# UNIVERSITE DE PARIS IX - DAUPHINE
# U.F.R. MATHEMATIQUES DE LA DECISION


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Année 2000 |__|__|__|__|__|__|__|__|__|__|

## THESE

pour obtenir le garde de

DOCTEUR EN SCIENCES DE L'UNIVERSITÉ PARIS IX

SPÉCIALITÉ MATHÉMATIQUES APPLIQUÉES

(arrêté du 30 mars 1992)

Présentée et soutenue publiquement par


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le 20 décembre 2000

**Sur le thème**

# Contrôle Stochastique Appliqué à la Finance

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**Remerciements**

Je tiens avant tout à remercier très chaleureusement Nizar Touzi et Ivar Ekeland. Nizar Touzi, pour avoir accepté d'encadrer ma thèse, pour sa très grande disponibilité et ses nombreux conseils, mais surtout parce que travailler avec lui a été, depuis le début, un réel plaisir. Ivar Ekeland, pour m'avoir permis de compléter mes études d'économie par des études en mathématiques appliquées et pour m'avoir fait l'honneur de présider ce jury.

Je remercie également Guy Barles, Jaksa Cvitaniĉ et Damien Lamberton pour avoir accepté de rédiger les rapports de cette thèse, ainsi que Nicole El Karoui et Elyes Jouini pour avoir accepté de participer à ce jury.

Toute ma reconnaissance va aussi à Christian Gouriéroux qui m'a permis d'effectuer ma recherche au sein du CREST.

Enfin, merci aux membres du CREST, et tout particulièrement à ceux avec lesquels j'ai eu l'occasion de passer de bons moments en dehors du labo, ainsi qu'à Fanda Traore et Nadine Guedj pour leur grande disponibilité et leur bonne humeur.

# Introduction générale

# Introduction

Une option est un contrat par lequel le vendeur s'engage à délivrer un paiement aléatoire $G$ à une date $T$ contre le versement initial d'une prime $p$ à la date 0.

Dans le contexte du modèle de marché complet de Black et Scholes (1973), le prix de l'option, encore appelé prime, est défini de manière unique par des arguments d'arbitrage.

L'idée sous-jacente est la suivante. Supposons qu'à partir d'une richesse initiale $x$, il soit possible d'adopter une stratégie financière $\nu$ telle que le portefeuille associé $X^{x,\nu}$ ait une valeur en $T$ égale à la valeur liquidative de l'option $G$ avec une probabilité égale à 1. Dans ce cas, l'option vaut tout simplement $x$. En effet, il est facile de vérifier que, si $p \neq x$, on peut construire des arbitrages (c'est-à-dire gagner de l'argent en $T$ presque sûrement avec une mise initiale nulle).

Malheureusement, en général, tous les actifs contingents $G$ ne sont pas réplicables : on ne peut pas toujours trouver une dotation initiale $x$ et une stratégie $\nu$ telles que $X_T^{x,\nu} = G$ avec probabilité 1. C'est le cas notamment lorsque le marché est incomplet ou lorsque l'achat, ou la vente, d'actifs est soumise à des coûts de transaction. Dans ce cas, si on se restreint aux conditions d'absence d'arbitrage, il existe une infinité de prix possibles (voir par exemple Harrison et Kreps 1979, Harrison et Pliska 1980, Delbaen et Schachermayer 1997, Jouini et Kallal 1995a et 1995b).

Il faut donc choisir un critère d'évaluation qui permette de déterminer un prix unique satisfaisant à la fois l'acheteur et le vendeur.

Il existe une large littérature sur ce sujet et cette thèse s'inscrit dans ce cadre.

On peut tout d'abord, étendre l'idée sous-jacente au modèle de Black-Scholes en cherchant la plus petite richesse initiale $x$ telle que l'on puisse trouver une stratégie $\nu$ pour laquelle

$$X_T^{x,\nu} \geq G .$$

Dans ce cas, il est clair que le vendeur ne prend aucun risque en vendant l'option au prix $x$ puisqu'il est capable de gérer un portefeuille, partant de $x$, dont la valeur en $T$ est supérieure à celle de l'option.

Ce critère d'évaluation est appelé *sur-réplication*. Il a été introduit dans le cadre de marchés incomplets sans coût de transaction par El Karoui et Quenez (1995) puis largement étudié, notamment par Kramkov (1996). Parmi les principaux articles concernant les modèles avec coûts de transaction on trouve Jouini et Kallal (1995a) et (1996), Soner, Shreve et Cvitanić (1995), Levental et Skorohod (1997) et Kabanov et Last (1999).

On peut également utiliser des critères d'évaluation fondés sur la *maximisation de fonction d'utilité*.

L'idée est également très simple. Supposons que le vendeur ait l'habitude de gérer son portefeuille en maximisant l'espérance de la valeur terminale d'une fonction d'utilité $U$ de sa richesse. Si sa dotation initiale est $x$, il résout le problème

$$V^0(x) := \sup_{\nu} EU(X_T^{x,\nu}) .$$

S'il vend l'option au prix $p$, son problème devient

$$V^G(x+p) := \sup_{\nu} EU(X_T^{x+p,\nu} - G) .$$

On peut alors considérer que le prix minimal acceptable pour le vendeur est celui pour lequel vendre ou ne pas vendre l'option lui est indifférent en terme d'utilité espérée. $p$, appelé prix de réserve, est alors la solution de

$$V^G(x+p) = V^0(x) .$$

Cette idée a été introduite par Hodges et Neuberger (1989) puis reprise, par Barles et Soner (1986) et Davis, Panas et Zariphopoulou (1993) dans des modèles avec coûts de transaction et par Rouge (1997) et Delbaen et al. (2000) dans le cadre de marchés incomplets sans friction.

Enfin, il existe un grand nombre d'autres approches mais qui souvent se ramènent à la précédente pour une fonction $U$ plus ou moins lisse. On peut se référer en particulier à Föllmer et Leukert (1999a) pour la gestion par quantile.

Dans la **première partie**, on s'intéresse au critère de *sur-réplication.* Dans le **Chapitre I**, on étudie le problème de *cible stochastique* suivant.

On considère le processus $Z_{t,z}^{\nu} = (X_{t,x}^{\nu},\, Y_{t,x,y}^{\nu})$, où $X$ et $Y$ sont à valeurs respectivement dans $I\!R^d$ et $I\!R$ et sont les solutions du système d'équations différentielles stochastiques

$$
\begin{aligned}
dX(s) &= \rho\left(s, X(s), \nu(s)\right) ds + \alpha'\left(s, X(s), \nu(s)\right) dW(s) \\
&\quad + \int_{\Sigma} \beta\left(s, X(s-), \nu(s), \sigma\right) v(ds, d\sigma) \\
dY(s) &= r\left(s, Z(s), \nu(s)\right) ds + a'\left(s, Z(s), \nu(s)\right) dW(s) \\
&\quad + \int_{\Sigma} b\left(s, Z(s-), \nu(s), \sigma\right) v(ds, d\sigma) \\
Z(t) &= (x, y)\ .
\end{aligned}
$$

Ici, $v$ et $\Sigma$ sont respectivement une mesure aléatoire et un espace de marques associés à un processus de point marqué et $W$ est un mouvement Brownien standard à valeurs dans $I\!R^d$ indépendant de $v$. Etant donnée une fonction $g$ de $I\!R^d$ dans $I\!R$, on explicite

$$
\Gamma(t,x) \;:=\; \left\{ y \in I\!R \;:\; \exists\, \nu,\ Y_{t,x,y}^{\nu}(T) \geq g\left(X_{t,x}^{\nu}(T)\right) \right\}\ ,
$$

l'ensemble des conditions initiales telles qu'il existe un contrôle $\nu$ pour lequel $Y_{t,x,y}^{\nu}(T) \geq g\left(X_{t,x}^{\nu}(T)\right)$. On impose des conditions sur les dynamiques telles que pour tout $y' \geq y \in \Gamma(t,x)$, $y' \in \Gamma(t,x)$. Cela nous permet de caractériser cet ensemble en montrant que la fonction valeur associée à sa borne inférieure

$$
u(t,x) \;:=\; \inf \left\{ y \in I\!R \;:\; \exists\, \nu \in \mathcal{U},\ Y_{t,x,y}^{\nu}(T) \geq g\left(X_{t,x}^{\nu}(T)\right) \right\} \qquad (0.1)
$$

est une solution de viscosité discontinue d'une certaine équation aux dérivées partielles de type variationnelle avec des conditions aux bords obtenues également au sens de la viscosité. Dans de nombreux cas, cette caractérisation s'avère suffisante (voir les exemples étudiés ainsi que les références données dans le Chapitre I).

Ce modèle est une généralisation des modèles usuels de finance mathématique. On peut par exemple supposer que $X^\nu$ correspond au processus de prix d'actifs financiers, $Y^\nu$ à celui d'un portefeuille et que le contrôle $\nu$ correspond à une stratégie de portefeuille. On peut noter que la dépendence de $X$ en $\nu$ permet notamment d'appliquer ce modèle à des problèmes de grand investisseur dans lesquels la stratégie financière $\nu$ peut avoir un impact sur l'évolution des cours.

Ce problème a déjà été traité par Soner et Touzi (2000a) dans le cas d'une diffusion Brownienne sans saut (voir également Soner et Touzi 2000b). Le fait d'ajouter des sauts permet, d'une part, de tenir compte d'un certain nombre de modèles financiers contenant des processus à sauts et, d'autre part, d'appliquer nos résultats à des problèmes de couverture en assurance comme nous le montrons à travers un exemple.

Dans la littérature, ce type de problème a été généralement étudié via des arguments de dualité qui permettent de se ramener à un problème écrit sous forme standard (voir par exemple Cvitanić, Pham et Touzi 1999a et également le **Chapitre II** de cette thèse). Cependant, il n'est pas toujours possible de trouver une formulation duale (c'est le cas par exemple dans le problème de couverture avec contraintes Gamma étudié par Soner et Touzi 1998). Par ailleurs, le nouveau principe de programmation dynamique dû à Soner et Touzi (2000b) permet de dériver les équations aux dérivées partielles associées au problème de cible (0.1) de manière plus directe. C'est cette approche que nous utilisons.

On peut noter que les résultats obtenus par Soner et Touzi (2000a) reposent en grande partie sur la continuité des trajectoires de $Z$. L'introduction de sauts génère un certain nombre de difficultés supplémentaires qui apparaissent essentiellement dans les preuves des propriétés de sous-solution (à l'intérieur du domaine et au bord). Toutefois, on montre comment l'introduction de temps d'arrêt permet d'isoler la composante discontinue et ainsi, en quelque sorte, de gérer séparément les parties continues et discontinues. On applique des arguments similaires à ceux de Soner et Touzi (2000a) à la première composante et on introduit de nouveaux arguments pour traiter les sauts.

Dans le **Chapitre II**, on étudie également un problème de sur-réplication mais cette fois dans un modèle financier avec coûts de transaction proportionnels. Ce n'est pas un cas particulier de celui étudié dans le Chapitre I (même si on montre comment on peut s'en approcher) dans la mesure où les dynamiques et les contrôles sont de nature très différente. En effet, dans les modèles avec coût de transaction, on est obligé de différencier dans le portefeuille les composantes $(X^i)_i$ investies dans les actifs $(S^i)_i$ correspondants. Lorsque les coûts de transaction sont proportionnels, chaque composante est solution d'une équation différentielle stochastique de la forme

$$\begin{aligned} X_t^i &= x^i + \int_0^t X_{s-}^i \frac{dS_s^i}{S_{s-}^i} + \sum_{j=1}^d L_t^{ji} - (1+\lambda^{ij}) L_t^{ij}, \\ X_0^i &= x^i \ , \quad i = 1, \ldots, d \ , \end{aligned} \tag{0.2}$$

où $L^{ij}$ est un contrôle correspondant au montant cumulé des transferts en argent depuis le compte investi en actif $S^i$ vers celui investi en actif $S^j$ et $\lambda^{ij}$ est le coût proportionnel payé pour chaque transfert.

On suppose que la valeur liquidative de l'option s'écrit $g(S_T)$ où $g$ est une fonction de $\mathbb{R}^d$ dans $\mathbb{R}^d$. Ainsi, on tient compte des cas où le paiement aléatoire est réalisé par livraison d'actions (par exemple, $g(S_T) = (0, \ldots, 0, S_T^i, 0, \ldots, 0)'$, i.e. on doit livrer l'actif $S^i$). Puisque les valeurs terminales du portefeuille et de l'option sont vectorielles, il est nécessaire de se doter d'une relation d'ordre partielle $\succeq$ sur $\mathbb{R}^d$ qui permette de les comparer. La relation introduite par Kabanov (1999) a une interprétation économique très naturelle. En effet, on aura $x \succeq y$ s'il est possible, en tenant compte des coûts de transaction, de construire un portefeuille $x_p$ à partir de $x$ tel que $x_p^i \geq y^i$ pour tout $i = 1, \ldots, d$. C'est-à-dire que, partant de $x$, on peut au moins constituer le portefeuille $y$.

Le problème de sur-réplication s'écrit alors

$$\inf \left\{ w \in \mathbb{R} \ : \ \exists\, L,\ X_T^{w\mathbf{1}_1, L} \succeq g(S_T) \right\} \ , \tag{0.3}$$

avec $\mathbf{1}_1 = (1, 0 \ldots, 0)'$. Il correspond à un problème de cible stochastique multivariée avec frictions.

Lorsque $S$ suit une diffusion Brownienne, on montre que la solution de ce problème existe et coïncide avec celle de

$$\inf\left\{w \in I\!\!R \ : \ \exists\, L \ \text{ constant sur } (0,T],\ X_T^{w\mathbf{1_1},\mathbf{L}} \succeq g(S_T)\right\} \ .$$

La stratégie optimale est donc la suivante. On constitue un portefeuille en $t = 0$, i.e., en général, $dL_0 = \Delta L_0 = L_0 - L_{0-} \neq 0$. Puis, on maintient sa composition (en nombre d'actions) constante jusqu'en $T$, i.e. $L_t = L_0$ pour tout $0 \leq t \leq T$. C'est ce que l'on appelle une stratégie buy-and-hold. Le prix de sur-réplication est égal à celui de la stratégie de couverture buy-and-hold la moins chère.

On donne également une formule fermée permettant de calculer ce prix. C'est une extension au cas où $S$ est à valeurs dans $I\!\!R^d$ du résultat de Cvitanić, Pham et Touzi (1999b) (voir également Soner, Shreve et Cvitanić 1995 et Levental and Skorohod 1997).

On donne deux possibilités de démonstration. La première utilise, comme dans le Chapitre I, la programmation dynamique directe. On relie le problème de cible avec frictions à une famille de problèmes de cible auxiliaires définis sur des marchés fictifs sans coûts de transaction. On utilise ensuite la programmation dynamique directe sur les problèmes auxiliaires et on montre que le supremum des fonctions valeurs associées est sur-solution discontinue de viscosité d'une équation de Hamilton-Jacobi-Bellman. On peut ainsi obtenir une borne inférieure pour le prix de sur-réplication en travaillant directement sur l'équation de H.-J.-B. On montre finalement que celle-ci correspond au coût de la stratégie de sur-réplication buy-and-hold la moins chère.

La seconde utilise la formulation duale du problème comme dans Cvitanić, Pham et Touzi (1999b). On utilise la programmation dynamique associée au problème de contrôle standard obtenu par dualité, on dérive une équation de H.-J.-B. puis on conclut comme dans la première preuve.

Dans les deux cas, l'aspect multidimensionnel génère un certain nombre de difficultés techniques qui ne sont pas apparentes dans le modèle unidimensionnel de Cvitanić, Pham et Touzi (1999b) : la paramétrisation du marché fictif et de la formulation duale nécessite une étude préalable du polaire du cône (convexe fermé), dit de solvabilité, introduit par Kabanov (1999) ; les arguments de viscosité et la caractérisation de la stratégie de sur-réplication buy-and-hold la moins chère obtenus dans Cvitanić, Pham et Touzi (1999b) doivent être généralisés au cadre multivarié. Tout ceci nécessite un usage intensif d'outils d'analyse convexe.

Dans la **deuxième partie**, on étudie le problème d'*évaluation par fonction d'utilité* en présence de coûts de transaction proportionnels. On suppose que $X$ suit la dynamique (0.2) où maintenant $S$ est une semi-martingale. Dans un premier temps, on s'intéresse aux problèmes d'optimisation

$$\begin{aligned} V^0(x,\eta) &= \sup_L EU^\eta\left(\ell(X_T^{x,L})\right) , \\ V^G(x,\eta) &= \sup_L EU^\eta\left(\ell(X_T^{x,L} - G)\right) , \end{aligned}$$

où $U^\eta$ est une fonction d'utilité et $\ell(x)$ est la valeur liquidative du portefeuille $x$, c'est-à-dire le montant en actif non-risqué maximal que l'on peut obtenir à partir de $x$ après liquidation des actifs risqués détenus dans le portefeuille.

Lorsque $G = 0$ et $U^\eta$ définie sur $(-k, \infty)$ ce type de problème est bien connu. L'approche classique consiste à étudier un problème dual plus simple à résoudre. On montre qu'il y a existence dans ce dernier et on en déduit l'existence dans le problème initial. Cette technique a été introduite par Cox et Huang (1989) et Karatzas, Lehoczky et Shreve (1987). En marché incomplet sans friction, les résultats les plus généraux ont été obtenus par Kramkov et Schachermayer (2000). Pour les modèles avec coûts de transaction proportionnels, on peut se référer par exemple à Cvitanić et Wang (1999) qui traitent le cas d'un seul actif risqué, et à Deelstra, Pham et Touzi (2000) qui étudient des fonctions définies de $\mathbb{R}^d$ dans $\mathbb{R}$ dans le cadre de marchés multivariés.

Toutefois, lorsque $U^\eta$ est définie sur $\mathbb{R}$, les techniques habituelles ne fonctionnent plus. Le seul article traitant ce cas pour des fonctions d'utilité quelconques est Schachermayer (2000) dans le cadre de marchés incomplets sans friction. Il n'y pas de résultat pour les modèles avec coûts de transaction.

L'introduction de l'actif contingent $G \neq 0$ engendre elle aussi certaines difficultés techniques. Dans Delbaen et al. (2000), la forme particulière de la fonction d'utilité, exponentielle, et l'absence de frictions permettent de les contourner par un simple changement de mesure de probabilité. Une telle technique n'est que partiellement applicable dans notre modèle.

Dans le **Chapitre IV**, on prouve qu'il y a existence pour ce problème lorsque $G$ est borné et $U^\eta(x) = -e^{-\eta x}$. Ceci nous permet de caractériser le portefeuille optimal en terme d'actif contingent (ou limite d'actifs contingents) attaignable(s) et également de donner une formulation duale pour le prix de réserve

$$p(x,\eta) \ := \ \inf\left\{w \in I\!R \ : \ V^G(x + w\mathbf{1}_1, \eta) \geq V^0(x,\eta)\right\} \ .$$

Comme dans Schachermayer (2000), ce résultat est obtenu en approchant le problème initial par une suite de problèmes $V_n$ définis sur une suite de fonctions dont les domaines sont bornés inférieurement et pour lesquels l'approche par dualité classique fonctionne.

La difficulté consiste à prouver la convergence des fonctions valeurs et des stratégies optimales associées aux problèmes auxiliaires vers la fonction valeur et la stratégie optimale du problème initiale. Comme dans Schachermayer (2000), on montre que l'optimum, même s'il ne correspond pas toujours à un actif attaignable, peut néanmoins être approché par une suite de stratégies bornées inférieurement. L'attaignabilité est obtenue sous certaines conditions sur $S$.

Les techniques utilisées ne semblent pas reposer de manière cruciale sur la forme spéciale de la fonction exponentielle comme c'est le cas dans Delbaen et al. (2000). Les résultats d'existence et de dualité devraient donc pouvoir être étendus, au moins lorsque $G$= 0, à une classe plus générale de fonctions d'utilité définies sur $I\!R$ et voir même à des fonctions non strictement concaves en combinant les approches de Schachermayer (2000) et de Deelstra, Pham et Touzi (2000) avec la notre.

On peut noter que pour $U^\eta(x) = -e^{-\eta x}$, $\eta$ coïncide avec l'aversion absolue à l'égard du risque $-(U^\eta)'/(U^\eta)''$. Cela signifie que plus $\eta$ est grand et moins l'agent choisit une stratégie risquée. A la limite, lorsque ce paramètre tend vers l'infini, l'agent devrait donc choisir une stratégie sans risque, c'est-à-dire une stratégie pour laquelle le portefeuille associé ne contient aucun actif risqué.

Ainsi, dans le problème auquel $V^0$ est associé, on peut penser que, lorsque l'aversion tend vers l'infinie, la stratégie optimale consiste à se débarrasser immédiatement de tous les actifs risqués du portefeuille. La valeur terminale du portefeuille optimal devrait donc tendre vers $\ell(x)$. De même, $p(x, \eta)$ devrait tendre vers un prix pour lequel l'agent ne prend plus aucun risque en vendant l'option, c'est-à-dire vers le prix de *sur-réplication*.

Cette conjecture a été émise en premier par Barles et Soner (1996) puis prouvée par Bouchard (1999) dans un cadre markovien par le biais d'une étude asymptotique d'équations de H.-J.-B. associées au problème. Rouge (1997) et Delbaen et al. (2000) ont obtenu le même résultat pour des modèles de marchés incomplets sans friction.

Dans le **Chapitre IV**, on montre comme corollaire des formulations duales associées à $V^G$ et $V^0$, que $p(x, \eta)$ tend effectivement vers le prix de sur-réplication augmenté de la valeur liquidative $\ell(x)$ (cette dernière composante provenant du fait que l'on compare $V^G$ à $V^0$). Ce résultat est généralisé pour des actifs contingents $G$ plus généraux et pour des stratégies financières moins contraintes dans le **Chapitre V**.

L'argument économique précédent devrait être valable quel que soit le choix de la fonction d'utilité. Dans le **Chapitre VI**, on le généralise pour une famille de fonctions comprenant notamment les fonctions CRRA[1] en plus des fonctions CARA[2] qui correspondent aux fonctions de types exponentielles étudiées dans les Chapitres IV et V. Nous pensons que ce résultat peut être obtenu par des arguments similaires dans le cadre de marchés incomplets sans friction.

[1] constant relative risk aversion, $-x(U^\eta)''/(U^\eta)' =$ constante

[2] constant absolute risk aversion, $-(U^\eta)''/(U^\eta)' =$ constante

Dans la **troisième partie**, on s'intéresse à la résolution numérique de problèmes de contrôle stochastique de la forme

$$\sup_{\nu} E\left[f(X_T^{\nu})\right]$$

où $X^{\nu}$ est processus contrôlé par $\nu$.

Une approche relativement classique lorsque $X^{\nu}$ suit une diffusion consiste à caractériser la fonction valeur associée au problème comme solution (dans un certain sens) d'une équation aux dérivées partielles puis à essayer d'en déduire les contrôles optimaux par des théorèmes de vérification (voir par exemple Fleming et Soner 1992).

Malheureusement, cette approche conduit souvent à résoudre des équations non-linéaires en grande dimension, ce qui est très coûteux, voir parfois impossible, d'un point de vue numérique.

On montre, à travers un exemple, comment l'utilisation conjointe de techniques de Monte-Carlo et de régression non-paramétrique permet d'obtenir une bonne approximation des contrôles optimaux. Les résultats obtenus ne sont que préliminaires et se basent sur une étude encore en cours.

# Organisation générale de la thèse

Le coeur de cette thèse est organisée selon les trois parties introduites précédemment.

Dans la première partie, chaque chapitre est introduit séparément car les problèmes étudiés sont de natures légèrement différentes. Dans la seconde partie, les différents chapitres font l'objet d'une introduction commune. La troisième partie ne contient qu'un seul chapitre.

Les notations générales sont définies juste avant le premier chapitre. Les références sont données à la fin de la thèse.

Le **plan général** de cette thèse est le suivant :

# Contents

# Notations

Etant donnés $x$ et $y$ dans $\mathbb{R}^n$, on note par $xy$ le produit scalaire naturel associé et par $|\cdot|$ la norme correspondante.

Etant donnés $x \in \mathbb{R}^n$ et $\eta > 0$, $B(x, \eta)$ est la boule de rayon $\eta$ associée à $|\cdot|$. On note $\bar{B}$ la boule fermée associée à $B$.

Etant donné un vecteur $x \in \mathbb{R}^n$, sa $i$-ème composante est dénotée $x^i$. $\mathbb{M}^{n,p}$ désigne l'ensemble de toutes les matrices réelles ayant $n$ lignes et $p$ colonnes. Etant donnée une matrice $M \in \mathbb{M}^{n,p}$, on note $M^{ij}$ la composante correspondant à la $i$-ème ligne et à la $j$-ème colonne. $\mathbb{M}_+^{n,p}$ désigne le sous-ensemble de $\mathbb{M}^{n,p}$ dont les éléments ont toutes leurs composantes positives. Si $n = p$, on note simplement $\mathbb{M}^n$ et $\mathbb{M}_+^n$ pour $\mathbb{M}^{n,n}$ et $\mathbb{M}_+^{n,n}$. Comme $\mathbb{M}^{n,p}$ peut être identifié à $\mathbb{R}^{np}$, on définit la norme sur $\mathbb{M}^{n,p}$ comme celle des éléments associés de $\mathbb{R}^{np}$.

La transposition est dénotée par $'$. Etant donnée une matrice carrée $M \in \mathbb{M}^n$, on note $\mathrm{Tr}[M] := \sum_{i=1}^n M^{ii}$ sa trace.

Etant donnés $n$ réels $x_1, \ldots, x_n$, on note $\mathrm{Vect}[x_i,\ i = 1, \ldots, n]$ le vecteur de $\mathbb{R}^n$ défini par les composantes $x_1, \ldots, x_n$. Pour tout $x \in \mathbb{R}^n$, $\mathrm{diag}[x]$ désigne la matrice diagonale de $\mathbb{M}^n$ dont le $i$-ème terme diagonal est $x^i$. Etant donnée une matrice $M \in \mathbb{M}^{n,p}$, on note $\bar{M}$ la matrice dans $\mathbb{M}^{n+1,p}$ obtenue à partir de $M$ en ajoutant une première ligne de 1. On utilise la même notation sur $\mathbb{R}^n$.

On note $\mathbf{1}_i$ le vecteur de $\mathbb{R}^n$ défini par $\mathbf{1}_i^j = 1$ si $j = i$ et 0 sinon.

Etant donnée une fonction $C^2$, $\varphi$, de $\mathbb{R}^n$ dans $\mathbb{R}^p$, on note $D\varphi$ la matrice Jacobienne de $\varphi$, i.e. $(D\varphi)^{ij} = \partial\varphi^i/\partial x^j$. Si $x = (y, z)$, $D_y\varphi$ désigne la matrice Jacobienne (partielle) de $\varphi$ par rapport à $y$. Dans le cas $p = 1$, on note $D^2\varphi$ la matrice Hessienne de $\varphi$, i.e. $(D^2\varphi)^{ij} = \partial^2\varphi/\partial x^i\partial x^j$. Si $x = (y, z)$, on définit les matrices $D^2_{yy}\varphi$, $D^2_{zz}\varphi$ et $D^2_{yz}\varphi$ de la même manière.

Etant donne une fonction $u$, on notera $u^*$ (resp. $u_*$) sont enveloppe semi-continue supérieurement (resp. semi-continue inférieurement).

Etant donné un espace de probabilité filtré $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t,\ 0 \le t \le T\})$, un réel $p \ge 0$ et $K \subset \mathbb{R}^d$, on note $L^p(K, \mathcal{F}_t)$ l'ensemble de toutes les variables aléatoires $\mathcal{F}_t$-mesurables à valeurs dans $K$ dont la norme $L^p$ est finie. Cette notation est étendue de manière naturelle aux temps d'arrêts. Pour $p = 0$, $L^0(K, \mathcal{F}_t)$ est l'ensemble de toutes les variables aléatoires $\mathcal{F}_t$-mesurables à valeurs dans $K$. Si $K$ est donné par le contexte, on utilise parfois la notation $L^p(t)$. Si $t = T$, on écrit simplement $L^p$ ou $L^p(K)$ pour $K \subset \mathbb{R}^d$. On dénote par $\mathcal{M}(P)$ l'ensemble des $P$-martingales adaptées à $\mathbb{F}$.

En général, si rien n'est précisé, toutes les inégalités entre variables aléatoires sont à considérer dans le sens presque sûr.

On utilisera la convention $\inf \emptyset = +\infty$.

# Part A.

# Stochastic targets, viscosity and applications in finance

# Chapter I.

# Stochastic targets with mixed diffusion processes and viscosity solutions

Let $Z^{\nu}_{t,z}$ be a $\mathbb{R}^{d+1}$-valued mixed diffusion process controlled by $\nu$ with initial condition $Z^{\nu}_{t,z}(t) = z$. In this paper, we characterize the set of initial conditions such that $Z^{\nu}_{t,z}$ can be driven above a given stochastic target at time $T$ by proving that the corresponding value function is a discontinuous viscosity solution of a variationnal partial differential equation. As applications of our main result, we study two examples : a problem of optimal insurance under self-protection and a problem of option hedging under jumping stochastic volatility where the underlying stock pays a random dividend at a fixed date.

# 1 Introduction

Let $Z^\nu_{t,z}$ be $\mathbb{R}^{d+1}$-valued process controlled by $\nu$ with initial condition $Z^\nu_{t,z}(t) = z$. A general stochastic target problem consists in finding the set of initial conditions $z$ such that there exists a control process $\nu$, belonging to a well defined set of admissible controls, for which $Z^\nu_{t,z}(T)$ reaches a given target, say for example a Borel subset of $\mathbb{R}^{d+1}$.

In this paper, we consider a general mixed diffusion model for the process $Z^\nu_{t,z} = (X^\nu_{t,x}, Y^\nu_{t,x,y})$, where $X$ is $\mathbb{R}^d$-valued and $Y$ is $\mathbb{R}$-valued. We address the problem of finding the minimal initial data $y$ such that $Y^\nu_{t,y,x}(T) \geq g(X^\nu_{t,x}(T))$ for some admissible control $\nu$, where $g$ is a $\mathbb{R}^d \mapsto \mathbb{R}$ measurable function. We prove that the associated value function is a discontinuous viscosity solution of a well suited variationnal partial differential equation with boundary condition obtained in a viscosity sense, i.e. we generalize the results obtained in the Brownian diffusion case by Soner and Touzi (2000a).

This problem has been addressed in financial mathematics in the super-replication literature (the first component of $Z$ is the portfolio process and the others, for instance, the prices of risky assets). In contrast with the previous literature, we do not make use of duality arguments which allow to reduce it to a standard control problem. Instead, we use a direct dynamic programming principle directly stated on our stochastic target problem. The reason is that a duality formulation is not always available (this is the case, for instance, for the problem of super-replication under Gamma contraints studied by Soner and Touzi 1998).

Since our goal is only to characterize the value function in terms of PDE's, we do not try to prove any comparison result for the associated variationnal partial differential equation. Such a characterization as already proved to be sufficient in many applications (see for instance Bouchard and Touzi 1999). Moreover, unicity and continuity are shown to hold in two applications of our main result. In the first one, we study a problem of optimal insurance under self-protection. In the second, we study a problem of option hedging under jumping stochastic volatility where the underlying asset pays a dividend at a fixed date $t_1$ and where the dividend revision process has jumps.

The rest of the paper is organized as follow. The general model is described in Section 2. In Section 3, we state our main results. The proofs are reported in Section 4 to 7. Finally, in Section 8, we provide some applications.

