will need both... see later. Anyway, we have the relation

\[
\hat{w}(t, y) = \hat{w}(t, x, y) - J(x, x, -y, y) 
\]
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We will need both... see later. Anyway, we have the relation

\[
\begin{align*}
  w(t, x(x, -y)) &= \hat{w}(t, x, y) - J(x(x, -y), y) \\
  \Rightarrow \quad & \text{One inequality (the other way round): With initial stock price} \\
  & x(x, -y), \text{ wealth } \hat{w}(t, x, y) - J(x(x, -y), y) \text{ and 0 stock,}
\end{align*}
\]
Super-hedging price

□ \( w(0, X_{0-}) \) is the inf over \( V_{0-} \) such that one super-hedges for some \((a, b, \nu)\), starting from \( Y_{0-} = 0 \).

□ \( \hat{w}(0, X_{0-}, Y_{0-}) \) is the inf over \( V_{0-} \) such that one super-hedges for some \((a, b, \nu)\), starting from \( Y_{0-} \in \mathbb{R} \).

□ We will need both... see later. Anyway, we have the relation

\[
\begin{align*}
\hat{w}(t, x, y) &= w(t, x, -y) = \hat{w}(t, x, y) - \mathcal{I}(x(x, -y), y) \\
& \Rightarrow \text{One inequality (the other way round)}: \text{With initial stock price} \ x(x, -y), \text{wealth} \ \hat{w}(t, x, y) - \mathcal{I}(x(x, -y), y) \text{and 0 stock, buying} \ y \text{stocks at} \ t \ \text{leads to}
\end{align*}
\]

\[
\begin{align*}
V_{0-} &= \hat{w}(t, x, y) - \mathcal{I}(x(x, -y), y) \rightarrow \hat{w}(t, x, y) \\
x_{0-} &= x(x, -y) \rightarrow x(x(x, -y), y) = x \\
Y_{0-} &= 0 \rightarrow y.
\end{align*}
\]
**Dynamic programming principle for stochastic targets**

- **Geometric Dynamic Programming Principle**: Let \( \theta \) be a stopping time.
  - GDP1: if \( V_0^- > \hat{w}(0, X_0^-, Y_0^-) \) then \( V_\theta \geq \hat{w}(\theta, X_\theta, Y_\theta) \) for some \((a, b, \nu)\).
  - GDP2: if \( V_\theta > \hat{w}(\theta, X_\theta, Y_\theta) \) for some \((a, b, \nu)\), then \( V_0^- \geq \hat{w}(0, X_0^-, Y_0^-) \).

This basically means that, for \( V_0^- = \hat{w}(0, X_0^-, Y_0^-) \), we can find \((a, b, \nu)\) such that \( dV = d\hat{w}(\cdot, \cdot, \cdot) \) but not better (i.e. with \( > \)). In particular, we should have

\[ f(x) \frac{\partial \hat{w}(t, x, y)}{\partial x} + \frac{\partial \hat{w}(t, x, y)}{\partial y} = yf(x) \]

otherwise the control \( b \) allows to violate the DPP. The solution leaves on a submanifold... (not easy to handle!!)
Dynamic programming principle for stochastic targets

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  - GDP2: if $V_{\theta} > \hat{w}(\theta, X_{\theta}, Y_{\theta})$ for some $(a, b, \nu)$, then $V_{0-} \geq \hat{w}(0, X_{0-}, Y_{0-})$.

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f(x)\partial_x \hat{w}(t, x, y) + \partial_y \hat{w}(t, x, y) = yf(x)
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Dynamic programming principle for stochastic targets

- **Geometric Dynamic Programming Principle**: Let $\theta$ be a stopping time.
  
  - **GDP1**: if $V_0 > \hat{w}(0, X_0, Y_0)$ then $V_\theta \geq \hat{w}(\theta, X_\theta, Y_\theta)$ for some $(a, b, \nu)$.
  
  - **GDP2**: if $V_\theta > \hat{w}(\theta, X_\theta, Y_\theta)$ for some $(a, b, \nu)$, then $V_0 \geq \hat{w}(0, X_0, Y_0)$.

- This basically means that, for $V_0 = \hat{w}(0, X_0, Y_0)$, we can find $(a, b, \nu)$ such that
  
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  $$f(x) \partial_x \hat{w}(t, x, y) + \partial_y \hat{w}(t, x, y) = yf(x)$$

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Geometric dynamic programming transferred from \( \hat{w} \) to \( w \) by using
\[
w(t, x(x, -y)) = \hat{w}(t, x, y) - I(x(x, -y), y).
\]
Geometric dynamic programming transferred from $\hat{w}$ to $w$ by using

$$w(t, x(x, -y)) = \hat{w}(t, x, y) - \mathcal{J}(x(x, -y), y).$$

GDP : (i) If $V_0 - w(0, X_0 - ) > 0$, then $\exists (a, b, \nu)$ and $Y_0 \in \mathbb{R}$ s.t.

$$V_\theta \geq w(\theta, x(X_\theta, -Y_\theta)) + \mathcal{J}(x(X_\theta, -Y_\theta), Y_\theta),$$

for all $\theta \geq t$, where $(X_0, V_0) = (x(X_0 -, Y_0), V_0 - + \mathcal{J}(X_0 -, Y_0))$. 
Geometric dynamic programming transferred from \( \hat{w} \) to \( w \) by using

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w(t, x(x, -y)) = \hat{w}(t, x, y) - \mathcal{I}(x(x, -y), y).
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\[\mathcal{GDP}:\]

(i) If \( V_{0-} > w(0, X_{0-}) \), then \( \exists (a, b, \nu) \) and \( Y_0 \in \mathbb{R} \) s.t.

\[
V_{\theta} \geq w(\theta, x(X_{\theta}, -Y_{\theta})) + \mathcal{I}(x(X_{\theta}, -Y_{\theta}), Y_{\theta}),
\]

for all \( \theta \geq t \), where \( (X_0, V_0) = (x(X_{0-}, Y_0), V_{0-} + \mathcal{I}(X_{0-}, Y_0)) \).

(ii) If \( V_{0-} < w(0, X_{0-}) \) then \( \nexists (a, b, \nu), Y_0 \) and \( \theta \geq t \) s.t.

\[
V_{\theta} > w(\theta, x(X_{\theta}, -Y_{\theta})) + \mathcal{I}(x(X_{\theta}, -Y_{\theta}), Y_{\theta}),
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Geometric dynamic programming transferred from \( \hat{w} \) to \( w \) by using

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GDP: (i) If \( V_0 - > w(0, X_0 -) \), then \( \exists (a, b, \nu) \) and \( Y_0 \in \mathbb{R} \) s.t.

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V_\theta \geq w(\theta, x(X_\theta, -Y_\theta)) + j(x(X_\theta, -Y_\theta), Y_\theta),
\]

for all \( \theta \geq t \), where \((X_0, V_0) = (x(X_0 -, Y_0), V_0 - + j(X_0 -, Y_0))\).

(ii) If \( V_0 - < w(0, X_0 -) \) then \( \not\exists (a, b, \nu), Y_0 \) and \( \theta \geq t \) s.t.

\[
V_\theta > w(\theta, x(X_\theta, -Y_\theta)) + j(x(X_\theta, -Y_\theta), Y_\theta),
\]

with \((X_0, V_0) = (x(X_0 -, Y_0), V_0 - + j(X_0 -, Y_0))\).

\( \Rightarrow \) This will kill the singularity issue!
If \( v = w(t, x) \) the GDP “implies”

\[
d\mathcal{E}_t := dV_t - dw(t, x(X_t, -Y_t)) - d\mathcal{J}(x(X_t, -Y_t), Y_t) = 0,
\]

where \((X_t, Y_t, V_t) = (x(x, y), y, v + \mathcal{J}(x, y))\).
If $v = w(t, x)$ the GDP “implies”

$$d\mathcal{E}_t := dV_t - dw(t, x(X_t, -Y_t)) - d\mathcal{J}(x(X_t, -Y_t), Y_t) = 0,$$

where $(X_t, Y_t, V_t) = (x(x, y), y, v + \mathcal{J}(x, y))$.

**Key property**:

$$d\mathcal{E} = [Y - \tilde{Y}][\left(\mu - f'f_\alpha^2/2\right)(X)dt + \sigma(X)dW]$$

$$+ \hat{F}[w](\cdot, x(X, -Y), Y)dt$$

in which

$$\tilde{Y} := Y + \frac{x(X, -Y) - X}{f(X)} + \partial_x w(\cdot, x(X, -Y))\frac{f(x(X, -Y))}{f(X)}$$
By identifying the $dW$ and $dt$ terms, we obtain the PDE:

$$0 = \hat{F}[w](\cdot, \hat{y})$$
By identifying the $dW$ and $dt$ terms, we obtain the PDE:

$$0 = \hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y})\partial_x [w + \mathcal{J}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx} [w + \mathcal{J}]$$

where

$$\hat{\mu}(\cdot, y) := \frac{1}{2} [\partial_{xx} x \sigma^2](x(\cdot, y), -y) \quad \text{and} \quad \hat{\sigma}(\cdot, y) := (\sigma \partial_x x)(x(\cdot, y), -y).$$

and

$$\hat{y}(t, x) := x^{-1}(x, x + f(x)\partial_x w(t, x)).$$
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Terminal condition

$$w(T-, \cdot) = G(\cdot) := \inf \{yx(x, y) + g_0(x(x, y)) : y = g_1(x(x, y))\}.$$
By identifying the $dW$ and $dt$ terms, we obtain the PDE:

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Terminal condition

$$w(T-, \cdot) = G(\cdot) := \inf \{yx(x, y) + g_0(x(x, y)) : y = g_1(x(x, y))\}.$$  

To be taken in the discontinuous viscosity sense for the relaxed semi-limits associated to problems with bounded controls.
Verification

Assume that \( w \) is a smooth solution of

\[
\hat{F}[w](\cdot, \hat{y}) = -\partial_tw - \mu(\cdot, \hat{y})\partial_x[w + \mathcal{J}] - \frac{1}{2}\sigma(\cdot, \hat{y})^2\partial^2_{xx}[w + \mathcal{J}] = 0
\]

with terminal condition

\[
w(T-, \cdot) = G(\cdot).
\]
We can use the strategy

- Make an initial jump of size
  \[ Y_0 = \hat{y}(0, X_{0-}) = x^{-1}(X_{0-}, X_{0-} + f(X_{0-})\partial_x w(0, X_{0-})). \]
- Follow \((a, b)\) such that \( Y = \hat{y}(\cdot, x(X, -Y)). \)
- \( V_{T-} = G(x(X_{T-}, -Y_{T-})) + J(x(X_{T-}, -Y_{T-}), Y_{T-}). \)
- Liquidate \( Y_{T-} : V_T = G(X_T) \) and \( Y_T = 0. \)
Verification

- Assume that $w$ is a smooth solution of

$$\hat{F}[w](\cdot, \hat{y}) = -\partial_t w - \hat{\mu}(\cdot, \hat{y}) \partial_x [w + \mathcal{J}] - \frac{1}{2} \hat{\sigma}(\cdot, \hat{y})^2 \partial_{xx} [w + \mathcal{J}] = 0$$

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$$w(T-, \cdot) = G(\cdot).$$

- We can use the strategy
  - Make an initial jump of size
    $$Y_0 = \hat{y}(0, X_{0-}) = x^{-1}(X_{0-}, X_{0-} + f(X_{0-}) \partial_x w(0, X_{0-})).$$
  - Follow $\langle a, b \rangle$ such that $Y = \hat{y}(\cdot, x(X, -Y)).$
  - $V_{T-} = G(x(X_{T-}, -Y_{T-})) + \mathcal{J}(x(X_{T-}, -Y_{T-}), Y_{T-}).$
  - Liquidate $Y_{T-}$: $V_T = G(X_T)$ and $Y_T = 0.$

$\Rightarrow$ Jumps only at 0 and $T!$
Viscosity solution approach

\[ \square \textbf{Proposition} : \] Let \( \sigma \) and \( \mu \) be adapted, bounded, and a.s. right-continuous at 0. Assume that

\[ Z_t := \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \geq 0 \]

a.s., for all \( t \leq t_0 \). Then, \( \sigma_0 = 0 \) and \( \mu_0 \geq 0 \).
Viscosity solution approach

- **Proposition**: Let $\sigma$ and $\mu$ be adapted, bounded, and a.s. right-continuous at 0. Assume that

$$Z_t := \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \geq 0$$

a.s., for all $t \leq t_0$. Then, $\sigma_0 = 0$ and $\mu_0 \geq 0$.

Proof. Take $dQ/dP = \mathcal{E}(-n \int_0^\cdot \sigma_s dW_s)$, so that $dZ_s = (\mu_s - n|\sigma_s|^2)ds + \sigma_s dW_s^Q$. 
Proposition: Let $\sigma$ and $\mu$ be adapted, bounded, and a.s. right-continuous at 0. Assume that

$$Z_t := \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \geq 0$$

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Proof. Take $dQ/dP = \mathcal{E}(-n \int_0^\cdot \sigma_s dW_s)$, so that $dZ_s = (\mu_s - n|\sigma_s|^2)ds + \sigma_s dW_s^Q$. In particular,

$$\frac{1}{t} \mathbb{E}^Q[\int_0^t (\mu_s - n|\sigma_s|^2)ds] = \frac{1}{t} \mathbb{E}^Q[Z_t] \geq 0.$$
Viscosity solution approach

\□ Proposition \textbf{ :} Let $\sigma$ and $\mu$ be adapted, bounded, and a.s. right-continuous at 0. Assume that

$$Z_t := \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \geq 0$$

a.s., for all $t \leq t_0$. Then, $\sigma_0 = 0$ and $\mu_0 \geq 0$.