# 2 The Model

Let $T > 0$ be a finite time horizon, $\Sigma$ a Borel subset of $I\!R_+$ and $v(dt, d\sigma) = v^1(dt, d\sigma) + \ldots + v^d(dt, d\sigma)$ the sum of independent integer valued $\Sigma$-marked right-continuous point processes defined on a complete probability space ($\Omega$, $\mathcal{F}$, $P$). Let $W$ be a $I\!R^d$-valued standard Brownian motion defined on ($\Omega$, $\mathcal{F}$, $P$) such that $W$ and $v$ are independent. We denote by $I\!F = \{\mathcal{F}(t), 0 \leq t \leq T\}$ the $P$-completed filtration generated by $(W_{\cdot}, v(\cdot, d\sigma))$ and we assume that $\mathcal{F}(0)$ is trivial.
The random measure $v(dt, d\sigma)$ is assumed to have predictable $(P, I\!F)$-intensity kernel $m(d\sigma)dt$ such that $\int_\Sigma m(d\sigma) < \infty$ and we denote by $\tilde{v}(dt, d\sigma) = v(dt, d\sigma) - m(d\sigma)dt$ the associated compensated random measure.

Let $U$ be a compact subset of $I\!R^d$ with non empty interior and $\mathcal{U}$ be the set of all $I\!F$-predictable processes $\nu = \{\nu(t), 0 \leq t \leq T\}$ valued in $U$.

Given a control process $\nu \in \mathcal{U}$, $t \in [0, T]$ and $z = (x, y) \in I\!R^d \times I\!R$, we define the controlled process $Z^\nu_{t,z} = (X^\nu_{t,x}, Y^\nu_{t,z})$ as the solution on $[t, T]$ of the stochastic differential system :

$$\begin{aligned} dX(s) &= \rho\left(s, X(s), \nu(s)\right) ds + \alpha'\left(s, X(s), \nu(s)\right) dW(s) \\ &\quad + \int_\Sigma \beta\left(s, X(s-), \nu(s), \sigma\right) v(ds, d\sigma) \\ dY(s) &= r\left(s, Z(s), \nu(s)\right) ds + a'\left(s, Z(s), \nu(s)\right) dW(s) \\ &\quad + \int_\Sigma b\left(s, Z(s-), \nu(s), \sigma\right) v(ds, d\sigma) \\ Z(t) &= (x, y) \end{aligned} \tag{2.1}$$

where $\rho$, $\alpha$, $\beta$, $r$, $b$ and $a$ are continuous with respect to $(t, \nu, \sigma) \in [0, T] \times U \times \Sigma$, Lipschitz in $t$, Lipschitz and linearly growing in the variable $z$, uniformly in the variables $(t, \nu, \sigma)$, and bounded with respect to $\sigma$. This guarantees existence and uniqueness of a strong solution $Z^{\nu}_{t,z}$ to the stochastic differential system (2.1) for each control process $\nu \in \mathcal{U}$.

Let $g : I\!R^d \longrightarrow I\!R$ be a measurable function. Our stochastic target problem is :

$$u(t,x) \ := \ \inf \Gamma(t,x)$$

where

$$\Gamma(t,x) \ := \ \left\{ y \in I\!R \ : \ \exists\, \nu \in \mathcal{U},\ Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right) \ \right\} .$$

Here, $g\left(X^{\nu}_{t,x}(T)\right)$ can be viewed as a stochastic target.

**Remark 2.1** Notice that

$$P\left[v(\Sigma \setminus \mathrm{supp}(m), [0,T]) > 0\right] = 0 .$$

Hence, we may assume, without loss of generality, that $\Sigma = \mathrm{supp}(m)$.

Assume that the infimum in the definition of $u$ is attained and let $y = u(t,x)$. Then, we can find some $\nu \in \mathcal{U}$ such that $Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right)$. Hence, if we start with $y' > y$, we should be able to find some $\nu' \in \mathcal{U}$ such that $Y^{\nu'}_{t,x,y'}(T) \geq g\left(X^{\nu'}_{t,x}(T)\right)$. If this property does not hold (which can be the case in a jump diffusion model) we are not able to characterize the set $\Gamma(t,x)$ by it lower bound $u(t,x)$.

Hence we assume that, for all $(t,x,y,y') \in [0,T] \times I\!R^d \times I\!R \times I\!R$,

$$y' \ \geq \ y \quad \text{and} \quad y \ \in \ \Gamma(t,x) \ \Longrightarrow \ y' \ \in \ \Gamma(t,x) .$$

By standard comparison arguments for stochastic differential equations, it will hold in particular if $b$ is independent of $y$ (see e.g. Protter 1990). It will also hold in most financial applications as soon as there is a non-risky asset.

Under the above assumption, for each $[0,T]$-valued stopping time $\theta$ and control $\nu \in \mathcal{U}$, $u(\theta, X^{\nu}_{t,x}(\theta))$ corresponds to the minimal condition, when starting at time $\theta$, such that the stochastic target can be reached at time $T$. This means that, if $u$ is finite, given $y > u(t,x)$, we can find a control $\nu$ such that $Y^{\nu}_{t,x,y}(\theta) \geq u(\theta, X^{\nu}_{t,x}(\theta))$ for any $[0,T]$-valued stopping time $\theta$ (see Proposition 4.1 below). Assume that $u$ is smooth and denote by $Du$ the gradiant of $u$ with respect to $x$. Applying Itô's Lemma to $u$ shows that the only way to control the Brownian part of $Y^{\nu}_{t,x,y}(\cdot) - u(\cdot, X^{\nu}_{t,x}(\cdot))$ is to define $\nu(\cdot)$ in a Markovian way by $\nu(\cdot) = \psi(\cdot, X^{\nu}_{t,x}(\cdot), Y^{\nu}_{t,x,y}(\cdot), Du(\cdot, X^{\nu}_{t,x}(\cdot)))$ where, for all $(t,x,y) \in [0,T] \times I\!R^d \times I\!R$,

$$\psi(t,x,y,\cdot) \text{ is the inverse of the mapping } \nu \mapsto \alpha^{-1}(t,x,\nu)a(t,x,y,\nu).$$

Hence, we assume that, either, $\alpha$ is invertible and $\psi$ is well defined, or, $a = \alpha = 0$.

# 3 The main results

We differentiate the mixed diffusion case to the pure jump case.

## 3.1 The general mixed diffusion case

In this part, we assume that $\psi$ is well defined (see the previous section) and that $U$ is convex. We introduce the support function $\delta_U$ of the closed convex set $U$ :

$$\delta_U(\zeta) \;:=\; \sup_{\nu \in U} \zeta^{'} \nu \;\;,\;\; \zeta \in I\!R^d \;,$$

and $\tilde{U}_1$ the restriction to the unit sphere of the effective domain of $\delta_U$ :

$$\tilde{U}_1 \;:=\; \left\{ \zeta \in I\!R^d \;,\; |\zeta| = 1 \;,\; \delta_U(\zeta) \in I\!R^d \right\} \;.$$

Clearly, $\tilde{U}_1$ is equal to the unit sphere of $I\!R^d$. We use this notation since part of our results hold without the compacity assumption on $U$.

Notice that $U$ and $\mathrm{Int}(U)$ may be characterized in terms of $\tilde{U}_1$ :

$$\nu \in U \Longleftrightarrow \chi_U(\nu) \;\geq\; 0 \quad \text{and} \quad \nu \in \mathrm{Int}(U) \Longleftrightarrow \chi_U(\nu) \;>\; 0 \;,$$

where

$$\chi_U(\nu) := \inf_{\zeta \in \tilde{U}_1} \left( \delta_U(\zeta) - \zeta' \nu \right) .$$

**Remark 3.1** The mapping $\nu \in U \mapsto \chi_U(\nu)$ is continuous. This follows from the compactness of $\tilde{U}_1$.

Given a smooth function $\varphi$ on $[0,T] \times I\!R^d$, $\nu \in U$ and $\sigma \in \Sigma$, we define the operators :

$$\begin{aligned}
\mathcal{L}^\nu \varphi(t,x) &:= r(t,x,\varphi(t,x),\nu) - \frac{\partial \varphi}{\partial t}(t,x) - \rho(t,x,\nu)' D\varphi(t,x) \\
&\quad - \frac{1}{2}\text{Trace}\left( D^2\varphi(t,x)\alpha'(t,x,\nu)\alpha(t,x,\nu) \right) \\
\mathcal{G}^{\nu,\sigma}\varphi(t,x) &:= b(t,x,\varphi(t,x),\nu,\sigma) - \varphi\left(t, x + \beta(t,x,\nu,\sigma)\right) + \varphi(t,x) \\
\mathcal{T}^\nu \varphi(t,x) &:= \min\left\{ \inf_{\sigma \in \Sigma} \mathcal{G}^{\nu,\sigma}\varphi(t,x) \; ; \; \chi_U(\nu) \right\} \\
\mathcal{H}^\nu \varphi(t,x) &:= \min\left\{ \mathcal{L}^\nu \varphi(t,x) \; ; \; \mathcal{T}^\nu \varphi(t,x) \right\}
\end{aligned}$$

where $D\varphi$ and $D^2\varphi$ denote respectively the gradiant and Hessian matrix of $\varphi$ with respect to $x$. We also define :

$$\begin{aligned}
\widehat{\mathcal{G}}^\sigma \varphi(t,x) &:= \mathcal{G}^{\nu,\sigma}\varphi(t,x) \quad \text{for } \nu = \psi(t,x,\varphi(t,x),D\varphi(t,x)) \, , \\
\widehat{\mathcal{T}} \varphi(t,x) &:= \mathcal{T}^{\nu}\varphi(t,x) \quad \text{for } \nu = \psi(t,x,\varphi(t,x),D\varphi(t,x)) \, , \\
\widehat{\mathcal{H}} \varphi(t,x) &:= \mathcal{H}^{\nu}\varphi(t,x) \quad \text{for } \nu = \psi(t,x,\varphi(t,x),D\varphi(t,x)) \, ,
\end{aligned}$$

and we naturally extend all these operators to functions independent of $t$ by replacing $t$ by $T$ in the definition of $\nu$.

We can now state our main results.

**Theorem 3.1** *Assume that $u^*$ and $u_*$ are finite. Then, the value function $u$ is a discontinuous viscosity solution on $(0,T) \times I\!R^d$ of :*

$$\widehat{\mathcal{H}}\varphi(t,x) = 0 \, . \tag{3.1}$$

The nonlinear PDE reported in the above theorem does not provide a complete characterization of the value function $u$. To further characterize it, we need to specify the terminal condition. From the definition of $u$ it is clear that $u(T,x) = g(x)$ but we know that $u$ may be discontinuous in $T$. Therefore we introduce :

$$\bar{G}(x) \ := \ \limsup_{t\uparrow T, x'\to x} u(t,x') \quad \text{and} \quad \underline{G}(x) \ := \ \liminf_{t\uparrow T, x'\to x} u(t,x') \qquad x \in I\!R^d .$$

**Theorem 3.2** *Let the conditions of Theorem 3.1 hold. Then, if $\underline{G}$ is finite, it is a viscosity supersolution on $I\!R^d$ of*

$$\min\left\{\varphi(x) - g_*(x) \ ; \ \widehat{\mathcal{T}}\varphi(x)\right\} \ = \ 0 \ , \tag{3.2}$$

*and, if $\bar{G}$ is finite, it is a viscosity subsolution on $I\!R^d$ of*

$$\min\left\{\varphi(x) - g^*(x) \ ; \ \widehat{\mathcal{T}}\varphi(x)\right\} \ = \ 0 \ . \tag{3.3}$$

**Remark 3.2** We can retrieve the results of Soner and Touzi (2000a) by setting $b = \beta = 0$.
We do not have a general comparison result for (3.1)-(3.2)-(3.3). Anyway, we show through examples that this characterization may be sufficient in financial applications (see our Example Section, the Example Section in Soner and Touzi 2000a, Bouchard and Touzi 1999, Soner and Touzi 1998, and, Touzi 2000).

Theorems 3.1 and 3.2 are proved in Sections 4 and 5.

## 3.2 The pure jump model

We now assume that $\alpha = a = 0$. We can rewrite Theorem 3.1 in a simpler way without assuming that $U$ is convex.

**Theorem 3.3** *Assume that $u^*$ and $u_*$ are finite. Then, the value function $u$ is a discontinuous viscosity solution on $(0,T) \times I\!R^d$ of :*

$$\sup_{\nu\in U} \min\left\{\mathcal{L}^\nu\varphi(t,x) \ ; \ \inf_{\sigma\in\Sigma} \mathcal{G}^{\nu,\sigma}\varphi(t,x)\right\} \ = \ 0 \ . \tag{3.4}$$

Notice that since $\alpha = 0$, we have

$$\mathcal{L}^\nu\varphi(t,x) \ := \ r(t,x,\varphi(t,x),\nu) - \frac{\partial\varphi}{\partial t}(t,x) - \rho(t,x,\nu)' D\varphi(t,x) \ .$$

**Theorem 3.4** *Let the conditions of Theorem 3.3 hold and assume that* $\underline{G}$ *and* $\bar{G}$ *are finite. Then :*

$$H_*(\underline{G}(x)) := \min\left\{\underline{G}(x) - g_*(x) \; ; \; \sup_{\nu\in U} \inf_{\sigma\in\Sigma} \mathcal{G}^{\nu,\sigma}\underline{G}(x)\right\} \geq 0 \; , \; x \in \mathbb{R}^d \quad (3.5)$$

$$H^*(\bar{G}(x)) := \min\left\{\bar{G}(x) - g^*(x) \; ; \; \sup_{\nu\in U} \inf_{\sigma\in\Sigma} \mathcal{G}^{\nu,\sigma}\bar{G}(x)\right\} \leq 0 \; , \; x \in \mathbb{R}^d \; . \quad (3.6)$$

In contrast to Theorem 3.2, the boundary condition is obtained in the classical sense (in opposition to the viscosity sense). This comes from the fact that it does not contain any derivative term.

In general, we are not able to prove that $\underline{G} = \bar{G}$, and, even if $g$ is continuous, we have no general comparison result for continuous functions satisfying both (3.5) and (3.6). Nevertheless, the intuition is that if $g$ is continuous and $\bar{G} = \underline{G} =: G$, then $G$ should be interpreted as the smallest solution of (3.5)-(3.6). In this case, and under mild assumptions, we can construct explicitly a sequence of functions that converges $G$.

The existence of a smallest solution for (3.5) is easily obtained under (3.7) below.

**Proposition 3.1** *Assume that there exists a strictly increasing function* $h$ *on* $\mathbb{R}$ *such that for all* $(x,\nu,\sigma) \in \mathbb{R}^d \times U \times \Sigma$, *the mapping*

$$y \longmapsto y + b(T,x,y,\nu,\sigma) - h(y) \;\; \textit{is increasing.} \quad (3.7)$$

*Assume further that there exists a finite function* $f$ *satisfying* $H_*(f) \geq 0$ *on* $\mathbb{R}^d$.

*Then, there exists a lower-semicontinuous function* $\ell$ *such that* $H_*(\ell) \geq 0$ *on* $\mathbb{R}^d$ *and such that* $\ell \leq f$ *for all function* $f$ *satisfying* $H_*(f) \geq 0$ *on* $\mathbb{R}^d$, *i.e.* (3.5) *admits a smallest solution which is lower-semicontinuous. Moreover, we have* $H_*(\ell(x)) = 0$ *for all* $x \in \mathbb{R}^d$.

**Remark 3.3** (3.7) implies in particular that for all finite function $f$

$$y \longmapsto y + \sup_{\nu\in U} \inf_{\sigma\in\Sigma} b(T,x,y,\nu,\sigma) - f(x+\beta(t,x,\nu,\sigma))$$

is strictly increasing. Hence, given $(y_1,y_2) \in \mathbb{R}^2$ and a finite function $f$ such that

$$\begin{aligned} y_1 + \sup_{\nu\in U} \inf_{\sigma\in\Sigma} b(T,x,y_1,\nu,\sigma) - f(x+\beta(t,x,\nu,\sigma)) &\geq 0 \\ y_2 + \sup_{\nu\in U} \inf_{\sigma\in\Sigma} b(T,x,y_2,\nu,\sigma) - f(x+\beta(t,x,\nu,\sigma)) &\leq 0, \end{aligned}$$

(3.7) implies that $y_1 \geq y_2$. Moreover, if such $y_1$ and $y_2$ exists, from the continuity of $b$ in $y$, uniformly in $(\nu,\sigma)$, we can find some $y$ (which is unique) such that

$$y + \sup_{\nu\in U} \inf_{\sigma\in\Sigma} b(T,x,y,\nu,\sigma) - f(x+\beta(t,x,\nu,\sigma)) \;=\; 0\,.$$

**Remark 3.4** In Section 7, we construct explicitly a sequence of functions converging to $\ell$, provided that it exists.

We now provide sufficient conditions under which we can explicitly characterize the boundary condition. The assumptions of the following proposition are quite strong but it gives the intuition for the general case.

**Proposition 3.2** *Let the conditions of Theorem 3.4 hold. Assume that $g$ is continuous and that there exists a continuous smallest solution $\ell$ of* (3.5). *Assume further that there exists a neighborhood $V$ of $T$ and a classical supersolution $w$ of* (3.4) *on $V \times \mathbb{R}^d$ such that, for all $x \in \mathbb{R}^d$, $\lim_{t\uparrow T,\, x'\to x} w(t,x')$ $= w(T,x) = \ell(x)$ and for all $(t,x) \in V \times \mathbb{R}^d$*

$$y \mapsto y + \sup_{\nu\in U} \inf_{\sigma\in\Sigma} b(t,x,y,\nu,\sigma) - w(t,x+\beta(t,x,\nu,\sigma)) \text{ is strictly increasing.} \tag{3.8}$$

*Then, $\underline{G} = \bar{G} = \ell$.*

**Remark 3.5** If we combine the conditions of Propositions 3.1 and 3.2, we obtain that $\underline{G}=\bar{G}= \ell$ where $\ell$ is the solution of $H_*(\ell) = 0$.

**Proof.** Fix $(t,x) \in V \times I\!R^d$ and set $z := (x,y)$ where $y := w(t,x)$. $w$ satisfies on $V \times I\!R^d$ :

$$\sup_{\nu\in U} \min\left\{\mathcal{L}^\nu w(t,x)\ ;\ \inf_{\sigma\in\Sigma} \mathcal{G}^{\nu,\sigma} w(t,x)\right\} \ \geq \ 0\ , \tag{3.9}$$

Define for all $n \in I\!N \setminus \{0\}$, the sequence of stopping times :

$$\begin{aligned} \theta_1 &:= T \wedge \inf\left\{s > t\ :\ \Delta Z^\nu_{t,z}(s) \neq 0\right\} \\ \theta_{n+1} &:= T \wedge \inf\left\{s > \theta_n\ :\ \Delta Z^\nu_{t,z}(s) \neq 0\right\}\ , \end{aligned}$$

where the control process $\nu$ is defined in a markovian way as $\nu(\cdot) := \hat{\nu}(\cdot, X^\nu_{t,x}(\cdot), Y^\nu_{t,z}(\cdot))$ and $\hat{\nu}(t,x)$ is the argmax in (3.9) for all $(t,x) \in V \times I\!R^d$. Using (3.9), the fact that $y = w(t,x)$ and standard comparison results for stochastic differential equation, we get that $Y^\nu_{t,z}(\theta_1 -) \geq w(\theta_1 -, X^\nu_{t,x}(\theta_1 -))$. By (3.8) and (3.9) again we obtain that $Y^\nu_{t,z}(\theta_1) \geq w(\theta_1, X^\nu_{t,x}(\theta_1))$. Using a recursive argument, we get that, for all $i \geq 1$, $Y^\nu_{t,z}(\theta_i) \geq w(\theta_i, X^\nu_{t,x}(\theta_i))$. This proves that : $Y^\nu_{t,z}(T) \geq w(T, X^\nu_{t,x}(T)) \geq \ell(X^\nu_{t,x}(T)) \geq g(X^\nu_{t,x}(T))$. Hence, by definition of $u$, for all $(t,x) \in V \times I\!R^d$, $w(t,x) \geq u(t,x)$ and $\ell(x) = \lim_{t\uparrow T, x'\to x} w(t,x')$ $\geq \limsup_{t\uparrow T, x'\to x} u(t,x) = \bar{G}(x) \geq \underline{G}(x)$, where the last inequality is obtained by definition. The result is finally obtained by noticing that, by definition of $\ell$ and Theorem 3.4, $\underline{G} \geq \ell$. □

Finally, we give some conditions under which we can easily prove the continuity of the smallest solution for (3.5).

**Proposition 3.3** *Under the conditions of Proposition 3.1, if $g$ is uniformly continuous and $b$ and $\beta$ are independent of $(x,y)$, then the smallest solution $\ell$ of* (3.5) *is uniformly continuous.*

The rest of the paper is devoted to the proofs of these results as well as some applications. Theorem 3.3 is proved as Theorem 3.1 up to some small modifications that we explain at the end of the different parts of the proof of this last theorem. Theorem 3.4 is proved in Section 6. Propositions 3.1 and 3.3 are proved in Section 7. In Section 8, we provide some applications.

# 4 Proof of Theorem 3.1

The proof of the viscosity properties stated in the previous section is mainly based on a direct dynamic programming principle proved in Soner and Touzi (2000b).

**Proposition 4.1** *Fix* $(t,x) \in [0,T] \times I\!R^d$.
(DP1) *Let* $y \in I\!R$ *and* $\nu \in \mathcal{U}$ *be such that* $Y^\nu_{t,x,y}(T) \geq g\left(X^\nu_{t,x}(T)\right)$ . *Then, for all stopping time* $\theta \geq t$, *we have :*

$$Y^\nu_{t,x,y}(\theta) \quad \geq \quad u\left(\theta, X^\nu_{t,x}(\theta)\right) \quad .$$

(DP2) *Set* $y := u(t,x)$ *and let* $\theta \geq t$ *be an arbitrary stopping time. Then, for all* $\nu \in \mathcal{U}$ *and* $\eta > 0$ *:*

$$P\left[Y^\nu_{t,x,y-\eta}(\theta) > u\left(\theta, X^\nu_{t,x}(\theta)\right)\right] \quad < \quad 1 \; .$$

**Proof.** (DP1) is a direct consequence of Theorem 3.1 in Soner and Touzi (2000b). To see that (DP2) holds, fix $(t,y,x,\eta,\theta)$ as in the proposition an assume that

$$P\left[Y^\nu_{t,x,y-\eta}(\theta) > u\left(\theta, X^\nu_{t,x}(\theta)\right)\right] \quad = \quad 1 \; .$$

for some $\nu \in \mathcal{U}$. Then by Theorem 3.1 in Soner and Touzi (2000b) and definition of $u$, $y - \eta \geq u(t,x)$ and we get a contradiction since $y = u(t,x)$ and $\eta > 0$. □

We now state the easy lemma that will be useful in the subsequent proofs.

**Lemma 4.1** *Fix* $(t,z) \in [0,T] \times I\!R^{d+1}$ *and let* $(\nu_n)_n$ *be a sequence in* $\mathcal{U}$. *Then for all sequence* $(t_n, t'_n, z_n)_n$ *(with* $t_n \leq t'_n$*) converging to* $(t,t,z)$,

$$\sup_{t_n \leq s \leq t'_n} Z^{\nu_n}_{t_n,z_n}(s) \quad \longrightarrow \quad z \qquad \text{in } L^2 \; .$$

**Proof.** Let $C$ be a generic constant that may take different values. Using the Lipschitz and linear growth condition on the coefficients of the diffusion of $Z$, it is easily checked that for all $t_n \le t \le t'_n$ :

$$E|Z^{\nu_n}_{t_n,z_n}(t)|^2 \;\le\; C\left(1+|z_n|^2+E\int_{t_n}^{t}|Z^{\nu_n}_{t_n,z_n}(s)|^2 ds\right) .$$

By Fubini's theorem and Gronwalls Lemma, this yields :

$$E|Z^{\nu_n}_{t_n,z_n}(t)|^2 \;\le\; C(1+|z_n|^2) . \tag{4.1}$$

Now, using the conditions on the coefficients again, we get :

$$E|\sup_{t_n\le s\le t'_n}|Z^{\nu_n}_{t_n,z_n}(s)-z||^2 \;\le\; |z_n-z|^2+CE\int_{t_n}^{t'_n}|Z^{\nu_n}_{t_n,z_n}(s)|^2 ds$$

and by (4.1) :

$$E|\sup_{t_n\le s\le t'_n}|Z^{\nu_n}_{t_n,z_n}(s)-z||^2 \;\le\; |z_n-z|^2+C(t'_n-t_n)(1+|z_n|^2) .$$

The proof is concluded by sending $n$ to $\infty$. □

## 4.1 Viscosity supersolution property

Fix $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ and let $\varphi\in C^2([0,T],\mathbb{R}^d)$ be such that :

$$0 \;=\; (u_*-\varphi)(t_0,x_0) \;=\; \min(u_*-\varphi)$$

<u>1st Step .</u> Let $(t_n,x_n)$ be a sequence in $[0,T]\times\mathbb{R}^d$ such that :

$$(t_n,x_n)\longrightarrow(t_0,x_0)\quad\text{and}\quad u(t_n,x_n)\longrightarrow u_*(t_0,x_0)\quad\text{as } n\longrightarrow\infty .$$

Set $y_0:=\varphi(t_0,x_0)$, $z_0:=(x_0,y_0)$, $y_n:=u(t_n,x_n)+1/n$, $z_n:=(x_n,y_n)$, $\beta_n:=y_n-\varphi(t_n,x_n)$ and notice that

$$\beta_n \;\longrightarrow\; 0 \quad\text{as } n\longrightarrow\infty .$$

From the definition of the value function and the fact that $y_n>u(t_n,x_n)$, there exists some $\nu_n$ in $\mathcal{U}$ such that $Y^{\nu_n}_{t_n,z_n}(T)\ge g\left(X^{\nu_n}_{t_n,x_n}(T)\right)$ . Let $\theta_n$ be some $(t_n, T)$-valued stopping time to be defined later (see 2nd Step), then (DP1) of Proposition 4.1 yields :

$$Y^{\nu_n}_{t_n,z_n}(\theta_n) \geq u\left(\theta_n, X^{\nu_n}_{t_n,x_n}(\theta_n)\right) \quad .$$

Since $u \geq u_* \geq \varphi$, it follows that :

$$Y^{\nu_n}_{t_n,z_n}(\theta_n) \geq \varphi\left(\theta_n, X^{\nu_n}_{t_n,x_n}(\theta_n)\right) \quad .$$

By definition of process $Y^{\nu_n}_{t_n,z_n}$ together with Itô's Lemma, this provides :

$$\begin{aligned}
0 \leq \beta_n &+ \int_{t_n}^{\theta_n} r\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - r\left(s, X^{\nu_n}_{t_n,x_n}(s), \varphi(s, X^{\nu_n}_{t_n,x_n}(s)), \nu_n(s)\right) ds \\
&+ \int_{t_n}^{\theta_n} \mathcal{L}^{\nu_n(s)}\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right) ds \qquad (4.2)\\
&+ \int_{t_n}^{\theta_n} \left[a\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \alpha\left(s, X^{\nu_n}_{t_n,x_n}(s), \nu_n(s)\right) D\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right]' dW(s) \\
&+ \int_{t_n}^{\theta_n} \int_\Sigma \Big(b\left(s, Z^{\nu_n}_{t_n,z_n}(s-), \nu_n(s), \sigma\right) \\
&\qquad - b\left(s, X^{\nu_n}_{t_n,x_n}(s-), \varphi(s, X^{\nu_n}_{t_n,x_n}(s-)), \nu_n(s), \sigma\right)\Big) v(ds, d\sigma) \\
&+ \int_{t_n}^{\theta_n} \int_\Sigma \mathcal{G}^{\nu_n(s),\sigma}\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s-)\right) v(ds, d\sigma) \,.
\end{aligned}$$

<u>2nd Step</u>. We now choose a suitable sequence of stopping times $\theta_n$. Let $\eta$ be such that $0 < \eta < T - t_0$. Denote by $B_0$ the $\mathbb{R}^{d+2}$-ball of radius $\eta$ centered at $(t_0, z_0)$. From our assumptions on the coefficients $\beta$ and $b$, we have :

$$\gamma \;:=\; \sup_{B_0 \times U \times \Sigma} \left(|\beta(t,x,\nu,\sigma)| + |b(t,x,y,\nu,\sigma)|\right) \;<\; \infty \,.$$

We then define the ball :

$$\mathcal{N} \;:=\; B((t_0, z_0), \eta + 2\gamma) \,,$$

as well as the sequence of stopping times :

$$\tau_n \;:=\; \inf\left\{s \geq t_n \;:\; \left(s, Z^{\nu_n}_{t_n,z_n}(s)\right) \notin \mathcal{N}\right\} \,.$$

Clearly, since $\gamma$ describes the largest immediate jump of process $Z^{\nu}_{t,z}$ for $(t,z) \in B_0$, we have $\tau_n > t_n$ for all $n \geq 0$. Also from Lemma 4.1, since $(t_n, z_n) \longrightarrow$

$(t_0, z_0)$, the process $Z^{\nu}_{t_n,z_n}(.)$ converges $P$-a.s. (possibly along some subsequence) to $Z^{\nu}_{t_0,z_0}(.)$ uniformly on compact subsets. This implies that

$$\liminf_{n\to\infty}(\tau_n - t_n) \;\geq\; \frac{1}{2}(\tau_0 - t_0) \;>\; 0 \quad . \tag{4.3}$$

We now define the sequence $(\theta_n)_{n\geq 1}$ by

$$\theta_n \;:=\; \tau_n \wedge (t_n + h_n) \;,$$

where $h_n$ is defined as follows.
(i) If the set $\{n \geq 0 : \beta_n = 0\}$ is finite, then there exists a subsequence renamed $(\beta_n)_{n\geq 0}$ such that $\beta_n \neq 0$ for all $n$. So we may assume that $\beta_n \neq 0$ for all $n$ and we set $h_n := \sqrt{\beta_n}$.
(ii) If the set $\{n \geq 0 : \beta_n = 0\}$ is not finite, then there exists a subsequence renamed $(\beta_n)_{n\geq 0}$ such that $\beta_n = 0$ for all $n$. So we may assume that $\beta_n = 0$ for all $n$ and we set $h_n := 1/n$.

Notice that in both cases $\beta_n/h_n \to 0$ as $n$ tends to $\infty$.

3rd Step. We now define a family of equivalent probability measures which will be used in the 4th Step. For $(\nu, \sigma) \in U \times \Sigma$, we define :

$$\chi(\nu,\sigma) \;:=\; 1\!\!\mathrm{I}_{\{\mathcal{G}^{\nu,\sigma}\varphi(t_0,x_0)\leq 0\}} \;.$$

Fix some integer $k \geq 1$, and define for all $t_n \leq t \leq T$ :

$$\begin{aligned} M^k_n(t) \;:=\; \mathcal{E}\Bigg( & \int_{t_n}^{t\wedge\theta_n} -k(a - \alpha D\varphi)'\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) dW(s) \\ & + \int_{\Sigma}(k\;\chi(\nu_n(s),\sigma) + k^{-1} - 1)\tilde{v}(ds, d\sigma)\Bigg) \;. \end{aligned}$$

(recall that $\tilde{v}$ is the compensated measure associated to $v$, see Section 2). Notice that

$$-k(a - \alpha D\varphi)'\left(s\wedge\theta_n, Z^{\nu_n}_{t_n,z_n}(s\wedge\theta_n), \nu_n(s\wedge\theta_n)\right)$$

is bounded for all $s \geq t_n$. Hence, we clearly have $E\left[M^k_n(T)\right] = 1$, and we can define a probability measure $Q^k_n$ equivalent to $P$ by :

$$dQ^k_n/dP := M^k_n(T) \;.$$

By Girsanov's Theorem

$$\int_{t_n}^{\cdot}\int_{\Sigma}\Big(v(ds,d\sigma)-(k\,\chi(\nu_n(s),\sigma)+k^{-1})m(d\sigma)ds\Big)$$

is a $Q_n^k-$ martingale, and

$$\int_{t_n}^{\cdot\wedge\theta_n} dW(s)+k(a-\alpha D\varphi)\left(s,Z_{t_n,z_n}^{\nu_n}(s),\nu_n(s)\right)ds$$

is a $Q_n^k$- stopped Brownian motion.