Proof. Take $dQ/dP = \mathcal{E}( -n \int_0^t \sigma_s dW_s )$, so that $dZ_s = (\mu_s - n|\sigma_s|^2) ds + \sigma_s dW_s^Q$. In particular,

$$\frac{1}{t} \mathbb{E}^Q[ \int_0^t (\mu_s - n|\sigma_s|^2) ds ] = \frac{1}{t} \mathbb{E}^Q[Z_t] \geq 0.$$ 

By sending $t \to 0$, we obtain: $\mu_0 - n|\sigma_0|^2 \geq 0$, for all $n \geq 0$. \hfill \Box
Take $\varphi$ such that $\min(w - \varphi) = (w - \varphi)(t_0, x_0) = 0$. Start from $V_{t_0} = w(t_0, x_0) = \varphi(t_0, x_0)$. 
Take \( \varphi \) such that \( \min(w - \varphi) = (w - \varphi)(t_0, x_0) = 0 \). Start from \( V_{t_0} = w(t_0, x_0) = \varphi(t_0, x_0) \).

Then, “there exists” \( (a, b, \nu) \) and \( Y_{t_0} \in \mathbb{R} \) s.t.

\[
V_\theta \geq w(\theta, x(X_\theta, -Y_\theta)) + \mathcal{I}(x(X_\theta, -Y_\theta), Y_\theta),
\]

for all \( \theta \geq t_0 \), where \( (X_{t_0}, V_{t_0}) = (x(X_{t_0}, Y_{t_0}), V_{t_0} + \mathcal{I}(X_{t_0}, Y_{t_0})) \).
Take $\varphi$ such that $\min(w - \varphi) = (w - \varphi)(t_0, x_0) = 0$. Start from $V_{t_0} = w(t_0, x_0) = \varphi(t_0, x_0)$.

Then, "there exists" $(a, b, \nu)$ and $Y_{t_0} \in \mathbb{R}$ s.t.

$$V_{\theta} \geq w(\theta, x(X_{\theta}, -Y_{\theta})) + I(x(X_{\theta}, -Y_{\theta}), Y_{\theta}),$$

for all $\theta \geq t_0$, where $(X_{t_0}, V_{t_0}) = (x(X_{t_0}, -Y_{t_0}), V_{t_0} + I(X_{t_0}, Y_{t_0}))$. Since $w \geq \varphi$,

$$V_{\theta} \geq \varphi(\theta, x(X_{\theta}, -Y_{\theta})) + I(x(X_{\theta}, -Y_{\theta}), Y_{\theta}).$$
Take $\varphi$ such that $\min(w - \varphi) = (w - \varphi)(t_0, x_0) = 0$. Start from $V_{t_0} = w(t_0, x_0) = \varphi(t_0, x_0)$.

Then, “there exists” $(a, b, \nu)$ and $Y_{t_0} \in \mathbb{R}$ s.t.

$$V_{\theta} \geq w(\theta, x(X_{\theta}, -Y_{\theta})) + \mathcal{I}(x(X_{\theta}, -Y_{\theta}), Y_{\theta}),$$

for all $\theta \geq t_0$, where $(X_{t_0}, V_{t_0}) = (x(X_{t_0} - Y_{t_0}), V_{t_0} + \mathcal{I}(X_{t_0} - Y_{t_0}), Y_{t_0}))$.

Since $w \geq \varphi$,

$$V_{\theta} \geq \varphi(\theta, x(X_{\theta}, -Y_{\theta})) + \mathcal{I}(x(X_{\theta}, -Y_{\theta}), Y_{\theta}).$$

Apply the above to $Z := V - [\varphi(\cdot, x(X, -Y)) + \mathcal{I}(x(X, -Y), Y)]$. 
Take \( \varphi \) such that \( \min(w - \varphi) = (w - \varphi)(t_0, x_0) = 0 \). Start from \( V_{t_0} = w(t_0, x_0) = \varphi(t_0, x_0) \).

Then, “there exists” \((a, b, \nu)\) and \( Y_{t_0} \in \mathbb{R} \) s.t.

\[
V_\theta \geq w(\theta, x(X_\theta, -Y_\theta)) + \mathcal{I}(x(X_\theta, -Y_\theta), Y_\theta),
\]

for all \( \theta \geq t_0 \), where \((X_{t_0}, V_{t_0}) = (x(X_{t_0}, Y_{t_0}), V_{t_0} + \mathcal{I}(X_{t_0}, Y_{t_0}))\).

Since \( w \geq \varphi \),

\[
V_\theta \geq \varphi(\theta, x(X_\theta, -Y_\theta)) + \mathcal{I}(x(X_\theta, -Y_\theta), Y_\theta).
\]

Apply the above to \( Z := V - [\varphi(\cdot, x(X, -Y)) + \mathcal{I}(x(X, -Y), Y)] \).

Then, \( \hat{F}[\varphi](t_0, x_0, \hat{y}(t_0, x_0)) \geq 0 \).
\[\textbf{Proposition}:\] Let \(\sigma\) and \(\mu\) be adapted, bounded. Assume that there exists a stopping time \(\theta > t_0\) such that

\[\sigma 1_{[t_0,\theta]} = 0 \quad \text{and} \quad \mu 1_{[t_0,\theta]} \geq 0.\]

Then

\[\int_0^\theta \mu_s \, ds + \int_0^\theta \sigma_s \, dW_s \geq 0.\]
Take \( \varphi \) such that \( \max(w - \varphi) = (w - \varphi)(t_0, x_0) = 0 \) with \( (w - \varphi)(t, x) < 0 \) for \( (t, x) \neq (t_0, x_0) \). Assume that

\[
\hat{F}[\varphi](t_0, x_0, \hat{y}(t_0, x_0)) > 0.
\]
Take \( \varphi \) such that \( \max(w - \varphi) = (w - \varphi)(t_0, x_0) = 0 \) with \( (w - \varphi)(t, x) < 0 \) for \( (t, x) \neq (t_0, x_0) \). Assume that

\[
\hat{F}[\varphi](t_0, x_0, \hat{y}(t_0, x_0)) > 0.
\]

Then there exists a neighborhood \( B \) of \( (t_0, x_0) \) such that \( \hat{F}[\varphi](\cdot, \hat{y}) \geq 0 \).
Take $\varphi$ such that $\max(w - \varphi) = (w - \varphi)(t_0, x_0) = 0$ with $(w - \varphi)(t, x) < 0$ for $(t, x) \neq (t_0, x_0)$. Assume that

$$\hat{F}[\varphi](t_0, x_0, \hat{y}(t_0, x_0)) > 0.$$  

Then there exists a neighborhood $B$ of $(t_0, x_0)$ such that $\hat{F}[\varphi](\cdot, \hat{y}) \geq 0$. Start from $V_{t_0-} = w(t_0, x_0) - \varepsilon = \varphi(t_0, x_0) - \varepsilon$ where $-2\varepsilon := \max_B(w - \varphi)$. Let $\theta$ be the exist time of $B$. 
Take $\varphi$ such that $\max (w - \varphi) = (w - \varphi)(t_0, x_0) = 0$ with $(w - \varphi)(t, x) < 0$ for $(t, x) \neq (t_0, x_0)$. Assume that

$$\hat{F}[\varphi](t_0, x_0, \hat{y}(t_0, x_0)) > 0.$$ 

Then there exists a neighborhood $B$ of $(t_0, x_0)$ such that $\hat{F}[\varphi](\cdot, \hat{y}) \geq 0$. Start from $V_{t_0^-} = w(t_0, x_0) - \varepsilon = \varphi(t_0, x_0) - \varepsilon$ where $-2\varepsilon := \max_B (w - \varphi)$. Let $\theta$ be the exist time of $B$. Then, using the controls of the verification argument applied with $\varphi$,

$$V_{\theta} \geq \varphi(\theta, x(X_\theta, -Y_\theta)) + \mathcal{J}(x(X_\theta, -Y_\theta), Y_\theta) - \varepsilon$$

$$\geq w(\theta, x(X_\theta, -Y_\theta)) + \mathcal{J}(x(X_\theta, -Y_\theta), Y_\theta) + 2\varepsilon - \varepsilon$$

$$> w(\theta, x(X_\theta, -Y_\theta)) + \mathcal{J}(x(X_\theta, -Y_\theta), Y_\theta).$$
Proposition: Comparison holds.
**Proposition** : Comparison holds.

This implies uniqueness and convergence of monotone finite difference numerical schemes.
A simple example: Bachelier model

- Model: $X_t = \mu t + \sigma W_t$ and $f(X) = f \in (0, \infty)$. 

- Hedging strategy: $Y = \partial_x w(\cdot, X - fY)$ with $\Delta Y = \partial_x w(0, X_0 - f)$. 

- Call hedging: 
  - Cash settlement: $G(x) = g_0(x) = [x - K]_+$. 
  - With delivery: $G(x) = \min\{y(x + yf) - K, x + yf \geq K\} = (x + f - K)_+ + 1_{K > x}$. 

This is the usual heat equation!!!
A simple example: Bachelier model

- Model: $X_t = \mu t + \sigma W_t$ and $f(X) = f \in (0, \infty)$.

- In this case, $x(x, \delta) = x + f \delta$, $J(x, \delta) = \frac{1}{2} \delta^2 f$, and the pde is

$$-\partial_t w - \frac{1}{2} \sigma^2 \partial^2_{xx} w = 0$$

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Hedging strategy: \( Y = \partial_x w(\cdot, X - fY) \) with \( \Delta Y_0 = \partial_x w(0, X_{0-}) \).
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- Interpretation:
  - We have \( x(X_t, -Y_t) = x(\mu t + \sigma W_t + Y_t f, -Y_t) = \mu t + \sigma W_t \), i.e. moves on price due to trading will cancel when the position is closed.
  - Cost of trading is compensated by the impact on prices:

\[
-\delta 0 - \frac{1}{2} \delta^2 f + \delta (0 + \mu t + \sigma W_t + \delta f) - \frac{1}{2} \delta^2 f = \delta (\mu t + \sigma W_t).
\]
A simple example: Bachelier model

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- Cash settlement: $G(x) = g_0(x) = [x - K]^+$
- With delivery:

$$G(x) = \min \left\{ y(x + yf) - K1_{\{x+yf\geq K\}} : y = 1_{\{x+yf\geq K\}} \right\}$$

$$= (x + f - K)^+1_{\{K>x\}} + (x + f - K)1_{\{x\geq K\}}$$
Chapter 3 - Hedging of covered options
Super-hedging problem

□ Fix a claim $g$ :

• At 0, the trader asks for receiving an initial amount of stocks $Y_0$ and of cash such that cash $+ Y_0 X_0 =$ premium.
• At $T$, the trader delivers $Y_T$ stocks plus some cash such that cash $+ Y_T X_T = g(X_T)$. 
Super-hedging problem

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- Avoids big impact at 0 and $T$. 

Super-hedging price = minimal initial cash so that $V_T \geq g(X_T)$.

⇒ In this case $V$ is exactly the relevant quantity. We will not need jumps anymore...

We set $v(0, X_0) := \inf \{ v = c + Y_0 X_0 : c, Y_0, (a, b) \text{ s.t. } V_T \geq g(X_T) \}$. 

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Hedging and pricing - informal derivation

Let us assume that we use the delta-hedging rule:

\[ V = v(\cdot, X), \quad Y = \partial_x v(\cdot, X). \]
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Then, equating the \( dt \) terms implies

\[ \frac{1}{2} a^2 f(X) = \partial_t v(\cdot, X) + \frac{1}{2} (\sigma + af)^2 \partial_{xx} v(\cdot, X), \]

By definition of \( \gamma_a \) and a little bit of algebra:

\[ -\partial_t v(\cdot, X) - \frac{1}{2} \sigma^2 (1 - f \partial_{xx} v(\cdot, X)) \partial_{xx} v(\cdot, X) = 0. \]
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\[ \gamma^a := \frac{a}{\sigma(X) + f(X)a} = \partial_{xx}^2 v(\cdot, X) \in \mathbb{R} \setminus \{1/f\} \]
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By definition of \( \gamma^a \) and a little bit of algebra:

\[ \left[ -\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 v)} \partial_{xx}^2 v \right] (\cdot, X) = 0. \]
The pricing pde should be
\[-\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx}^2 v)} \partial_{xx}^2 v = 0 \text{ on } [0, T) \times \mathbb{R},\]
\[v(T-, \cdot) = g \text{ on } \mathbb{R}.\]
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$$v(T-, \cdot) = g \quad \text{on } \mathbb{R}.$$

**Singular pde:**
- Can find smooth solutions s.t. $1 > f \partial_{xx}^2 v$, cf. below.
- In general, needs to take care of $1 \neq f \partial_{xx}^2 v$
- One possibility: add a gamma constraint $\partial_{xx}^2 v \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f \partial_{xx}^2 v > 1$ does not make sense.
Hedging with a gamma constraint

□ By a change of variable, we write the dynamics in the form:

\[ dY = \gamma^a(X) dX + \mu^a_b(X) dt \quad \text{and} \quad dX = \sigma^a(X) dW + \mu^a_b(X) dt. \]

□ We now define \( v \) with respect to the gamma constraint

\[ \gamma^a(X) \leq \bar{\gamma}(X) \]

with

\[ f\bar{\gamma} < 1 - \varepsilon, \quad \varepsilon > 0. \]
Pricing pde:

$$\min \left\{ -\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} v)} \partial_{xx}^2 v , \bar{\gamma} - \partial_{xx}^2 v \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R}.$$ 

Propagation of the gamma constraint at the boundary:

$$v(T-, \cdot) = \hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

$$\min \{ \varphi - g , \bar{\gamma} - \partial_{xx}^2 \varphi \} = 0.$$ 

See Soner and Touzi 00, and Cheridito, Soner and Touzi 05.
Super-solution property

Use a weak formulation approach and results on small time behavior of double stochastic integrals, see Soner and Touzi 00 and Cheridito, Soner and Touzi 05.

It is based on the Geometric DPP (Soner and Touzi) : if

\[ V_0 > v(0, X_0) \]

then we can find \((a, b, Y_0)\) such that

\[ V_\theta \geq v(\theta, X_\theta) \]

for any stopping time \(\theta\) with values in \([0, T]\).
Sub-solution property

- Main difficulty: cannot establish the reverse Geometric DPP, i.e.

If \((a, b, Y_0)\) are such that

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- **Problem**: 
  - at \(\theta\) we have a position \(Y_\theta\) that may not match with the position \(\hat{Y}_\theta\) associated to \(v(\theta, X_\theta)\). Can not jump from \(Y_\theta\) to \(\hat{Y}_\theta\)...
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- At \(\theta\) we have a position \(Y_\theta\) that may not match with the position \(\hat{Y}_\theta\) associated to \(v(\theta, X_\theta)\). Cannot jump from \(Y_\theta\) to \(\hat{Y}_\theta\).
- Cannot go smoothly to it as it will move \(X\) because of the impact, and therefore \(\hat{Y}\) (sort of fixed point problem), compare with Cheridito, Soner, and Touzi 05.
The smoothing approach

In place, we use a smoothing/verification approach initiated by B. and Nutz 13 (inspired from Jensen’s and Krylov’s ideas).
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1. Using the concavity of the PDE, create a sequence $w^\delta_\iota$ of smooth super-solutions that converges to a viscosity solution $w$. 

Conclusion: $v$ is the (unique) viscosity solution.
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Conclusion : $v$ is the (unique) viscosity solution.
How to construct the smooth super-solution (in a nutshell)

Consider a viscosity solution to the PDE (with $F$ convex and non-decreasing)

$$0 = -\partial_t w - F(\partial_{xx}^2 w).$$
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Take the quadratic inf-convolution

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Then, it is semi-concave and

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How to construct the smooth super-solution (in a nutshell)

Consider a viscosity solution to the PDE (with $F$ convexe non-decreasing)

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Then, it is semi-concave and

$$0 = -\partial_t w^\iota - F((\partial_{xx}^2 w^\iota)^{abse}), \text{ a.e.}$$

Then, smooth it out and use the fact that $-F$ is concave and non-increasing

$$0 = \int \left( -\partial_t w^\iota - F((\partial_{xx}^2 w^\iota)^{abse}) \right) (t', x') \phi_\delta(t' - t, x' - x) dt' dx',$$

$$\leq -\partial_t w_\delta^\iota(t, x) - F(\partial_{xx}^2 w_\delta^\iota)(t, x).$$
A-priori estimates

- Assume that \( \partial_{xx}^2 g \leq 1/f - \varepsilon \) for some \( \varepsilon > 0 \).
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1. Assume that $\partial_{xx}^2 g \leq 1/f - \varepsilon$ for some $\varepsilon > 0$.

2. Set $F(x, z) := \sigma(x)^2 z/(1 - f(x)z)$. Let $\varphi$ be a solution of
   
   $$-\partial_t \varphi - F(\cdot, \partial_{xx}^2 \varphi) = 0$$

   and let $\varpi := F(\cdot, \partial_{xx}^2 \varphi)$. 
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$$\partial_t \varpi = \partial_z F(\cdot, \partial_{xx}^2 \varphi) \partial_t \partial_{xx}^2 \varphi$$
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  and let $\omega := F(\cdot, \partial_{xx}^2 \varphi)$. Then, $-\partial_t \partial_{xx}^2 \varphi - \partial_{xx}^2 \omega = 0$, and
  
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and let $\varpi := F(\cdot, \partial_{xx}^2 \varphi)$. Then, $-\partial_t \partial_{xx}^2 \varphi - \partial_{xx}^2 \varpi = 0$, and

$$\partial_t \varpi = -\partial_z F(\cdot, \partial_{xx}^2 \varphi) \partial_{xx}^2 \varpi$$

which means that

$$\varpi(t, x) = \mathbb{E} \left[ \varpi(T, \tilde{X}_T) \right]$$

with $d\tilde{X}_s = \sqrt{2\partial_z F(\tilde{X}_s, \partial_{xx}^2 \varphi(s, \tilde{X}_s))} dW_s$, $\tilde{X}_t = x.$
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  which means that
  \[
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  \]
  with $d\tilde{X}_s = \sqrt{2\partial_z F(\tilde{X}_s, \partial_{xx}^2 \varphi(s, \tilde{X}_s))} dW_s$, $\tilde{X}_t = x$. 

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\[
- \partial_t \varphi - F(\cdot, \partial_{xx}^2 \varphi) = 0
\]

and let \( \varpi := F(\cdot, \partial_{xx}^2 \varphi) \). Then, \( -\partial_t \partial_{xx}^2 \varphi - \partial_{xx} \varpi = 0 \), and

\[
\partial_t \varpi = -\partial_z F(\cdot, \partial_{xx}^2 \varphi) \partial_{xx} \varpi
\]

which means that

\[
\frac{\sigma^2(x)}{1 - f(x)\partial_{xx}^2 \varphi(t, x)} \partial_{xx}^2 \varphi(t, x) = \mathbb{E} \left[ \frac{\sigma^2(\tilde{X}_T)}{1 - f(\tilde{X}_T)\partial_{xx}^2 g(\tilde{X}_T)} \partial_{xx}^2 g(\tilde{X}_T) \right]
\]

with \( d\tilde{X}_s = \sqrt{2\partial_z F(\tilde{X}_s, \partial_{xx}^2 \varphi(s, \tilde{X}_s))} dW_s, \tilde{X}_t = x \).

\( \Rightarrow \) \( \partial_{xx}^2 \varphi \leq 1/f - \varepsilon_g \) with \( \varepsilon_g > 0 \).
\(\square\) **Proposition** : Assume that \(\partial_{xx} g \leq 1/f - \varepsilon\) for some \(\varepsilon > 0\) (+ smoothness conditions). Then, \(v\) is a smooth solution of

\[
0 = -\partial_t v - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial_{xx} v)} \partial_{xx}^2 v
\]

and \(\partial_{xx} v \leq 1/f - \varepsilon_g\) for some \(\varepsilon_g > 0\).
Small impact expansion

We replace $f$ by $\epsilon f$, $\epsilon > 0$. 

Small impact expansion

We replace \( f \) by \( \epsilon f \), \( \epsilon > 0 \). Then, the PDE becomes

\[
0 = -\partial_t v^\epsilon - \frac{1}{2} \frac{\sigma^2}{1 - \epsilon f \partial_{xx} v^\epsilon} \partial_{xx}^2 v^\epsilon.
\]
Small impact expansion

We replace $f$ by $\epsilon f$, $\epsilon > 0$. Then, the PDE becomes

$$0 = -\partial_t v^\epsilon - \frac{1}{2} \frac{\sigma^2}{(1 - \epsilon f \partial_{xx} v^\epsilon)} \partial_{xx}^2 v^\epsilon.$$ 

□ Proposition:

$$v^\epsilon(0, x) = v^0(0, x) + \frac{\epsilon}{2} \mathbb{E} \left[ \int_0^T [\sigma^2 f |\partial_x^2 v^0|^2] (s, \tilde{X}_s) ds \right] + o(\epsilon)$$

where, $\tilde{X}$ is the solution on $[0, T]$ of

$$\tilde{X} = x + \int_t^T \sigma(\tilde{X}_s) dW_s.$$
Proof: Since

\[ 0 = -\partial_t v^\epsilon - \frac{1}{2} \frac{\sigma^2}{(1 - \epsilon f \partial_{xx}^2 v^\epsilon)} \partial_{xx}^2 v^\epsilon, \]

we have

\[ 0 = -\partial_t v^\epsilon - \frac{1}{2} \sigma^2 \partial_{xx}^2 v^\epsilon - \frac{\epsilon}{2} \sigma^2 f |\partial_{xx}^2 v^\epsilon|^2 - o(\epsilon) \]

\[ = -\partial_t v^0 - \frac{1}{2} \sigma^2 \partial_{xx}^2 v^0. \]
There exists a constant $C > 0$ such that

$$|V_T^\epsilon - g(X_T^\epsilon)| \leq C\epsilon^2$$

in which

$$V_0^\epsilon = v^0(0, X_0) + \epsilon \Delta v(0, X_0)$$
$$Y^\epsilon = \partial_x v^0(0, X_0) + \epsilon \partial_x \Delta v(0, X_0),$$

with

$$\Delta v(0, x) := \frac{1}{2} \mathbb{E} \left[ \int_0^T \sigma^2 f \left| \partial_{xx} v^0 \right|^2 (s, \tilde{X}_s) ds \right].$$
Numerical illustration
- Constant impact and constraint.
- Bachelier model: \( dX_t = 0.2\ dW_t \).
- Butterfly option: \( g(x) = (x + 1)^+ - 2x^+ + (x - 1)^+ \), \( T = 2 \).
- Covered option.

**Figure** — Left: Dashed line: \( f = 0.5, \bar{\gamma} = 1.75 \); solid line: \( f = 0, \bar{\gamma} = 1.75 \); dotted line: \( f = 0, \bar{\gamma} = +\infty \).
Towards a duality

Observe that:

\[
0 = -\partial_t v - \frac{1}{2} \frac{\sigma^2}{1 - f \partial_{xx}^2 v} \partial_{xx}^2 v
\]

\[
= \inf_{s \in \mathbb{R}} \left( -\partial_t v - \frac{1}{2} s^2 \partial_{xx}^2 v + \frac{\gamma}{2} (s - \sigma)^2 \right).
\]
Towards a duality

Observe that:

\[ 0 = -\partial_t v - \frac{1}{2} \sigma^2 \partial^2_{xx} v - \frac{1}{2} f \partial^2_{xx} v \]

\[ = \inf_{s \in \mathbb{R}} \left( -\partial_t v - \frac{1}{2} s^2 \partial^2_{xx} v + \frac{\gamma}{2} (s - \sigma)^2 \right). \]

Then

\[ v(0, x) = \bar{v}(0, x) := \sup_{s \in \mathcal{A}_2} \mathbb{E} \left[ g(\bar{X}_T) - \int_0^T \frac{\gamma(\bar{X}_t^s)}{2} (s_t - \sigma(\bar{X}_t^s))^2 dt \right] \]

with

\[ \bar{X}_t^s := x + \int_0^t s_t dW_t. \]

\[ \Rightarrow \text{Dual formulation!} \]
Chapter 4 - Understanding the dual formulation
We now consider the relaxed formulation with path dependent coefficients:

\[
Y_{a,\mathcal{B}} = Y_0 + \int_0^\cdot a_t dW_t - \mathcal{B}
\]

\[
X_{a,\mathcal{B}} = x_{\wedge 0} + \int_0^\cdot (\sigma_t + a_t f_t)(X_{a,\mathcal{B}}) dW_t,
\]

\[
V_{a,\mathcal{B}} = V_0 + \int_0^T Y_{a,\mathcal{B}} dX_{a,\mathcal{B}} + \int_0^T \frac{1}{2} f_t(X_{a,\mathcal{B}}) a_t^2 dt = g(X_{a,\mathcal{B}}).
\]

where

- \( x \in C([0, T]) \),
- \( \sigma, f : [0, T] \times C([0, T]) \mapsto \mathbb{R} \) are non-anticipative,
- The controls are now \((a, \mathcal{B})\) where \(\mathcal{B}\) is an adapted bounded variation process.
Relaxed formulation

We now consider the relaxed formulation with path dependent coefficients:

\[
Y^{a,\mathcal{B}} = Y_0 + \int_0^\cdot a_t dW_t - \mathcal{B}
\]

\[
X^{a,\mathcal{B}} = x^{\land 0} + \int_0^\cdot (\sigma_t + a_t f_t)(X^{a,\mathcal{B}})dW_t,
\]

\[
V^{a,\mathcal{B}}_T = V_0 + \int_0^T Y^{a,\mathcal{B}}_t dX^{a,\mathcal{B}}_t + \int_0^T \frac{1}{2} f_t(X^{a,\mathcal{B}})a_t^2 dt = g(X^{a,\mathcal{B}}).
\]

where

- \( x \in C([0, T]) \),
- \( \sigma, f : [0, T] \times C([0, T]) \mapsto \mathbb{R} \) are non-anticipative,
- The controls are now \((a, \mathcal{B})\) where \(\mathcal{B}\) is an adapted bounded variation process.

The above corresponds to the dynamics of \(X^{a,\mathcal{B}}\) under its “martingale measure”.

Assuming hedging holds...

Assume we have a hedging strategy $(\hat{a}, \hat{B})$ for a path dependent payoff $g$, then

$$V_0 = \mathbb{E}_{\hat{Q}^{\hat{a}, \hat{B}}} \left[ g(X^{\hat{a}, \hat{B}}) - \int_0^T \frac{1}{2} f_t(X^{\hat{a}, \hat{B}}) \hat{a}_t^2 \, dt \right]$$
Assuming hedging holds...