Moreover, arguing like in Lemma 4.1, it is not difficult to see that for all $k$, there exists some $C>0$ such that

$$\begin{aligned}&\sup_{n\geq 1}E|M_n^k(T)|^2\leq C \\ &E(|M_n^k(T)-1|)^2\leq C(t_n+h_n-t_n)\to 0 \quad \text{as } n\to\infty\ .\end{aligned} \tag{4.4}$$

Hence, for all $k$,

$$M_n^k(T)\ \longrightarrow\ 1\ \ P-\text{a.s.}\quad \text{as } n\to\infty \tag{4.5}$$

(after possibly passing to a subsequence that we relabel as the original one).

We shall denote by $E_n^k$ the expectation operator under $Q_n^k$.

4th Step. Taking expectation in (4.2) under $Q_n^k$ and noticing that all the integrands are bounded on $[t_n,\theta_n]\!]$ leads to

$$\begin{aligned}
0\leq\beta_n&+E_n^k\int_{t_n}^{\theta_n} r\left(s,Z_{t_n,z_n}^{\nu_n}(s),\nu_n(s)\right)-r\left(s,X_{t_n,x_n}^{\nu_n}(s),\varphi(s,X_{t_n,x_n}^{\nu_n}(s)),\nu_n(s)\right)ds\\
&+E_n^k\int_{t_n}^{\theta_n}\mathcal{L}^{\nu_n(s)}\varphi\left(s,X_{t_n,x_n}^{\nu_n}(s)\right)ds\\
&-kE_n^k\int_{t_n}^{\theta_n}\left|a\left(s,Z_{t_n,z_n}^{\nu_n}(s),\nu_n(s)\right)-\alpha\left(s,X_{t_n,x_n}^{\nu_n}(s),\nu_n(s)\right)D\varphi\left(s,X_{t_n,x_n}^{\nu_n}(s)\right)\right|^2ds\\
&+E_n^k\int_{t_n}^{\theta_n}\int_{\Sigma}\left(k\chi(\nu_n(s),\sigma)+k^{-1}\right)b\left(s,Z_{t_n,x_n}^{\nu_n}(s),\nu_n(s),\sigma\right)m(d\sigma)ds\\
&-E_n^k\int_{t_n}^{\theta_n}\int_{\Sigma}\left(k\chi(\nu_n(s),\sigma)+k^{-1}\right)b\left(s,X_{t_n,x_n}^{\nu_n}(s),\varphi(s,X_{t_n,x_n}^{\nu_n}(s)),\nu_n(s),\sigma\right)m(d\sigma)ds\\
&+E_n^k\int_{t_n}^{\theta_n}\int_{\Sigma}\left(k\chi(\nu_n(s),\sigma)+k^{-1}\right)\mathcal{G}^{\nu_n(s),\sigma}\varphi\left(s,X_{t_n,x_n}^{\nu_n}(s)\right)m(d\sigma)ds\ .
\end{aligned}$$

Now, dividing the last inequality by $h_n$, sending $n$ to infinity, and using (4.4)-(4.5), we get by uniform integrability (after possibly passing to a subsequence)

$$
\begin{aligned}
0 \leq E \Bigg[ & \lim_{n\to\infty} \frac{1}{h_n} \int_{t_n}^{\theta_n} r\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - r\left(s, X^{\nu_n}_{t_n,x_n}(s), \varphi(s, X^{\nu_n}_{t_n,x_n}(s)), \nu_n(s)\right) ds \\
& + \frac{1}{h_n} \int_{t_n}^{\theta_n} \mathcal{L}^{\nu_n(s)} \varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right) ds \\
& - k \frac{1}{h_n} \int_{t_n}^{\theta_n} \left| a\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \alpha\left(s, X^{\nu_n}_{t_n,x_n}(s), \nu_n(s)\right) D\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right|^2 ds \\
& + \frac{1}{h_n} \int_{t_n}^{\theta_n} \int_\Sigma \left(k\chi(\nu_n(s),\sigma) + k^{-1}\right) b\left(s, Z^{\nu_n}_{t_n,x_n}(s), \nu_n(s), \sigma\right) m(d\sigma) ds \\
& - \frac{1}{h_n} \int_{t_n}^{\theta_n} \int_\Sigma \left(k\chi(\nu_n(s),\sigma) + k^{-1}\right) b\left(s, X^{\nu_n}_{t_n,x_n}(s), \varphi(s, X^{\nu_n}_{t_n,x_n}(s)), \nu_n(s), \sigma\right) m(d\sigma) ds \\
& + \frac{1}{h_n} \int_{t_n}^{\theta_n} \int_\Sigma \left(k\chi(\nu_n(s),\sigma) + k^{-1}\right) \mathcal{G}^{\nu_n(s),\sigma} \varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right) m(d\sigma) ds \Bigg] .
\end{aligned}
$$

We now use the following result whose proof is reported later on.

**Lemma 4.2** *Let $\psi(t,z,\nu,\sigma)\ :\ [0,T]\times I\!R^{d+1}\times U\times\Sigma \longrightarrow I\!R$ be a locally Lipschitz function in $(t,z)$ uniformly in $(\nu,\sigma)\in U\times\Sigma$. Then, for $t_n, z_n, h_n, \theta_n$ defined as above, we have :*

$$
\frac{1}{h_n} \int_{t_n}^{\theta_n} \int_\Sigma \left[\psi\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s), \sigma\right) - \psi\left(t_0, z_0, \nu_n(s), \sigma\right)\right] m(d\sigma) ds \longrightarrow 0
$$

*$P-$a.s. along some subsequence as $n$ tends to $\infty$.*

From the previous lemma, we get :

$$
0 \;\leq\; \liminf_{n\to\infty} \frac{1}{h_n} \int_{t_n}^{\theta_n} H_k(t_0, z_0, \nu_n(s)) ds \;, \tag{4.6}
$$

where

$$
\begin{aligned}
H_k(t_0,z_0,\nu) &= \mathcal{L}^\nu \varphi(t_0,x_0) - k\left|a(t_0,z_0,\nu) - \alpha(t_0,x_0,\nu) D\varphi(t_0,x_0)\right|^2 \\
&+ \int_\Sigma \left(k\chi(\nu,\sigma) + k^{-1}\right) \mathcal{G}^{\nu,\sigma}\varphi(t_0,x_0)\, m(d\sigma) .
\end{aligned}
$$

We next use the following standard argument due to Lions (1983) : denoting by $\mathcal{H}_k(t_0,z_0)$ the closed convex hull of the set $H_k(t_0,z_0,U)$, it follows from (4.3) and (4.6) that :

$$
0 \;\leq\; \sup \mathcal{H}_k(t_0,z_0) \;=\; \sup_{\nu\in U} H_k(t_0,z_0,\nu) .
$$

Since $U$ is compact, there exists some $\nu_k$ which attains the supremum on the right-hand side for each integer $k \geq 1$. By possibly passing to a subsequence, we can assume that the sequence $(\nu_k)_k$ converges to some $\hat{\nu} \in U$. Then, by sending $k$ to infinity, we see that :

$$\begin{aligned}
0 &\leq \mathcal{L}^{\hat{\nu}}\varphi\,(t_0, z_0, \hat{\nu}) \\
0 &\leq \mathcal{G}^{\hat{\nu},\sigma}\varphi\,(t_0, z_0, \hat{\nu}, .)^- \quad m(d\sigma) - \text{a.e.} \\
0 &\leq -\left|a\,(t_0, z_0, \hat{\nu}) - \alpha\,(t_0, x_0, \hat{\nu})\, D\varphi\,(t_0, x_0)\right|^2 \ .
\end{aligned}$$

The proof is completed by recalling that the function $\mathcal{G}^{\hat{\nu},\sigma}\varphi$ is continuous in variable $\sigma$. □

**Remark 4.1** The proof is similar in the pure jump case.

**Proof of Lemma 4.2.** Let $C$ be a generic constant. Since $\psi$ is locally Lipschitz in $(t, z)$ uniformly in $(\nu, \sigma)$, and $m(\Sigma) < \infty$, we have :

$$\begin{aligned}
&\frac{1}{h_n}\int_{t_n}^{\theta_n}\int_{\Sigma}\left|\psi\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s), \sigma\right) - \psi\,(t_0, z_0, \nu_n(s), \sigma)\right| m(d\sigma)ds \\
\leq\ & C\,\frac{1}{h_n}\int_{t_n}^{\theta_n}\left(|s - t_0| + |Z^{\nu_n}_{t_n,z_n}(s) - z_0|\right) ds \\
\leq\ & C\left(h_n + |t_n - t_0| + \sup_{t_n \leq s \leq t_n+\theta_n}|Z^{\nu_n}_{t_n,z_n}(s) - z_0|\right) \\
\leq\ & C\left(h_n + |t_n - t_0| + |z_n - z_0| + \sup_{t_n \leq s \leq t_n+h_n}|Z^{\nu_n}_{t_n,z_n}(s) - z_n|\right) \ .
\end{aligned}$$

The proof is completed by using Lemma 4.1. □

## 4.2 Viscosity subsolution property

Let $\varphi \in C^2([0, T] \times \mathbb{R}^d)$ and $(t_0, x_0)$ be a strict global maximizer of $u^* - \varphi$. Without loss of generality, we may assume that $(u^* - \varphi)(t_0, x_0) = 0$.

We argue by contradiction. Set $y_0 := \varphi(t_0, x_0)$, $z_0 := (x_0, y_0)$, and assume that

$$2\varepsilon \ := \ \widehat{\mathcal{H}}\varphi(t_0, x_0) \ > \ 0 \ .$$

Then, from our continuity assumptions, there exists some $\eta > 0$, such that for all $(t,x) \in B_0 := B((t_0,x_0),2\eta)$ and $\delta \in [-\eta,\eta]$ :

$$\widehat{\mathcal{H}}(\varphi+\delta)(t,x) \quad > \quad \varepsilon \,. \tag{4.7}$$

Let $(t_n,x_n)_{n\geq 0}$ be a sequence such that :

$$(t_n,x_n) \;\longrightarrow\; (t_0,x_0) \quad \text{and} \quad u(t_n,x_n) \;\longrightarrow\; u^*(t_0,x_0)$$

as $n$ tends to $\infty$. Set $y_n := u(t_n,x_n) - n^{-1}$, $z_n := (x_n,y_n)$ and notice that

$$u(t_n,x_n) - n^{-1} - \varphi(t_n,x_n) \;\text{ tends to }\; 0 \;\text{ as }\; n \;\text{ tends to }\; \infty \,. \tag{4.8}$$

Since $(t_n,z_n) \longrightarrow (t_0,z_0)$, we may assume without loss of generality that $(t_n,z_n) \in B_1 := B((t_0,z_0),\eta)$. In order to alleviate the notations, we shall denote :

$$Z_n(.) \;=\; (X_n(.),Y_n(.)) \;\;:=\;\; Z^{\hat{\nu}_n}_{t_n,z_n}(.)$$

the state process with initial data $(t_n,z_n)$ and feedback control process $\hat{\nu}_n(\cdot) := \psi(\cdot, X_n(.),\, Y_n(\cdot),\, D\varphi(\cdot,X_n(\cdot)))$ (existence of $Z_n$ follows from our Lipschitz and linear growth assumption on the coefficients of the diffusion uniformly in $\nu$, see Section 2). Notice that from (4.7) and the characterization of $U$ in terms of its support function (see Section 3)

$$[(s,X_n(s)) \in B_0 \text{ and } |Y_n(s) - \varphi(s,X_n(s))| \leq \eta] \;\;\Longrightarrow\;\; \hat{\nu}_n(s) \in U \,. \tag{4.9}$$

Define the stopping times :

$$\begin{aligned}
\theta_n^j \;&:=\; T \wedge \inf\{s > t_n \;:\; \Delta Z_n(s) \neq 0\} \;, \\
\theta_n^d \;&:=\; T \wedge \inf\{s > t_n \;:\; |Y_n(s) - \varphi(s,X_n(s))| \geq \eta\} \;.
\end{aligned}$$

Denote by $Z_n^c$ the continuous part of $Z_n$. Since $\theta_n^j$ is the time of the first jump of $Z_n$, we have :

$$Z_n(s \wedge \theta_n^j\text{-}) \;\;=\;\; Z_n^c(s \wedge \theta_n^j) \;, \qquad s \geq t_n. \tag{4.10}$$

Finally, define the sequences :

$$\tau_n \;:=\; T \wedge \inf\{s > t_n \;:\; (s,X_n(s)) \notin B_0\} \quad , \quad \theta_n \;:=\; \tau_n \wedge \theta_n^j \wedge \theta_n^d$$

together with the random set $\mathcal{J}_n := \{\omega \in \Omega \ : \ \tau_n < \theta_n^j \wedge \theta_n^d\}$. Notice that from the definition of $\theta_n$, (4.7) and (4.9) for all $s \geq t_n$,

$$\begin{aligned}
\hat{\nu}_n(s\wedge\theta_n{}^-) \in U \\
\varepsilon &< r\left(s\wedge\theta_n{}^-, Z_n(s\wedge\theta_n{}^-), \nu_n(s\wedge\theta_n{}^-)\right) + \mathcal{L}^{\nu_n(s\wedge\theta_n{}^-)}\varphi\left(s\wedge\theta_n{}^-, X_n(s\wedge\theta_n{}^-)\right) \\
&\quad -r\left(s\wedge\theta_n{}^-, X_n(s\wedge\theta_n{}^-), \varphi\left(s\wedge\theta_n{}^-, X_n(s\wedge\theta_n{}^-)\right), \nu_n(s\wedge\theta_n{}^-)\right) \qquad (4.11) \\
\varepsilon &< \inf_{\sigma\in\Sigma} b\left(s\wedge\theta_n{}^-, Z_n(s\wedge\theta_n{}^-), \nu_n(s\wedge\theta_n{}^-), \sigma\right) \\
&\quad +\mathcal{G}^{\nu_n(s\wedge\theta_n{}^-),\sigma}\varphi\left(s\wedge\theta_n{}^-, X_n(s\wedge\theta_n{}^-)\right) \\
&\quad -b\left(s\wedge\theta_n{}^-, X_n(s\wedge\theta_n{}^-), \varphi\left(s\wedge\theta_n{}^-, X_n(s\wedge\theta_n{}^-)\right), \nu_n(s\wedge\theta_n{}^-), \sigma\right) \ .
\end{aligned}$$

By (4.10), applying Itô's Lemma to $\varphi$ on $[t_n, \theta_n)$ leads to

$$\begin{aligned}
\varphi(\theta_n-, X_n(\theta_n-)) &= \varphi(t_n, x_n) + \int_{t_n}^{\theta_n} r(s, X_n^c(s), \varphi(s, X_n^c(s)), \hat{\nu}_n(s))ds \\
&\quad - \int_{t_n}^{\theta_n} \mathcal{L}^{\hat{\nu}_n(s)}\varphi(s, X_n^c(s))ds \\
&\quad + D\varphi(s, X_n^c(s))'\alpha(s, X_n^c(s), \hat{\nu}_n(s))dW(s) \ ,
\end{aligned}$$

where by definition of $Y_n$, $y_n$ and $\hat{\nu}_n$

$$\begin{aligned}
Y_n(\theta_n-) &= y_n + \int_{t_n}^{\theta_n} r(s, Z_n^c(s), \hat{\nu}_n(s))ds + a'(s, Z_n^c(s), \hat{\nu}_n(s))dW(s) \\
&= u(t_n, x_n) - n^{-1} \\
&\quad + \int_{t_n}^{\theta_n} r(s, Z_n^c(s), \hat{\nu}_n(s))ds + D\varphi(s, X_n^c(s))'\alpha(s, X_n^c(s), \hat{\nu}_n(s))dW(s) \ .
\end{aligned}$$

Then, by a standard comparison result on the dynamics of $\varphi(\cdot, X_n(\cdot))$ and $Y_n(\cdot)$, the definition of $\hat{\nu}_n$, $\theta_n$ and (4.11), we obtain :

$$Y_n(\theta_n-) - \varphi(\theta_n-, X_n(\theta_n-)) \ \geq \ u(t_n, x_n) - \frac{1}{n} - \varphi(t_n, x_n) \ > \ -\eta \qquad (4.12)$$

where the last inequality is obtained by taking some sufficiently large $n$ and using (4.8).

We now provide a contradiction to (DP2) at stopping time $\theta_n$ for some large $n$. We study separately the case where $\omega \in \mathcal{J}_n$ and the case where $\omega \in$

$\mathcal{J}_n^c$.

Case 1, on $\mathcal{J}_n$: Define

$$-\zeta \ := \ \sup_{(t,x)\in\partial_p B_0} (u^* - \varphi)(t,x) \tag{4.13}$$

where $\partial_p B_0$ stands for the parabolic boundary of $B_0$, i.e. $\partial_p B_0 := [t_0 - 2\eta, t_0 + 2\eta] \times \partial B(x_0, 2\eta) \cup \{t_0 + 2\eta\} \times \bar{B}(x_0, 2\eta)$. Since $(t_0, x_0)$ is a strict global maximizer of $u^* - \varphi$, we have $\zeta > 0$.

Recall that on $\mathcal{J}_n$, $\theta_n^j > \theta_n = \tau_n$. Hence, from (4.10), $Z_n(\cdot \wedge \theta_n)$ is continuous on $\mathcal{J}_n$. Together with (4.12) and the fact that $u \le u^*$, this leads to

$$\begin{aligned}
[Y_n(\theta_n) - u(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n} \ &= \ [Y_n(\tau_n) - u(\tau_n, X_n(\tau_n))]\, 1\!\!1_{\mathcal{J}_n} \\
&\ge \ \big[\varphi(\tau_n, X_n(\tau_n)) - u^*(\tau_n, X_n(\tau_n)) \\
&\qquad + u(t_n, x_n) - n^{-1} - \varphi(t_n, x_n)\big]\, 1\!\!1_{\mathcal{J}_n} \ .
\end{aligned}$$

Since by continuity, $(\tau_n, X_n(\tau_n)) \in \partial_p B_0$, on $\mathcal{J}_n$, (4.13) implies that

$$[Y_n(\theta_n) - u(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n} \ \ge \ \big[\zeta + u(t_n, x_n) - n^{-1} - \varphi(t_n, x_n)\big]\, 1\!\!1_{\mathcal{J}_n} \ .$$

Using (4.8) and assuming that $n$ is large enough, this proves such that :

$$[Y_n(\theta_n) - u(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n} \ \ge \ (\zeta/2) 1\!\!1_{\mathcal{J}_n} \quad \text{for some } \zeta > 0. \tag{4.14}$$

Case 2, on $\mathcal{J}_n^c$ : Recall that on $\mathcal{J}_n^c$, $\theta_n = (\theta_n^j \wedge \theta_n^d)$. From the definition of $\theta_n^d$, (4.10) and (4.12), we have

$$\begin{aligned}
[Y_n(\theta_n) - \varphi(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d < \theta_n^j} \ &= \ \big[Y_n^c(\theta_n^d) - \varphi(\theta_n^d, X_n^c(\theta_n^d))\big]\, 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d < \theta_n^j} \\
&= \ \eta 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d < \theta_n^j}
\end{aligned} \tag{4.15}$$

On the other hand, on $\mathcal{J}_n^c \cap \{\theta_n^d \ge \theta_n^j\}$

$$\begin{aligned}
& Y_n(\theta_n) - \varphi(\theta_n, X_n(\theta_n)) \\
&= Y_n(\theta_n^j -) - \varphi(\theta_n^j -, X_n(\theta_n^j -))
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Sigma b(\theta_n^j-, Z_n(\theta_n^j-), \hat{\nu}_n(\theta_n^j-), \sigma)v(\{\theta_n^j\}, d\sigma) \\
&+\int_\Sigma \mathcal{G}^{\hat{\nu}_n(\theta_n^j-),\sigma}\varphi(\theta_n^j-, X_n(\theta_n^j-))v(\{\theta_n^j\}, d\sigma) \\
&-\int_\Sigma b(\theta_n^j-, X_n(\theta_n^j-), \varphi(\theta_n^j-, X_n(\theta_n^j-)), \hat{\nu}_n(\theta_n^j-), \sigma)v(\{\theta_n^j\}, d\sigma)\ .
\end{aligned}
$$

Using (4.11) and (4.12), this proves that

$$
[Y_n(\theta_n) - \varphi(\theta_n, X_n(\theta_n))]\ 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d \geq \theta_n^j} \geq \left[u(t_n, x_n) - \frac{1}{n} - \varphi(t_n, x_n) + \varepsilon\right] 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d \geq \theta_n^j}\ .
$$

Finally, by (4.8), the fact that $u \leq u^* \leq \varphi$ and (4.15), this proves that we can find some $n$ such that :

$$
[Y_n(\theta_n) - u(\theta_n, X_n(\theta_n))]\ 1\!\!1_{\mathcal{J}_n^c} \geq (\varepsilon/2 \wedge \eta) 1\!\!1_{\mathcal{J}_n^c} \text{ for some } \varepsilon > 0 \text{ and } \eta > 0. \tag{4.16}
$$

Conclusion of the proof. We can now conclude the proof. By (4.14), (4.16) and assuming that $n$ is large enough, we get

$$
Y_n(\theta_n) - u(\theta_n, X_n(\theta_n)) \ \geq\ \frac{\varepsilon \wedge \zeta}{2} \wedge \eta\ >\ 0\ .
$$

Since, $y_n < u(t_n, x_n)$ and $\hat{\nu}_n(\cdot \wedge \theta_n) \in \mathcal{U}$, this contradicts (DP2). □

**Remark 4.2** The result is proved by similar arguments in the pure jump case. It suffices to define $\hat{\nu}_n$ as the constant process equal to $\hat{\nu}(t_0, x_0)$, where $\hat{\nu}(t_0, x_0)$ is the argmax of

$$
\max_{\nu \in U} \min \left\{ \mathcal{L}^\nu \varphi(t_0, x_0)\ ;\ \inf_{\sigma \in \Sigma} \mathcal{G}^{\nu,\sigma}\varphi(t_0, x_0) \right\}\ .
$$

# 5 Proof of the terminal condition

We split the proof of Theorem 3.2 in different lemmas.

## 5.1 Terminal condition for $\underline{G}$

**Lemma 5.1** *For all $x \in \mathbb{R}^d$, we have $\underline{G}(x) \geq g_*(x)$.*

**Proof.** Fix $x \in I\!R^d$ and let $(t_n, x_n)$ be a sequence in $(0,T) \times I\!R^d$ such that $u(t_n, x_n)$ tends to $\underline{G}(x)$ as $n$ tends to $\infty$. Set $y_n := u(t_n, x_n) + n^{-1}$. By definition of the stochastic control problem $u(t_n, x_n)$, there exists some control $\nu_n \in \mathcal{U}$ such that :

$$Y^{\nu_n}_{t_n,x_n,y_n}(T) \;\; \geq \;\; g\left(X^{\nu_n}_{t_n,x_n}(T)\right) \; .$$

Now, observe that $Z^{\nu_n}_{t_n,x_n,y_n}(T) \longrightarrow (x, \underline{G}(x))$ $P$-a.s. as $n \to \infty$ after possibly passing to a subsequence (see Lemma 4.1). Then, taking limits in the last inequality provides :

$$\underline{G}(x) \;\; \geq \;\; \liminf_{x' \to x} g\left(x'\right) \;\; = \;\; g_*(x) \; .$$

□

**Lemma 5.2** *Let $x_0 \in I\!R^d$ and $f \in C^2(I\!R^d)$ be such that :*

$$0 \;=\; (\underline{G} - f)(x_0) \;\; = \;\; \min_{x \in I\!R^d} (\underline{G} - f)(x) \; .$$

*Then,*

$$\widehat{\mathcal{T}} f\,(x_0) \;\; \geq \;\; 0 \; .$$

**Proof.** Let $f$ and $x_0$ be as in the above statement. Let $(s_n, \xi_n)_n$ be a sequence in $(0,T) \times I\!R^d$ satisfying :

$$(s_n, \xi_n) \;\longrightarrow\; (T, x_0)\;, \quad s_n < T \quad \text{and} \quad u_*(s_n, \xi_n) \;\longrightarrow\; \underline{G}(x_0)\; .$$

The existence of such a sequence is justified by the fact that we may always replace $u$ by $u_*$ in the definition of $\underline{G}$. For all $n \in I\!N$ and $k > 0$, we define :

$$\varphi^k_n(t,x) \;\; := \;\; f(x) - \frac{k}{2}|x - x_0|^2 + k\,\frac{T-t}{T-s_n} \; .$$

Since the function $\beta$ is continuous in $(t, x, \nu)$, bounded in $\sigma$ and $U$ is compact, we see that :

$$\eta \;\; := \;\; \sup\left\{|\beta(t,x,\sigma,\nu)| \;:\; \sigma \in \Sigma,\; \nu \in U,\; |t - t_0| + |x - x_0| \leq C\right\} \;<\; \infty \;,$$

where $C > 0$ is a given constant. Let $\bar{B}_0$ denote the closed ball of radius $\eta$ centered at $x_0$. Notice that for $t \in [s_n, T]$, we have $0 \le (T-t)(T-s_n)^{-1} \le 1$, and therefore :

$$\lim_{k\to 0} \limsup_{n\to\infty} \sup_{(t,x)\in[s_n,T]\times\bar{B}_0} |\varphi_n^k(t,x) - f(x)| \;=\; 0 \,. \tag{5.1}$$

Next, let $(t_n^k, x_n^k)$ be a sequence of local minimizers of $u_* - \varphi_n^k$ on $[s_n, T] \times \bar{B}_0$ and set $e_n^k := (u_* - \varphi_n^k)(t_n^k, x_n^k)$. We shall prove later that, after possibly passing to a subsequence :

$$\text{for all } \; k > 0 \,, \quad (t_n^k, x_n^k) \longrightarrow (T, x_0) \,, \tag{5.2}$$

$$\text{for all } \; k > 0 \,, \quad t_n^k \;<\; T \;\; \text{for sufficiently large } n \,, \tag{5.3}$$

$$u_*(t_n^k, x_n^k) \longrightarrow \underline{G}(x_0) = f(x_0) \;\; \text{as} \;\; n \to \infty \text{ and } k \to 0 \,. \tag{5.4}$$

First notice from (5.2) and a standard diagonalization argument, that we may assume that $x_n^k \in \mathrm{Int}\bar{B}_0$. Therefore, by (5.3), for all $k$, $(t_n^k, x_n^k)$ is a sequence of local minimizers on $[s_n, T) \times \mathrm{Int}\bar{B}_0$.

Also, notice that from (5.1), (5.2) and (5.4)

$$\text{for all } \; k > 0 \,, \;\; D\varphi_n^k(t_n^k, x_n^k) \;=\; Df(x_n^k) - k(x_n^k - x_0) \to Df(x_0) \,, \tag{5.5}$$

$$\text{and } \;\; \lim_{k\to 0} \lim_{n\to\infty} e_n^k \;=\; 0 \;\; . \tag{5.6}$$

Hence, for sufficiently large $n$, using Theorem 3.1, (5.3) and the fact that $(t_n^k, x_n^k)$ is a local minimizer for $u_* - \varphi_n^k$, we get,

$$\widehat{\mathcal{T}}(\varphi + e_n^k)(t_n^k, x_n^k) \;\ge\; 0 \quad \text{for all} \quad n \in I\!N, \; k > 0 \,.$$

The statement of the lemma is then obtained by taking limits as $n \to \infty$, then as $k \to 0$, and using (5.1), (5.2), (5.4), (5.5), (5.6) as well as the continuity of the involved functions.

In order to complete the proof, it remains to show that (5.2), (5.3) and (5.4) hold.

Notice, from the convergence assumption on $(s_n, \xi_n)$, that we can find some large integer $N$ (independent of $k$) such that for all $n \ge N$ and $k > 0$ :

$$(u_* - \varphi_n^k)(s_n, \xi_n) \;=\; u_*(s_n, \xi_n) - f(\xi_n) + \frac{k}{2}|\xi_n - x_0| - k \;\;\le\;\; -\frac{k}{2} \;<\; 0 \,.$$

On the other hand, by definition of the test function $f$,

$$(u_* - \varphi_n^k)(T,x) \;=\; \underline{G}(x) - f(x) + \frac{k}{2}|x-x_0|^2 \;\geq\; 0 \quad \text{for all} \quad x \in I\!R^d .$$

Comparing the two inequalities and using the definition of $(t_n^k, x_n^k)$ provides (5.3).

For all $k > 0$, let $x^k \in \bar{B}_0$ be the limit of some subsequence of $(x_n^k)_n$. Then by definition of $x_0$, we have :

$$\begin{aligned} 0 &\leq (\underline{G} - f)(x^k) - (\underline{G} - f)(x_0) \\ &\leq \liminf_{n\to\infty} (u_* - \varphi_n^k)(t_n^k, x_n^k) - (u_* - \varphi_n^k)(s_n, \xi_n) - \frac{k}{2}|x^k - x_0|^2 + k\frac{T - t_n^k}{T - s_n} - k . \end{aligned}$$

Since $s_n \leq t_n^k < T$, it follows from the definition of $(t_n^k, x_n^k)$ that :

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} (u_* - \varphi_n^k)(t_n^k, x_n^k) - (u_* - \varphi_n^k)(s_n, \xi_n) - \frac{k}{2}|x^k - x_0|^2 \\ &\leq -\frac{k}{2}|x^k - x_0|^2 \;\leq\; 0 , \end{aligned}$$

This proves that we must have $x^k = x_0$, and therefore (5.2) holds since the convergence of the sequence $(t_n^k)_n$ to $T$ is trivial. Notice that the two last terms in the previous inequality tend to 0. This proves that

$$\liminf_{n\to\infty} (u_* - \varphi_n^k)(t_n^k, x_n^k) \;=\; 0$$

which, together with (5.1) and (5.2), implies (5.4), after possibly passing to some subsequence. □

## 5.2 Terminal condition for $\bar{G}$

Let $f$ be in $C^2(I\!R^d)$ and $x_0 \in I\!R^d$ be such that

$$0 \;=\; (\bar{G} - f)(x_0) = \max_{I\!R^d} \,(\textit{strict})(\bar{G} - f) .$$

We assume that

$$\min\left\{ f(x_0) - g^*(x_0) \;;\; \widehat{\mathcal{T}} f\,(x_0) \right\} \;>\; 0 , \tag{5.7}$$

and we work towards a contradiction to (DP2) of Proposition 4.1 in the 4th Step of this proof.