Assume we have a hedging strategy \((\hat{a}, \hat{B})\) for a path dependent payoff \(g\), then

\[
V_0 = \mathbb{E}^{\hat{Q}, \hat{B}} \left[ g(\hat{X}, \hat{B}) - \int_0^T \frac{1}{2} f_t(\hat{X}, \hat{B}) \hat{a}_t^2 dt \right]
\]

\[
\leq \sup_{(a, B)} \mathbb{E}^{Q_a, B} \left[ g(X, B) - \int_0^T \frac{1}{2} f_t(X, B) a_t^2 dt \right].
\]
Assuming hedging holds...

Assume we have a hedging strategy \((\hat{a}, \hat{B})\) for a path dependent payoff \(g\), then

\[
V_0 = \mathbb{E}^{Q^{\hat{a}, \hat{B}}} \left[ g(X^{\hat{a}, \hat{B}}) - \int_0^T \frac{1}{2} f_t(X^{\hat{a}, \hat{B}}) \hat{a}_t^2 dt \right]
\]

\[
\leq \sup_{(a, B)} \mathbb{E}^{Q^{a, B}} \left[ g(X^{a, B}) - \int_0^T \frac{1}{2} f_t(X^{a, B}) a_t^2 dt \right].
\]

We need to retrieve

\[
\sup_s \mathbb{E} \left[ g(\tilde{X}_T^s) - \int_0^T \frac{1}{2} \gamma_t(\tilde{X}^s)(s_t - \sigma_t(\tilde{X}^s))^2 dt \right]
\]

with

\[
\tilde{X}^s := x_{\wedge 0} + \int_0^s s_t dW_t \quad \text{while} \quad X^{a, B} = x_{\wedge 0} + \int_0^\cdot (\sigma_t + a_t f_t)(X^{a, B})dW_t^{a, B}.
\]
Assuming hedging holds...

Assume we have a hedging strategy \((\hat{a}, \hat{B})\) for a path dependent payoff \(g\), then

\[
V_0 = \mathbb{E}^{\mathbb{Q}^{\hat{a}, \hat{B}}} \left[ g(X^{\hat{a}, \hat{B}}) - \int_0^T \frac{1}{2} f_t(X^{\hat{a}, \hat{B}}) \hat{a}_t^2 \, dt \right] 
\]

\[
\leq \sup_{(a, B)} \mathbb{E}^{\mathbb{Q}^a, \mathbb{B}} \left[ g(X^{a, \mathbb{B}}) - \int_0^T \frac{1}{2} f_t(X^{a, \mathbb{B}}) a_t^2 \, dt \right] .
\]

We need to retrieve

\[
\sup_s \mathbb{E} \left[ g(\bar{X}_T^s) - \int_0^T \frac{1}{2} \gamma_t(\bar{X}^s)(s_t - \sigma_t(\bar{X}^s))^2 \, dt \right]
\]

with

\[
\bar{X}^s := x \wedge 0 + \int_0^s s_t \, dW_t \quad \text{while} \quad X^{a, \mathbb{B}} = x \wedge 0 + \int_0^\cdot (\sigma_t + a_t f_t)(X^{a, \mathbb{B}}) \, dW^{a, \mathbb{B}}_t.
\]

Ok, up to change of variable : \(s_t = \sigma_t(X^{a, \mathbb{B}}) + a_t f_t(X^{a, \mathbb{B}})\).
Note that super-hedging does not permit to say anything...:

\[ V_0 \geq \mathbb{E}^{Q, \hat{\mathcal{B}}} \left[ g(X^{\hat{\mathcal{A}}, \hat{\mathcal{B}}}) - \int_0^T f_t(X^{\hat{\mathcal{A}}, \hat{\mathcal{B}}}) a_t^2 dt \right] \]
Note that super-hedging does not permit to say anything...:

\[ V_0 \geq E_{Q^a, \hat{\mathcal{B}}}^{a, \hat{\mathcal{B}}} \left[ g(X^{\hat{a}, \hat{\mathcal{B}}}) - \int_0^T f_t(X^{\hat{a}, \hat{\mathcal{B}}})a_t^2 dt \right] \]

\[ \not\geq \sup_{(a, \mathcal{B})} E_{Q^a, \mathcal{B}}^{a, \mathcal{B}} \left[ g(X^{a, \mathcal{B}}) - \int_0^T f_t(X^{a, \mathcal{B}})a_t^2 dt \right]. \]
Fundamental assumption

Set

\[ \tilde{v}(0, x) := \sup_{s} \mathbb{E} \left[ g(\tilde{X}^s_T) - \int_0^T \frac{1}{2} \gamma_t(\tilde{X}^s)(s_t - \sigma_t(\tilde{X}^s))^2 dt \right] \]

**Assumption:** \( \tilde{v}(t, x) \) admits a solution \( \hat{s}[t, x] \) (need weak...) + smoothness assumptions.
Differentiability of the gain function

- For differentiability, we use the notion of Dupire’s derivative.
Differentiability of the gain function

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- For a path $x$, set $x \oplus_t y := x + 1_{[t, T]}y$
Differentiability of the gain function

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- **Dupire derivative**: A function $\varphi$ is said to be horizontally differentiable if, for all $(t, x)$, its horizontal derivative

$$\partial_t \varphi(t, x) := \lim_{h \searrow 0} \frac{\varphi(t + h, x \wedge \cdot) - \varphi(t, x \wedge \cdot)}{h}$$

is well-defined.
Differentiability of the gain function

- For differentiability, we use the notion of Dupire’s derivative.

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- **Dupire derivative**: A function $\varphi$ is said to be horizontally differentiable if, for all $(t, x)$, its horizontal derivative

\[
\partial_t \varphi(t, x) := \lim_{h \searrow 0} \frac{\varphi(t + h, x_{t\wedge}) - \varphi(t, x_{t\wedge})}{h}
\]

is well-defined.

A function $\varphi$ is said to be vertically differentiable if, for all $(t, x)$, its vertical derivative

\[
\nabla_x \varphi(t, x) := \lim_{y \to 0, y \neq 0} \frac{\varphi(t, x \oplus_t y) - \varphi(t, x)}{y}
\]

is well-defined.
**Result #1:** The gain function

\[
J(t, x; s) := \mathbb{E} \left[ g(\tilde{X}^{t,x,s}) - \int_t^T \frac{1}{2} \gamma_r(\tilde{X}^s)(s_r - \sigma_r(\tilde{X}^s))^2 dr \right],
\]

\[
\tilde{X}^{t,x,s} := x \wedge t + \int_t^s s_r dW_r,
\]

admits a Dupire vertical derivative

\[
\nabla_x J(t, x; s) := \mathbb{E} [\mathcal{B}_T^s - \mathcal{B}_t^s]
\]

where \( \mathcal{B}^s \) is an adapted BV process.
Proof for constant coefficients: Recall

\[ \bar{X}^{t,x,s} := x_{\wedge t} + \int_t^\cdot s_r \, dW_r. \]
Proof for constant coefficients: Recall

\[ \bar{X}^{t,x,s} := x \wedge t + \int_t^\cdot s_r dW_r. \]

If

\[ J(t, x; s) := \mathbb{E} \left[ g(\bar{X}^{t,x,s}) - \int_t^T \frac{1}{2} \gamma (s_r - \sigma)^2 dr \right] , \]
Proof for constant coefficients : Recall

\[ \tilde{X}^{t,x,s} := x_{\wedge t} + \int_t^s r \, dW_r. \]

If

\[ J(t, x; s) := \mathbb{E} \left[ g(\tilde{X}^{t,x,s}) - \int_t^T \frac{1}{2}\gamma(s_r - \sigma)^2 \, dr \right], \]

then

\[ \nabla_x J(t, x; s) := \mathbb{E} \left[ \int_t^T \lambda_g(dr; \tilde{X}^{t,x,s}) \right] \]

where \( \lambda_g \) is the Fréchet derivative of \( g \) at \( \tilde{X}^{t,x,s} \) :

\[ g(x') - g(x) = \int_0^T (x'_t - x_t) \lambda_g(dt; x) + \|x - x'\| \epsilon(x', x) \]

with \( \epsilon(x', x) \to 0 \) as \( x' \to x \)
Proof for constant coefficients: Recall

\[ \bar{X}^{t,x,s} := x \wedge t + \int_t^s r \, dW_r. \]

If

\[ J(t, x; s) := \mathbb{E}\left[ g(\bar{X}^{t,x,s}) - \int_t^T \frac{1}{2} \gamma(s_r - \sigma)^2 \, dr \right], \]

then

\[ \nabla_x J(t, x; s) := \mathbb{E}\left[ \int_t^T \lambda_g(dr; \bar{X}^{t,x,s}) \right] = \mathbb{E}\left[ \int_t^T \lambda^o_g(dr; \bar{X}^{t,x,s}) \right], \]

where \( \lambda_g \) is the Fréchet derivative of \( g \) at \( \bar{X}^{t,x,s} \):

\[ g(x') - g(x) = \int_0^T (x'_t - x_t) \lambda_g(dt; x) + \|x - x'\| \epsilon(x', x) \]

with \( \epsilon(x', x) \to 0 \) as \( x' \to x \), and \( \lambda^o_g(\cdot; \bar{X}^{t,x,s}) \) is its dual predictable projection.
Calculus of variations

Result #2: By a simple calculus of variations argument,

$$\gamma(\hat{s}[t,x] - \sigma)(\tilde{X}^{t,x,\hat{s}[t,x]}) = \hat{a}[t,x]$$

where \((m[t,x],\hat{a}[t,x])\) is such that

$$m[t,x] + \int_t^T \hat{a}[t,x]_u dW_u = \hat{B}[t,x]_T - \hat{B}[t,x]_t.$$
Calculus of variations

**Result #2**: By a simple calculus of variations argument,

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where \((m[t, x], \hat{a}[t, x])\) is such that

\[
m[t, x] + \int_t^T \hat{a}[t, x] u dW_u = \hat{B}[t, x]_T - \hat{B}[t, x]_t.
\]

Recall that

\[
\nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{B}[t, x]_T - \hat{B}[t, x]_t \right].
\]
Proof for

\[
J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ g(\bar{X}^{t, x, \hat{s}[t, x]}) - \int_{t}^{T} \frac{1}{2} \gamma(\hat{s}[t, x]_r - \sigma)^2 dr \right],
\]

the first order condition implies (for all \(\delta\) adapted bounded):

\[
0 = \mathbb{E} \left[ \int_{t}^{T} \left( \int_{r}^{\infty} \delta_s dW_s \right) \lambda_g(\bar{X}^{t, x, \hat{s}[t, x]}) \right.
\]

\[
- \int_{t}^{T} \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r)(\bar{X}^{t, x, \hat{s}[t, x]}) dr
\]
Proof for

$$J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ g(\tilde{X}^{t,x,\hat{s}[t,x]} - \int_t^T \frac{1}{2} \gamma(\hat{s}[t, x]_r - \sigma)^2 dr \right] ,$$

the first order condition implies (for all $\delta$ adapted bounded)

$$0 = \mathbb{E} \left[ \int_t^T (\int_t^r \delta_s dW_s) \lambda_g^o (dr; \tilde{X}^{t,x,\hat{s}[t,x]}) - \int_t^T \delta_r \gamma_r(\hat{s}[t, x]_r - \sigma_r)(\tilde{X}^{t,x,\hat{s}[t,x]}) dr \right]$$
**Proof** for

\[ J(t, x; \hat{s}[t, x]) := \mathbb{E}\left[ g(\bar{X}^{t,x,\hat{s}[t,x]} - \int_t^T \frac{1}{2} \gamma(\hat{s}[t, x]_r - \sigma)^2 dr \right], \]

the first order condition implies (for all \( \delta \) adapted bounded):

\[ 0 = \mathbb{E}\left[ \int_t^T \left( \int_t^T \delta_s \, dW_s \right) \lambda^0_g (dr; \bar{X}^{t,x,\hat{s}[t,x]}) - \int_t^T \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r)(\bar{X}^{t,x,\hat{s}[t,x]}_r) dr \right] \]

\[ = \mathbb{E}\left[ \int_t^T \lambda^0_g (dr; \bar{X}^{t,x,\hat{s}[t,x]}) \int_t^T \delta_r dW_r - \int_t^T \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r)(\bar{X}^{t,x,\hat{s}[t,x]}_r) dr \right] \]
Proof for

\[ J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ g(\tilde{X}^{t,x,\hat{s}[t,x]}_T) - \int_t^T \frac{1}{2} \gamma(\hat{s}[t, x]_r - \sigma)^2 dr \right], \]

the first order condition implies (for all \( \delta \) adapted bounded):

\[
0 = \mathbb{E} \left[ \int_t^T \left( \int_t^r \delta_s dW_s \right) \lambda_g^\circ(dr; \tilde{X}^{t,x,\hat{s}[t,x]}) - \int_t^T \delta_r \gamma_r(\hat{s}[t, x]_r - \sigma_r)(\tilde{X}^{t,x,\hat{s}[t,x]}) dr \right]
\]

\[
= \mathbb{E} \left[ \int_t^T \lambda_g^\circ(dr; \tilde{X}^{t,x,\hat{s}[t,x]}) \int_t^T \delta_r dW_r - \int_t^T \delta_r \gamma_r(\hat{s}[t, x]_r - \sigma_r)(\tilde{X}^{t,x,\hat{s}[t,x]}) dr \right]
\]

Set \( \int_t^T \lambda_g^\circ(dr; \tilde{X}^{t,x,\hat{s}[t,x]} = m + \int_t^T \hat{a}[t, x]_r dW_r. \)
Proof for

\[ J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ g(\tilde{X}^{t, x, \hat{s}[t, x]} - \int_t^T \frac{1}{2} \gamma(\hat{s}[t, x]_r - \sigma)^2 dr \right], \]

the first order condition implies (for all \( \delta \) adapted bounded):

\[ 0 = \mathbb{E} \left[ \int_t^T \left( \int_t^r \delta_s dW_s \right) \lambda^\circ_g (dr; \tilde{X}^{t, x, \hat{s}[t, x]}) \right. \]

\[ - \int_t^T \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r) (\tilde{X}^{t, x, \hat{s}[t, x]} dr] \]

\[ = \mathbb{E} \left[ \int_t^T \hat{a}[t, x]_r \delta_r dr \right. \]

\[ - \int_t^T \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r) (\tilde{X}^{t, x, \hat{s}[t, x]} dr] \]

Set \( \int_t^T \lambda^\circ_g (dr; \tilde{X}^{t, x, \hat{s}[t, x]}) = m + \int_t^T \hat{a}[t, x]_r dW_r. \)
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\[ J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ g(\bar{X}^{t, x, \hat{s}[t, x]}) - \int_t^T \frac{1}{2} \gamma(\hat{s}[t, x]_r - \sigma)^2 dr \right], \]

the first order condition implies (for all \( \delta \) adapted bounded):

\[ 0 = \mathbb{E} \left[ \int_t^T \left( \int_t^r \delta_s dW_s \right) \lambda^\circ_g (dr; \bar{X}^{t, x, \hat{s}[t, x]}) \right. \]
\[ \quad - \int_t^T \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r)(\bar{X}^{t, x, \hat{s}[t, x]} dr] \]
\[ = \mathbb{E} \left[ \int_t^T \hat{a}[t, x]_r \delta_r dr \right. \]
\[ \quad - \int_t^T \delta_r \gamma_r (\hat{s}[t, x]_r - \sigma_r)(\bar{X}^{t, x, \hat{s}[t, x]} dr] \]

Set \( \int_t^T \lambda^\circ_g (dr; \bar{X}^{t, x, \hat{s}[t, x]}) = m + \int_t^T \hat{a}[t, x]_r dW_r. \)
Result #2: By a simple calculus of variations argument,

\[ \gamma (\hat{s}[t, x] - \sigma) (\bar{X}^{t, x, \hat{s}[t, x]} ) = \hat{a}[t, x] \]

where \((m[t, x], \hat{a}[t, x])\) is the element of \(\mathbb{R} \times A_2\) such that

\[ m[t, x] + \int_t^T \hat{a}[t, x] dW_u = \hat{B}[t, x]_T - \hat{B}[t, x]_t. \]
Result #2: By a simple calculus of variations argument,

$$\gamma(\hat{s}[t, x] - \sigma)(\bar{X}^{t,x,\hat{s}[t,x]}) = \hat{a}[t, x]$$

where \((m[t, x], \hat{a}[t, x])\) is the element of \(\mathbb{R} \times A_2\) such that

$$m[t, x] + \int_t^T \hat{a}[t, x]_u d\mathcal{W}_u = \hat{B}[t, x]_T - \hat{B}[t, x]_t.$$ 

Since, \(\nabla_x J(\cdot, \bar{X}^{t,x,\hat{s}[t,x]; \hat{s}[t, x]}) := \mathbb{E} \left[ \hat{B}[t, x]_T - \hat{B}[t, x]. | \mathcal{F} \right]\),
**Result #2**: By a simple calculus of variations argument,

\[ \gamma(\hat{s}[t, x] - \sigma)(\tilde{X}^{t, x, \hat{s}[t, x]}) = \hat{a}[t, x] \]

where \((m[t, x], \hat{a}[t, x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[ m[t, x] + \int_t^T \hat{a}[t, x]_u dW_u = \hat{B}[t, x]_T - \hat{B}[t, x]_t. \]

Since,

\[ \nabla_x J(\cdot, \tilde{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]) := \mathbb{E} \left[ \hat{B}[t, x]_T - \hat{B}[t, x]. | \mathcal{F} \right], \]

\[ \hat{Y}[t, x] := m[t, x] + \int_t^T \hat{a}[t, x]_u dW_u - (\hat{B}[t, x] - \hat{B}[t, x]_t) \]

satisfies

\[ \hat{Y}[t, x] = \mathbb{E} \left[ \hat{B}[t, x]_T - \hat{B}[t, x]. | \mathcal{F} \right] - (\hat{B}[t, x] - \hat{B}[t, x]_t) \]

\[ = \nabla_x J(\cdot, \tilde{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]). \]
Concavity of the value function

**Result #3**: Set

\[
\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x \wedge t + 1_{\{t\}}(y^2 - x_t))dy^2 dy^1,
\]

then \( y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y) \) is concave (\( \bar{v} - \Gamma \) is Dupire concave).
Concavity of the value function

Result #3: Set

\[ \Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x \land y + 1_{\{t\}}(y^2 - x_t))dy^2 dy_1, \]

then \( y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y) \) is concave (\( \bar{v} - \Gamma \) is Dupire concave).

Cf constant coefficients + Markov:

\[ \bar{v}(t, x) = \sup_s \mathbb{E}[\bar{v}(t + h, \bar{X}^{t, x, s}) - \int_t^{t+h} \frac{\gamma}{2} (s_r - \sigma)^2 dr] \]

implies

\[
\bar{v}(t, x) - \frac{\gamma}{2} x_t^2
\]

\[ = \sup_s \mathbb{E}[\bar{v}(t + h, \bar{X}^{t, x, s}) - \frac{\gamma}{2} (\bar{X}_{t+h}^{t, x, s})^2 - \int_t^{t+h} \gamma(-s_r \sigma + \frac{1}{2} |\sigma|^2) dr]. \]
Proof in a simpler situation: Assume that, for all $s$, $h > 0$,

$$
\varphi(t, x) \geq \mathbb{E}[\varphi(t + h, \tilde{X}_{t+h}^{t,x,s})],
$$

where

$$
\tilde{X}_{t+h}^{t,x,s} = x + \int_s^{t+h} s_s dW_s.
$$
\[ \text{Proof} \] in a simpler situation: Assume that, for all \( s, \ h > 0, \)

\[ \varphi(t, x) \geq \mathbb{E}[\varphi(t + h, \tilde{X}_{t+h}^{t,x,s})], \]

where

\[ \tilde{X}_{t+h}^{t,x,s} = x + \int_{t}^{t+h} s_s dW_s. \]

Take \( x = \lambda x^1 + (1 - \lambda)x^2 \) and \( s \) s.t.

\[ \mathbb{P}[\tilde{X}_{t+h}^{t,x,s} = x^1] = \lambda = 1 - \mathbb{P}[\tilde{X}_{t+h}^{t,x,s} = x^2]. \]
□ Proof in a simpler situation: Assume that, for all $s$, $h > 0$,

$$
\varphi(t, x) \geq \mathbb{E}[\varphi(t + h, \tilde{X}_{t+h}^{t,x,s})],
$$

where

$$
\tilde{X}_{t+h}^{t,x,s} = x + \int_t^{t+h} s_s dW_s.
$$

Take $x = \lambda x^1 + (1 - \lambda)x^2$ and $s$ s.t.

$$
\mathbb{P}[\tilde{X}_{t+h}^{t,x,s} = x^1] = \lambda = 1 - \mathbb{P}[\tilde{X}_{t+h}^{t,x,s} = x^2].
$$

Then,

$$
\varphi(t, x) \geq \lambda \varphi(t + h, x^1) + (1 - \lambda) \varphi(t + h, x^2),
$$

\[ \text{Proof} \] in a simpler situation: Assume that, for all \( s, h > 0, \)

\[ \varphi(t, x) \geq \mathbb{E}[\varphi(t + h, \bar{X}_{t+h}^{t,x,s})], \]

where

\[ \bar{X}_{t+h}^{t,x,s} = x + \int_{t}^{t+h} s_s dW_s. \]

Take \( x = \lambda x^1 + (1 - \lambda) x^2 \) and \( s \) s.t.

\[ \mathbb{P}[\bar{X}_{t+h}^{t,x,s} = x^1] = \lambda = 1 - \mathbb{P}[\bar{X}_{t+h}^{t,x,s} = x^2]. \]

Then,

\[ \varphi(t, x) \geq \lambda \varphi(t + h, x^1) + (1 - \lambda) \varphi(t + h, x^2), \]

and let \( h \to 0 : \)

\[ \varphi(t, x) \geq \lambda \varphi(t, x^1) + (1 - \lambda) \varphi(t, x^2), \]
□ Proof in a simpler situation: Assume that, for all $s$, $h > 0$,

$$\varphi(t, x) \geq \mathbb{E}[\varphi(t + h, \tilde{X}_{t+h}^{t,x,s})],$$

where

$$\tilde{X}_{t+h}^{t,x,s} = x + \int_{t}^{t+h} s_sdW_s.$$ 

Take $x = \lambda x^1 + (1 - \lambda)x^2$ and $s$ s.t.

$$\mathbb{P}[\tilde{X}_{t+h}^{t,x,s} = x^1] = \lambda = 1 - \mathbb{P}[\tilde{X}_{t+h}^{t,x,s} = x^2].$$

Then,

$$\varphi(t, x) \geq \lambda \varphi(t + h, x^1) + (1 - \lambda)\varphi(t + h, x^2),$$

and let $h \to 0$ :

$$\varphi(t, x) \geq \lambda \varphi(t, x^1) + (1 - \lambda)\varphi(t, x^2),$$

$\Rightarrow \varphi$ is concave.
Differentiability of the value function

Result #4: \( \bar{v} \) admits a continuous vertical Dupire derivative given by

\[
\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) = \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right] \quad (= \hat{Y}[t, x]_t) \]

\]
Differentiability of the value function

**Result #4**: $\tilde{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \tilde{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) = \mathbb{E} \left[ \hat{B}[t, x]_T - \hat{B}[t, x]_t \right] (= \hat{Y}[t, x]_t)$$

because $(t, x)$ maximizes $(t', x') \mapsto \tilde{v}(t', x') - J(t', x'; \hat{s}[t, x])$, i.e. $0 \in \partial_y (v(t, x \oplus_t y) - J(t, x \oplus_t y; \hat{s}[t, x])) = \partial_y v(t, x \oplus_t y) - \nabla_x J(t, x; \hat{s}[t, x]).$
Differentiability of the value function

**Result #4**: \( \bar{\nu} \) admits a continuous vertical Dupire derivative given by

\[
\nabla_x \bar{\nu}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) = \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right] (= \hat{Y}[t, x]_t)
\]

because \((t, x)\) maximizes \((t', x') \mapsto \bar{\nu}(t', x') - J(t', x'; \hat{s}[t, x])\), i.e. \(0 \in \partial_y(\nu(t, x \oplus_t y) - J(t, x \oplus_t y; \hat{s}[t, x])) = \partial_y \nu(t, x \oplus_t y) - \nabla_x J(t, x; \hat{s}[t, x])\).

And (Meyer-Tanaka + martingale property - just need \(C_r^{0,1}\))

\[
\bar{\nu}(t', \bar{X}^{t, x, \hat{s}[t, x]} = \bar{\nu}(t, x) + \int_t^{t'} \nabla_x \bar{\nu}(r, \bar{X}^{t, x, \hat{s}[t, x]}) d\bar{X}^{t, x, \hat{s}[t, x]}
\]

\[
+ \int_t^{t'} \frac{1}{2} \gamma_r(\bar{X}^{t, x, \hat{s}[t, x]})(s_r - \sigma_r(\bar{X}^{t, x, \hat{s}[t, x]}))^2 dr.
\]
More generally

Let $Z$ be a $(\mathbb{F}, \mathbb{P})$-continuous semi-martingale such that $\mathbb{E}^{\mathbb{P}}[\|Z\|^2] < \infty$. Let $\phi$ be a non-anticipative map in $C^0_{r,1}$. Assume that there exists $R \in C^1_{r,2}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1_{\{t\}}y)$ is concave for all $t$),
2. $\phi - \ell$ is non-increasing in time ($(\phi - \ell)(t + h, x_{\wedge t}) \leq (\phi - \ell)(t, x_{\wedge t})$).

Moreover, if $Z$ and $\phi \cdot (Z)$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process $B$, then $\phi \cdot (Z) = \phi_0(Z_0) + \int \nabla x \phi_t(Z) \, dZ_t + B$, on $[0, T]$. Compare with Cont and Fournier (2013), Saporito (2017) for the Functional Itô-Meyer-Tanaka, Russo and Vallois (1996), and Gozzi and Russo (2006) for $C^1$ functionals of semi-martingales.
More generally
Let $Z$ be a $(\mathbb{F}, \mathbb{P})$-continuous semi-martingale such that $\mathbb{E}^\mathbb{P}[\|Z\|^2] < \infty$. Let $\phi$ be a non-anticipative map in $C^{0,1}_r$. Assume that there exists $R \in C^{1,2}_r$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1_{\{t\}}y)$ is concave for all $t$),
2. $\phi - \ell$ is non-increasing in time ($((\phi - \ell)(t + h, x \wedge t) \leq (\phi - \ell)(t, x \wedge t)$).