1st Step. Fix some arbitrarily small scalar $c > 0$ and define on $[0,T] \times I\!R^d$ :

$$\varphi(t,x) \;:=\; f(x) + c|x - x_0|^2 + (T-t)^{\frac{1}{2}}$$

Notice that for all $x \in I\!R^d$ :

$$\frac{\partial \varphi(t,x)}{\partial t} \longrightarrow -\infty \quad \text{as} \quad t \longrightarrow T\,. \tag{5.8}$$

Recall that $r$, $D\varphi$, $D^2\varphi$, $\alpha$ and $\rho$ are continuous and therefore locally bounded. Hence, we may assume by (5.8) that for all $(t,x,y)$ in a suitable neighborhood of $(T, x_0, f(x_0))$

$$r(t,z,\nu) - r(t,x,\varphi(t,x),\nu) + \mathcal{L}^\nu \varphi(t,x) \;>\; 0 \qquad \text{for all } \nu \in U. \tag{5.9}$$

Now, notice that $\beta$ is bounded uniformly in $\sigma$ on any neighborhood of $(T, x_0)$ and that $\varphi(T,x_0) = f(x_0)$. Hence, by (5.7) and by taking a sufficiently small $c$, we may assume that $\varphi$ satisfies :

$$\varphi(T,x_0) - g^*(x_0) > 0 \quad \text{and} \quad \inf_{\sigma\in\Sigma} \widehat{\mathcal{G}}^\sigma(\varphi)(T,x_0) > 0\ .$$

Then, by upper-semicontinuity of $g^*$, continuity of $\varphi$, and continuity of $b$ and $\beta$ uniformly in $\sigma$ and $\nu$, there exists some $\varepsilon > 0$ and some $\eta > 0$ such that for all $(t,x) \in \bar{B}_0 := [T-\eta, T] \times \bar{B}(x_0,\eta)$ and $\delta \in [-\eta,\eta]$ :

$$\varphi(t,x) + \delta - g^*(x) > \varepsilon \quad \text{and} \quad \inf_{\sigma\in\Sigma} \widehat{\mathcal{G}}^\sigma(\varphi+\delta)(t,x) > \varepsilon\ .$$

Finally, by using (5.9) and by taking a sufficiently small $\eta$, we may assume that for all $(t,x,\delta) \in \bar{B}_0 \times [-\eta,\eta]$ :

$$\min\left\{\varphi(t,x) + \delta - g^*(x)\ ;\ \widehat{\mathcal{H}}(\varphi+\delta)(t,x)\right\} \;>\; \varepsilon\,. \tag{5.10}$$

2nd Step. Let $(s_n,\xi_n)$ be a sequence in $[T-\eta/2, T) \times \bar{B}(x_0,\eta) \subset \bar{B}_0$ satisfying :

$$(s_n,\xi_n) \longrightarrow (T,x_0)\ , \quad s_n < T \quad \text{and} \quad u^*(s_n,\xi_n) \longrightarrow \bar{G}(x_0)\ .$$

Let $(t_n, x_n)$ be a maximizer of $(u^* - \varphi)$ on $[s_n, T]\times \bar{B}(x_0, \eta) \subset \bar{B}_0$. For all $n$, let $(t_n^k, x_n^k)_k$ be a subsequence in $[s_n, T] \times \bar{B}(x_0, \eta)$ satisfying :

$$(t_n^k, x_n^k) \longrightarrow (t_n, x_n) \quad \text{and} \quad u(t_n^k, x_n^k) \longrightarrow u^*(t_n, x_n) \ .$$

We shall prove later that

$$(t_n, x_n) \longrightarrow (T, x_0) \quad \text{and} \quad u^*(t_n, x_n) \longrightarrow \bar{G}(x_0) \tag{5.11}$$

and that there exists a subsequence of $(t_n^k, x_n^k)_{k,n}$, relabelled $(t'_n, x'_n)$, satisfying :

$$(t'_n, x'_n) \to (T, x_0) \quad \text{and} \quad u(t'_n, x'_n) \to \bar{G}(x_0) \ \ , \text{where for all } n,\ t'_n < T \ . \tag{5.12}$$

<u>3rd Step.</u> Consider the sequence $(t'_n, x'_n)$ of the 2nd Step. Set $y'_n := u(t'_n, x'_n) - n^{-1}$, $z'_n := (x'_n, y'_n)$ and notice that

$$u(t'_n, x'_n) - n^{-1} - \varphi(t'_n, x'_n) \ \text{ tends to } \ 0 \ \text{ as } \ n \ \text{ tends to } \ \infty \ . \tag{5.13}$$

Hence $(t'_n, z'_n) \longrightarrow (T, x_0, f(x_0))$ and we may assume without loss of generality that $(t'_n, x'_n) \in \mathrm{Int}\bar{B}_0$ and that $|y'_n - \varphi(t'_n, x'_n)| \leq \eta$ for all $n$. In order to alleviate the notations, we shall denote :

$$Z_n(.) \ = \ (X_n(.), Y_n(.)) \ \ := \ \ Z_{t_n, z_n}^{\hat{\nu}_n}(.)$$

the state process with initial data $(t'_n, z'_n)$ and control process $\hat{\nu}_n(\cdot) := \psi(\cdot, X_n(.),$ $Y_n(\cdot), D\varphi(\cdot, X_n(\cdot)))$. Recall that by (5.12), $t'_n < T$ for all $n$ and define the stopping times :

$$\begin{aligned}
\theta_n^j &:= T \wedge \inf\{s > t'_n \ : \ \Delta Z_n(s) \neq 0\} \ , \\
\theta_n^d &:= T \wedge \inf\{s > t'_n \ : \ |Y_n(s) - \varphi(s, X_n(s))| \geq \eta\} \\
\tau_n &:= T \wedge \inf\{s > t'_n \ : \ (s, X_n^c(s)) \notin B_0\} \ , \\
\theta_n &:= \tau_n \wedge (\theta_n^j \wedge \theta_n^d) \ .
\end{aligned}$$

together with the random set $\mathcal{J}_n := \{\omega \in \Omega \ : \ \tau_n < (\theta_n^j \wedge \theta_n^d)\}$. Finally, define

$$-\zeta := \sup_{x \in \partial \bar{B}(x_0, \eta)} (\bar{G} - f)(x) \ .$$

Since $x_0$ is a strict maximizer and $(\bar{G}-f)(x_0)=0$, we have $\zeta>0$.

4th Step. We can now prove the required contradiction. Arguing like in Subsection 4.2 and using (5.10) as well as (5.13), it is easily checked that we can find some $n$ such that :

$$\begin{aligned} & Y_n(\theta)-u(\theta_n, X_n(\theta_n) \;\geq\; \zeta/2 1\!\!\mathrm{I}_{\mathcal{J}_n}+(\varepsilon/2\wedge\eta) 1\!\!\mathrm{I}_{\mathcal{J}_n^c} \;>\; 0 \\ \text{and} \quad & \hat{\nu}_n(\cdot\wedge\theta_n)\in\mathcal{U} \text{ on } [t_n,T] . \end{aligned}$$

Since, $y_n' < u(t_n', x_n')$ this leads to the required contradiction of (DP2).

5th Step. It remains to prove (5.11) and (5.12). Clearly, $t_n\to T$. Let $\hat{x}\in[x_0-\eta, x_0+\eta]$ be such that $x_n\to\hat{x}$, along some subsequence. Then, by definition of $f$ and $x_0$ :

$$\begin{aligned} 0 &\geq (\bar{G}-f)(\hat{x})-(\bar{G}-f)(x_0) \\ &\geq \limsup_{n\to\infty}(u^*-\varphi)(t_n,x_n)+c|\hat{x}-x_0|^2-(u^*-\varphi)(s_n,\xi_n) \\ &\geq c|\hat{x}-x_0|^2\geq 0 , \end{aligned}$$

where the third inequality is obtained by definition of $(t_n,x_n)$. Then, $\hat{x}=x_0$ and, by continuity of $\varphi$, $u^*(t_n,x_n)\to\bar{G}(x_0)$. This also proves that

$$\lim_n\lim_k(t_n^k,x_n^k)=(T,x_0) \quad\text{and}\quad \lim_n\lim_k u(t_n^k,x_n^k)=\bar{G}(x_0) . \tag{5.14}$$

Now assume that $\mathrm{card}\{(n,k)\in I\!N\times I\!N \;:\; t_n^k=T\}=\infty$. Since $u(T,\cdot)=g(\cdot)$, there exists a subsequence, relabelled $(t_n^k,x_n^k)$, such that :

$$\limsup_n\limsup_k u(t_n^k,x_n^k)\leq g^*(x_0) .$$

Since by assumption $g^*(x_0)<f(x_0)=\bar{G}(x_0)$, this leads to a contradiction with (5.14). Hence, $\mathrm{card}\{(n,k)\in I\!N\times I\!N \;:\; t_n^k=T\}<\infty$, and, using a diagonalization argument, we can construct a subsequence $(t_n',x_n')_n$ of $(t_n^k,x_n^k)_{n,k}$ satisfying (5.12). □

# 6 Boundary condition in the pure jump model

In this section we always assume $a = \alpha = 0$, i.e. we consider the pure jump model. We prove that the boundary condition may always be considered in the classical sense (in opposition to the viscosity sense). This comes from the fact that, in the pure jump model, the boundary equation does not depend on the derivatives of the test function.

We first characterize the boundary condition in the viscosity sense.

**Theorem 6.1** *Let the conditions of Theorem 3.3 hold. Then, if $\underline{G}$ is finite, it is a viscosity supersolution on $\mathbb{R}^d$ of*

$$\min\left\{\varphi(x) - g_*(x)\ ;\ \sup_{\nu\in U}\inf_{\sigma\in\Sigma}\mathcal{G}^{\nu,\sigma}\varphi(x)\right\} \;=\; 0\ , \tag{6.1}$$

*and, if $\bar{G}$ is finite, it is a viscosity subsolution on $\mathbb{R}^d$ of*

$$\min\left\{\varphi(x) - g^*(x)\ ;\ \sup_{\nu\in U}\inf_{\sigma\in\Sigma}\mathcal{G}^{\nu,\sigma}\varphi(x)\right\} \;=\; 0\ . \tag{6.2}$$

**Proof.** This is obtained by similar arguments as in the proof of Theorem 3.2 by using Theorem 3.3. □

The characterization in the classical sense is obtained from the previous theorem together with the following lemma.

**Lemma 6.1** *Let $f$ be a finite lower-semicontinuous function on $\mathbb{R}^d$. Then, for all $x_0 \in \mathbb{R}^d$, $\eta > 0$ and compact subset $K$ of $\mathbb{R}^d$ such that $B(x_0,\eta) \subset K$, there exists a sequence $(x_n^\lambda, \varphi_n^\lambda)_{\{n\in\mathbb{N},\ \lambda>0\}} \in B(x_0,\eta)\times C^0(K)$ such that for all $\lambda > 0$ :*

$$\min_{B(x_0,\eta)}(f-\varphi_n^\lambda)(x) \;=\; (f-\varphi_n^\lambda)(x_n^\lambda) \;\geq\; 0\ ,$$
$$\lim_{n\to\infty}\left(x_n^\lambda, f(x_n^\lambda), \varphi_n^\lambda(x_n^\lambda)\right) = (x_0, f(x_0), f(x_0))\ .$$

*Moreover for all sequence $x_n \in K$ such that $x_n \longrightarrow x$,*

$$\liminf_{\lambda\to 0}\liminf_{n\to\infty}\varphi_n^\lambda(x_n) \geq f(x)\ .$$

**Proof.** 1st Step. Fix $f$, $x_0$ and $\eta$ as in the previous lemma. From Lemma 3.5 in Reny (1999), there exists a sequence $(f_n)_n \in C^0(K)$ such that :

$$f_n(x) \;\leq\; f(x) \quad \forall\, x \in K \quad \text{and} \quad \liminf_{n\to\infty} f_n(x_n) \;\geq\; f(x) \quad \forall\, x_n \to x \;. \tag{6.3}$$

For all $(n, \lambda) \in I\!N \times (0, \infty)$, define on $K$ :

$$\varphi_n^\lambda(x) \;:=\; f_n(x) - \lambda|x_0 - x| - \lambda|f(x_0) - f_n(x)| \;\leq\; f(x)\;.$$

Clearly, $\varphi_n^\lambda \in C^0(K)$ and, by (6.3), $f_n(x_0) \longrightarrow f(x_0)$ as $n \longrightarrow \infty$. This proves that for all $\lambda > 0$ :

$$\lim_{n\to\infty} \varphi_n^\lambda(x_0) \;=\; f(x_0)\;. \tag{6.4}$$

Let $x_n^\lambda$ be a minimizer in $\bar{B}(x_0, \eta)$ of $f - \varphi_n^\lambda$. Then :

$$0 \;\leq\; \liminf_{n\to\infty} (f - \varphi_n^\lambda)(x_0) - (f - \varphi_n^\lambda)(x_n^\lambda) \;=\; \liminf_{n\to\infty} -(f - \varphi_n^\lambda)(x_n^\lambda)\;. \tag{6.5}$$

By construction of $\varphi_n^\lambda$ and since $f_n \leq f$ by (6.3), this yields :

$$0 \;\leq\; \liminf_{n\to\infty} (\varphi_n^\lambda - f_n)(x_n^\lambda) \;=\; \limsup_{n\to\infty} -\lambda|x_0 - x_n^\lambda| - \lambda|f(x_0) - f_n(x_n^\lambda)| \;\leq\; 0\;.$$

Therefore, for all $\lambda > 0$, $x_n^\lambda \longrightarrow x_0$, $f_n(x_n^\lambda) \longrightarrow f(x_0)$ and $\varphi_n^\lambda(x_n^\lambda) \longrightarrow f(x_0)$ as $n$ tends to $\infty$. By (6.5), this also proves that $\liminf_{n\to\infty} f(x_n^\lambda) \leq \lim_{n\to\infty} \varphi_n^\lambda(x_n^\lambda) = f(x_0)$. Hence, by lower-semicontinuity of $f$, we get that $f(x_n^\lambda) \to f(x_0)$ (possibly after passing to a subsequence). Finally notice that, since $x_0 \in B(x_0, \eta)$ and $B(x_0, \eta)$ is open, we may assume that, for all $n$, $x_n^\lambda \in B(x_0, \eta)$.
2nd Step. Let $x_n$ be a sequence in $K$ such that $x_n \longrightarrow x$ for some $x \in K$. Notice that by footnote 35 in Reny (1999), we may assume that $f_n$ is uniformly bounded from below in $n$ on $K$. Since $f$ is finite, $f_n$ is uniformly bounded in $n$ on $K$. Then, by construction of $\varphi_n^\lambda$ and (6.3) : $\liminf_{\lambda\to 0} \liminf_{n\to\infty} \varphi_n^\lambda(x_n) \geq f(x)$.
□

We conclude the first part of the proof of Theorem 3.4 by the following lemma

**Lemma 6.2** *Assume that $\underline{G}$ is finite, then for all $x \in I\!R^d$ :*

$$\underline{G}(x) + \sup_{\nu \in U} \inf_{\sigma \in \Sigma} b(T, x, \underline{G}(x), \nu, \sigma) - \underline{G}(x + \beta(T, x, \nu, \sigma)) \;\; \geq \;\; 0 \; .$$

**Proof.** Fix $x_0 \in I\!R^d$, $\eta > 0$ and a compact subset $K$ of $I\!R^d$ such that $B(x_0, \eta) \subset K$. By Lemma 6.1, there exists a sequence $(x_n^\lambda, \varphi_n^\lambda)_{\{n \in I\!N,\, \lambda > 0\}} \in B(x_0, \eta) \times C^0(K)$ such that for all $\lambda > 0$ :

$$\begin{aligned} \min_{B(x_0,\eta)} (\underline{G} - \varphi_n^\lambda)(x) \;\; &= \;\; (\underline{G} - \varphi_n^\lambda)(x_n^\lambda) \;\; \geq \;\; 0 \; , \\ \lim_{n \to \infty} (x_n^\lambda, \underline{G}(x_n^\lambda), \varphi_n^\lambda(x_n^\lambda)) &= (x_0, \underline{G}(x_0), \underline{G}(x_0)) \; , \end{aligned}$$

and such that for all sequence $x_n \in K$ such that $x_n \longrightarrow x$, $\liminf_{\lambda \to 0} \liminf_{n \to \infty} \varphi_n^\lambda(x_n) \geq \underline{G}(x)$.

Define $e_n^\lambda := (\underline{G} - \varphi_n^\lambda)(x_n^\lambda)$ and notice that $e_n^\lambda$ tends to 0. By Theorem 6.1, for all $(n, \lambda) \in I\!N \times (0, \infty)$, there exists some $\nu_n^\lambda \in U$ such that for all $\sigma \in \Sigma$ :

$$\underline{G}(x_n^\lambda) + b(T, x_n^\lambda, \underline{G}(x_n^\lambda), \nu_n^\lambda, \sigma) - \varphi_n^\lambda(x_n^\lambda + \beta(T, x_n^\lambda, \nu_n^\lambda, \sigma)) - e_n^\lambda \;\; \geq \;\; 0 \; .$$

Notice that since $\beta$ is continuous and bounded in $\sigma$, we may assume, possibly by changing $K$, that, for all $(n, \lambda, \nu, \sigma) \in I\!N \times (0, \infty) \times U \times \Sigma$, $x_n^\lambda + \beta(T, x_n^\lambda, \nu_n^\lambda, \sigma) \in K$. Then, applying the above convergence results, taking limsup as $n$ tends to $\infty$ and then sending $\lambda$ to 0 in the previous inequality, we get, after possibly passing to a subsequence, that for all $\sigma \in \Sigma$ :

$$\underline{G}(x_0) + b(T, x_0, \underline{G}(x_0), \hat{\nu}, \sigma) - \underline{G}(x_0 + \beta(T, x_0, \hat{\nu}, \sigma)) \;\; \geq \;\; 0 \; ,$$

for some $\hat{\nu}$ in the compact set $U$. □

**Lemma 6.3** *Assume that $\bar{G}$ is finite, then for all $x \in I\!R^d$ :*

$$H^*(\bar{G}(x)) \;\; \leq \;\; 0 \; .$$

**Proof.** The last result is obtained by using Theorem 6.1, similar arguments as in the proof of Lemmas 6.1 and 6.2, and by recalling that $b$ and $\beta$ are Lipschitz continuous uniformly in $(\nu, \sigma)$. □

# 7 Existence and continuity of the smallest solution of (3.5)

**Lemma 7.1** *Under the conditions of Proposition 3.1,* (3.5) *admits a smallest solution $\ell$. Moreover, $\ell$ is lower-semicontinuous and satisfies $H_*(\ell(x)) = 0$ for all $x \in \mathbb{R}^d$.*

**Remark 7.1** The proof of the last Lemma is based on the construction of an increasing sequence of lower-semicontinuous functions that converges to $\ell$.

**Proof.** Consider the sequence :

$$v^0 \ := \ g_* \qquad \text{and } \forall\, n \in \mathbb{N} \qquad v^{n+1} \ := \ \max\{\, g_* \ ; \ \mathcal{M}v^n \,\} \qquad , \tag{7.1}$$

where for all $x \in \mathbb{R}^d$

$$\mathcal{M}v^n(x) := $$
$$\inf\left\{ y \in \mathbb{R} \ : \ y + \sup_{\nu\in U}\inf_{\sigma\in\Sigma} b(T,x,y,\nu,\sigma) - v^n(x+\beta(T,x,\nu,\sigma)) \ \geq \ 0 \right\} .$$

<u>1st Step.</u> We first prove that $(v^n)_n$ is a sequence of lower-semicontinuous functions bounded from above by any finite solution of (3.5), and satisfying

$$\min\left\{ v^n(x) - g_*(x) \ ; \ \mathcal{G}(v^n, v^{n-1})(x) \right\} = 0 \tag{7.2}$$

for all $n \geq 1$ and $x \in \mathbb{R}^d$, where we use the notation

$$\mathcal{G}(v^n, v^{n-1})(x) = v^n(x) + \sup_{\nu\in U}\inf_{\sigma\in\Sigma} b(T,x,v^n(x),\nu,\sigma) - v^{n-1}(x+\beta(T,x,\nu,\sigma)) \ .$$

Let $w$ be any finite solution of (3.5). Then $w \geq g_*$ and for all $x \in \mathbb{R}^d$ :

$$w(x) + \sup_{\nu\in U}\inf_{\sigma\in\Sigma} b(T,x,w(x),\nu,\sigma) - w(x+\beta(T,x,\nu,\sigma)) \ \geq \ 0 \ .$$

Recalling that $v^1 = \max\ \{g_* \ ; \ \mathcal{M}v^0 = \mathcal{M}g_*\}$, it is easily checked by using Remark 3.3 that, for all $x \in \mathbb{R}^d$ :

$$\min\left\{ v^1(x) - g_*(x) \ ; \ \mathcal{G}(v^1, v^0)(x) \right\} = 0 \ . \tag{7.3}$$

The lower-semicontinuity of $v^1$ is then obtained by (7.3), Remark 3.3 as well as the continuity properties of $b$ and $\beta$ and the lower-semicontinuity of $g_*$. The fact that $v^1(x) \leq w(x)$ is also obtained by Remark 3.3. Finally, all these results may be easily extended to some arbitrary $n \geq 1$ by using a recursive argument.

2nd Step. We now prove that the sequence $(v^n)_n$ is increasing. Fix $n \geq 1$. Assume that $v^n \geq v^{n-1}$. Then, by Remark 3.3, $\mathcal{M}v^n \geq \mathcal{M}v^{n-1}$. This implies that $v^{n+1} = \max\{g_* \, ; \, \mathcal{M}v^n\} \geq \max\{g_* \, ; \, \mathcal{M}v^{n-1}\} = v^n$. The desired result is obtained by noticing that $v^1 \geq g_* = v^0$ .

3rd Step. Notice that by Step 1 and Step 2, for all $x \in I\!R^d$, the sequence $(v^n(x))_n$ is increasing and bounded from above. Hence, it converges to $\ell(x) := \sup_n v^n(x)$. Since by Step 1, for all $n$, $v^n$ is lower-semicontinuous, this proves that $\ell$ is lower-semicontinuous. Finally, using Step 1 again, we see that $\ell \leq w$ for any solution $w$ of (3.5).

4th Step. We conclude the proof by showing that $\ell$ satisfies $H_*(\ell(x)) = 0$ for all $x \in I\!R^d$. Assume that we can find some $\hat{x} \in I\!R^d$ such that :

$$H_*(\ell(\hat{x})) \quad =: \quad 2\varepsilon \; > \; 0$$

for some $\hat{\nu} \in U$. Recall that for all $x$, $\beta(T, x, \cdot, \cdot)$ is bounded. Then, from the convergence result in Step 3 and the uniform continuity property of $b$, there exists some $n \in I\!N$ such that for all $\sigma \in \Sigma$ :

$$\min\left\{v^n(\hat{x}) - g_*(\hat{x}) \; ; \; \mathcal{G}(v^n, v^{n-1})(x)\right\} \;\; > \;\; \varepsilon$$

and we get a contradiction to (7.2) by taking the infimum over $\sigma$ in the last inequality. This proves that $H_*(\ell(x)) = 0$ on $I\!R^d$. □

**Lemma 7.2** *Under the conditions of Proposition 3.3, the smallest solution $\ell$ of* (3.5) *is uniformly continuous.*

**Proof.** Consider the increasing sequence $(v^n)_n$ of the proof of Lemma 7.1. We claim that $(v^n)_n$ is uniformly equicontinuous. This is sufficient for $\ell$ to be uniformly continuous since $(v^n)_n$ converges pointwise to $\ell$. Indeed, for all $\varepsilon > 0$, there exists some $\eta(\varepsilon)$ such that for all $n$, $(x,y) \in \mathbb{R}^{2d}$ with $|x-y| \leq \eta(\varepsilon)$,

$$|v^n(x) - v^n(y)| \leq \varepsilon$$

and therefore

$$\begin{aligned} |\ell(x) - \ell(y)| &\leq \lim_{n\to\infty} |\ell(x) - v^n(x)| + |v^n(x) - v^n(y)| + |v^n(y) - \ell(y)| \\ &\leq \lim_{n\to\infty} |\ell(x) - v^n(x)| + \varepsilon + |v^n(y) - \ell(y)| = \varepsilon \,. \end{aligned}$$

We now prove that $(v^n)_n$ is uniformly equicontinuous. Since $g$ is uniformly continuous, for all $\varepsilon > 0$, there exists some $\eta(\varepsilon) > 0$ such that for all $x_0 \in \mathbb{R}^d$, $x \in B(x_0, \eta(\varepsilon))$ implies that $g(x) \in B(g(x_0), \varepsilon)$. Fix $x_0 \in \mathbb{R}^d$ and recall from the previous proof that

$$\min\left\{v^1(x_0) - g(x_0)\ ;\ \mathcal{G}(v^1, g)(x_0)\right\} \;=\; 0 \;,$$

with $\mathcal{G}$ defined as in the previous proof. Then, for all $x \in B(x_0, \eta(\varepsilon))$

$$\min\left\{v^1(x_0) - g(x) + \varepsilon\ ;\ \mathcal{G}(v^1, g)(x) + v^1(x_0) - v^1(x) + \varepsilon\right\} \;\geq\; 0 \;.$$

By construction of $v^1$ and Remark 3.3, this proves that $v^1(x) \leq v^1(x_0) + \varepsilon$. Also notice that

$$\min\left\{v^1(x_0) - g(x) - \varepsilon\ ;\ \mathcal{G}(v^1, g)(x) + v^1(x_0) - v^1(x) - \varepsilon\right\} \;\leq\; 0 \;,$$

and then $v^1(x) \geq v^1(x_0) - \varepsilon$. From the arbitrariness of $x_0$, this proves that $\{g^0 = v^0, v^1\}$ is uniformly equicontinuous. The uniform equicontinuity of $(v^n)_n$ is obtained by a recursive argument. □

# 8 Applications

In this section, we will always assume that the assumptions of Section 2 hold except when the contrary is explicitly specified.

## 8.1 Optimal insurance and self-protection strategies

We denote by $\mathcal{U}$ the set of all $I\!\!F$-predictable processes $\nu = \{\nu(t),\ 0 \le t \le T\}$ valued in $U := U_1 \times [0,1]$. Fix $z := (t,x,y) \in [0,T] \times (0,\infty) \times I\!\!R$. We assume that the dynamics of $Y^{\nu,z}$ and $X^{\nu,z}$ is given by

$$\begin{aligned} dY_s^\nu &= Y_s^\nu r ds - c(\nu_s^1)ds - \pi(\nu_s^2, X_s^\nu)ds - \int_\Sigma (1-\nu_s^2) b(X_s^\nu, \sigma) v(ds, d\sigma) \\ dX_s^\nu &= \nu_s^1 ds \end{aligned}$$

together with the initial condition $(X_t^{\nu,z}, Y_t^{\nu,z}) = (x,y)$.

**Remark 8.1** This dynamics is derived from that of Section 2 by setting $\rho(t,x,\nu) = \nu^1$, $r(t,x,y,\nu) = ry - c(\nu^1) - \pi(\nu^2, x)$, $\alpha = a = \beta = 0$ and $b(t,x,y,\nu,\sigma) = -(1-\nu^2)b(x,\sigma)$.

The economic interpretation of the above model is the following. Consider the problem of an agent who wants to protect part of his wealth from a depreciation due to a random event modelled by a point process $v(\cdot,\cdot)$ associated with the mark-space $\Sigma$.

He has the choice between insurance and self-protection. The level of self-protection is modelled by the controlled process $X^\nu$. The nonnegative insurance premium $\pi$ is paid continuously and depends on the level of insurance $\nu^2 \in [0,1]$ and self-protection $X^\nu$. We suppose that $\pi$ , defined on $[0,1] \times [0,\infty)$, is Lipschitz continuous, nondecreasing with respect to its first variable and nonincreasing with respect to its second variable. We assume that loss $b$ is decreasing with $x$, that there exists a level $\hat{x} \in I\!\!R^+$ such that, for all $x \ge \hat{x}$, $b(x,\cdot) = 0$ and that $b(x,\sigma) > 0$ for all $x < \hat{x}$ and $\sigma \in \Sigma$.

The wealth of the agent $Y^\nu$ may be invested in a nonrisky asset with instantaneous appreciation rate $r > 0$. $Y^\nu$ is used to pay the insurance premium, the non insured losses $(1-\nu^2)b(X^\nu,\cdot)$ and to invest in order to increase the level of self-protection $X^\nu$. The instantaneous level of investment is modelled by the $U^1$-valued control process $\nu^1$, where $U^1 = [0, \bar{\nu}^1]$ (with $\bar{\nu}^1 > 0$), and the associated instantaneous costs is $c(\nu^1)$ where $c(U^1)$ is bounded, $c(0) = 0$ and

$c(\nu^1) > 0$ on $(0, \bar{\nu}^1]$.

The aim of the agent is to compute the minimal initial wealth needed in order to guarantee the non negativity of the terminal wealth $Y_T^\nu$, and therefore, the value function of the associated super-replication problem is :

$$u(t,x) \;:=\; \inf\Big\{y \in I\!R \;:\; \exists\, \nu \in \mathcal{U},\; Y_T^{\nu,(t,x,y)} \geq 0 \;\Big\}\;.$$

Using Theorem 3.3 we can easily prove that :

**Theorem 8.1** *The value function $u$ is the unique continous viscosity solution on $(0,T) \times (0,\hat{x})$ of*

$$\varphi(t,x)r - \pi(1,x) - \frac{\partial\varphi(t,x)}{\partial t}(t,x) - \tilde{c}\left(\frac{\partial\varphi(t,x)}{\partial x}\right) = 0\;.$$

*where*

$$\tilde{c}\left(\frac{\partial\varphi(t,x)}{\partial x}\right) = \inf_{\nu^1 \in U^1}\left(c(\nu^1) + \nu^1\frac{\partial\varphi(t,x)}{\partial x}\right)\;.$$

*Moreover,* $\lim_{t\to T} u(t,x) = 0$*, for all* $x \in (0,\hat{x})$*, and* $\lim_{x\uparrow\hat{x}} u(t,x) = 0$*, for all* $t \in [0,T]$*.*

**Remark 8.2** Fix $(t,x) \in [0,T] \times [0,\hat{x})$. Direct computation shows that $f(t,x)$ $:= \pi(1,x)/r\;(1 - \exp(-r(T-t))$ is the minimal initial capital needed in order to pay full insurance on $[t,T]$ if the level of self-protection remains equal to $x$, i.e. $\nu^1 = 0$. Therefore, if $f(t,x) > u(t,x)$, it is less expensive to invest in self-protection and the problem is basically a problem of optimal rate of investment. From an economic point of view, $u(t,x)$ may be considered as a upper-bound for the discounted price of full insurance.

**Proof.** We first prove that for all $(t,x) \in [0,T] \times [0,\hat{x}]$ :

$$0 \;\leq\; u(t,x) \;\leq \min\left(f(t,x)\;,\; \frac{[\hat{x}-x]^+}{\bar{\nu}^1}\Big(\pi(1,x) + c(\bar{\nu}^1)\Big)\right)\;, \qquad (8.1)$$

where $f(t,x)$ is defined as in the above Remark. It is clear from the dynamic of $Y^\nu$ that $u \geq 0$. To see that $u(t,x) \leq \frac{[\hat{x}-x]^+}{\bar{\nu}^1}$ $(\pi(1,x)+ c(\bar{\nu}^1))$, consider the strategy where $(\nu^1(s), \nu^2(s)) = (\bar{\nu}^1 1\!\mathrm{I}_{s-t \leq [\hat{x}-x]^+/\bar{\nu}^1},\ 1\!\mathrm{I}_{s-t \leq [\hat{x}-x]^+/\bar{\nu}^1})$ for $s \in [t, T]$ and notice that $X^\nu$ is non decreasing with $X^\nu(t + [\hat{x} - x]^+/\bar{\nu}^1) = \hat{x}$. Then, using the fact that $\pi$ is nonincreasing with respect to $x$, it is easily checked starting with $[\hat{x} - x]^+/\bar{\nu}^1\ (\pi(1, x)+\ c(\bar{\nu}^1))$ is more than we need to adopt a full insurance strategy up to $T\wedge\ [\hat{x} - x]^+/\bar{\nu}^1$ and then a full self-protection strategy with no insurance from $T\wedge\ [\hat{x} - x]^+/\bar{\nu}^1$ up to $T$.

*Boundary conditions.* This is a direct consequence of (8.1).

*Supersolution property.* From (8.1), $u_*$ is finite and, by Theorem 3.3, $u_*$ is a viscosity supersolution on $(0, T) \times (0, \hat{x})$ of

$$\sup_{\nu \in U} \min \left\{ \varphi(t, x)r - c(\nu^1) - \pi(\nu^2, x) - \frac{\partial \varphi(t, x)}{\partial t} - \nu^1 \frac{\partial \varphi(t, x)}{\partial x} \ ; \right.$$
$$\left. \inf_{\sigma \in \Sigma} -(1 - \nu^2) b(x, \sigma) \right\} = 0 \ .$$

Since $\inf_{\sigma \in \Sigma} -(1 - \nu^2) b(x, \sigma) < 0$ if $x < \hat{x}$ and $\nu^2 < 1$, this proves that $u_*$ is also a viscosity supersolution on $(0, T) \times (0, \hat{x})$ of

$$\sup_{\nu^1 \in U^1} \ \varphi(t, x)r - c(\nu^1) - \pi(1, x) - \frac{\partial \varphi(t, x)}{\partial t} - \nu^1 \frac{\partial \varphi(t, x)}{\partial x} = 0 \ .$$

*Subsolution property.* From (8.1), $u^*$ is finite. Then, the fact that $u^*$ is a viscosity subsolution on $(0, T) \times (0, \hat{x})$ of

$$\sup_{\nu^1 \in U^1} \ \varphi(t, x)r - c(\nu^1) - \pi(1, x) - \frac{\partial \varphi(t, x)}{\partial t} - \nu^1 \frac{\partial \varphi(t, x)}{\partial x} \quad = \quad 0 \ ,$$

is obtained by arguing as in Subsection 4.2.