Then, there exists a non-increasing predictable process $A$ starting at $0$ such that:

$$
\phi(Z) - \int_0^1 \frac{1}{2} \nabla^2_x R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^1 \nabla_x \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).
$$

Moreover, if $Z$ and $\phi(Z)$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process $B$, then $\phi(Z) = \phi_0(Z_0) + \int_0^1 \nabla_x \phi_t(Z) dZ_t + B$, on $[0, T]$.

More generally

Let $Z$ be a $(\mathbb{F}, \mathbb{P})$-continuous semi-martingale such that $\mathbb{E}^\mathbb{P}[\|Z\|^2] < \infty$. Let $\phi$ be a non-anticipative map in $C^0_{\mathbb{F}, 1}$. Assume that there exists $R \in C^1_{\mathbb{F}, 2}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1_{\{t\}}y)$ is concave for all $t$),
2. $\phi - \ell$ is non-increasing in time ($((\phi - \ell)(t + h, x_{\land t}) \leq (\phi - \ell)(t, x_{\land t})$).

Then, there exists a non-increasing predictable process $A$ starting at 0 such that

$$
\phi(Z) - \int_0^\cdot \frac{1}{2} \nabla_x^2 R_r(Z) d\langle Z \rangle_r = \phi_0(Z) + \int_0^\cdot \nabla_x \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).
$$

Moreover, if $Z$ and $\phi(Z) - B$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process $B$, then

$$
\phi(Z) = \phi_0(Z_0) + \int_0^\cdot \nabla_x \phi_t(Z) dZ_t + B, \text{ on } [0, T].
$$

More generally

Let $Z$ be a $(\mathbb{F}, \mathbb{P})$-continuous semi-martingale such that $\mathbb{E}_{}^\mathbb{P}[\|Z\|^2] < \infty$. Let $\phi$ be a non-anticipating map in $C^{0,1}_{\mathbb{F}}$. Assume that there exists $R \in C^{1,2}_{\mathbb{F}}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that:

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, x + 1\{t\}y)$ is concave for all $t$),
2. $\phi - \ell$ is non-increasing in time ($((\phi - \ell)(t + h, x_{\wedge} t) \leq (\phi - \ell)(t, x_{\wedge} t)$).

Then, there exists a non-increasing predictable process $A$ starting at 0 such that

$$\phi(Z) - \int_0^\cdot \frac{1}{2} \nabla^2_x R_r(Z)d\langle Z \rangle_r = \phi_0(Z) + \int_0^\cdot \nabla_x \phi_r(Z)dZ_r + A + \ell(\cdot) - \ell(0).$$

Moreover, if $Z$ and $\phi_.(Z) - B$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process $B$, then

$$\phi_.(Z) = \phi_0(Z_0) + \int_0^\cdot \nabla_x \phi_t(Z)dZ_t + B, \text{ on } [0, T].$$

Remark: see also B. Bouchard and X. Tan, A quasi-sure optional decomposition and super-hedging result on the Skorokhod space, arXiv:2004.11105, for the case where $\phi$ is not $C^1$ in space.
In our case: \( \bar{v} - \Gamma \) is Dupire-concave (see above).
In our case: $\tilde{v} - \Gamma$ is Dupire-concave (see above).
Moreover (with bounded coefficients):

$$\tilde{v}(t, x)$$

$$= \sup_{s} \mathbb{E}[\tilde{v}(t + h, \tilde{X}^{t, x, s}) - \int_{t}^{t+h} \frac{1}{2} \gamma_r(\tilde{X}^{t, x, s})(s_r - \sigma_r(\tilde{X}^{t, x, s}))^2 dr]$$

$$\geq \mathbb{E}[\tilde{v}(t + h, x_{\wedge t}) - \int_{t}^{t+h} \frac{1}{2} \gamma_r(x_{\wedge t})|\sigma_r(x_{\wedge t})|^2 dr] \ (s \equiv 0)$$

$$\geq \tilde{v}(t + h, x_{\wedge t}) - Ch.$$
In our case: \( \tilde{v} - \Gamma \) is Dupire-concave (see above).

Moreover (with bounded coefficients):

\[
\tilde{v}(t, x)
= \sup_{\bar{s}} \mathbb{E}[\tilde{v}(t + h, X^{t, x, \bar{s}})] - \int_t^{t+h} \frac{1}{2} \gamma_r(\bar{X}^{t, x, \bar{s}})(\bar{s}_r - \sigma_r(\bar{X}^{t, x, \bar{s}}))^2 dr
\]

\[
\geq \mathbb{E}[\tilde{v}(t + h, x_{\land t})] - \int_t^{t+h} \frac{1}{2} \gamma_r(x_{\land t})|\sigma_r(x_{\land t})|^2 dr \quad (\bar{s} \equiv 0)
\]

\[
\geq \tilde{v}(t + h, x_{\land t}) - Ch.
\]

\( \Rightarrow \) non-increasing in time up to \( t \mapsto \ell(t) = Ct. \)
\[ \bar{v} - \Gamma \text{ is Dupire-concave (see above).} \]

Moreover (with bounded coefficients):

\[
\bar{v}(t, x) = \sup_{\bar{s}} \mathbb{E}[\bar{v}(t, x, \bar{X}^{t, x, \bar{s}}) - \int_t^{t+h} \frac{1}{2} \gamma_r(\bar{X}^{t, x, \bar{s}})(\bar{s}_r - \sigma_r(\bar{X}^{t, x, \bar{s}}))^2 dr]
\]

\[
\geq \mathbb{E}[\bar{v}(t + h, x_{\wedge t}) - \int_t^{t+h} \frac{1}{2} \gamma_r(x_{\wedge t})|\sigma_r(x_{\wedge t})|^2 dr] \quad (\bar{s} \equiv 0)
\]

\[
\geq \bar{v}(t + h, x_{\wedge t}) - Ch.
\]

⇒ non-increasing in time up to \( t \mapsto \ell(t) = Ct. \)

Finally, the DPP

\[
\bar{v}(t, x) = \sup_{\bar{s}} \mathbb{E}[\bar{v}(t, x, \bar{X}^{t, x, \bar{s}}) - \int_t^{t+h} \frac{1}{2} \gamma_r(\bar{X}^{t, x, \bar{s}})(\bar{s}_r - \sigma_r(\bar{X}^{t, x, \bar{s}}))^2 dr]
\]

implies that

\[
\left( \bar{v}(s, \bar{X}^{t, x, \hat{s}[t, x]}) - \int_t^s \frac{1}{2} \gamma_r(\bar{X}^{t, x, \hat{s}[t, x]})(\hat{s}[t, x]_r - \sigma_r(\bar{X}^{t, x, \hat{s}[t, x]}))^2 dr \right)_{s \geq t}
\]

is a martingale.
Proof for $\phi$ Dupire-concave (i.e. $y \mapsto \phi(t, x + 1_{\{t\}}y)$ is concave for all $t$) and non-increasing in time.
Proof for $\phi$ Dupire-concave (i.e. $y \mapsto \phi(t, x + 1\{t\}y)$ is concave for all $t$) and non-increasing in time.

Fix $t^n_i = ih^n$ and set $Z^n := \sum_i Z^n_{t^n_i}1_{[t^n_i, t^n_{i+1})}$.
Proof for $\phi$ Dupire-concave (i.e. $y \mapsto \phi(t, x + 1_{\{t\}}y)$ is concave for all $t$) and non-increasing in time.

Fix $t^n_i = ih^n$ and set $Z^n := \sum_i Z^n_i 1_{[t^n_i, t^n_{i+1})}$. Then,

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n) = \phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_{i+1}}(Z^n_{\land t^n_i}) + \phi_{t^n_{i+1}}(Z^n_{\land t^n_i}) - \phi_{t^n_i}(Z^n).
$$
Proof for $\phi$ Dupire-concave (i.e. $y \mapsto \phi(t, x + 1_{\{t\}}y)$ is concave for all $t$) and non-increasing in time.

Fix $t^n_i = ih^n$ and set $Z^n := \sum_i Z^n_{t^n_i} 1_{[t^n_i, t^n_{i+1})}$. Then,

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n) = \phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_{i+1}}(Z^n_{t^n_i}) + \phi_{t^n_{i+1}}(Z^n_{t^n_i}) - \phi_{t^n_i}(Z^n).
$$

By Meyer-Tanaka formula:

$$
\exists K^n_{t^n_i} \text{non-increasing s.t.} \\
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n_{t^n_i}) = \int_{t^n_i}^{t^n_{i+1}} \nabla_x \phi_{t^n_{i+1}}(Z^n_{t^n_i}) + K^n_{t^n_i} - K^n_{t^n_i} \leq 0.
$$
Proof for $\phi$ Dupire-concave (i.e. $y \mapsto \phi(t, x + 1_{\{t\}}y)$ is concave for all $t$) and non-increasing in time.

Fix $t^n_i = ih^n$ and set $Z^n := \sum_i Z^n_{t^n_i}1_{[t^n_i, t^n_{i+1})}$. Then,

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n) = \phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_{i+1}}(Z^n_{\wedge t^n_i}) + \phi_{t^n_{i+1}}(Z^n_{\wedge t^n_i}) - \phi_{t^n_i}(Z^n) \leq 0
$$

By Meyer-Tanaka formula: $\exists K^n$ non-increasing s.t.

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_{i+1}}(Z^n_{\wedge t^n_i}) = \int_{t^n_i}^{t^n_{i+1}} \nabla_x \phi_{t^n_{i+1}}(Z^n_{\wedge t^n_i} \oplus t^n_{i+1} (Z_r - Z^n_{t^n_i})) dZ_r + K^n_{t^n_{i+1}} - K^n_{t^n_i}
$$
Proof for $\phi$ Dupire-concave (i.e. $y \mapsto \phi(t, x + 1_{\{t\}}y)$ is concave for all $t$) and non-increasing in time.

Fix $t^n_i = ih^n$ and set $Z^n := \sum_i Z^n_{t^n_i}1_{t^n_i, t^n_{i+1}}$. Then,

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n) = \phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_{i+1}}(Z^n_{t^n_i}) + \phi_{t^n_{i+1}}(Z^n_{t^n_i}) - \phi_{t^n_i}(Z^n) \leq 0
$$

By Meyer-Tanaka formula : $\exists K^n$ non-increasing s.t.

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n_{t^n_i}) = \int_{t^n_i}^{t^n_{i+1}} \nabla x \phi_{t^n_{i+1}}(Z^n_{t^n_i} \oplus t^n_{i+1}(Z_r - Z^n_{t^n_i}))dZ_r + K^n_{t^n_{i+1}} - K^n_{t^n_i}
$$

Hence,

$$
\phi_{t^n_{i+1}}(Z^n) - \phi_{t^n_i}(Z^n) = \int_{t^n_i}^{t^n_{i+1}} \nabla x \phi_{t^n_{i+1}}(Z^n_{t^n_i} \oplus t^n_{i+1}(Z_r - Z^n_{t^n_i}))dZ_r + \tilde{K}^n_{t^n_{i+1}} - \tilde{K}^n_{t^n_i} \leq 0
$$
Construction of the hedging strategy

**Result #4**: $\tilde{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \tilde{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right] = \hat{Y}[t, x]_t.$$

And (Meyer-Tanaka + martingale property - just need $C^{0,1}$)

$$\tilde{v}(t', \tilde{X}^{t, x, \hat{s}[t, x]}) = \tilde{v}(t, x) + \int_t^{t'} \nabla_x \tilde{v}(r, \tilde{X}^{t, x, \hat{s}[t, x]})d\tilde{X}^{t, x, \hat{s}[t, x]}_r$$

$$+ \int_t^{t'} \frac{1}{2} \gamma_r(\tilde{X}^{t, x, \hat{s}[t, x]})(s_r - \sigma_r(\tilde{X}^{t, x, \hat{s}[t, x]}))^2 dr.$$
Construction of the hedging strategy

**Result #4**: $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := E\left[\hat{B}[t, x]_T - \hat{B}[t, x]_t\right] = \hat{Y}[t, x]_t.$$

And (Meyer-Tanaka + martingale property - just need $C^{0,1}$)

$$\bar{v}(t', \tilde{X}^{t, x, \hat{s}[t, x]}) = \bar{v}(t, x) + \int_t^{t'} \hat{Y}[t, x]_r d\tilde{X}_r^{t, x, \hat{s}[t, x]}$$

$$+ \int_t^{t'} \frac{1}{2} \gamma_r(\tilde{X}^{t, x, \hat{s}[t, x]})(s_r - \sigma_r(\tilde{X}^{t, x, \hat{s}[t, x]}))^2 dr.$$  

where

$$\hat{Y}[t, x] = m[t, x] + \int_t^\cdot \hat{a}[t, x]_u d\mathcal{W}_u - (\hat{B}[t, x] - \hat{B}[t, x]_t).$$
Recall that $\tilde{v}(T, \cdot) = g$
Recall that $\bar{v}(T, \cdot) = g$ and

$$g(\bar{X}^x, \hat{s}[x]) = \bar{v}(T, \bar{X}^x, \hat{s}[x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}^r[x]$$

$$+ \int_0^T \frac{1}{2} \gamma_r(\bar{X}^x, \hat{s}[x])(s_r - \sigma_r(\bar{X}^x, \hat{s}[x]))^2 dr,$$

$$\hat{Y}[x] = m[x] + \int_0^T \hat{a}[x]_r d\mathcal{W}_r - \hat{\mathcal{B}}[x].$$
Recall that $\bar{v}(T, \cdot) = g$ and
\[
g(\bar{X}^{x, \hat{s}[x]} ) = \bar{v}(T, \bar{X}^{x, \hat{s}[x]} ) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^{x, \hat{s}[x]}_r \\
+ \int_0^T \frac{1}{2} \gamma_r(\bar{X}^{x, \hat{s}[x]})(s_r - \sigma_r(\bar{X}^{x, \hat{s}[x]}))^2 dr,
\]
\[
\hat{Y}[x] = m[x] + \int_0^\cdot \hat{a}[x]_r dW_r - \hat{B}[x].
\]
Recall that $\hat{s}[x] = \sigma(\bar{X}^{x, \hat{s}[x]} ) + \hat{a}[x] f(\bar{X}^{x, \hat{s}[x]} )$
Recall that $\bar{v}(T, \cdot) = g$ and

$$g(\bar{X}^{x,\hat{s}[x]}_T) = \bar{v}(T, \bar{X}^{x,\hat{s}[x]}_T) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^{x,\hat{s}[x]}_r$$

$$+ \int_0^T \frac{1}{2} \gamma_r(\bar{X}^{x,\hat{s}[x]}_r)(s_r - \sigma_r(\bar{X}^{x,\hat{s}[x]}_r))^2 dr,$$

$$\hat{Y}[x] = m[x] + \int_0^T \hat{a}[x]_r dW_r - \hat{B}[x].$$

Recall that $\hat{s}[x] = \sigma(\bar{X}^{x,\hat{s}[x]}_x) + \hat{a}[x] f(\bar{X}^{x,\hat{s}[x]}_x)$ so that

$$\bar{X}^{x,\hat{s}[x]}_T = x_0 + \int_0^T \hat{s}[x]_r dW_r = x_0 + \int_0^T (\sigma_r(\bar{X}^{x,\hat{s}[x]}_x) + \hat{a}[x]_r f_r(\bar{X}^{x,\hat{s}[x]}_x)) dW_r.$$
Recall that $\bar{v}(T, \cdot) = g$ and

$$
\begin{align*}
g(\tilde{X}^{x, \hat{\sigma}[x]}(T, \tilde{X}^{x, \hat{\sigma}[x]}) &= \bar{v}(0, x) + \int_{0}^{T} \hat{Y}[x]_r d\tilde{X}^{x, \hat{\sigma}[x]}_r \\
+ \int_{0}^{T} \frac{1}{2} \gamma_r(\tilde{X}^{x, \hat{\sigma}[x]})(s_r - \sigma_r(\tilde{X}^{x, \hat{\sigma}[x]}))^2 dr,
\end{align*}
$$

$$
\hat{Y}[x] = m[x] + \int_{0}^{T} \hat{a}[x]_r d\mathcal{W}_r - \hat{\mathcal{B}}[x].
$$

Recall that $\hat{\sigma}[x] = \sigma(\tilde{X}^{x, \hat{\sigma}[x]}(T, \tilde{X}^{x, \hat{\sigma}[x]}) + \hat{a}[x]f(\tilde{X}^{x, \hat{\sigma}[x]})$ so that

$$
\tilde{X}^{x, \hat{\sigma}[x]} = x \wedge 0 + \int_{0}^{T} \hat{\sigma}[x]_r d\mathcal{W}_r = X^{x, \hat{a}[x], \hat{\mathcal{B}}[x]}. 
$$
Recall that $\bar{v}(T, \cdot) = g$ and

$$\begin{align*}
g(X^x, \hat{a}[x], \hat{B}[x]) &= \bar{v}(T, \bar{X}^x, \hat{s}[x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r dX^x_r, \\
&\quad + \int_0^T \frac{1}{2} \gamma_r(\bar{X}^x, \hat{s}[x])(s_r - \sigma_r(\bar{X}^x, \hat{s}[x]))^2 dr, \\
\hat{Y}[x] &= m[x] + \int_0^\cdot \hat{a}[x]_r dW_r - \hat{B}[x].
\end{align*}$$

Recall that $\hat{s}[x] = \sigma(\bar{X}^x, \hat{s}[x]) + \hat{a}[x] f(\bar{X}^x, \hat{s}[x])$ so that

$$\bar{X}^x, \hat{s}[x] = x^0 + \int_0^\cdot \hat{s}[x]_r dW_r = X^x, \hat{a}[x], \hat{B}[x].$$
Recall that $\bar{v}(T, \cdot) = g$ and

$$g(X^x, \hat{a}[x], \hat{B}[x]) = \bar{v}(T, \bar{X}^x, \hat{s}[x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r dX^x_r, \hat{a}[x], \hat{B}[x]$$

$$+ \int_0^T \frac{1}{2} \gamma_r(\bar{X}^x, \hat{s}[x])(\hat{s}_r - \sigma_r(\bar{X}^x, \hat{s}[x]))^2 dr,$$

$$\hat{Y}[x] = m[x] + \int_0^x \hat{a}[x]_r dW_r - \hat{B}[x].$$

Recall that $\hat{s}[x] = \sigma(\bar{X}^x, \hat{s}[x]) + \hat{a}[x] f(\bar{X}^x, \hat{s}[x])$ so that

$$\bar{X}^x, \hat{s}[x] = x \wedge 0 + \int_0^x \hat{s}[x]_r dW_r = X^x, \hat{a}[x], \hat{B}[x].$$

Moreover,

$$\hat{s}[x] - \sigma(\bar{X}^x, \hat{s}[x]) = \hat{a}[x] f(\bar{X}^x, \hat{s}[x]) = \hat{a}[x] f(X^x, \hat{a}[x], \hat{B}[x]).$$
Recall that $\bar{v}(T, \cdot) = g$ and

$$g(X^x, \hat{a}[x], \hat{B}[x]) = \bar{v}(T, \bar{X}^x, \hat{\sigma}[x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r dX^x_r, \hat{a}[x], \hat{B}[x]$$

$$+ \int_0^T \frac{1}{2} f_r(X^x, \hat{a}[x], \hat{B}[x])|\hat{a}[x]|^2 dr,$$

$$\hat{Y}[x] = m[x] + \int_0^\cdot \hat{a}[x]_r dW_r - \hat{B}[x].$$

Recall that $\hat{s}[x] = \sigma(\bar{X}^x, \hat{\sigma}[x]) + \hat{a}[x] f(\bar{X}^x, \hat{\sigma}[x])$ so that

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$$+ \int_0^T \frac{1}{2} f_r(X^x, \hat{a}[x], \hat{B}[x]) |\hat{a}[x]_r|^2 dr,$$

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$$\bar{X}^x, \hat{s}[x] = x \wedge 0 + \int_0^\cdot \hat{s}[x]_r dW_r = X^x, \hat{a}[x], \hat{B}[x].$$

Moreover,

$$\hat{s}[x] - \sigma(\bar{X}^x, \hat{s}[x]) = \hat{a}[x] f(\bar{X}^x, \hat{s}[x]) = \hat{a}[x] f(X^x, \hat{a}[x], \hat{B}[x]).$$

$\Rightarrow \hat{s}[x]$ provides $(\hat{a}[x], -\hat{B}[x])$ which is the hedging strategy starting from $V_0 = \bar{v}(0, x)$ and $Y_0 = \nabla_x \bar{v}(0, x)$.  

Absolute continuity of $\hat{\mathcal{B}}[x]$?

- Example of the constant coefficients case:

$$\hat{\mathcal{B}}[x] = \int_0^\cdot \lambda^\circ_g (dr; \bar{X}^{x,\hat{s}[x]}).$$
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$$\hat{\mathcal{B}}[x] = \int_0^\cdot \lambda_g^\circ (dr; X^x, \hat{s}[x]).$$

Assume that $g(x) = \tilde{g} \left( \int_0^T x_r \rho_r dr \right)$ for some process $\rho$ and $g$ differentiable.
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Assume that $g(x) = \tilde{g} \left( \int_0^T x_r \rho_r dr \right)$ for some process $\rho$ and $g$ differentiable. Then,

$$\lambda_g(ds; \tilde{X}^x,\hat{s}[x]) = \tilde{g}' \left( \int_0^T \tilde{X}^x,\hat{s}[x] \rho_r dr \right) \rho_s ds.$$
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and

$$\lambda^\circ_g(ds; \bar{X}^{x,\hat{s}[x]}) = \mathbb{E}\left[\tilde{g}'\left(\int_0^T \bar{X}^{x,\hat{s}[x]} \rho_r dr\right) \rho_s |\mathcal{F}_s\right] ds.$$
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and

$$\lambda^\circ_g(ds; \bar{X}_x,\hat{s}[x]) = \mathbb{E} \left[ \tilde{g}' \left( \int_0^T \bar{X}_r,\hat{s}[x] \rho_r dr \right) \rho_s |\mathcal{F}_s \right] ds.$$

In particular, $\hat{\mathcal{B}}[x]$ is absolutely continuous.
Sufficient conditions for existence I: strong existence

- From now, we assume for simplicity that all coefficients are bounded.
Sufficient conditions for existence I: strong existence

- From now, we assume for simplicity that all coefficients are bounded.

- The problem is:

\[
\bar{v}(0, x) = \sup_{s} \mathbb{E}\left[ g(\bar{X}^{x, s}) - \int_{0}^{T} \frac{1}{2} \gamma_r(\bar{X}^{x, s})(s_r - \sigma_r(\bar{X}^{x, s}))^2 dr \right]
\]
Sufficient conditions for existence I: strong existence

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which implies that, for some $C > 0$, one can restrict to controls so that

$$\mathbb{E}[\int_0^T s_r^2 dr] \leq C.$$
Sufficient conditions for existence I : strong existence

- From now, we assume for simplicity that all coefficients are bounded.

- The problem is:

\[
\bar{v}(0, x) = \sup_{s} \mathbb{E}[g(\tilde{X}^{x,s}) - \int_{0}^{T} \frac{1}{2} \gamma_{r}(\tilde{X}^{x,s})(s_{r} - \sigma_{r}(\tilde{X}^{x,s}))^{2} dr]
\]

which implies that, for some \( C > 0 \), one can restrict to controls so that

\[
\mathbb{E}[\int_{0}^{T} s_{r}^{2} dr] \leq C.
\]

By Mazur’s Theorem, if \((s^{n})_{n \geq 1}\) is a maximizing sequence then one can find \((\tilde{s}^{n})_{n \geq 1}\) s.t.

\[
\tilde{s}^{n} \in \text{conv}(s^{k}, k \geq n)
\]

that converges in \( L^{2} \) to some \( \hat{s} \).
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Sufficient conditions for existence I: strong existence

☐ From now, we assume for simplicity that all coefficients are bounded.

☐ The problem is:

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which implies that, for some $C > 0$, one can restrict to controls so that

$$\mathbb{E} \left[ \int_0^T s_r^2 dr \right] \leq C.$$

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$$\tilde{s}^n \in \text{conv}(s^k, k \geq n)$$

that converges in $L^2$ to some $\hat{s}$. In particular, $\bar{X}^{x,\tilde{s}^n} \rightarrow \bar{X}^{x,\hat{s}}$.

If $g$ and $(s,x) \mapsto -\gamma_r(x)(s - \sigma_r(x))^2$ are concave, then existence holds.
The problem is:

\[ \bar{v}(0, x) = \sup_s \mathbb{E}[g(\bar{X}^{x,s}) - \int_0^T \frac{1}{2} \gamma_r(\bar{X}^{x,s})(s_r - \sigma_r(\bar{X}^{x,s}))^2 dr] \]
Sufficient conditions for existence II: weak existence

The problem is:

\[ \tilde{v}(0, x) = \sup_{s} \mathbb{E}[g(\tilde{X}^{x,s}) - \int_{0}^{T} \frac{1}{2} \gamma_r(\tilde{X}^{x,s})(s_r - \sigma_r(\tilde{X}^{x,s}))^2 dr] \]

For using typical results ensuring tightness, one would need a penalty of the form

\[ \gamma_r(\tilde{X}^{x,s})(s_r - \sigma_r(\tilde{X}^{x,s}))^{2+\iota} \]

with \( \iota > 0 \)!
Assume that
\[ y \in \mathbb{R} \mapsto (v - \bar{\Gamma}_{\varepsilon_0}(t, x \oplus_t y)) \text{ is concave for all } (t, x) \in [0, T] \times D([0, T]). \]

with
\[ \bar{\Gamma}_{\varepsilon_0}(t, x) := \bar{\Gamma}_0(t, x) - \varepsilon_0 x_t^2, \]

for some \( \varepsilon_0 > 0 \). Cf. Chapter 3 when \( g \) satisfies such a condition in the Markovian setting.
Assume that
\[ y \in \mathbb{R} \mapsto (v - \bar{\Gamma}_{\varepsilon_0})(t, x \oplus_t y) \] is concave for all \((t, x) \in [0, T] \times D([0, T])\).

with
\[ \bar{\Gamma}_{\varepsilon_0}(t, x) := \bar{\Gamma}_0(t, x) - \varepsilon_0 x^2, \]
for some \(\varepsilon_0 > 0\). Cf. Chapter 3 when \(g\) satisfies such a condition in the Markovian setting.