*Continuity and uniqueness.* Recall that $r > 0$ and $\pi(1, \cdot)$ is Lipschitz continuous. Moreover, from the compacity of $U^1$ and the boundedness of $c(U^1)$ it is easily checked that $\tilde{c}$ is uniformly Lipschitz. Therefore, the result is a direct application of theorem 4.8 p.100 in Barles (1994). □

## 8.2 Option hedging under stochastic volatility and dividend revision process

We consider a financial market with a non risky asset, normalized to unity, and a risky asset $S$ that pays a dividend $S_{t_1}\delta(X_{t_1}) \in \mathcal{F}(t_1)$ at time $t_1 \in (0, T]$.

We assume that the dividend anticipation process $X$ may be modified along the time. The problem consists in finding the minimal initial capital needed in order to hedge the contingent claim $\psi(S_T)$ where $\psi$ is a $\mathbb{R}$-valued function, continuous and bounded from below. We assume that the dynamics of $S$ and $X$ are given on $[0,T]$ by :

$$\begin{aligned} dS_t &= S_{t\text{-}}\left(\alpha(t,S_t,X_t)dW(t) - \delta(X_t)\mathbb{1}_{t=t_1}\right) \\ dX_t &:= \int_\Sigma X_{t\text{-}} b(t,S_{t\text{-}},X_{t\text{-}},\sigma)v(dt,d\sigma) \end{aligned}$$

where $v$ is a random point process associated with the mark-space $\Sigma$, $\delta$ is continuous, valued in $[0,\bar{\delta}]$ with $\bar{\delta} < 1$, and $b$ takes values in $(-1,1)$. We also assume that for all $(t,s,x) \in [0,T] \times (0,\infty)^2$ :

$(i)$ there exists some $\sigma_1$ and $\sigma_2 \in \Sigma$ such that $b(t,s,x,\sigma_1)b(t,s,x,\sigma_2) < 0$.

$(ii)$ $\bar{\alpha}(t,s) := \sup_{x\in(0,\infty)} \alpha(t,s,x) < \infty$
*and* $\underline{\alpha}(t,s) := \inf_{x\in(0,\infty)} \alpha(t,s,x) \geq \varepsilon$ for some $\varepsilon > 0$ .

Let $\nu$ be a progressively measurable $\mathbb{F}$-predictable process valued in a convex compact set $U$ with non empty interior corresponding to the proportion of wealth $Y^\nu$ invested in the risky asset. Then, under the self-financing condition, the dynamics of $Y^\nu$ on $[0,T]$ is given by

$$dY_t^\nu := \nu_t Y_{t\text{-}}^\nu \left(\frac{dS_t}{S_{t\text{-}}} + \delta(X_t)\mathbb{1}_{t=t_1}\right) = \nu_t Y_t^\nu \alpha(t,S_t,X_t)dW(t) .$$

Given, $(t,z) = (t,s,x,y) \in [0,T] \times (0,\infty)^2 \times \mathbb{R}$, we denote by $(S^{(t,z)}$, $X^{(t,z)}$, $Y^{\nu,(t,z)})$ the previously introduced processes with initial conditions $(S_t^{(t,z)}$, $X_t^{(t,z)}$, $Y_t^{\nu,(t,z)}) = (s,x,y)$.

The value function associated with the target problem is defined on $[0,T]$ $\times$ $(0,\infty)^2$ by :

$$u(t,s,x) := \inf\left\{y \in \mathbb{R} \ : \ Y_T^{\nu,(t,s,x,y)} \geq \psi(S_T^{(t,s,x)}) \ , \text{ for some } \nu \in \mathcal{U}\right\} .$$

**Remark 8.3** Using standard arguments it is easily checked that for all $\nu \in \mathcal{U}$ and $(t,z) \in [0,T] \times (0,\infty)^2 \times I\!R$, $Y^{\nu,(t,z)}$ is a super-martingale. Therefore, from the definition of $u$ and the fact that $\psi$ is bounded from below, $u$ is also bounded from below.

Notice that our continuity assumption of Section 2 does not hold in this model because of the term $\delta(X_t)1\!\!1_{t=t_1}$ in the dynamic of $S$. We show in Lemma 8.1 that this difficulty may be avoided.

We first introduce some notations. For all $(t,s,x) \in [0,T] \times (0,\infty)^2$, we set

$$\begin{aligned}
\tilde{u}_*(t,s,x) &:= \liminf_{(s',x')\to(s,x)} u(t,s',x') \;, \\
\tilde{u}^*(t,s,x) &:= \limsup_{(s',x')\to(s,x)} u(t,s',x') \;, \\
\underline{G}_t(s,x) &:= \liminf_{t'\uparrow t,\ (s',x')\to(s,x)} u(t',s',x') \;, \\
\bar{G}_t(s,x) &:= \limsup_{t'\uparrow t,\ (s',x')\to(s,x)} u(t',s',x') \;.
\end{aligned}$$

Then we have the

**Lemma 8.1** *Assume that $u^*$ is finite on $[0,t_1]$ and $\bar{G}_{t_1}$ is finite, then Theorem 3.1 holds for $u$ on $(0,t_1)$. Moreover, Theorem 3.2 holds for $\underline{G}_{t_1}$ and $\bar{G}_{t_1}$ with $g_*((s,x)) = \tilde{u}_*(t_1, s(1-\delta(x)), x)$, $g^*((s,x)) = \tilde{u}^*(t_1, s(1-\delta(x)), x)$ and $T = t_1$.*

**Proof.** The proof is similar to that of Theorem 3.1, so we only explain how to adapt it. First notice that (DP1) and (DP2) hold in our framework and that there is no discontinuity on the functions driving the dynamic of $S$ on $(0,t_1)$. Since by Remark 8.3, $u_*$ is finite, Theorem 3.1 holds for $u$ on $(0,t_1)$. We now consider the boundary conditions.

Supersolution. First notice that, by Remark 8.3, $\underline{G}_{t_1}$ is finite. From (DP1), for all $(t,s,x) \in [0,t_1) \times (0,\infty)^2$ and $y > u(t,s,x)$, there exists some $\nu \in \mathcal{U}$ such that :

$$Y_{t_1}^{\nu,(t,s,x,y)} \;\geq\; u(t_1, S_{t_1}^{(t,s,x)}, X_{t_1}^{(t,s,x)}) \;=\; u(t_1, S_{t_1\text{-}}^{(t,s,x)}(1-\delta(X_{t_1}^{(t,s,x)})), X_{t_1}^{(t,s,x)})$$

So the proof of the supersolution property is similar to that of Subsection 5.1. It suffices to replace $T$ by $t_1$, $g((s,x))$ by $u(t_1, s - s\delta(x), x)$ and consider the continuous part of the state process $(S^{(t,s,x)},\ X^{(t,s,x)})$.

Subsolution. From (DP2), for all $(t,s,x) \in (0,t_1) \times (0,\infty)$, $y < u(t,s,x)$ and $\nu \in \mathcal{U}$ such that :

$$P\left(Y_{t_1}^{\nu,(t,s,x,y)} \ > \ u(t_1, S_{t_1^-}^{(t,s,x)}(1-\delta(X_{t_1}^{(t,s,x)})), X_{t_1}^{(t,s,x)})\right) < 1$$

Hence, we may apply the same kind of contradiction as in the subsolution property of Subsection 5.2. Here again, it suffices to replace $T$ by $t_1$, $g((s,x))$ by $u(t_1, s - s\delta(x), x)$ and consider the continuous part of the state process $(S^{(t,s,x)},\ X^{(t,s,x)})$. □

We can now state the main result of this subsection.

**Theorem 8.2** *Assume that $u^*$ is finite. Then, the value function $u$ is a discontinuous viscosity solution on $(0,t_1) \times (0,\infty)$ and on $(t_1,T) \times (0,\infty)$ of :*

$$\min\left\{-\frac{\partial\varphi}{\partial t}(t,s) - \frac{1}{2}s^2\hat{\alpha}\frac{\partial^2\varphi}{\partial s^2}(t,s)\ ;\ \chi_U\left(\frac{s}{\varphi(t,s)}\frac{\partial\varphi}{\partial s}(t,s)\right)\right\} = 0 \tag{8.2}$$

*where*

$$\hat{\alpha}(t,s) := \left(\bar{\alpha}^2 1\!\!1_{\frac{\partial^2\varphi}{\partial s^2}\geq 0} + \underline{\alpha}^2 1\!\!1_{\frac{\partial^2\varphi}{\partial s^2}\leq 0}\right)(t,s)\ .$$

*Assume further that $\bar{G}_{t_1}$ and $\bar{G}_T$ are finite, then $\underline{G}_{t_1}$ and $\bar{G}_{t_1}$ are viscosity super and subsolutions on $(0,\infty)$ of*

$$\min\left\{\varphi(s) - \sup_{x\in(0,\infty)}\tilde{u}_*(t_1, s(1-\delta(x)))\ ;\ \chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} \ = \ 0\ , \tag{8.3}$$

$$\min\left\{\varphi(s) - \sup_{x\in(0,\infty)}\tilde{u}^*(t_1, s(1-\delta(x)))\ ;\chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} \ = \ 0\ , \tag{8.4}$$

*and $\underline{G}_T$, $\bar{G}_T$ are viscosity super and subsolutions on $(0,\infty)$ of*

$$\min\left\{\varphi(s) - \psi(s)\ ;\ \chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} \ = \ 0\ . \tag{8.5}$$

**Remark 8.4** Assume that we can prove a comparison theorem for (8.2)-(8.5), then $u$ is continuous on $(t_1, T)$ and we can replace $\tilde{u}_*$ and $\tilde{u}^*$ by $u$ in (8.3) and (8.4). We may even expect to have a comparison theorem for (8.2)-(8.3)-(8.4). In this case, we may be able to estimate $u$ numerically. It suffices to compute $u$ on $[t_1, T]$ and then use its value in $t_1$ to approximate it on $[0, t_1)$ by using the boundary conditions (8.3)-(8.4). See Touzi (2000) for conditions under which (8.2)-(8.5) admits a comparison principle

**Proof.** First notice that, by Remark 8.3, $u_*$, $\underline{G}_{t_1}$ and $\underline{G}_T$ are finite. We only prove that $u_*$ is a viscosity supersolution of (8.2) on $(t_1, T)$. The other results are proved similarly by using Theorems 3.1, 3.2 and Lemma 8.1.
1st Step . We first prove that $u_*$ is independent of $x$. Fix $(t_0, s_0, x_0) \in (t_1, T) \times (0, \infty)^2$ and a $C^2((t_1, T) \times (0, \infty)^2)$ function $\varphi$ such that $(t_0, s_0, x_0)$ is a strict local minimum for $u_* - \varphi$.

Assume that $\varphi$ is locally strictly increasing in $x$ at $(t_0, s_0, x_0)$. Then, for all $C \geq 0$, $(t_0, s_0, x_0)$ is a strict local minimum for $u_* - \tilde{\varphi}$ where $\tilde{\varphi}$ is defined on $(t_1, T) \times (0, \infty)^2$ by $\tilde{\varphi}(t, s, x) := \varphi(t, s, x - C(x - x_0)^2)$.

By Theorem 3.1, this proves that $\tilde{\varphi}$ satisfies :

$$\inf_{\sigma \in \Sigma} \tilde{\varphi}(t_0, s_0, x_0) - \tilde{\varphi}(t_0, s_0, x_0 + x_0 b(t_0, s_0, x_0, \sigma)) \quad \geq \quad 0$$

Hence,

$$\varphi(t_0, s_0, x_0) \geq \sup_{\sigma \in \Sigma} \varphi(t_0, s_0, x_0 + x_0 b(t_0, s_0, x_0, \sigma) - C(x_0 b(t_0, s_0, x_0, \sigma))^2) \; . (8.6)$$

From assumption (ii) there exists some $\tilde{\sigma} \in \Sigma$ such that $b(t_0, s_0, x_0, \tilde{\sigma}) > 0$. Since $\varphi$ is $C^1$ and locally strictly increasing in $x$ at $(s_0, x_0)$, we can find some sufficiently small $C > 0$ such that

$$\varphi(s_0, x_0 + x_0 b(t_0, s_0, x_0, \tilde{\sigma}) - C(x_0 b(t_0, s_0, x_0, \tilde{\sigma}))^2) \quad > \quad \varphi(s_0, x_0)$$

which contradicts (8.6). Hence, $(\partial\varphi/\partial x)(t_0, s_0, x_0) \leq 0$.

We can prove similarly that $(\partial\varphi/\partial x)(t_0, s_0, x_0) \geq 0$. Hence, $u_*$ is a viscosity supersolution of $\partial\varphi/\partial x \geq 0$ and $-\partial\varphi/\partial x \geq 0$. By Remark 8.3 and Lemmas

5.3 and 5.4 in Cvitanić, Pham and Touzi (1999b), this proves that $u_*$ is independent of $x$.

2nd Step . We now prove that $u_*$ is a viscosity supersolution of (8.2) on $(t_1, T)$. Recall that $u_*$ is independent of $x$. Fix $(t_0, s_0) \in (t_1, T) \times (0, \infty)$ and a $C^2((t_1, T) \times (0, \infty))$ function $\varphi$ such that $(t_0, s_0)$ is a local minimum for $u_* - \varphi$. By Theorem 3.1, for all $x \in (0, \infty)$, $\varphi$ satisfies :

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(t_0, s_0) - \frac{1}{2} s_0^2 \alpha^2(t_0, s_0, x) \frac{\partial^2 \varphi}{\partial s^2}(t_0, s_0) \; ; \; \chi_U \left( \frac{s_0}{\varphi(t_0, s_0)} \frac{\partial \varphi}{\partial s}(t_0, s_0) \right) \right\} \geq 0$$

Consider a maximizing sequence $x_n$ of $\alpha^2(t_0, s_0, \cdot)(\partial^2 \varphi / \partial s^2)(t_0, s_0)$ as $n$ tends to $\infty$. Then, the previous inequality also holds at $x_n$ for all $n$ and the desired result is obtained by sending $n$ to $\infty$ and using the continuity of $\alpha$ with respect to $x$. □

Notice that in the case where $\delta = 0$, the model reduces to a stochastic volatility one where the volatility is driven by a pure jump process. In this last case we have the

**Theorem 8.3** *Assume that $\delta = 0$. Assume further that $u^*$ is finite. Then, the value function $u$ is a discontinuous viscosity solution on $(0, T) \times (0, \infty)$ of :*

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(t, s) - \frac{1}{2} s^2 \hat{\alpha} \frac{\partial^2 \varphi}{\partial s^2}(t, s) \; ; \; \chi_U \left( \frac{s}{\varphi(t, s)} \frac{\partial \varphi}{\partial s}(t, s) \right) \right\} = 0$$

*where*

$$\hat{\alpha}(t, s) := \left( \bar{\alpha}^2 \mathbb{1}_{\frac{\partial^2 \varphi}{\partial s^2} \geq 0} + \underline{\alpha}^2 \mathbb{1}_{\frac{\partial^2 \varphi}{\partial s^2} \leq 0} \right)(t, s) \; .$$

*Assume further that $\bar{G}_T$ is finite, then $\underline{G}_T$ and $\bar{G}_T$ are viscosity super and subsolutions on $(0, \infty)$ of*

$$\min \left\{ \varphi(s) - \psi(s) \; ; \; \chi_U \left( \frac{s}{\varphi(s)} \frac{\partial \varphi}{\partial s}(s) \right) \right\} \quad = \quad 0 \; .$$

**Proof.** The result is obtained by the same arguments as in the previous proof.

□

# Chapter II.

# Explicit solution to the problem of super-replication with proportional transaction costs

We[1] consider a multivariate financial market with transaction costs as in Kabanov (1999). We study the problem of finding the minimal initial capital needed to hedge, without risk, European-type contingent claims. We prove that the value of this stochastic control problem is given by the cost of the cheapest buy-and-hold strategy. This is an extension of the already known result in the one-dimensional case. We show how the result can be obtained with and without making use of the dual formulation obtained in Kabanov (1999).

[1]This chapter is based on Bouchard and Touzi (1998) and Bouchard (1999)

# 1 Introduction

In the context of the Black and Scholes one-dimensional financial market with proportional transaction costs, Davis and Clark (1994) conjectured that the minimal initial wealth needed to super-replicate a European call option is just the price of one share of the underlying asset. In other words, the cheapest buy-and-hold strategy solves the super-replication problem. The conjecture was proved by analytic methods by Soner, Shreve and Cvitanić (1995) and (independently, and for more general models and contingent claims) by Levental and Skorohod (1997) by probabilistic methods.

In a one-dimensional Markov diffusion model, a simple proof of this conjecture was provided by Cvitanić, Pham and Touzi (1999) for general contingent claims. Their approach relies on the dual formulation of the super-replication cost (Jouini and Kallal 1995 and Cvitanić and Karatzas 1996) which reduces the problem to a singular stochastic control problem in standard form.

In a recent paper, Kabanov (1999) provided an extension of the dual formulation of the super-replication problem to the case of currency markets with proportional transaction costs. This framework is a natural multidimensional version of the models discussed above. The multivariate super-replication problem under transaction costs presents some important difficulties which are not apparent in the one-dimensional model. In particular, we were not able to extend directly Cvitanić, Pham and Touzi's (1999) proof to this context.

The approach of Bouchard and Touzi (1999) consists in relating the super-replication problem to some convenient auxiliary super-replication problems defined on fictitious financial markets without transaction costs. Definition of such fictitious financial markets is obtained by use of the solvency cone introduced by Kabanov (1999). We can then use a dynamic programming principle stated directly on the auxiliary problems as in Soner and Touzi (1998, 2000a) and prove that such a dynamic programming equation allows to characterize the value of the auxiliary control problem as a viscosity super-solution of a suitable Hamilton-Jacobi-Bellman partial differential equation. The remaining arguments are similar to Cvitanić, Pham and Touzi (1999).

An other approach consists in adapting the argument of Cvitanić, Pham and Touzi (1999) to our framework. It relies on a new parameterization introduced in Bouchard (1999).

The paper is organized as follows. We describe the model and the super-replication problem in Section 2. The main results of the paper are stated in Section 3 with a partial argument. In Section 4, we introduce a parameterization of the polar associated with the solvency cone of Kabanov (1999). In Section 5, we define auxiliary stochastic control problems which are interpreted as super-replication problems on fictitious financial markets without transaction costs. Then, we prove our main result by using the direct dynamic programming principle. In Section 6, we show how to adapt the arguments of Cvitanić, Pham and Touzi (1999) to our multidimensional framework. Finally, Section 7 contains some examples.

# 2 Financial market and super-replication

Let $T$ be a finite time horizon and $(\Omega, \mathcal{F}, P)$ be a complete probability space supporting a $d$-dimensional Brownian motion $\{B(t),\ 0 \leq t \leq T\}$. We shall denote by $\mathbb{F} = \{\mathcal{F}(t),\ 0 \leq t \leq T\}$ the $P$-augmentation of the filtration generated by $B$.

## 2.1 Financial market with proportional transaction costs

We consider a financial market which consists of one bank account, with constant price process $S^0$, normalized to unity, and $d$ risky assets $S := (S^1, \ldots, S^d)'$. The price process $S = \{S(t),\ 0 \leq t \leq T\}$ is an $\mathbb{R}^d$-valued stochastic process defined by the following stochastic differential system :

$$dS(t) \;=\; \text{diag}[S(t)]\sigma(t, S(t))dB(t), \qquad 0 \;<\; t \;\leq\; T\,. \tag{2.1}$$

Here $\sigma(.,.)$ is an $\mathbb{M}^d$-valued function. We shall assume all over the paper that the function $\text{diag}[s]\sigma(t,s)$ satisfies the usual Lipschitz and linear growth conditions in order for the process $S$ to be well-defined and that $\sigma(t,s)$ is invertible for all $(t,s) \in [0,T] \times (0,\infty)^d$. We also assume that

$$P\left[S(u) \in A \mid \mathcal{F}(t)\right] \quad > \quad 0, \; 0 \leq t < u \leq T \tag{2.2}$$

for all open subset $A$ of $(0,\infty)^d$.

**Remark 2.1** A sufficient condition for (2.2) to be verified is that, for all $(t,s) \in [0,T] \times (0,\infty)^d$, matrix $\sigma(t,s)$ satisfies the uniform ellipticity condition

$$\exists\, \varepsilon > 0 \;:\; |\sigma(t,s)\xi| \;\geq\; \varepsilon|\xi|^2 \quad \text{for all} \quad \xi \in I\!R^d \,. \tag{2.3}$$

Notice that, with our notations, $\bar{S} := (S^0, S^1, \ldots, S^d)'$.

**Remark 2.2** As usual, the assumption that the interest rate of the bank account is zero could be easily dispensed with by discounting. Also, there is no loss of generality in defining $S$ as a local martingale since we can always reduce the model to this context by appropriate change of measure (under mild conditions on the initial coefficients).

A *trading strategy* is an $I\!M_+^{d+1}$-valued process $L$, with initial value $L(0-) = 0$, such that $L^{ij}$ is $I\!F-$adapted, right-continuous, and nondecreasing for all $i,j = 0,\ldots,d$. Here, $L^{ij}$ describes the cumulative amount of funds transferred from asset $i$ to asset $j$.

Proportional transaction costs in this financial market are described by matrix $\lambda \in I\!M_+^{d+1}$. This means that transfers from asset $i$ to asset $j$ are subject to proportional transaction costs $\lambda^{ij}$ for all $i,j = 0,\ldots,d$.

**Remark 2.3** We may always assume that

$$(1+\lambda^{ij}) \;\leq\; (1+\lambda^{ik})(1+\lambda^{kj}) \quad \forall\, i,\; j,\; k \in \{0,\ldots,d\} \tag{2.4}$$

$$\lambda^{ii} \;=\; 0 \quad \forall\, i \in \{0,\ldots,d\} \;. \tag{2.5}$$

(2.5) can always be imposed and if we can find some $i$, $j$, $k \in \{0,\ldots,d\}$ such that (2.4) does not hold, we can always replace $\lambda^{ij}$ by

$$\tilde{\lambda}^{ij} := (1+\lambda^{ik})(1+\lambda^{kj}) - 1$$

since, to transfer money from $i$ to $j$, it is less expensive to make a transfer for $i$ to $k$ and then from $k$ to $j$.

Given an initial holdings vector $x \in I\!R^{d+1}$ and a trading strategy $L$, the portfolio holdings $X_x^L = (X_x^{i,L})_{i=0,\dots,d}$ are defined by the dynamics :

$$\begin{aligned} X_x^{i,L}(0-) &= x^i \\ dX_x^{i,L}(t) &= X_x^{i,L}(t)\,\frac{d\bar{S}^i(t)}{\bar{S}^i(t)} + \sum_{j=0}^{d}\left[dL^{ji}(t) - (1+\lambda^{ij})dL^{ij}(t)\right], \quad 0 \le t \le T \end{aligned}$$

for all $i = 0, \dots, d$. The dynamics of the portfolio holdings process can be written alternatively in terms of the number of shares transferred from one asset to another. Set $\ell_k^{ij}(t) := \int_0^t dL^{ij}(r)/\bar{S}^k(r)$ for all $i, j = 0, \dots, d$ and $k \in \{i, j\}$. Here, $\ell_i^{ij}$ (resp. $\ell_j^{ij}$) is the cumulated transfer from asset $i$ to asset $j$ in terms of number of shares of asset $i$ (resp. $j$). Then, from the previous dynamics, we get :

$$X_x^{i,L}(t) = x^i + \bar{S}^i(t)\sum_{j=0}^{d}\left(\ell_i^{ji}(t) - (1+\lambda^{ij})\ell_i^{ij}(t)\right), \qquad 0 \le t \le T .$$

## 2.2 Definition of the super-replication problem

Following Kabanov (1999), we define the *solvency region* :

$$K := \left\{ x \in I\!R^{d+1} \;:\; \exists\, a \in I\!M_+^{d+1},\; x^i + \sum_{j=0}^{d}(a^{ji} - (1+\lambda^{ij})a^{ij}) \ge 0;\; 0 \le i \le d \right\} .$$

The elements of $K$ can be interpreted as the vectors of portfolio holdings such that the no-bankruptcy condition is satisfied: the liquidation value of the portfolio holdings $x$, through some convenient transfers, is nonnegative. Another economic interpretation is that the portfolio holdings $-x$ can be reached from zero initial portfolio holdings through some convenient transfers.

Clearly, the set $K$ is a closed convex cone containing the origin. Following Kabanov (1999), we introduce the partial ordering $\succeq$ induced by $K$, defined by :

$$\text{for all } x_1, x_2 \in I\!R^{d+1}, \quad x_1 \succeq x_2 \;\text{ if and only if }\; x_1 - x_2 \in K .$$

A trading strategy $L$ is said to be *admissible* for the initial holdings $x \in I\!R^{d+1}$ if the no-bankruptcy condition

$$X_x^L(t) \succeq 0 \qquad 0 \le t \le T \tag{2.6}$$

holds. We shall denote by $\underline{\mathcal{A}}(x)$ the set of all admissible trading strategies.

A *contingent claim* is a $(d+1)$-dimensional $\mathcal{F}(T)$-measurable random variable $g(S(T))$. Here, $g$ maps $(0,\infty)^d$ into $I\!R^{d+1}$ and satisfies

$$g \text{ is lower semicontinuous and } g(s) \succeq 0 \text{ for all } s \text{ in } (0,\infty)^d \ . \tag{2.7}$$

In the rest of the paper, we shall identify a contingent claim with its pay-off function $g$. For all $i = 0, \ldots, d$, the random variable $g^i(S(T))$ represents a target position in asset $i$.

The *super-replication problem* of contingent claim $g$ is then defined by

$$v(0, S(0)) := \inf\left\{w \in I\!R \ : \ \exists\, L \in \underline{\mathcal{A}}(w\mathbf{1}_0),\ X^L_{w\mathbf{1}_0}(T) \succeq g(S(T))\right\} \ , \tag{2.8}$$

i.e. $v(0, S(0))$ is the minimal initial capital which allows to hedge the contingent claim $g$ through some admissible trading strategy.

**Remark 2.4** Given an initial portfolio holding $x \in I\!R$. We can generalize the definition of $v$ by setting :

$$v(0, S(0), x) := \inf\left\{w \in I\!R \ : \ \exists\, L \in \underline{\mathcal{A}}(x + w\mathbf{1}_0),\ X^L_{x+w\mathbf{1}_0}(T) \succeq g(S(T))\right\} \ ,$$

i.e. $v(0, S(0), x)$ is the minimal initial cash increment needed to hedge the contingent claim $g$ when starting with the initial holding $x$. This generalized problem can also be solved by the techniques developed in this paper (see the survey paper of Touzi 1999).

# 3 Solution of the super-hedging problem

Following Kabanov (1999), we introduce the positive polar of $K$ :

$$K^* := \{\xi \in I\!R^{d+1} \ : \ \xi x \ge 0, \ \ \forall x \in K\} \ ,$$

and we denote by $\underline{K}^*$ the subset of $I\!R^d$ :

$$\underline{K}^* := \{r \in I\!R^d \ : \ \bar{r} \in K^*\} \ ,$$

i.e. $\{1\} \times \underline{K}^*$ is the section of the positive polar cone $K^*$ with the hyperplane $\{\xi \in I\!R^{d+1} \ : \ \xi^0 = 1\}$. The partial ordering $\succeq$ can be characterized in terms of $\underline{K}^*$ by

$$x_1 \ \succeq \ x_2 \quad \text{if and only if} \quad \bar{r}(x_1 - x_2) \geq 0 \quad \text{ for all } \ r \in \underline{K}^* \ .$$

Notice that $\underline{K}^*$ is not empty since $\mathbf{1} = \sum_{i=1}^d \mathbf{1}_i \in \underline{K}^*$. This is easily checked from the definition of $K$ (by summing up the $d+1$ inequalities, and using the fact that the transfer matrix $a$ as well as the transaction costs matrix $\lambda$ have non-negative entries).

**Remark 3.1** An important property of the polar cone $K^*$ is that $K^* \setminus \{0\} \subset (0, \infty)^{d+1}$. This claim will be proved in Lemma 4.1. It follows that $\underline{K}^* \subset (0, \infty)^d$.

**Remark 3.2** The set $\underline{K}^*$ is a compact subset of $I\!R^d$. This claim will be justified in Remark 4.2 of the next section.

Next, we introduce the functions :

$$G(z) \ := \ \sup_{r \in \underline{K}^*} \bar{r} g \left( \text{diag}[r]^{-1} z \right) \qquad \text{for all } \ z \text{ in } (0, \infty)^d,$$

and

$$\hat{g}(s) \ := \ \sup_{r \in \underline{K}^*} G^{conc}(\text{diag}[r] s) \qquad \text{for all } \ s \text{ in } (0, \infty)^d,$$

where $G^{conc}$ is the concave envelope of $G$.

The main result of this paper requires the following condition.

**Assumption 3.1** $\lambda^{ij} + \lambda^{ji} > 0$ *for all* $i, j = 0, \ldots, d$, $i \neq j$.

**Remark 3.3** Under (2.4)-(2.5), Assumption 3.1 is necessary and sufficient for $K^*$ to have non-empty interior. This remark will be justified in Subsection 7.1.

**Theorem 3.1** *Under Assumption 3.1, the solution of the super-replication problem is given by*

$$v(0, S(0)) \quad = \quad \hat{g}(S(0)) \; .$$

The proof of the last result will be provided in the subsequent sections of the paper. We now give an economic interpretation of the result.

Let $w \geq 0$ be some initial capital. A buy-and-hold strategy is an admissible strategy $L$ in $\underline{\mathcal{A}}(w\mathbf{1}_0)$ such that the number of shares of the $i-$th asset $X^{i,L}_{w\mathbf{1}_0}(t)/\bar{S}^i(t)$ is constant over the time interval $[0, T]$, for all $i = 0, \ldots, d$. In other words, the number of shares induced by the strategy $L$ is unchanged during the time interval $[0, T]$, so that no transfers are operated after time 0. The cost of the cheapest buy-and-hold strategy is clearly given by :

$$h(S(0)) \quad := \quad \inf \Big\{ w \in I\!\!R \; : \; \exists \, \Delta \in I\!\!R^{d+1}, \; w\mathbf{1}_0 \succeq \mathrm{diag}[\bar{S}(0)]\Delta \\ \text{and } \mathrm{diag}[\bar{z}]\Delta \succeq g(z), \; \text{for all } z \text{ in } (0,\infty)^d \Big\} \; .$$

**Theorem 3.2** *For all $s$ in $(0,\infty)^d$, we have $\hat{g}(s) = h(s)$.*

Finally, as a direct consequence of the last two Theorems :

**Corollary 3.1** *Under Assumption 3.1, the value of the super-replication problem is the cost of the cheapest buy-and-hold strategy. Moreover, existence holds for the optimization problem (2.8).*

**Proof of Theorem 3.2** (i) We first prove that $\hat{g}(s) \leq h(s)$. Consider some arbitrary scalar $w > h(s)$. By definition of $h(s)$, there exists some $\Delta$ in $I\!\!R^{d+1}$ such that, for all $z$ in $(0,\infty)^d$,

$$\bar{r}'(w\mathbf{1}_0 - \mathrm{diag}[\bar{s}]\Delta) \geq 0 \quad \text{ and } \quad \bar{r}(\mathrm{diag}[\bar{z}]\Delta - g(z)) \geq 0 \quad \text{ for all } r, \; r' \; \in \underline{K}^* \; .$$

Using the fact that $\bar{r}^0 = 1$, this provides

$$w + \Delta(\bar{z} - \mathrm{diag}[\bar{r}']\bar{s}) \; \geq \bar{r} g\left(\mathrm{diag}[r]^{-1} z\right) \quad \text{ for all } r, \; r' \; \in \underline{K}^*,$$

and taking supremum over $r$, we get

$$w + \Delta(\bar{z} - \mathrm{diag}[\bar{r}']\bar{s}) \quad \geq G(z) \qquad \text{for all } z \ \in \ (0,\infty)^d \text{ and } \bar{r}' \ \in \underline{K}^*.$$

This proves that $w \geq G^{conc}(\mathrm{diag}[r']s)$ for all $r'$ in $\underline{K}^*$, and therefore $w \geq \hat{g}(s)$. The required inequality follows from the arbitrariness of $w > h(s)$.