We claim that (for \(s\) a maximizing sequence - encoded into \(\mathbb{P}_n\))
\[
\lim_{\theta \to 0} \delta(\theta) = 0, \quad \text{with} \quad \delta(\theta) := \lim_{n \to \infty} \sup_{\sigma, \tau \in \mathcal{T}, \sigma \leq \tau \leq \sigma + \theta} \mathbb{E}^{\mathbb{P}_n} [||\bar{X}_\tau^{s} - \bar{X}_\sigma^{s}||^2].
\]
Assume that
\[ y \in \mathbb{R} \mapsto (v - \bar{\Gamma}_\varepsilon)(t, x \oplus_t y) \] is concave for all \((t, x) \in [0, T] \times D([0, T])\).

with
\[ \bar{\Gamma}_\varepsilon(t, x) := \bar{\Gamma}_0(t, x) - \varepsilon_0 x_t^2, \]
for some \(\varepsilon_0 > 0\). Cf. Chapter 3 when \(g\) satisfies such a condition in the Markovian setting.

We claim that (for a maximizing sequence - encoded into \(\mathbb{P}_n\))
\[ \lim_{\theta \searrow 0} \delta(\theta) = 0, \quad \text{with} \quad \delta(\theta) := \limsup_{n \to \infty} \sup_{\sigma, \tau \in \mathcal{T}, \sigma \leq \tau \leq \sigma + \theta} \mathbb{E}^{\mathbb{P}_n} \left[ \left| \bar{X}_\tau^s - \bar{X}_\sigma^s \right|^2 \right]. \]

If not, \(\exists \theta_n \to 0\), and \((\sigma_n, \tau_n)_n\) s.t.
\[ 2c := \liminf_{n} \mathbb{E}^{\mathbb{P}_n} \left[ \int_{\sigma_n}^{\tau_n} |\bar{s}_s|^2 \, ds \right] > 0. \]
Set
\[ \phi := v - \bar{\Gamma}_{\varepsilon_0} \quad \text{and} \quad \xi_n := \mathbb{E}_{\sigma_n}^{\mathbb{P}} [\phi(\tau_n, \bar{X}^5) - \phi(\tau_n, (\bar{X}^5 \oplus_{\sigma_n} (\bar{X}^5_{\tau_n} - \bar{X}^5_{\sigma_n})))_{\sigma_n \wedge .}] . \]
Set
\[ \phi := v - \bar{\Gamma}_{\bar{\epsilon}_0} \quad \text{and} \quad \xi_n := \mathbb{E}_{\sigma_n}^{\mathbb{P}_n} \left[ \phi(\tau_n, \tilde{X}^s) - \phi(\tau_n, (\tilde{X}^s \oplus_{\sigma_n} (\tilde{X}^s_{\tau_n} - \tilde{X}^s_{\sigma_n}))_{\sigma_n \wedge}) \right]. \]

Then,
\[
\mathbb{E}_{\sigma_n}^{\mathbb{P}_n} \left[ v(\tau_n, \tilde{X}^s) - \frac{1}{2} \int_{\tau_n}^{\sigma_n} \gamma_s(s, \tilde{X}^s_s) s_s^2 ds \right] \\
= \mathbb{E}_{\sigma_n}^{\mathbb{P}_n} \left[ \phi(\tau_n, (\tilde{X}^s \oplus_{\sigma_n} (\tilde{X}^s_{\tau_n} - \tilde{X}^s_{\sigma_n}))_{\sigma_n \wedge}) \right] - \frac{1}{2} \int_{\tau_n}^{\sigma_n} \bar{\epsilon}_0 s_s^2 ds + \bar{\Gamma}_{\epsilon_0} (\sigma_n, \tilde{X}^s) + \xi_n \\
\leq \phi(\sigma_n, \tilde{X}^s) + C \theta_n - \frac{\bar{\epsilon}_0}{2} \mathbb{E}_{\sigma_n}^{\mathbb{P}_n} \left[ \int_{\sigma_n}^{\tau_n} s_s^2 ds \right] + \bar{\Gamma}_{\epsilon_0} (\sigma_n, \tilde{X}^s) + \xi_n \\
= v(\sigma_n, \tilde{X}^s) + C \theta_n - \frac{\bar{\epsilon}_0}{2} \mathbb{E}_{\sigma_n}^{\mathbb{P}_n} \left[ \int_{\sigma_n}^{\tau_n} s_s^2 ds \right] + \xi_n.
\]
Hence,

\[
\mathbb{E}^{p_n} \left[ v(\tau_n, \tilde{X}^s) - \frac{1}{2} \int_{\sigma_n}^{\tau_n} \gamma_s(s, \tilde{X}^s)(s_s - \sigma_s(\tilde{X}^s))^2 ds \right] \\
\leq \mathbb{E}^{p_n} \left[ v(\sigma_n, \tilde{X}^s) \right] + C(\theta_n)^{\frac{1}{2}} - \varepsilon_0 c + \xi_n.
\]
Hence,

\[
\mathbb{E}^{p^n}[v(\tau_n, \bar{X}^s) - \frac{1}{2} \int_{\sigma_n}^{\tau_n} \gamma_s(s, \bar{X}_s^s)(s_s - \sigma_s(\bar{X}^s))^2 \, ds] \leq \mathbb{E}^{p^n}[v(\sigma_n, \bar{X}^s)] + C(\theta_n)^{\frac{1}{2}} - \varepsilon_0 c + \xi_n.
\]

while the DPP implies that

\[
\lim_{n \to \infty} \mathbb{E}^{p^n}[v(\tau_n, X) - \int_{\sigma_n}^{\tau_n} \gamma_s(s, \bar{X}_s^s)(s_s - \sigma_s(\bar{X}^s))^2 \, ds] = \lim_{n \to \infty} \mathbb{E}^{p^n}[v(\sigma_n, X)].
\]
Hence,

\[
\mathbb{E}^p_n \left[ v(\tau_n, \bar{X}^s) - \frac{1}{2} \int_{\tau_n}^{\tau_n} \gamma_s(s, \bar{X}^s)(s_s - \sigma_s(\bar{X}^s))^2 ds \right]
\leq \mathbb{E}^p_n \left[ v(\sigma_n, \bar{X}^s) \right] + C(\theta_n)^{\frac{1}{2}} - \varepsilon_0 c + \xi_n.
\]

while the DPP implies that

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\]

Contradiction of

\[
2c := \lim \inf_n \mathbb{E}^p_n \left[ \int_{\tau_n}^{\tau_n} |s_s|^2 ds \right] > 0.
\]

\Rightarrow the optimization sequence is tight!
• How to prove by a pure probabilistic approach that

\[ y \in \mathbb{R} \mapsto (v - \bar{\Gamma}_{\varepsilon_0})(t, x \oplus t y) \]

is concave for all \((t, x) \in [0, T] \times D([0, T])\).

with

\[ \bar{\Gamma}_{\varepsilon_0}(t, x) := \bar{\Gamma}_0(t, x) - \varepsilon_0 x_t^2, \]

for some \(\varepsilon_0 > 0\), by using just the properties of the terminal data \(g\)?
Open question

- **Conclusion**: In a fairly general path-dependent setting, solving the dual problem provides one solution to the hedging problem.
Open question

*Conclusion:* In a fairly general path-dependent setting, solving the dual problem provides one solution to the hedging problem.

*Open question:* In the Markovian setting, and under smoothness conditions, the super-hedging price is the only hedging price. How to prove this in the path-dependent case by simply using probabilistic arguments?
General take away message

- One can construct models taking into account market impact and illiquidity costs and still allowing for perfect hedging.
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- Stochastic target technics allows one to derive the associated pde (in the viscosity solution sens).
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General take away message

- One can construct models taking into account market impact and illiquidity costs and still allowing for perfect hedging.

- Stochastic target technics allows one to derive the associated pde (in the viscosity solution sens).

- In this model, covered and un-covered options are of very different nature.

- The question of understanding the non-Markovian case is still quite open!


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Appendix - Itô’s Lemma for $C^{0,1}$ functions.
Given two measurable continuous $X$ and $Y$,

$$[X, Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)ds, \ t \geq 0,$$

whenever this limit is well defined for the uniform convergence in probability on compact sets.

A measurable continuous process $A$ is a weak zero energy process if $[A, N] = 0$ a.s. for all continuous local martingale $N$.

$X$ is a weak Dirichlet process if it admits the decomposition $X = M + A$ in which $M$ is a continuous local martingale and $A$ is a weak zero energy process.
Preliminaries

□ Given two measurable continuous $X$ and $Y$,

$$[X, Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)ds, \quad t \geq 0,$$

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$X$ is a weak Dirichlet process if it admits the decomposition $X = M + A$ in which $M$ is a continuous local martingale and $A$ is a weak zero energy process.
Remark: If $X$ is $Y$-integrable and $Y$ is a semimartingale then

$$\int_0^t X_s dY_s = \lim_{\varepsilon \downarrow 0} \int_0^t X_s \frac{Y_s + \varepsilon - Y_s}{\varepsilon} ds, \quad t \geq 0.$$
Assumptions

Let $X$ be a continuous and adapted weak Dirichlet process, such that $[X]_t < \infty$ a.s. for all $t \geq 0$. 
Assumptions

- Let $X$ be a continuous and adapted weak Dirichlet process, such that $[X]_t < \infty$ a.s. for all $t \geq 0$.

- There exists a measurable family of non-negative measures $(\mu(\cdot; t, x), (t, x) \in [0, T] \times D([0, T]))$ and $\eta > 0, \beta \geq 0$ satisfying

\[
\varphi(t, x) - \varphi(t, x') = O \left( \int_{[0,t]} |x_s - x'_s| \mu(ds; t, x) + \|x_{t\wedge} - x'_{t\wedge}\|^{1+\eta} (1 + \|x\|^{\beta} + \|x'\|^{\beta}) \right)
\]
Assumptions

Let $X$ be a continuous and adapted weak Dirichlet process, such that $[X]_t < \infty$ a.s. for all $t \geq 0$.

There exists a measurable family of non-negative measures $(\mu(\cdot; t, x), (t, x) \in [0, T] \times D([0, T])$ and $\eta > 0, \beta \geq 0$ satisfying

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for $(x, x')$ s.t. $x_t = x'_t$ ($\Rightarrow$ always true in the not path dependent case).
Theorem

Assume that $\varphi$ and $\nabla_x \varphi$ are “uniformly continuous”. Then,

(i) There exists a weak zero energy process $B$ such that
$$\varphi(t, X) = \varphi(0, X) + \int_0^t \nabla_x \varphi(s, X) \, dM_s + B_t \, P - \text{a.s.} \quad \forall t \leq T.$$ 

(ii) If $A$ has bounded variations, then
$$\varphi(t, X) = \varphi(0, X) + \int_0^t \nabla_x \varphi(s, X) \, dX_s + B'_t \, P - \text{a.s.} \quad \forall t \leq T,$$
where $B'_t := B - \int \nabla_x \varphi(s, X) \, dA_s$ is a weak energy process.

(iii) If $X$ and $\varphi(\cdot, X)$ are both martingales, then
(ii) holds with $B' \equiv 0$. 

□
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(ii) If \( A \) has bounded variations, then

\[
\varphi(t, X) = \varphi(0, X) + \int_0^t \nabla_x \varphi(s, X) dX_s + B'_t \quad \mathbb{P} - \text{a.s. } \forall \ t \leq T,
\]

where \( B' := B - \int_0^t \nabla_x \varphi(s, X) dA_s \) is a weak energy process.
Theorem

Assume that $\varphi$ and $\nabla_x \varphi$ are “uniformly continuous”. Then,

(i) There exists a weak zero energy process $B$ such that

$$
\varphi(t, X) = \varphi(0, X) + \int_0^t \nabla_x \varphi(s, X) dM_s + B_t \ P - \text{a.s. } \forall \ t \leq T.
$$

(ii) If $A$ has bounded variations, then

$$
\varphi(t, X) = \varphi(0, X) + \int_0^t \nabla_x \varphi(s, X) dX_s + B'_t \ P - \text{a.s. } \forall \ t \leq T,
$$

where $B' := B - \int_0^t \nabla_x \varphi(s, X) dA_s$ is a weak energy process.

(iii) If $X$ and $\varphi(\cdot, X)$ are both martingales, then (ii) holds with $B' \equiv 0$. 
Idea of the proof:

- We want to show that $\phi(t, X) - \int_0^t \nabla_x \phi(s, X) \, dM_s =: B_t$ is a zero energy process.

- Need to check that $[B, N] = 0$ for all (bounded) continuous martingale $N$, i.e.
  $$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t (B_s + \varepsilon - B_s)(N_s + \varepsilon - N_s) \, ds = 0.$$
Idea of the proof:
We want to show that

\[ \varphi(t, X) - \int_0^t \nabla_x \varphi(s, X) dM_s =: B_t \]

is a zero energy process.
□ Idea of the proof:

We want to show that

$$\varphi(t, X) - \int_0^t \nabla_x \varphi(s, X) dM_s =: B_t$$

is a zero energy process. Need to check that

$$[B, N] = 0$$

for all (bounded) continuous martingale $N$
Idea of the proof:
We want to show that

$$\varphi(t, X) - \int_0^t \nabla_x \varphi(s, X) dM_s =: B_t$$

is a zero energy process. Need to check that

$$[B, N] = 0$$

for all (bounded) continuous martingale $N$, i.e.

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (B_{s+\varepsilon} - B_s)(N_{s+\varepsilon} - N_s) ds = 0.$$
Let \( X \) be a continuous martingale with independent increments. Then,

\[
\Phi(X) = \mathbb{E}[\Phi(X)] + \int_0^T \mathbb{E}[\lambda_X([t, T]; X)|\mathcal{F}_t]dX_t.
\]