(ii) We now prove the converse inequality. Since $\underline{K}^*$ is a compact subset of $\mathbb{R}^d$ and $G^{conc}$ is continuous, there exists some $\hat{r} \in \underline{K}^*$ such that

$$\hat{g}(s) \quad = \quad G^{conc}(\mathrm{diag}[\hat{r}]s) \ . \tag{3.1}$$

Recall that the concave envelope is characterized by :

$$G^{conc}(z) = \min\left\{c \in \mathbb{R} \ : \ \exists\, \zeta \in \mathbb{R}^d,\ c + \zeta(z' - z) \ \geq \ G(z'),\ \forall z' \in (0,\infty)^d\right\} \ .$$

Also, it is well-known that the solution of the above optimization problem is given by any element of the subgradient $\partial G^{conc}(z)$ of the concave function $G^{conc}$ at $z$. Hence

$$\forall\, \zeta \in \partial G^{conc}\,(\mathrm{diag}[\hat{r}]s) \ , \ \ \hat{g}(s) + \zeta(z - \mathrm{diag}[\hat{r}]s) \geq G(z) \ ; z \in (0,\infty)^d \ . \tag{3.2}$$

We claim that :

$$\exists\, \hat{\zeta} \in \partial G^{conc}\,(\mathrm{diag}[\hat{r}]s) \ \ : \ \mathrm{diag}[s]\hat{\zeta}(r - \hat{r}) \leq 0 \text{ for all } r \in \underline{K}^* \ . \tag{3.3}$$

We leave the proof of the previous claim to part (iii). Now, let $\hat{\Delta}$ be the vector of $\mathbb{R}^{d+1}$ defined by $\hat{\Delta}^i = \hat{\zeta}^i$ for $i = 1, \ldots, d$ and $\hat{\Delta}^0 = \hat{g}(s) - \hat{\zeta}\mathrm{diag}[\hat{r}]s$. Then from (3.2), $\hat{\Delta}\bar{z} \geq G(z)$ for all $z \in (0,\infty)^d$, and by definition of $G$, we see that $\hat{\Delta}\bar{z} \geq \bar{r}g(\mathrm{diag}[r]^{-1}z)$ for all $(r,z) \in \underline{K}^* \times (0,\infty)^d$. By a trivial change of variables, this provides

$$\bar{r}\left(\mathrm{diag}[\bar{z}]\hat{\Delta} - g(z)\right) \quad \geq \quad 0 \qquad \text{for all } \ r \in \underline{K}^* \ \text{ and } \ z \in (0,\infty)^d \ .$$

By definition of the normalized polar $\underline{K}^*$, this proves that :

$$\mathrm{diag}[\bar{z}]\hat{\Delta} \quad \succeq \quad g(z) \qquad \text{for all } \ z \in (0,\infty)^d \ .$$

Now, rewriting (3.2) in terms of $\hat{\Delta}$ and using (3.3) yields :

$$\hat{g}(s)\mathbb{1}_0 \ \succeq \ \mathrm{diag}[\bar{s}]\hat{\Delta} \ ,$$

which together with the previous inequality implies that $\hat{g}(s) \geq h(s)$.
(iii) In order to conclude the proof, it remains to verify (3.3). Let $\varepsilon$ be an arbitrary parameter in $(0,1)$ and $r \in \underline{K}^*$. Since $\underline{K}^*$ is convex, we have

$$\begin{aligned}\hat{g}(s) &\geq G^{conc}\left(\mathrm{diag}[(1-\varepsilon)\hat{r} + \varepsilon r]s\right) \\ &= \hat{g}(s) + \varepsilon(r - \hat{r})\mathrm{diag}[s]\zeta_\varepsilon \ ,\end{aligned}$$

where $\zeta_\varepsilon$ is an element of $\partial G^{conc}(z_\varepsilon)$ for some $z_\varepsilon$ lying in the interval defined by the bounds $\mathrm{diag}[\hat{r}]s$ and $\mathrm{diag}[\hat{r} + \varepsilon(r - \hat{r})]s$. Since $G^{conc}$ is concave and $z_\varepsilon$ converges to the interior point $\mathrm{diag}[\hat{r}]s$, the sequence $(\zeta_\varepsilon)$ converges to some $\hat{\zeta}$ $\in \partial G^{conc}(\mathrm{diag}[\hat{r}]s)$. Then claim (3.3) is obtained by passing to the limit in the last inequality. □

# 4 Parameterization of the initial model

By direct computation, it is easily checked that :

$$K^* \ = \ \bigcap_{i,j=0}^{d} H^{ij} \quad \text{where} \quad H^{ij} \ = \ \left\{ \xi \in I\!R_+^{d+1} : \xi^j - (1+\lambda^{ij})\xi^i \leq 0 \right\} \ ,$$

see Kabanov (1999). In the sequel, we shall denote $\partial H^{ij} := \{\xi \in I\!R_+^{d+1} : \xi^j - (1+\lambda^{ij})\xi^i = 0\}$.

**Lemma 4.1** (i) *$K^*$ is a closed convex polyhedral cone.*
(ii) *Let $\xi$ be a nonzero element of $K^*$. Then $\xi^i > 0$ for all $i = 0, \ldots, d$.*

**Proof.** Part (i) follows from Rockafellar (1970), p.171, Theorem 19.1 and the fact that $K^*$ is a finite intersection of half-hyperplanes. To see that part (ii) holds, consider some $\xi \in K^*$ with $r^i > 0$ for some $i = 0, \ldots, d$. Then since $\xi \in H^{ji}$ for all $j = 0, \ldots, d$ we have $\xi^i - (1+\lambda^{ji})\xi^j \leq 0$. □

Since $K^*$ is a closed convex polyhedral cone, it is finitely generated (see Rockafellar 1970, p.171, Thm 19.1). Then, we can define a generating family $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$, i.e.

$$K^* \;=\; \left\{\xi \in I\!R^{d+1} \;:\; \xi = \sum_{i=1}^{n} y^i \tilde{e}_i \;\text{ for some }\; y \in I\!R^n_+\right\} .$$

By part (ii) of Lemma 4.1, the generating vectors can be normalized by

$$\tilde{e}_i^0 \;=\; 1 \quad \text{for all} \quad i = 1, \ldots, n \,. \tag{4.1}$$

Therefore, denoting by $e_i$ the $I\!R^d$ vector defined by the $d$ last components of $\tilde{e}_i$, we can rewrite the generating family as $\{\bar{e}_1, \ldots, \bar{e}_n\}$.

**Example 4.1** In the one-dimensional case $d = 1$, the generating vectors of $K^*$ are given by :

$$\bar{e}_1 \;=\; (1, 1 + \lambda^{01}) \quad \text{and} \quad \bar{e}_2 \;=\; (1, (1 + \lambda^{10})^{-1}).$$

This is the case studied by Cvitanić, Pham and Touzi (1999).

For the two-dimensional case $d = 2$, we also have explicitly the vectors of the generating family under some condition on the transaction costs matrix ; see Section 7. In the general case, we do not have an explicit form of the family of generators. However, the main result of this paper does not require this information.

**Remark 4.1** Since $K^*$ has non-empty interior by Remark 3.3, the rank of the family $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is $d + 1$. In particular, $n \geq d + 1$.

In order to have a parameterization of $\underline{K}^*$, we define the following function $f$ mapping $(0, \infty)^n$ into $I\!R^d$ by

$$f^i(y) \;=\; \left(\sum_{j=1}^{n} y^j \bar{e}_j^0\right)^{-1} \sum_{j=1}^{n} y^j \bar{e}_j^i \;=\; \left(\sum_{j=1}^{n} y^j\right)^{-1} \sum_{j=1}^{n} y^j e_j^i \,; \qquad i = 1, \ldots, d,$$

so that the set $\underline{K}^*$ can be written in terms of the function $f$ :

$$\underline{K}^* \;=\; \{f(y) \;:\; y \in (0, \infty)^n\} \,.$$

The following result is the key stone of our analysis.

**Lemma 4.2** *Let Assumption 3.1 hold. Then, for all $y$ in $(0,\infty)^n$, the rank of the Jacobian matrix $Df(y)$ is $d$.*

**Proof.** For all $k = 1,\ldots,n$, we introduce the matrices $N_k := [e_1,\ldots,e_k]$. Recall that $\bar{N}_k$ is obtained from $N_k$ by adding the first row of one, so that $\bar{N}_k := [\bar{e}_1,\ldots,\bar{e}_k]$. Then, direct computation shows that

$$\left(\sum_{i=1}^n y^i\right) Df(y) = N_n - N_n\tilde{y} \quad \text{where } \tilde{y} = \left(\sum_{i=1}^n y^i\right)^{-1} y.$$

It follows that $Df(y)$ and $N_n - N_n\tilde{y}$ have the same rank.

(i) We first show that $\text{rank}[N_n - N_n\tilde{y}] = \text{rank}[N_n]$. To see this, observe that the $i-$th column of $N_n - N_n\tilde{y}$ is given by $\tilde{e}_i := e_i - \sum_{j=1}^n \tilde{y}^j e_j = (1-\tilde{y}^i)e_i - \sum_{i\neq j, j=1}^n \tilde{y}^j e_j$. Since $\sum_{i=1}^n \tilde{y}^i = 1$ and $\tilde{y}^i > 0$ for all $i$, we have $(1-\tilde{y}^i) > 0$. Then the families $\{\tilde{e}_i,\ i = 1,\ldots,n\}$ and $\{e_i,\ i = 1,\ldots,n\}$ have same rank.

(ii) We now prove that $\text{rank}[N_n] = d$ which concludes the proof. By Remark 4.1, the matrix $\bar{N}_{d+1}$ is invertible, after possibly changing the order of the $\bar{e}_i$'s. Then, clearly

$$\text{rank}[N_n] = \text{rank}[N_{d+1}] = \text{rank}[J\bar{N}_{d+1}] \quad \text{with } J = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \\ 0 & & I_d \end{pmatrix}$$

and $I_d$ is the identity matrix of $\mathbb{R}^d$. The required result follows from the fact that $\bar{N}_{d+1}$ is invertible and $\text{rank}[J] = d$.

□

**Remark 4.2** From Lemma 4.1 (ii), $f^i(y) > 0$ for all $y \in (0,+\infty)^n$. Moreover, since $\bar{f} \in H^{0i}$ for all $i = 1,\ldots,d$, it follows that $f^i(y) = \bar{f}^i(y) \leq (1+\lambda^{0i})\bar{f}^0(y) = (1+\lambda^{0i})$. Hence $f$ is bounded and

$$0 < f^i(y) \leq (1+\lambda^{0i}) \quad \text{for all } i = 1,\ldots,d \text{ and } y \in (0,\infty)^n.$$

Also, from the expression of the Jacobian matrix given in the above proof, it is easily checked that $Df(y)\text{diag}[y]$ is bounded.

# 5 Proof by direct dynamic programming

In this section, we follow the arguments of Bouchard and Touzi (1999), i.e. we make use of the direct dynamic programming equation obtained by Soner and Touzi (2000a) for stochastic target problems.

## 5.1 Auxiliary control problems

In order to alleviate notations, we introduce the functions :

$$F(y) \ := \ \mathrm{diag}[f(y)]$$

Fix some arbitrary parameter $\mu > 0$. For all $y_0 > 0$, we define the continuous function $\alpha^{y_0}$ on $[0,T] \times (0,\infty)^{d+n} \times I\!M^{n,d} \times I\!R^n$ as

$$\alpha^{y_0}(t,s,y,a,b) := \begin{cases} A(t,s,y,a,b) & \text{if } \sum_{i=1}^d \sum_{j=1}^n \left( |\ln \frac{s^i}{S^i(0)}| + |\ln \frac{y^j}{y_0^j}| \right) < \mu \\ \text{constant} & \text{otherwise,} \end{cases} \tag{5.1}$$

where

$$\begin{aligned} A(t,s,y,a,b) \ = \ & \sigma(t,s)^{-1} F(y)^{-1} \{ Df(y)\mathrm{diag}[y]b \\ & + \frac{1}{2} \mathrm{Vect} \left[ \mathrm{Tr} \left( D^2 f^i(y) \mathrm{diag}[y] a a' \mathrm{diag}[y] \right), i = 1, \ldots, d \right] \\ & + \mathrm{Vect} \left[ (Df(y)\mathrm{diag}[y] a \sigma'(t,s))_{ii}, i = 1, \ldots, d \right] \} \ . \end{aligned}$$

Let $\mathcal{D}$ be the set of all bounded progressively measurable processes $(a,b) = \{(a(t), b(t)), 0 \le t \le T\}$ where $a$ and $b$ are valued respectively in $I\!M^{n,d}$ and $I\!R^n$. For all $y$ in $(0,\infty)^n$ and $(a,b)$ in $\mathcal{D}$, we introduce the controlled process $Y_y^{(a,b)}$ defined on $[0,T]$ defined as the solution of the stochastic differential equation

$$\begin{aligned} & dY(t) = \mathrm{diag}[Y(t)] \left[ (b(t) + a(t)\alpha^y(t, S(t), Y(t), a(t), b(t))) \, dt + a(t) dB(t) \right] \\ & Y(0) = y. \end{aligned} \tag{5.2}$$

We do not write the dependence of $Y_y^{(a,b)}$ with respect to $\mu$ to alleviate the notations. Notice that, since $\alpha^{y_0}(t,s,y,a,b)$ is a random Lipschitz function of $y$, the process $Y_y^{(a,b)}$ is well defined on $[0,T]$.

For each $(a,b)$ in $\mathcal{D}$, we define the process $Z_y^{(a,b)}$ by

$$Z_y^{(a,b)}(t) = F(Y_y^{(a,b)}(t))S(t), \qquad \text{for all } 0 \le t \le T. \tag{5.3}$$

Let $\phi$ be a progressively measurable process valued in $I\!R^{d+1}$ and satisfying

$$\sum_{i=1}^{d} \int_0^T |\phi^i(t)|^2 d\langle Z_y^{i,(a,b)}(t)\rangle < \infty \,. \tag{5.4}$$

Then, given $w \ge 0$, we introduce the process $W_{w,y}^{(a,b)^\phi}$ defined by

$$W_{w,y}^{(a,b)^\phi}(t) = w + \int_0^t \phi(r) d\bar{Z}_y^{(a,b)}(r), \qquad 0 \le t \le T, \tag{5.5}$$

and we denote by $\mathcal{B}^{(a,b)}(w,y)$ the set of all such processes $\phi$ satisfying the additional condition

$$W_{w,y}^{(a,b)^\phi}(t) \;\; \ge \;\; 0, \qquad 0 \le t \le T. \tag{5.6}$$

We finally define the auxiliary stochastic control problems

$$\begin{aligned} u^{(a,b)}(0,y,F(y)S(0)) \;\; := \;\; \inf \Big\{ & w \in I\!R \; : \; \exists \phi \in \mathcal{B}^{(a,b)}(w,y) \,, \\ & W_{w,y}^{(a,b)^\phi}(T) \ge \bar{f}\left(Y_y^{(a,b)}(T)\right) g\left(S(T)\right) \Big\} \end{aligned} \,, \tag{5.7}$$

and

$$u(0,y,F(y)S(0)) := \sup_{(a,b)\in\mathcal{D}} u^{(a,b)}(0,y,F(y)S(0)) \,. \tag{5.8}$$

Before stating the main result of this section, we provide an economic interpretation of the auxiliary control problems $u^{(a,b)}$. For all $(a,b) \in \mathcal{D}$, the process $Z_y^{(a,b)}$ describes the price process of $d$ risky assets in a fictitious financial market without transaction costs. The process $\phi$ is a portfolio strategy on the fictitious financial market : $\phi^i$ is the number of shares of risky asset $i$ held at each time, for $i = 1, \ldots, d$. The process $W_{w,y}^{(a,b)^\phi}$ describes the wealth induced by portfolio strategy $\phi$ and initial wealth $w$, under the self-financing condition. Hence, $u^{(a,b)}$ is the super-replication problem for a conveniently modified contingent claim on the auxiliary market. The main feature of the price process $Z_y^{(a,b)}$ is the following. Let $x$, $x' \in I\!R^{d+1}$ be two vectors of portfolio holdings such that $x'^i = x^i + \sum_{i,j}(a^{ji} - (1+\lambda^{ij})a^{ij})$, $i = 0, \ldots, d$ for some transfer matrix $a \in \mathbb{M}_+^{d+1}$. Then

$$\mathrm{diag}[\bar{S}(t)]^{-1}(x' - x)\bar{Z}_y^{(a,b)}(t) \quad \leq \quad 0 \ ,$$

i.e. portfolio rebalancement on the fictitious financial market without transaction costs is cheaper than on the initial market with transaction costs.

The above formal discussion provides an intuitive justification of the following connection between the control problems $u$ and $v$.

**Proposition 5.1** *For all $y$ in $(0, \infty)^n$, we have*

$$v(0, S(0)) \quad \geq \quad u(0, y, F(y)S(0)) \ .$$

**Proof.** Let $(a, b) \in \mathcal{D}$, $w > v(0, S(0))$, $y \in (0, \infty)^n$, set $x := w\mathbf{1}_0$, and consider some portfolio strategy $L \in \underline{\mathcal{A}}(x)$ such that $X_x^L(T) \succeq g(S(T))$, i.e.

$$X_x^L(T) - g(S(T)) \in K \tag{5.9}$$

Set $\ell_k^{ij}(t) := \int_0^t dL^{ij}(r)/\bar{S}^k(r)$ for all $i, j = 0, \ldots, d$, $k \in \{i, j\}$ and $0 \leq t \leq T$. Here $\ell_i^{ij}(t)$ (resp. $\ell_j^{ij}(t)$) is the cumulated transfer from asset $i$ to asset $j$ in terms of number of shares of asset $i$ (resp. $j$). In terms of $\ell$, the wealth process can be written in :

$$X_x^{i,L}(t) = x^i + \bar{S}^i(t) \sum_{j=0}^{d} \left(\ell_i^{ji}(t) - \ell_i^{ij}(t)(1 + \lambda^{ij})\right), \ i = 0, \ldots, d, \ 0 \leq t \leq T.$$

For all $i = 0, \ldots, d$, define

$$\phi^i(t) := \sum_{j=0}^{d} \left(\ell_i^{ji}(t) - \ell_i^{ij}(t)(1 + \lambda^{ij})\right) \quad \text{for } 0 \leq t \leq T,$$

so that

$$X_x^{i,L}(t) = x^i + \bar{S}^i(t)\phi^i(t), \qquad i = 0, \ldots, d, \ 0 \leq t \leq T.$$

Also observe that $\phi^i(0-) = 0$ since $L(0-) = 0$. Notice that

$$\begin{aligned}
\bar{Z}_y^{(a,b)}(t)d\phi(t) \quad &= \quad \sum_{i,j=0}^{d} \left(dL^{ji}(t)\bar{f}^i(Y_y^{(a,b)}(t)) - dL^{ji}(t)(1 + \lambda^{ji})\bar{f}^j(Y_y^{(a,b)}(t))\right) \\
&= \quad \sum_{i,j=0}^{d} dL^{ji}(t) \left(\bar{f}^i(Y_y^{(a,b)}(t)) - (1 + \lambda^{ji})\bar{f}^j(Y_y^{(a,b)}(t))\right) \\
&\leq \quad 0 \ ,
\end{aligned}$$

where we used the fact that $f(Y_y^{(a,b)}(.)) \in \underline{K}^*$, $L^{ji}$ is non-decreasing and the expression of $\{1\} \times \underline{K}^*$ as intersection of the half hyperplanes $H^{ij}$. Since $\phi$ is a bounded variation process and $\phi(0-) = 0$, this proves that :

$$\bar{Z}_y^{(a,b)}(t)\phi(t) \;\leq\; \int_0^t \phi(r) d\bar{Z}_y^{(a,b)}(r) \;=\; W_{w,y}^{(a,b)^\phi}(t) - w.$$

Then

$$\begin{aligned} W_{w,y}^{(a,b)^\phi}(t) &\geq w + \phi(t)\mathrm{diag}[\bar{f}(Y_y^{(a,b)}(t))]\bar{S}(t) \\ &= w + \sum_{i=0}^{d} \phi^i(t)\bar{f}^i(Y_y^{(a,b)}(t))\bar{S}^i(t) \\ &= X_x^L(t)\bar{f}(Y_y^{(a,b)}(t)) \quad \text{for } 0 \leq t \leq T . \end{aligned} \tag{5.10}$$

Since $f(Y_y^{(a,b)}(.)) \in \underline{K}^*$, it follows from (5.9) that

$$\bar{f}(Y_y^{(a,b)}(T))(X_x^L(T) - g(S(T))) \geq \; 0 \; ,$$

and then

$$W_{w,y}^{(a,b)^\phi}(T) \;\geq\; \bar{f}(Y_y^{(a,b)}(T))g\left(S(T)\right) \; .$$

Now, we claim that $\phi \in \mathcal{B}^{(a,b)}(w,y)$. Then the last inequality proves that $w \geq u^{a,b}(0, y, F(y)S(0))$ and therefore $v(0, S(0)) \geq u(0, y, F(y)S(0))$ from the arbitrariness of $w > v(0, S(0))$, $y \in (0,\infty)^n$ and $(a,b) \in \mathcal{D}$.

Hence, in order to conclude the proof, it remains to show that $\phi \in \mathcal{B}^{(a,b)}(w,y)$. Using (5.10), the admissibility condition (2.6) and the fact that $f(Y_y^{(a,b)}(.)) \in \underline{K}^*$, we see that

$$W_{w,y}^{(a,b)^\phi}(t) \;\geq\; 0 \quad 0 \leq t \leq T \; .$$

Finally, for all $i = 1, \ldots, d$

$$\begin{aligned} \int_0^T |\phi^i(t)|^2 d\left\langle Z_y^{i,(a,b)}(t)\right\rangle &= \int_0^T |\phi^i(t)S^i(t)|^2 \sum_{j=1}^{d} \left(\left\{F(Y_y^{(a,b)}(t))\sigma(t,S(t))\right\}^{ij}\right. \\ &+ \left.\left\{Df(Y_y^{(a,b)}(t))\mathrm{diag}[Y_y^{(a,b)}(t)]a(t)\right\}^{jj}\right)^2 dt \\ &< +\infty \; , \end{aligned}$$

since $\phi$ is a bounded variation process and $\sigma$, $a$, $f(y)$ and $Df(y)\mathrm{diag}[y]$ are bounded; see Remark 4.2. $\square$

We conclude this section by the following result which explains the reason for adopting the parameterization of the control process $Y_y^{(a,b)}$ in (5.2). Consider the stopping time

$$\theta_y^{(a,b)} := \inf\left\{ t > 0 \ : \ \sum_{j=1}^{n}\sum_{i=1}^{d}\Big(|\ln(S^i(t)/S^i(0))| + |\ln(Y_y^{j,(a,b)}(t)/y^j)|\Big) \geq \mu \right\},$$

and the exponential

$$M_y^{(a,b)}(t) := \mathcal{E}\left(-\int_0^t \alpha^y(r, S(r), Y_y^{(a,b)}(r), a(r), b(r))dB(r)\right) \qquad t \geq 0 .$$

As defined, function $\alpha^y$ is bounded. Then, the process $\left\{M_y^{(a,b)}(t), t \geq 0\right\}$ is well-defined and is a martingale. We then introduce the probability measure $Q_y^{(a,b)}$ equivalent to $P$ by :

$$\frac{dQ_y^{(a,b)}}{dP} = M_y^{(a,b)}(T) .$$

**Lemma 5.1** *The stopped process* $\left\{Z_y^{(a,b)}(t \wedge \theta_y^{(a,b)}), t \geq 0\right\}$ *is a* $Q_y^{(a,b)}$*- martingale.*

**Proof.** Define the process

$$B_y^{(a,b)}(t) := B(t) + \int_0^t \alpha^y(r, S(r), Y_y^{(a,b)}(r), a(r), b(r))dr \quad 0 \leq t \leq T.$$

Then by definition of the stopping time $\theta_y^{(a,b)}$ and Girsanov's theorem, we see that the process $\left\{B_y^{(a,b)}(t \wedge \theta_y^{(a,b)}),\ t \geq 0\right\}$ is a Brownian motion under $Q_y^{(a,b)}$. Applying Itô's lemma on the stochastic interval $[0, T \wedge \theta_y^{(a,b)}]$, we get by direct computation

$$\begin{aligned} dZ_y^{(a,b)}(t) &= \mathrm{diag}[Z_y^{(a,b)}(t)]\Big(\sigma(t, F(Y_y^{(a,b)}(t))^{-1}Z_y^{(a,b)}(t)) \\ &+ F^{-1}(Y_y^{(a,b)}(t))Df(Y_y^{(a,b)}(t))\mathrm{diag}[Y_y^{(a,b)}(t)]a(t)\Big)\ dB_y^{(a,b)}(t) . \end{aligned}$$

The required result follows from the fact that the diffusion term in the above stochastic differential equation is bounded on the stochastic interval $[0, \theta_y^{(a,b)}]$. $\square$

**Remark 5.1** In the one-dimensional case, Cvitanić, Pham and Touzi (1999) solved the super-replication problem under transaction costs by means of the dual formulation of the problem. The key stone of their analysis was to define processes $Z_y^{(a,b)}$ which are martingales under $Q_y^{(a,b)}$. In our multidimensional framework, this property holds only up to the stopping time $\theta_y^{(a,b)}$. We show in the next section how to adapt their arguments by considering of subset of $\mathcal{D}$.

**Remark 5.2** Notice that process $Z_y^{(0,0)}$ defined by $Z_y^{(0,0)}(t) = F(y)S(t)$ for all $0 \leq t \leq T$ is a $P$-martingale.

## 5.2 Direct dynamic programming

We first extend the definition of the processes $\left(W^{(a,b)\phi}, Y^{(a,b)}, Z^{(a,b)}\right)$ to the case where the time origin is defined by some $t$ in $[0,T]$. Let $(w,y,z) \in \mathbb{R}_+ \times (0,\infty)^n \times (0,\infty)^d$. We define the process $\left(W_{t,w,y,z}^{(a,b)\phi}, Y_{t,y,z}^{(a,b)}, Z_{t,y,z}^{(a,b)}\right)$ by the dynamics (5.5), (5.2), (5.3) and the initial condition $(W_{t,w,y,z}^{(a,b)\phi}(t)$, $Y_{t,y,z}^{(a,b)}(t)$, $Z_{t,y,z}^{(a,b)}(t)) = (w,y,z)$. We define accordingly the set of admissible controls $\mathcal{B}^{(a,b)}(t,w,y,z)$, the stopping time $\theta_{t,y,z}^{(a,b)}$ and the probability measure $Q_{t,y,z}^{(a,b)}$.

The dynamic stochastic control problems associated with (5.7) and (5.8) are then given by :

$$\begin{aligned} u^{(a,b)}(t,y,z) &:= \inf\Big\{w \in \mathbb{R} \;:\; \exists \phi \in \mathcal{B}^{(a,b)}(t,w,y,z), \\ &\qquad W_{t,w,y,z}^{(a,b)\phi}(T) \geq \bar{f}(Y_{t,y,z}^{(a,b)}(T))g\left(F(Y_{t,y,z}^{(a,b)}(T))^{-1}Z_{t,y,z}^{(a,b)}(T)\right)\Big\}\;, \end{aligned}$$

$$u(t,y,z) \;:=\; \sup_{(a,b)\in\mathcal{D}} u^{(a,b)}(t,y,z)\;.$$

The following result is adapted from Soner and Touzi (1998, 2000a).

**Proposition 5.2** (Dynamic programming) *Fix some* $(t,y,z) \in [0,T)\times(0,\infty)^{n+d}$ *and* $(a,b) \in \mathcal{D}$. *Then for all scalar* $w > u(t,y,z)$, *there exists some control* $\phi$ *in* $\mathcal{B}^{(a,b)}(t,w,y,z)$ *such that*

$$W_{t,w,y,z}^{(a,b)\phi}(\theta) \;\;\geq\;\; u\left(\theta, Y_{t,y,z}^{(a,b)}(\theta), Z_{t,y,z}^{(a,b)}(\theta)\right)$$

*for all $[t,T]$-valued stopping time $\theta$.*

**Proof.** Fix $w > u(t,y,z)$ and $(a,b) \in \mathcal{D}$. By definition of the control problem $u^{(a,b)}$, there exists $\phi \in \mathcal{B}^{(a,b)}(t,w,y,z)$ such that

$$W^{(a,b)^\phi}_{t,w,y,z}(T) \geq \bar{f}(Y^{(a,b)}_{t,y,z}(T))g\left(F(Y^{(a,b)}_{t,y,z}(T))^{-1}Z^{(a,b)}_{t,y,z}(T)\right) . \tag{5.11}$$

Fix some $[t,T]$-valued stopping time $\theta$. Since for each $(a,b) \in \mathcal{D}$ the coefficients of (5.2) are (random) Lipschitz, there exists a unique solution to (5.2). Then, clearly :

$$Y^{(a,b)}_{t,y,z}(T) \quad = \quad Y^{(a,b)}_{\theta,Y^{(a,b)}_{t,y,z}(\theta),Z^{(a,b)}_{t,y,z}(\theta)}(T) .$$

From the definition of $W^{(a,b)^\phi}_{t,w,y,z}$ and $Z^{(a,b)}_{t,y,z}$, the same property holds for these processes. Then, by direct substitution, we see that $\phi$ is an admissible super-replicating strategy for the contingent claim, when starting with the initial conditions $\left(\theta, W^{(a,b)^\phi}_{t,w,y,z}(\theta), Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)\right)$ for $P$-almost every $\omega \in \Omega$. By definition of the dynamic stochastic control problem $u$, this proves that :

$$W^{(a,b)^\phi}_{t,w,y,z}(\theta) \geq u^{(a,b)}(\theta, Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)) .$$

Since $\left(W^{(a,b)^\phi}_{t,w,y,z}(\theta), Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)\right)$ depends on $(a,b)$ only through the stochastic interval $[t,\theta]$, we may take supremum on the right-hand side, and we get the required result from the arbitrariness of $w$, $(a,b)$ and $\theta$. □

**Corollary 5.1** *Fix some $(t,y,z) \in [0,T)\times(0,\infty)^{n+d}$ and consider some scalar $w > u(t,y,z)$. Then for all $(a,b)$ in $\mathcal{D}$ and $t \leq r \leq T$, we have*

$$w \geq E^{Q^{(a,b)}_{t,y,z}}\left[u\left(r \wedge (t+\theta^{(a,b)}_{t,y,z}), Y^{(a,b)}_{t,y,z}(r \wedge (t+\theta^{(a,b)}_{t,y,z})), Z^{(a,b)}_{t,y,z}(r \wedge (t+\theta^{(a,b)}_{t,y,z}))\right)\right] .$$

**Proof.** By the dynamic programming equation of the previous proposition, for all $(a,b)$ in $\mathcal{D}$, there exists some control $\phi$ in $\mathcal{B}^{(a,b)}(t,w,y,z)$ such that $W^{(a,b)^\phi}_{t,w,y,z}(r \wedge (t+\theta^{(a,b)}_{t,y,z})) \geq u^{(a,b)}(r \wedge (t+\theta^{(a,b)}_{t,y,z}),\ Y^{(a,b)}_{t,y,z}(r \wedge (t+\theta^{(a,b)}_{t,y,z})),\ Z^{(a,b)}_{t,y,z}(r \wedge (t+\theta^{(a,b)}_{t,y,z})))$ for all $t \leq r \leq T$. Then the required result is obtained by taking expectations under $Q^{(a,b)}_{t,y,z}$ and using Lemma 5.1 together with the admissibility conditions (5.4)-(5.6) and Fatou's lemma. □

## 5.3 Viscosity property

We denote by $u_*$ the *lower semicontinuous envelope* of $u$ :

$$u_*(t,y,z) = \liminf_{(t',y',z')\to(t,y,z)} u(t',y',z').$$

We shall use the notation

$$\Gamma^a(t,y,z) \ := \ \mathrm{diag}[z]\left(\sigma(t,F(y)^{-1}z) + F(y)^{-1}Df(y)\mathrm{diag}[y]a\right) .$$

Then, for all control $(a,b)\in\mathcal{D}$ and $(t,y,z)\in[0,T)\times(0,\infty)^{n+d}$, the dynamics of the process $(Y^{(a,b)}_{t,y,z}, Z^{(a,b)}_{t,y,z})$ on the stochastic interval $[t,t+\theta^{(a,b)}_{t,y,z}]$ is given by :

$$dY^{(a,b)}_{t,y,z}(r) \ = \ \mathrm{diag}[Y^{(a,b)}_{t,y,z}(r)]\left(b(r)dr + a(r)dB^{(a,b)}_{t,y,z}(r)\right) \tag{5.12}$$

$$dZ^{(a,b)}_{t,y,z}(r) \ = \ \Gamma^{a(r)}\left(r, Y^{(a,b)}_{t,y,z}(r), Z^{(a,b)}_{t,y,z}(r)\right)dB^{(a,b)}_{t,y,z}(r) \ , \tag{5.13}$$

where $B^{(a,b)}_{t,y,z}$ is a Brownian motion under the equivalent probability measure $Q^{(a,b)}_{t,y,z}$; see proof of Lemma 5.1.

**Proposition 5.3** *Function $u_*(t,y,z)$ is a lower semicontinuous viscosity supersolution of the Hamilton-Jacobi-Bellman equation :*

$$\inf_{(a,b)\in\mathbb{M}^{n,d}\times\mathbb{R}^n} -\mathcal{L}^a\varphi \ - \mathcal{G}^{a,b}\varphi \ = \ 0 \quad on \quad [0,T)\times(0,\infty)^{n+d}, \tag{5.14}$$

*where*

$$\mathcal{L}^a\varphi(t,y,z) \ = \ D_t\varphi(t,y,z) + \frac{1}{2}\mathrm{Tr}\left[\Gamma^{a\prime}D^2_{zz}\varphi\Gamma^a\right](t,y,z)$$

$$\mathcal{G}^{a,b}\varphi(t,y,z) \ = \ \mathrm{diag}[y]bD_y\varphi(t,y,z) + \frac{1}{2}\mathrm{Tr}\left[D^2_{yy}\varphi(t,y,z)\mathrm{diag}[y]aa'\mathrm{diag}[y]\right] + \mathrm{Tr}\left[\Gamma^a(t,y,z)a'\mathrm{diag}[y]D^2_{yz}\varphi(t,y,z)\right] .$$

*Moreover, for all $(y,z)$ in $(0,\infty)^{n+d}$, we have*

$$u_*(T,y,z) \geq \bar{f}(y)g\left(F(y)^{-1}z\right). \tag{5.15}$$

**Proof.** We first prove (5.15). Let $\varepsilon$ be an arbitrary positive scalar and $(t,y,z)\in[0,T)\times(0,\infty)^{n+d}$. Set $w := u(t,y,z)$. By definition of the control problem $u^{(0,0)}$, there exists some control $\phi$ in $\mathcal{B}^{(0,0)}(t,w+\varepsilon,y,z)$ such that

$$W^{(0,0)^\phi}_{t,w+\varepsilon,y,z}(T) \geq \bar{f}(y)g\left(F(y)^{-1}Z^{(0,0)}_{t,y,z}(T)\right) .$$

Since $W^{(0,0)^{\phi}}_{t,w+\varepsilon,y,z}$ is a nonnegative $P$-local martingale, it is a $P-$supermartingale (see Remark 5.2), and we get by taking expectations under $P$ and sending $\varepsilon$ to zero :

$$u(t,y,z) \geq E\left[\bar{f}(y)g\left(F(y)^{-1}Z^{(0,0)}_{t,y,z}(T)\right)\right]. \tag{5.16}$$

The required result is obtained by sending $t$ to $T$ and using Fatou's lemma as well as the lower semi-continuity of $g$.

We now prove (5.14). Fix $(t,y,z) \in [0,T) \times (0,\infty)^{n+d}$ and some control $(a,b) \in \mathcal{D}$ such that the process $(a,b)$ is constant on a neighborhood of $t$. Let $\varphi$ be an arbitrary $C^2([0,T) \times (0,\infty)^{n+d})$ function such that

$$0 = (u_* - \varphi)(t,y,z) = \min(u_* - \varphi) .$$

Let $(t_k, y_k, z_k)_{k\geq 1}$ be a sequence in $[0,T) \times (0,\infty)^{n+d}$ satisfying

$$(t_k, y_k, z_k) \longrightarrow (t,y,z) \quad \text{and} \quad u(t_k,y_k,z_k) \longrightarrow u_*(t,y,z) \quad \text{as} \quad k \to +\infty .$$

Set $w_k := u(t_k,y_k,z_k) + \frac{1}{k}$ and $\beta_k := w_k - \varphi(t_k,y_k,z_k)$ and observe that

$$\beta_k \longrightarrow 0 \qquad \text{as} \qquad k \to +\infty .$$

For ease of notation, we set $\theta_k := \theta^{(a,b)}_{t_k,y_k,z_k}$ and $Q_k := Q^{(a,b)}_{t_k,y_k,z_k}$. We introduce the stopping time

$$h_k := \theta_k \wedge (\sqrt{\beta_k} + h 1_{\{\beta_k \neq 0\}}) \quad \text{for some } h > 0 .$$

Observe that for all $h > 0$

$$h \wedge \theta_k \longrightarrow h \wedge \theta^{(a,b)}_{t,y,z} \;>\; 0 \;\text{ as }\; k \to +\infty. \tag{5.17}$$

This follows from the fact that $(Y^{(a,b)}_{t_k,y_k,z_k}, Z^{(a,b)}_{t_k,y_k,z_k}) \longrightarrow (Y^{(a,b)}_{t,y,z}, Z^{(a,b)}_{t,y,z})$ for $P$-a.e. $\omega \in \Omega$, uniformly on compact subsets (see Protter (1990) Theorem 37 p246). From Corollary 5.1, it follows that

$$w_k \;\geq\; E^{Q_k}\left[u\left(t_k + h_k, Y^{(a,b)}_{t_k,y_k,z_k}(t_k+h_k), Z^{(a,b)}_{t_k,y_k,z_k}(t_k+h_k)\right)\right] .$$

Since $u \geq u_* \geq \varphi$, we may replace $u$ by $\varphi$ in the previous inequality and we get by Itô's lemma

$$\beta_k - E^{Q_k}\left[\int_{t_k}^{t_k+h_k}\left(\mathcal{L}^a\varphi + \mathcal{G}^{a,b}\varphi\right)\left(r, Y^{(a,b)}_{t_k,y_k,z_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r)\right)dr\right] \geq 0. \quad (5.18)$$

We now consider two cases.

<u>*1st case.*</u> Suppose that the set $\{k \geq 1 : \beta_k = 0\}$ is finite. Then there exists a subsequence renamed $(\beta_k)_{k\geq 1}$ such that $\beta_k \neq 0$ for all $k \geq 1$. We use the same argument as in Subsection I.4.1. Dividing by $\sqrt{\beta_k}$ and sending $k$ to infinity, we get (up to a subsequence) by uniform integrability

$$\liminf_{k\to+\infty} -\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}(\mathcal{L}^a\varphi + \mathcal{G}^{a,b}\varphi)(r, Y^{(a,b)}_{t_k,s_k,y_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r))dr \geq 0 \; .$$

The required result is a direct consequence of (5.17) and the following Lemma whose proof will be carried out later.

**Lemma 5.2** *Let $\psi : [0,T) \times (0,\infty)^{n+d} \to I\!R$ be locally Lipschitz in $(t,y,z)$ then*

$$\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}\left[\psi(r, Y^{(a,b)}_{t_k,y_k,z_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r)) - \psi(t,y,z)\right]dr \to 0 \quad as \quad k \to +\infty$$

*along some subsequence.*

<u>2nd case.</u> If the set $\{k \geq 1 : \beta_k = 0\}$ is not finite, then there exists a subsequence renamed $(\beta_k)_{k\geq 1}$ such that $\beta_k = 0$ for all $k \geq 1$. Then, we follow the same line of arguments as in first case, by dividing equation (5.18) by $h$ and sending $h$ to zero.

**Proof of Lemma 5.2** Since $\psi(t,y,z)$ is locally Lipschitz in $(t,y,z)$, we have

$$\begin{aligned}
&\left|\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}\left[\psi(r, Y^{(a,b)}_{t_k,y_k,z_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r)) - \psi(t,y,z)\right]dr\right| \\
\leq\;& C\,\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}\left(|r-t| + |Z^{(a,b)}_{t_k,y_k,z_k}(r) - z| + |Y^{(a,b)}_{t_k,y_k,z_k}(r) - y|\right)dr \\
\leq\;& C\,\frac{h_k}{\sqrt{\beta_k}}\left(h_k + |t_k - t| + \sup_{t_k\leq r\leq t_k+h_k}|Z^{(a,b)}_{t_k,y_k,z_k}(r) - z|\right. \\
+\;& \left.\sup_{t_k\leq r\leq t_k+h_k}|Y^{(a,b)}_{t_k,y_k,z_k}(r) - y|\right) \; ,
\end{aligned}$$

for some constant $C$. In order to obtain the required result, we shall prove that :

$$\sup_{t_k \le r \le t_k+h_k} |Z^{(a,b)}_{t_k,y_k,z_k}(r) - z| \longrightarrow 0 \quad \text{and} \quad \sup_{t_k \le r \le t_k+h_k} |Y^{(a,b)}_{t_k,y_k,z_k}(r) - y| \longrightarrow 0$$

as $k \to \infty$. We only report the proof of the second convergence result. The first one is obtained by the same line of arguments. Since $b$, $\alpha^{y_k}$, $a$ and $Y^{(a,b)}_{t_k,y_k,z_k}$ are bounded on $[t_k, t_k + h_k]$

$$|Y^{(a,b)}_{t_k,y_k,z_k}(r) - y| \;\le\; |y_k - y| + h_k C' + |\int_{t_k}^{s} \tilde{a}(\tau, Y^{(a,b)}_{t_k,y_k,z_k}(\tau)) dB(\tau)|$$

for some constant $C'$, where we denoted $\tilde{a}(t,y) = \mathrm{diag}[y]a(t)$. Therefore

$$\begin{aligned} \sup_{t_k \le r \le t_k+h_k} |Y^{(a,b)}_{t_k,y_k,z_k}(r) - y| \;&\le\; |y_k - y| + h_k C' \\ &\quad + \sup_{t_k \le r \le t_k+h_k} |\int_{t_k}^{r} \tilde{a}(\tau, Y^{(a,b)}_{t_k,y_k,z_k}(\tau)) dB(\tau)| \; . \end{aligned}$$

The first two terms on the right-hand side converge to zero. As for the third term, it follows from Doob's maximal inequality for submartingales that

$$\begin{aligned} &E\left[\left(\sup_{t_k \le r \le t_k+h_k} |\int_{t_k}^{r} \tilde{a}(\tau, Y^{(a,b)}_{t_k,y_k,z_k}(\tau)) dB(\tau)|\right)^2\right] \\ &\le 4E\left[\int_{t_k}^{t_k+h_k} |\tilde{a}\tilde{a}'(\tau, Y^{(a,b)}_{t_k,y_k,z_k}(\tau))| d\tau\right] \; . \end{aligned}$$

Since $a$ is bounded and $Y^{(a,b)}_{t_k,y_k,z_k}$ is bounded on $[t_k, t_k + h_k]$, uniformly in $k$, this proves that

$$\sup_{t_k \le r \le t_k+h_k} |Y^{(a,b)}_{t_k,y_k,z_k}(r) - y| \;\longrightarrow\; 0 \quad \text{as } k \to \infty \quad \text{in } L^2(P),$$

and therefore $P$-a.s. along some subsequence. □

**Remark 5.3** In the previous proof, we established inequality (5.16). Since $f(.)$ is valued in $\underline{K}^*$, (2.7) implies in particular that

$$u(t,y,z) \;\ge\; 0 \quad \text{for all} \quad (t,y,z) \in [0,T) \times (0,\infty)^{n+d} \; .$$

**Lemma 5.3** (i) Function $u_*(t,y,z)$ is independent of $y$.
(ii) Under Assumption 3.1, function $u_*(t,y,z)$ is nonincreasing in $t$ and concave in $z$.

**Proof.** The $y-$independence of function $u_*(t,y,z)$ is proved by sending $b_i$ to $\pm\infty$ ,$1 \leq i \leq n$, in equation (5.14), and using Lemmas 5.3 and 5.4 of Cvitanić, Pham and Touzi (1999). We now prove that $u_*$ is nonincreasing in $t$. Let $\hat{a}$ be any solution of

$$Df(y)\mathrm{diag}[y]\hat{a} \quad = \quad F(y)\sigma(t, F(y)^{-1}z) \ .$$

Notice that $\hat{a}$ is well-defined by Lemma 4.2. Then $\Gamma^{\hat{a}}(t,y,z) = 0$. Since $u_*$ is independent of its $y$ variable, equation (5.14) shows that $u_*$ is a viscosity supersolution of the equation $-\varphi_t = 0$. Then, it follows from Lemma 5.3 in Cvitanić, Pham and Touzi (1999) that $u_*$ is nonincreasing in $t$.

It remains to prove the concavity of $u_*$ in the $z$ variable. Let $(n,\xi)$ be an arbitrary element of $I\!N \times I\!R^d$ and define $\check{a}$ as a solution of

$$\frac{1}{n}Df(y)\mathrm{diag}[y]\check{a} = F(y)\left( \ \mathrm{diag}[z]^{-1}\xi \quad | \ \mathbf{0} \ \right) - F(y)\sigma(t, F(y)^{-1}z) \ ,$$

where $\mathbf{0}$ is the zero matrix of $I\!M^{d,d-1}$. Notice that $\check{a}$ is well defined by Lemma 4.2. Then, it is easily checked that

$$\mathrm{Tr}\left[\Gamma^{\check{a}}(t,y,z)'D^2_{zz}\varphi(t,y,z)\Gamma^{\check{a}}(t,y,z)\right] \quad = \quad n^2\xi' D^2_{zz}\varphi(t,y,z)\xi \ ,$$

and therefore $u_*$ is a viscosity supersolution of the equation $-\xi' D^2_{zz}\varphi\xi = 0$ for all $\xi \in I\!R^d$. Now, let $z_1$ and $z_2$ be two arbitrary elements of $(0,\infty)^d$. Then, the function

$$\psi(r) := u_*(t, r(z_1 - z_2) + z_2), \qquad r \in [0,1],$$

is a viscosity supersolution of $-\varphi'' = 0$. This is easily seen by an appropriate change of basis of $I\!R^d$ and Lemma 5.3 of Cvitanić, Pham and Touzi (1999). Moreover, $\psi$ is bounded from below by Remark 5.3. Then, by the same argument as in Proposition 5.2 of Cvitanić, Pham and Touzi (1999), it follows that $\psi$ is concave and therefore

$$\begin{aligned} u_*(t, rz_1 + (1-r)z_2) &= \psi(r) \ \geq \ r\psi(1) + (1-r)\psi(0) \\ &= ru_*(t, z_1) + (1-r)u_*(t, z_2) \end{aligned}$$

for all $r \in [0,1]$ and $t \in [0,T)$. The required result follows from the arbitrariness of $z_1$ and $z_2$ in $(0,\infty)^d$. □

## 5.4 Proof of the main theorem

We first prove that $v(0, S(0)) \geq \hat{g}(S(0))$. By Lemma 5.3, $u_*$ is nonincreasing on $[0,T)$ and independent of $y$. Then

$$u_*(0,y,z) = u_*(0,y',z) \geq u_*(t,y',z) \quad \text{for all } (t,y,y',z) \in [0,T) \times (0,\infty)^{2n+d}.$$

Since $u_*$ is lower semicontinuous, we get from the terminal condition (5.15) of Proposition 5.3

$$u_*(0,y,z) \geq \bar{f}(y')g(F(y')^{-1}z) \quad \text{for all } (y,y',z) \in (0,\infty)^{2n+d}.$$

Taking supremum over $y'$, this provides $u_*(0,y,z) \geq G(z)$. Again by Lemma 5.3, $u_*$ is concave in $z$, and therefore

$$u_*(0,y,z) \geq G^{conc}(z) \quad \text{for all } (y,z) \in (0,\infty)^{n+d}.$$

Now from Proposition 5.1 and the fact that $u \geq u_*$, we see that

$$\begin{aligned} v(0,S(0)) &\geq \sup_{y \in (0,\infty)^n} u(0,y,F(y)S(0)) \\ &\geq \sup_{y \in (0,\infty)^n} u_*(0,y,F(y)S(0)) \\ &\geq \sup_{y \in (0,\infty)^n} G^{conc}(F(y)S(0)) \ = \ \hat{g}(S(0)) \, . \end{aligned}$$

It remains to prove the converse inequality. If $\hat{g}(s) = +\infty$, then the result follows from the previous inequality. Next suppose that $\hat{g}(s) < +\infty$. From Theorem 3.2, there exists some $\Delta \in I\!R^{d+1}$ such that :

$$\hat{g}(S(0))\mathbf{1}_0 \ \succeq \ \text{diag}[\bar{S}(0)]\Delta \quad \text{and} \quad \text{diag}[\bar{S}(T)]\Delta \ \succeq \ g(S(T)) \, . \qquad (5.19)$$

From the left hand-side inequality, we see that there exists a matrix $a \in M_+^{d+1}$ such that for all $i = 0, \ldots, d$ :

$$\left(\hat{g}(S(0))\mathbf{1}_0 - \text{diag}[\bar{S}(0)]\Delta\right)^i + \sum_{j=0}^{d} \left(a^{ji} - (1+\lambda^{ij})a^{ij}\right) \quad \geq \quad 0.$$

Now define the trading strategy

$$L(t) \;=\; L(0) \;:=\; a \qquad 0 \leq t \leq T\;.$$

Then it is easily checked that $L \in \underline{\mathcal{A}}(\hat{g}(S(0))\mathbf{1}_0)$. From the right hand-side inequality of (5.19), it follows that $X^L_{\hat{g}(S(0))\mathbf{1}_0}(T) \succeq g(S(T))$. The required result then follows from the definition of the super-replication problem $v(0, S(0))$. □

# 6 Proof by the dual formulation

We now show how to adapt the arguments of Cvitanić, Pham and Touzi (1999) to our multidimensional framework. Since the technical arguments are similar to that of the previous section, we only sketch the main steps of the proof.

## 6.1 Auxiliary control problems

Fix $(t, y, z) \in [0, T] \times (0, \infty)^{n+d}$. We use the same notations and definitions as in Section 5 but we now define $\alpha(t, y, z, a, b)$ as

$$\begin{aligned}\alpha(t,y,z,a,b) \;=\;& \sigma(t,F(y)^{-1}z)^{-1}F(y)^{-1}\left\{Df(y)\text{diag}[y]b\right.\\ &+\frac{1}{2}\text{Vect}\left[\text{Tr}\left(D^2f^i(y)\text{diag}[y]aa'\text{diag}[y]\right), i=1,\ldots,d\right]\\ &\left.+\,\text{Vect}\left[\left(Df(y)\text{diag}[y]a\sigma'(t,F(y)^{-1}z)\right)_{ii}, i=1,\ldots,d\right]\right\}\;,\end{aligned}$$

(compare with (5.1)) and we define $Y^{(a,b)}$ and $Z^{(a,b)}$ as in the previous section where $\alpha^y$ is replaced by $\alpha$ (see (5.2)-(5.3)). We also introduce the subset of $\mathcal{D}$

$$\begin{aligned}\tilde{\mathcal{D}}(t,y,z) \;:=\;& \left\{(a,b)\in\mathcal{D} \;:\; \ln\left(\frac{(Y^{(a,b)}_{t,y,z})^i}{y^i}\right)\right.\\ &\left.\text{and } \ln\left(\frac{(Z^{(a,b)}_{t,y,z})^j}{z^j}\right) \text{ are bounded for all } i\leq n\;,\; j\leq d\right\}\;.\end{aligned}$$

**Remark 6.1** Fix $(t,y,z) \in [0,T] \times (0,\infty)^{n+d}$ and $(a,b)$ in $\mathcal{D}$. Set $\varepsilon > 0$ such that $1/\varepsilon < |y| + |z| < \varepsilon$ and define the positive stopping time :

$$\theta := \inf \left\{ s > t \ : \ |Y_{t,y,z}^{(a,b)}(s)| + |Z_{t,y,z}^{(a,b)}(t)| \notin [\frac{1}{\varepsilon}, \varepsilon] \right\} .$$

Then clearly $(a,b)1\!\!1_{[t,\theta]} \in \tilde{\mathcal{D}}(t,y,z)$ and therefore $\tilde{\mathcal{D}}(t,y,z) \neq \emptyset$.

For all $(a,b) \in \tilde{\mathcal{D}}(t,y,z)$, we introduce the exponential :

$$M_{t,y,z}^{(a,b)}(\tau) \ = \ \mathcal{E}\left( -\int_t^\tau \alpha(r, Y_{t,y,z}^{(a,b)}(r), Z_{t,y,z}^{(a,b)}(r), a(r), b(r)) dB(r) \right) \ , \ t \le \tau \le T \ .$$

By definition of $\alpha$ and $\tilde{\mathcal{D}}(t,y,z)$ and (2.3), $M_{t,y,z}^{(a,b)}$ is a martingale and we can define on $[t,T]$ the probability measure $Q_{t,y,z}^{(a,b)}$ equivalent to $P$ by :

$$\frac{dQ_{t,y,z}^{(a,b)}}{dP} \ := \ H_{t,y,z}^{(a,b)}(T) \ .$$

**Remark 6.2** Recall that this martingale property did not hold for the parameterization of the previous section (see Remark 5.1). The idea of introducing the subset $\tilde{\mathcal{D}}$ comes from Bouchard (1999).

Also notice that, by Girsanov's Theorem, the process :

$$B_{t,y,z}^{(a,b)}(\tau) \ := \ B(\tau) + \int_t^\tau \alpha(r, Y_{t,y,z}^{(a,b)}(r), Z_{t,y,z}^{(a,b)}(r), a(r), b(r)) dr, \quad t \le \tau \le T.$$

is a $Q_{t,y,z}^{(a,b)}$-Brownian motion. We shall denote by $E_{t,y,z}^{(a,b)}$ the expectation operator under the probability measure $Q_{t,y,z}^{(a,b)}$.

**Proposition 6.1** *For all* $(t,y,z)$

$$v(0, S(0)) \ \ge \ \sup_{y \in (0,\infty)} u(0, y, F(y) S(0))$$

*where*

$$u(t,y,z) := \sup_{(a,b) \in \tilde{\mathcal{D}}(t,y,z)} E_{t,y,z}^{(a,b)} \left[ \bar{f}(Y_{t,y,z}^{(a,b)}(T)) g\left( F\left( Y_{t,y,z}^{(a,b)}(T) \right)^{-1} Z_{t,y,z}^{(a,b)}(T) \right) \right] \quad (6.1)$$

**Proof.** This is a direct consequence of the dual formulation obtained in Kabanov and Last (1999) (see also Cvitanić, Pham and Touzi (1999)). □

In view of the results of Section 5, it suffices to show that $u_*$ (the lower semicontinuous envelope of $u$) is a viscosity supersolution of (5.14) with the boundary condition (5.15).

## 6.2 Viscosity property

We start with the following dynamic programming principle on our parameterized auxiliary control problem :

**Proposition 6.2** *Fix* $(t, y, z) \in [0, T] \times (0, \infty)^{n+d}$. *Then, for all* $(a, b) \in \tilde{\mathcal{D}}(t, y, z)$ *and for all* $[t, T]$*-valued stopping time* $\theta$ *:*

$$v(t, y, z) \geq \sup_{(a,b)\in\tilde{\mathcal{D}}(t,y,z)} E^{(a,b)}_{t,y,z}\left[v\left(\theta, Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)\right)\right].$$

**Proof.** The proof is standard and similar to that of Lemma 5.2 in Cvitanić, Pham and Touzi (1999). □

**Proposition 6.3** *Function* $u_*$ *is a viscosity supersolution of* (5.14) *with the boundary condition* (5.15).

The proof is the same as the one of Proposition 5.3 up to small modifications. We sketch it for completeness.

**Proof.** Boundary condition. Fix $(y, z) \in (0, \infty)^{n+d}$. Since, for all $t \in [0, T]$, $(a, b) = (0, 0) \in \tilde{\mathcal{D}}(t, y, z)$ and clearly $Y^{(0,0)}_{t,y,z}(\cdot) = y$, we get by (6.1)

$$u(t, y, z) \ \geq \ E^{(0,0)}_{t,y,z}\left[\bar{f}(y)g\left(F(y)^{-1}Z^{(0,0)}_{t,y,z}(T)\right)\right] .$$

The desired result is obtained by sending $t$ to $T$ and using Fatou's Lemma as well as the lower semi-continuity of $g$.

Viscosity property. Fix some $(t, y, z) \in [0, T) \times (0, \infty)^{n+d}$. Let $\varphi$ be an arbitrary $C^2([0, T) \times (0, \infty)^{n+d})$ function such that

$$0 = (v_* - \varphi)(t, y, z) = \min(v_* - \varphi) .$$

Fix $(a, b) \in \tilde{\mathcal{D}}(t, y, z)$ such that the process $(a, b)$ is constant on a neighborhood of $t$ (this is possible by Remark 6.1 and the continuity of process $Y^{(a,b)}$ for any constant $(a, b)$). Define :

$$\mathcal{N} := \left\{(t', y', z') \in [0, T] \times (0, \infty)^{n+d} : |(t', y', z') - (t, y, z)| \leq \zeta\right\}$$

for some $\zeta$ such that $0 < \zeta < y^i$ $(i = 1, \ldots, n)$ and $0 < \zeta < z^j$ $(j = 1, \ldots, d)$ . Let $(t_k, y_k, z_k)_{k\geq 1}$ be a sequence in $\mathcal{N}$ satisfying

$$(t_k, y_k, z_k) \to (t, y, z) \quad \text{and} \quad u(t_k, y_k, z_k) \to u_*(t, y, z) \quad \text{as} \quad k \to +\infty \ .$$

The rest of the proof follows line by line the proof of Proposition 5.3. It suffices to define $\theta_k$ as

$$\theta_k := \inf \left\{ s > 0 \ : \ \left( t_k + s, Y^{(a,b)}_{t_k,y_k,z_k}(t_k + s), Z^{(a,b)}_{t_k,y_k,z_k}(t_k + s) \right) \notin \mathcal{N} \right\} \ ,$$

and to use Proposition 6.2 instead of Corollary 5.1. □

# 7 Examples

## 7.1 On the generating family of $K^*$

The solution of the super-replication problem is given in terms of a variationnal problem involving the normalized polar cone $\underline{K}^*$. Hence, in order to compute explicitly the value function $v$, we need to characterize explicitly the generating family of the polyhedral cone $K^*$.

We first provide a sub-family of the generating family $\{e_1, \ldots, e_n\}$ of the polar cone $K^*$. Consider the $\mathbb{R}^{d+1}$ vectors :

$$\begin{aligned} z_i \ &: \ = (1 + \lambda^{0i}) \left( (1 + \lambda^{0i})^{-1}, \ldots, (1 + \lambda^{di})^{-1} \right) ; \qquad i = 0, \ldots, d \\ z'_i \ &: \ = (1 + \lambda^{i0})^{-1} \left( (1 + \lambda^{i0}), \ldots, (1 + \lambda^{id}) \right) ; \qquad i = 0, \ldots, d. \end{aligned}$$

Then, it is easily checked that $z_i$, $z'_i \in K^*$ for all $i = 0, \ldots, d$. Also, we have

$$z_i \ , \ \ z'_i \ \in \ \partial(K^*) \quad \text{for all} \quad i = 0, \ldots, d.$$

To see this, suppose that $z_i \in \text{Int}(K^*)$ for some $i$. Then, the vector $\hat{z}_i := z_i + \varepsilon \mathbf{1}_i \in K^*$ for some $\varepsilon > 0$. We end up with a contradiction by writing that $\hat{z}_i \in H^{0i}$. By the same argument, we get the result for the vectors $z'_i$.

Now observe that, by (2.4)-(2.5), $z_i \in \cap_{j\neq i} \partial H^{ji}$ and $z'_i \in \cap_{j\neq i} \partial H^{ij}$, for all $i = 0, \ldots, d$. Then, from the above discussion, it follows that the generating family $\{e_1, \ldots, e_n\}$ can be constructed by completing the family $\{z_i, z_i',\ i = 0, \ldots, d\}$. Also notice that the assumption

$$\lambda^{ij} + \lambda^{ji} > 0 \quad \text{for all } i \neq j \in \{0, \ldots, d\}$$

implies that

$$\partial H^{ij} \cap \partial H^{ji} \;\; = \;\; \{0\} \quad \forall\, i \neq j \in \{0, \ldots, d\}\ .$$

In particular, it is not difficult to see that

$$z \;\; := \;\; \sum_{i=0}^{d} z_i + z_i' \in \mathrm{Int} K^*$$

and therefore $\mathrm{Int}(K^*) \neq \emptyset$.

Now, consider the following conditions on the transaction costs matrix $\lambda$ :

$$\lambda^{ij} \;\; > \;\; 0 \ \text{ for all } i \neq j = 0, \ldots, d \tag{7.1}$$

and

$$(1 + \lambda^{ik})(1 + \lambda^{kj}) > (1 + \lambda^{ij}) \text{ for all } i, j, k = 0, \ldots, d \text{ with } i, j \neq k\ . \tag{7.2}$$

Condition 7.2 is used to provide a complete characterization of the set of generators in the two-dimensional case, see Example 7.3 below.

**Example 7.2** In the one-dimensional case $d = 1$, $z_o = z_1' = (1, (1 + \lambda^{10})^{-1})$ and $z_o' = z_1 = (1, (1 + \lambda^{01}))$. These are exactly the generating vectors of Example 4.1.

**Example 7.3** We consider here the two-dimensional case $d = 2$. In order to obtain a generating family of the polar cone $K^*$ by completing the family $\{z_i, z_i', i = 0, \ldots, 2\}$, we proceed as follows. Consider all vectors, with unit first component, defined by the intersection of hyperplanes $\partial H^{ij}$ and $\partial H^{kl}$. This provides all candidates for the required generating vectors. Such a candidate is effectively a generating vector if and only if it lies in $K^*$. By tedious calculation, it is easily checked that condition 7.2 rules out all such candidates except the vectors $z_i$ and $z_i'$ for $i = 0, \ldots, 2$. Hence $\{z_i, z_i',\ i = 0, \ldots, 2\}$ is a generating family of $K^*$.

In the general case, one can proceed as in Example 7.3 : define the candidate generating vectors as intersections of $d$ hyperplanes $\partial H^{i,j}$, then check whether such vectors lie in the polar cone $K^*$. In contrast with the two-dimensional case, condition 7.2 does not allow to characterize those candidates which are effectively in $K^*$, and we are unable to provide explicitly a generating family for the polar cone $K^*$.

However, notice that the characterization of $\hat{g}$ as the cost of the cheapest buy-and-hold strategy in Theorem 3.2 can also be used for the explicit computation of the value function of the super-replication problem; see paragraph 7.4 below.

## 7.2 Call and put options

Let $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ be two arbitrary constants and consider the real payoff function

$$\begin{aligned} g^o(s^1, s^2) &= -\kappa_1 1_{s^1 \geq \kappa_1} + \kappa_2 1_{s^2 < \kappa_2} \\ g^1(s^1, s^2) &= +s^1 1_{s^1 > \kappa_1} \\ g^2(s^1, s^2) &= -s^2 1_{s^2 \leq \kappa_2} \ . \end{aligned}$$

Then it is easily checked that

$$v\left(0, S^1(0), S^2(0)\right) = (1 + \lambda^{01}) S^1(0) + \kappa_2.$$

Notice that there is no compensation between the two options : the value function is equal to the sum of the super-replication costs of each option. The super-replicating strategy consists in buying one unit of stock 1 and keeping in cash $\kappa_2$.

## 7.3 Spread option

Let $\kappa \geq 0$ be an arbitrary constant and consider the payoff function

$$g^o(s^1, s^2) = -\kappa 1_{s^1 - s^2 \geq \kappa}$$

$$g^1(s^1, s^2) \;=\; +s^1 1_{s^1-s^2>\kappa}$$
$$g^2(s^1, s^2) \;=\; -s^2 1_{s^1-s^2\geq\kappa}\ .$$

Notice that we have defined $g^2$ in order to insure lower-semicontinuity of the payoff function and that only the signe of the inequality matters. Then, it is easily checked that

$$v\left(0, S^1(0), S^2(0)\right) = (1+\lambda^{01})S^1(0).$$

Hence, the cheapest super-replicating strategy consists in buying $(1+\lambda^{10})$ units of stock $S^1$.

## 7.4 Index call option

Let $(\kappa, d)$ be an arbitrary constant of $\mathbb{R}_+ \times \mathbb{N}\backslash\{0\}$, and consider the stock index $I(.)$ defined as

$$I(t) = \sum_{k=1}^{d} \alpha^k S^k(t) \quad \text{ for all } t \in [0,T],\ \sum_{k=1}^{d} \alpha^k = 1,\ \alpha^k > 0 \text{ for all } k = 1, \ldots, d,$$

and the European call option with pay-off function $[I(T)-\kappa]^+$. Then, the minimal super-replication cost is given by

$$v\left(0, S(0)\right) = \sum_{k=1}^{d} \alpha^k (1+\lambda^{0k})(1+\lambda^{k0})S^k(0),$$

which is the cost of replication of the index.

# Part B.

# Utility hedging and pricing under proportional transaction costs

# Introduction

We consider a general multivariate financial market with transaction costs as in Kabanov (1999) and we analyse the stochastic control problems of maximizing the expected utility of the liquidation value of terminal wealth diminished by some random claim $G$

$$V^G(x,\eta) := \sup_L EU^\eta\left(\ell(X_T^{x,L} - G)\right)$$

for a utility function of exponential form, i.e. $U^\eta(x) = -e^{-\eta x}$.

The existing literature in this framework only considers utility functions with bounded from below domains and, except for the recent papers of Deelstra, Pham and Touzi (2000) and Kabanov (1999), it concentrates on the univariate model (i.e. only one risky asset), see e.g. Davis, Panas and Zariphopoulou (1993), Cvitanić and Karatzas (1996) and Cvitanić and Wang (1999). Finally, it never considers the case where the terminal wealth is diminished by some random variable $G$.

In Chapter IV, we prove that existence and duality hold when suitably extending the set on which the maximization is performed. In particular, we show that, even if the optimum may not be attained, it can be approximated, in the sense that the terminal expected utility converge, by a sequence of attainable claims that are bounded from below. We also prove that the optimal terminal wealth is attainable if there exists some equivalent martingale measure with finite relative entropy.

The fact that the domain of $U^\eta$ is not bounded generates some additional difficulties. The usual duality arguments do not work and we are compelled to approximate $U^\eta$ by a sequence of utility functions with bounded from below domains as in Schachermayer (2000).

Contrary to Delbaen and al. (2000), we can not reduce to the case $G = 0$ by a simple change of measure. Moreover, we can not use existing results on the minimization of relative entropy as in the context of frictionless markets.

As a corollary of our main result, we prove a dual formulation for the utility based price

$$p(x,\eta) \quad := \quad \inf\left\{w \in I\!R \ : \ V^G(x + w\mathbf{1}_1, \eta) \geq V^0(x,\eta)\right\} \ ,$$

and we show that, if the seller is strongly risk-averse, his reservation price approaches the super-replication price increased by the liquidation value of the initial endowment. This result was first conjectured by Barles and Soner (1996) and proved by Bouchard (1999) in a Markovian setting via an asymptotic analysis of viscosity solutions of HJB equations. Rouge (1996) and Delbaen and al. (2000) proved it in the context of frictionless incomplete markets.

This result is generalized in **Chapter V**, for less restrictive conditions on the contingent claim $G$ and on the set of portfolio strategies, and in **Chapter VI**, for a class of functions including CARA and CRRA functions. In contrast with Chapter IV, we do not use the dual formulation associated to the utility maximization problem.

The general setting is introduced in **Chapter III**.

# Chapter III.

# Financial market and portfolio strategies under proportional transaction costs

In this chapter we generalize the financial market introduced in Chapter II to the semi-martingale case. We introduce the concepts and stochastic control problems that will be studied in Chapters IV to VI.

# 1 Financial market with costs

Let $T$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\leq T}, P)$ be a complete probability space with trivial initial $\sigma$-algebra, supporting a semimartingale $S := (S^1, \ldots, S^d)'$ with positive components. We normalize $S^1$ to be equal to 1.

**Remark 1.1** $S^1$ stands for the non-risky asset. As usual, the assumption that the interest rate of the bank account is zero could be easily dispensed with by discounting.

A *trading strategy* is a $\mathbb{F}-$adapted, right-continuous, nondecreasing process $L$ with initial condition $L(0-) = 0$ and taking values in $\mathbb{M}^d_+$, the set of square matrices with $d$ lines and non-negative entries.

Here, $L^{ij}$ describes the cumulative amount of funds transferred from asset $i$ to asset $j$. Proportional transaction costs in this financial market are described by matrix $\lambda \in \mathbb{M}^d_+$. Recall from Chapter II that we can always assume that

$$\begin{aligned}(1+\lambda^{ij}) &\leq (1+\lambda^{ik})(1+\lambda^{kj}) \quad \forall\, i,\ j,\ k \in \{0,\ldots,d\} \\ \lambda^{ii} &= 0 \quad \forall\, i \in \{0,\ldots,d\}\ ,\end{aligned}$$

(see Remark II.2.3).

Given an initial holdings vector $x \in \mathbb{R}^d$ and a strategy $L$, the portfolio holdings $X^{x,L}$ are defined by the dynamics :

$$X^i_t = x^i + \int_0^t X^i_{s-}\frac{dS^i_s}{S^i_{s-}} + \int_0^t \sum_{j=1}^d dL^{ji}_s - (1+\lambda^{ij})dL^{ij}_s, \quad t \leq T, \quad i = 1,\ldots,d\ .$$

We shall denote by $\mathcal{A}$ the set of all trading strategies $L$.

# 2 Solvency region and liquidation function

As in Chapter II, we define the *solvency region* :

$$K := \left\{ x \in \mathbb{R}^d \ :\ \exists\, a \in \mathbb{M}^d_+,\ x^i + \sum_{j=1}^d a^{ji} - (1+\lambda^{ij})a^{ij} \geq 0\ ;\ i = 1,\ldots,d \right\}$$

together with the induced partial ordering on $\mathbb{R}^d$ :

$$x_1 \succeq x_2 \quad \text{if} \quad x_1 - x_2 \in K .$$

We will also use the notation

$$x_1 \succ x_2 \quad \text{if for some } \varepsilon > 0 \quad x_1 \succeq x_2 + \varepsilon \mathbf{1}_1 .$$

We shall denote by $\underline{\mathcal{A}}$ the set of $L \in \mathcal{A}$ such that there exists some constant $\delta_L$ for which

$$X_t^{x,L} \succeq -\delta_L S_t \quad \text{for all } t \leq T .$$

Clearly, $\mathcal{A}$ and $\underline{\mathcal{A}}$ are independent of the initial dotation $x$.

We can now define the convex sets of attainable terminal wealth

$$\begin{aligned} \mathcal{X}(x) &:= \left\{ X \in L^0(\mathbb{R}^d, \mathcal{F}_T) \ : \ X = X_T^{x,L} \text{ for some } L \in \mathcal{A} \right\} , \\ \underline{\mathcal{X}}(x) &:= \left\{ X \in L^0(\mathbb{R}^d, \mathcal{F}_T) \ : \ X = X_T^{x,L} \text{ for some } L \in \underline{\mathcal{A}} \right\} . \end{aligned}$$

**Remark 2.1** For all $x \in \mathbb{R}^d$ and $L \in \mathcal{A}$, $X^{x,L} + c\mathbf{1}_1 = X^{x+c\mathbf{1}_1,L}$. This follows from the linear structure of the dynamic. Since $\mathcal{A}$ is independent of the initial dotation, this implies that $\mathcal{X}(x + c\mathbf{1}_1) = \mathcal{X}(x) + c\mathbf{1}_1$. The same remark holds for $\underline{\mathcal{X}}$.

Recall from Chapter II that $K$ is a polyhedral closed convex cone containing the origin and that $K^*$ denotes the positive polar of $K$, i.e.

$$K^* := \left\{ y \in \mathbb{R}^d \ : \ xy \geq 0, \ \forall \, x \in K \right\} .$$

Also recall that the elements of $K^*$ have non-negative components, that $K^*$ may be written

$$K^* \ = \ \left\{ y \in \mathbb{R}^d \ : \ y^j - (1 + \lambda^{ij}) y^i \leq 0 \ , \ \ 1 \leq i, j \leq d \right\} , \qquad (2.1)$$

and that the partial ordering $\succeq$ may be characterized in terms of $K^*$ by

$$x \succeq 0 \quad \Leftrightarrow \quad xy \geq 0 \quad \text{ for all } y \in K^* . \qquad (2.2)$$

**Remark 2.2** If $y \in K^*$, then $y^i = 0$, for some $i = 1, \dots, d$, implies that $y = 0$.

As in Bouchard (1999), we finally define the *liquidation function* $\ell$ mapping $I\!R^d$ into $I\!R$ :

$$\ell(x) \;:=\; \sup\{w \in I\!R \;:\; x \succeq w\mathbf{1}_1\} \;,$$

i.e. the maximal cash endowment (in terms of non risky asset) that one can get from $x$ by clearing all the positions in the risky assets.

Notice that

$$-\ell(-x) = \inf\{w \in I\!R \;:\; w\mathbf{1}_1 \succeq x\} \;,$$

may be interpreted as the smallest cash endowment needed in order to constitute $x$.

We close this subsection with some properties of $\ell$.

**Proposition 2.1** *For all $x$, $x' \in I\!R^d$ and for all $c \in I\!R$ :*
(i) $\ell(x) = \inf\{yx \,:\, y \in K^*$ *with* $y^1 = 1\}$.
(ii) $\ell(x)\mathbf{1}_1 \in \mathcal{X}(x)$, *i.e. the supremum in the definition of* $\ell(x)$ *is attained.*
(iii) $\ell(x + c\mathbf{1}_1) = \ell(x) + c$, $\ell(x + x') \geq \ell(x) + \ell(x')$.
(iv) $x \succeq x' \Leftrightarrow \ell(x - x') \geq 0 \Rightarrow \ell(x) \geq \ell(x')$. $\ell(x) > 0 \Leftrightarrow x \succeq \varepsilon\mathbf{1}_1$ *for some* $\varepsilon > 0$ .

The proofs are easy and can be found in Bouchard (1999). We sketch them for completeness.

**Proof.** (i) : (2.2) $\Rightarrow \ell(x) \leq \inf_{y\in K^*}(1/y^1)yx$, (2.2) and $\inf_{y\in K^*}(1/y^1)y[x - \inf_{y\in K^*}(1/y^1)yx] \geq 0 \Rightarrow \ell(x) \geq \inf_{y\in K^*}(1/y^1)yx$. (ii) follows from the closedness of the set $\{w \in I\!R \;:\; x \succeq w\mathbf{1}_1\} = \{w \in I\!R : (x - w\mathbf{1}_1)y \geq 0$ for all $y \in K^*\}$ where the equality is obtained by (2.2). (i) $\Rightarrow$ (iii). First part of (iv) follows from (i)-(2.2) (for $\Leftrightarrow$) and from (i) (for $\Rightarrow$). Second part of (iv) follows from the first one and (iii). $\square$

# 3 Utility maximization and utility based pricing

Let $G$ be a $I\!R^d$-valued $\mathcal{F}_T$-measurable random variable such that

$$G \succeq -cS_T$$

for some real $c$. We interpret $G$ as being a *contingent claim.*

Let $U^\eta$ be of exponential type, i.e.

$$U^\eta(x) = -e^{-\eta x} \quad , \quad x \in I\!R^d \; .$$

In the economic literature $\eta = -(U^\eta)''(x)/(U^\eta)'(x)$ ($x \in I\!R^d$) is called absolute risk aversion. The larger $\eta$ is, the more risk adverse the agent is.

In this part we address the problem of utility maximization :

$$V^G(x,\eta) \;\; = \;\; \sup_L EU^\eta \left(\ell(X_T^{x,L} - G)\right) \; ,$$

where the supremum is taken on a well defined set related to $\mathcal{A}$.

Defining $V^0$ as $V^G$ with $G = 0$, we also study the utility based price defined as in Hodges and Neuberger (1989) :

$$p(x,\eta) \;\; := \;\; \inf\left\{w \in I\!R \; : \; V^G(x + w\mathbf{1}_1, \eta) \geq V^0(x,\eta)\right\} \; .$$

It can be interpreted as the minimal initial cash endowment which induces a higher maximal expected utility with liability $G$ at the terminal date $T$. In the economics literature, $p(x,\eta)$ is known as the reservation price (for the seller) of the contingent claim $G$.

In particular, we study the convergence of the utility based price as the absolute risk aversion $\eta$ tends to $\infty$.

In Chapter VI, we generalize the convergence result on $p(x,\eta)$ obtained in Chapters IV and V to a parameterized family of utility functions $(U^\eta)_\eta$ for which the absolute risk aversion is also an increasing function of $\eta$.

# 4 Super-replication

We close this chapter with some results concerning the super-replication theory. They will be used in the subsequent chapters.

We first introduce the set of hedging endowments

$$\Gamma := \{x \in \mathbb{R}^d : \ X \succeq G \text{ for some } X \in \underline{\mathcal{X}}(x)\}.$$

For all $x \in \mathbb{R}^d$, the super-replication price $g(x)$ associated to $G$ is the lower bound of the set

$$\{w \in \mathbb{R} \ : \ x + w\mathbf{1}_1 \in \Gamma\} \ ,$$

i.e.

$$g(x) \ \ := \ \ \inf\{w \in \mathbb{R} \ : \ X \succeq G \text{ for some } X \in \underline{\mathcal{X}}(x + w\mathbf{1}_1)\} \ .$$

This is the minimal increase in term of non risky asset which allows to super-replicate $G$ without risk when starting with the initial dotation $x$.

Following Kabanov (1999), we define

$$\mathcal{D} \ \ := \ \ \left\{Z \in L^0(K^*, \mathbb{F}) \ : \ Z_0^1 = 1 \ , \ (Z^i S^i)_{i \le d} \in \mathcal{M}(P)\right\} \ .$$

$\mathcal{D}$ plays the same role as the set of equivalent martingale measures in frictionless financial markets.

**Remark 4.1** *From the normalization of $S^1 = 1$ and the martingale property of $Z^1 S^1$, it follows from $Z_0^1 > 0$ that $Z^1 \neq 0$. Then, by Remark 2.2, $Z^i > 0$ ($i \le d$) for all $Z \in \mathcal{D}$.*

The following results are due to Kabanov (1999) and Kabanov and Last (1999).

**Proposition 4.1** *For all $x \in \mathbb{R}^d$ and $L \in \underline{\mathcal{A}}$, the process $(Z_t X_t^{x,L})_t$ is a supermartingale.*

**Remark 4.2** *Fix $x \in \mathbb{R}^d$. Then, from the definition of $g(x)$, the martingale property of $Z^1 S^1 = Z^1$ and Proposition 4.1, it is easily checked that*

$$\sup_{Z \in \mathcal{D}} E\left(Z_T G - Z_0 x\right) \;\; \leq \;\; g(x)\ .$$

**Theorem 4.1** *Fix $x \in \mathbb{R}^d$. Assume that*

$\mathbf{M_1}$. *The process $S$ is continuous.*

$\mathbf{M_2}$. *There exists a probability measure $Q \sim P$ such that $S \in \mathcal{M}(Q)$.*

$\mathbf{M_3}$. *The cone $K$ is proper (i.e. $K \cap (-K) = \{0\}$).*

*Then, $\Gamma$ is closed and*

$$\Gamma \;\; = \;\; \left\{ x \in \mathbb{R}^d \; : \; \sup_{Z \in \mathcal{D}} E\left(Z_T G - Z_0 x\right) \leq 0 \right\}\ .$$

*It follows that :*

$$\sup_{Z \in \mathcal{D}} E\left(Z_T G - Z_0 x\right) \;\; = \;\; g(x)\ .$$

**Remark 4.3** Notice that $\mathbf{M_3} \Leftrightarrow \mathrm{Int}(K^*) \neq \emptyset \Leftrightarrow \lambda^{ij} + \lambda^{ji} > 0$ for all $1 \leq j \neq i \leq d$. See Remark II.3.3 for the last equivalence.

# Chapter IV.

# Exponential hedging and pricing under proportional transaction costs

In the framework introduced in Chapter III, we study the problem of utility maximization when the utility function is of exponential type. The optimization problem is related to a suitable dual stochastic control problem. We establish existence for both problems. We also prove a dual formulation for the *reservation price* and we study its asymptotic as the absolute risk aversion tends to infinity.

# 1 Utility maximization and pricing problems

In this chapter we address the problem of utility maximization for the exponential utility function :

$$U^\eta(x) \;\; := \;\; -e^{-\eta x} \quad \text{for all} \;\; x \in I\!R \, , \;\; \eta > 0,$$

i.e. we prove existence for the problem

$$V^G(x,\eta) \;\; = \;\; \sup_L EU^\eta \left(\ell(X_T^{x,L} - G)\right) \; .$$

where the supremum is taken on a well defined set related to $\mathcal{A}$ that we shall introduce in the following section.

Finally, we study the convergence of the utility based price,

$$p(x,\eta) \;\; := \;\; \inf \left\{ w \in I\!R \; : \; V^G(x + w\mathbf{1}_1, \eta) \geq V^0(x,\eta) \right\} \; ,$$

as the absolute risk aversion $\eta$ tends to $\infty$.

# 2 The duality result

## 2.1 Admissible strategies and dual problem

As in Schachermayer (2000), we need to well define the domain of random variables on which the supremum is taken in order to obtain existence for the utility optimization problem.

We first choose a convenient set of dual variables. Given the positive orthant of $\mathcal{X}(x)$ :

$$\mathcal{X}_+(x) \;\; := \;\; \{X \in \mathcal{X}(x) \; : \; X \succeq 0\} \; ,$$

and the convex cone

$$\mathbf{Y}_+ \;\; := \;\; \left\{ (y,Y) \in K^* \times L^0(K^*, \mathcal{F}_T) \; : \; EYX \leq yx \;\; \forall \, x \in K \; , X \in \mathcal{X}_+(x) \right\} \; ,$$

we define

$$\mathbf{Y}_e \ := \ \left\{(y,Y) \in \mathbf{Y}_+ \ : \ EY^1 = y^1 \ , \ Y^1 \ln Y^1 \in L^1\right\} \ .$$

We denote by $\mathbf{Y}_+^*$ (resp. $\mathbf{Y}_e^*$) the element $(y,Y) \in \mathbf{Y}_+$ (resp. $(y,Y) \in \mathbf{Y}_e$) such that $P(Y^1 = 0) = 0$. Notice that $(y,Y) \in \mathbf{Y}_+ \Rightarrow EY^1 \leq y^1$ since $\mathbf{1}_1 \in \mathcal{X}_+(\mathbf{1}_1)$.

We can now introduce the set of *admissible trading strategies*

$$\underline{\mathcal{X}}_e(x) := \{X \in \underline{\mathcal{X}}(x) \ : \ EYX \leq yx \quad \forall\, (y,Y) \in \mathbf{Y}_e\} \ .$$

Clearly,

$$\sup_{X \in \underline{\mathcal{X}}_e(x)} EU^\eta\left(\ell(X-G)\right) \ = \ \sup_{X \in \mathcal{X}_{U^\eta}(x)} EU^\eta\left(\ell(X-G)\right) \ , \tag{2.1}$$

where $\mathcal{X}_{U^\eta}(x)$ is the set of $X \in L^0(I\!R^d, \mathcal{F}_T)$ for which there exists a sequence $(X_k)_k \in \underline{\mathcal{X}}_e(x)$ such that

$$EU^\eta(\ell(X_k - G)) \to EU^\eta(\ell(X-G)) \ .$$

Since existence may fail to hold in $\underline{\mathcal{X}}_e(x)$, we define our optimization problem on $\mathcal{X}_{U^\eta}(x)$ :

$$V^G(x,\eta) \ := \ \sup_{X \in \mathcal{X}_{U^\eta}(x)} EU^\eta\left(\ell(X-G)\right) \ ,$$

i.e. we try to find $X \in L^0(I\!R^d, \mathcal{F}_T)$ such that the corresponding expected utility can be approximated by some admissible trading strategies.

As usual, we shall relate this problem to some dual optimization problem

$$W^G(x,\eta) := \inf_{(y,Y) \in \mathbf{Y}_e \setminus \{0\}} E\left(\tilde{U}^\eta(Y^1) + yx - YG\right) \ ,$$

where $\tilde{U}^\eta$ is the Fenchel transform of $-U^\eta(-\cdot)$, i.e.

$$\tilde{U}^\eta(y) \ := \ \sup_{x \in I\!R} \left(U^\eta(x) - xy\right) = U^\eta(I^\eta(y)) - yI^\eta(y) \ , \ y \geq 0 \ , \quad \text{with } I^\eta := (U^{\eta\prime})^{-1}.$$

By (2.1), we may replace $\mathcal{X}_{U^\eta}(x)$ by $\underline{\mathcal{X}}_e(x)$ in the definition of $V^G(x,\eta)$ and thus we will only need properties on $\underline{\mathcal{X}}_e(x)$.

**Proposition 2.1** *For all $x$, $x' \in I\!R^d$ and $c \in I\!R$ :*
(i) $\ell(x)\mathbf{1}_1 \in \underline{\mathcal{X}}_e(x)$ *(in particular by Proposition III.2.1 (iii)* $c\mathbf{1}_1 \in \underline{\mathcal{X}}_e(c\mathbf{1}_1)$*).*
(ii) $\{(x, X) \ : x \in I\!R^d$ *and* $X \in \underline{\mathcal{X}}_e(x)\}$ *is a convex cone.*
(iii) *If $x \succeq x'$ then for all $X' \in \underline{\mathcal{X}}_e(x')$, we can find some $X \in \underline{\mathcal{X}}_e(x)$ such that $X \succeq X'$.*
(iv) *For all $L \in \mathcal{A}$, $X_T^{x+c\mathbf{1}_1,L} = X_T^{x,L} + c\mathbf{1}_1$ and therefore $\underline{\mathcal{X}}_e(x+c\mathbf{1}_1) = \underline{\mathcal{X}}_e(x)$ $+ c\mathbf{1}_1$.*

**Proof.** The fact that $EY^1 = y^1$ for $(y, Y) \in \mathbf{Y}_e$ and Proposition III.2.1 (i)-(ii) imply (i). (ii) follows from the fact that $\{(x, X) \ : x \in I\!R^d$ and $X \in \underline{\mathcal{X}}(x)\}$ is a convex cone. To see that (iii) holds, let $X := X' + \ell(x - x')\mathbf{1}_1 \succeq X'$ (since $\ell(x - x') \geq 0$ by Proposition III.2.1 (iv)). Finally, $X \in \underline{\mathcal{X}}_e(x)$ by (i) and (ii). First part of (iv) follows from the linear structure of the dynamic and the normalisation of $S^1$ to unity. Second part follows from the first one and (i)-(ii). □

**Remark 2.1** Using Proposition III.2.1 (iii) and Proposition 2.1 (iv), it is easily checked that, for all $x \in I\!R^d$, $(c_1, c_2) \in I\!R \times I\!R$ and $\eta > 0$,

$$V^{G+c_2\mathbf{1}_1}(x + c_1\mathbf{1}_1, \eta) \quad = \quad e^{\eta(c_2-c_1)}V^G(x, \eta) \ .$$

In particular, by Proposition III.2.1 (iii) again, there is no loss of generality in assuming that $\ell(x)$ is arbitrarily large and that $G \succeq 0$ if $G \in L^\infty(K)$ (see Proposition III.2.1 (iv)).

## 2.2 Existence and duality for the utility maximization problem

**Assumptions:** The financial model satisfies :
$\mathbf{M_1}$. $S$ is continuous
$\mathbf{M_2}$. there exists some $Q \sim P$ such that $S \in \mathcal{M}(Q)$,
$\mathbf{M_3}$. $\mathrm{Int}(K^*) \neq \emptyset$.
We also assume that :

**C**. $G \in L^\infty(K)$,

and that the set of dual variables is such that :

**D**. $\mathbf{Y}_e^* \neq \emptyset$.

Finally, the main result of the next section requires that

$\mathbf{M_4}$. $H := dQ/dP$ is such that $EH \ln H < \infty$.

**Remark 2.2** It is not difficult to see that, under mild assuptions on the coefficients, $\mathbf{M_1}$-$\mathbf{M_2}$-**D**-$\mathbf{M_4}$ hold in the Brownian diffusion case.

**Remark 2.3** **C** is equivalent to $\bar{p}\mathbf{1}_1 \succeq G \succeq 0$ for some $\bar{p} > 0$.

**Remark 2.4** Assumptions $\mathbf{M_1}$-$\mathbf{M_3}$ are needed in order to apply the results of Kabanov and Last (1999) concerning the set of attainable contingent claims (see Section III.4) and to prove (iv) of Theorem 2.1 below.

**Remark 2.5** By **C**, definition of $\mathbf{Y}_+$ and (III.2.2) : $0 \leq EYG \leq y^1\bar{p}$ for all $(y,Y) \in \mathbf{Y}_+$.

**Remark 2.6** Suppose that $G - \tilde{G}\,\mathbf{1}_1 \in L^\infty(K)$ for some $\tilde{G}$ such that $e^{\eta\tilde{G}} \in L^1$. Let $\tilde{P}$ be the equivalent probability measure defined by $d\tilde{P} = \tilde{H}dP$ where $\tilde{H} := e^{\eta\tilde{G}}/\ Ee^{\eta\tilde{G}}$. We emphasise the dependence of $V^G$ with respect to $P$ by using the notation $V^G_P$. Then, arguing as in the Remark 2.1, we get

$$V^G_P(x,\eta) = V^{G-\tilde{G}\mathbf{1}_1}_{\tilde{P}}(x,\eta)Ee^{\eta\tilde{G}}$$

and therefore we may reduce to **C** if **D** holds under $\tilde{P}$.

**Remark 2.7** Since $\tilde{U}^\eta$ is bounded from below, $\mathbf{Y}_e \setminus \{0\} \neq \emptyset \Leftrightarrow |W^G(x,\eta)| < \infty$ for all $x \in I\!R^d$. In particular $\mathbf{D} \Rightarrow |W^G(x,\eta)| < \infty$. This follows from Remark 2.5 and the particular form of $\tilde{U}^\eta$.

**Theorem 2.1** *Let assumptions* $\mathbf{M_1}$-$\mathbf{M_3}$, **C** *and* **D** *hold. Then, for all* $x \in I\!R^d$ *and* $\eta > 0$ *:*

(i) Existence for $W^G$ : *There exists* $(y_*,Y_*) \in \mathbf{Y}_e^*$ *such that :*

$$W^G(x,\eta) \quad = \quad E\left(\tilde{U}^\eta(Y^1_*) + y_*x - Y_*G\right) \ .$$

*Moreover,*

$$E\left(\tilde{U}^\eta(Y_*^1) + y_* x - Y_* G\right) = \inf_{(y,Y)\in \mathbf{Y}_e^*,\ y^1=1} U^\eta\left(E\left(yx - YG + \frac{1}{\eta}Y^1 \ln Y^1\right)\right) ,$$

*where the last infimum is attained by* $(1/y_*^1)(y_*, Y_*)$.

(ii) Existence for $V^G$ : *The r.v.* $X_* := I^\eta(Y_*^1)\mathbf{1}_1 + G \in \mathcal{X}_{U^\eta}(x)$ *is optimal, i.e. :*

$$V^G(x,\eta) = EU^\eta\left(\ell(X_* - G)\right) .$$

(iii) Duality : $V^G(x,\eta) = W^G(x,\eta)$.

(iv) Attainability : *Assume further that* $\mathbf{M_4}$ *holds. Then there exists some*

$$X_* \in \mathcal{X}(x) \cap \mathcal{X}_{U^\eta}(x)$$

*that attains the supremum in* $V^G(x,\eta)$. *Moreover,*

$$EX_*Y \le xy \quad \text{for all } (y,Y) \in \mathbf{Y}_e .$$

The proof of this Theorem is reported in Section 4.

## 2.3 Existence and duality for the reservation price

We start this part with some properties of $V^G$. They will be useful for the proof of the duality result for the reservation price (in fact we will only use the implied properties for the mapping $c \in I\!R \mapsto V^G(x + c\mathbf{1}_1, \eta)$ $(x \in I\!R^d)$).

**Proposition 2.2** *Fix* $\eta > 0$.

*(i) If* $\ln|V^G(0,\eta)| < \infty$, *then* $V^G(\cdot,\eta)$ *is concave, continuous and strictly increasing for* $\succeq$ *on* $I\!R^d$ *(i.e.* $x' \succeq x + \varepsilon\mathbf{1}_1$ *for some* $\varepsilon > 0$ *implies* $V^G(x',\eta) > V^G(x,\eta)$*). Moreover,*

$$\lim_{\ell(x)\to\infty} V^G(x,\eta) = 0 \quad \forall\, x \in I\!R^d .$$

*(ii) If* $\mathbf{M_1}$-$\mathbf{M_3}$, $\mathbf{C}$ *and* $\mathbf{D}$ *hold, then* $\ln|V^G(0,\eta)| < \infty$.

**Proof.** (ii) follows from Theorem 2.1 (i)-(ii)-(iii) and Remark 2.5. We now prove (i).

*Continuity and concavity:* Concavity on $\mathbb{R}^d$ follows from Proposition 2.1 (ii) and concavity of $U^\eta(\ell(\cdot))$. We shall prove later that $\mathbb{R}^d = \mathrm{Int}(\mathrm{dom}(V^G(\cdot,\eta))$. Since $V^G(\cdot,\eta)$ is concave on $\mathbb{R}^d$, this implies that $V^G(\cdot,\eta)$ is continuous on $\mathbb{R}^d$.

*Increase and limit property:* By Proposition 2.1 (iii) and Remark 2.1,

$$V^G(x',\eta) \quad \geq \quad e^{-\eta\varepsilon} V^G(x,\eta) \ \ \forall\, x,\ x' \in \mathbb{R}^d,\ \varepsilon \geq 0\ , x' \succeq x + \varepsilon \mathbf{1}_1\ . \tag{2.2}$$

Recalling from Chapter III that $-\ell(-x)\mathbf{1}_1 \succeq x$ and $x \succeq \ell(x)\mathbf{1}_1$, we get by replacing successively $(x',\, \varepsilon,\, x)$ by $(x,\, \ell(x),\, 0)$ and $(0,\, \ell(-x),\, x)$ that

$$\ln|V^G(0,\eta)| < \infty \ \Rightarrow\ \ln|V^G(\cdot,\eta)| < \infty\ .$$

In particular,

$$\ln|V^G(0,\eta)| < \infty \ \Rightarrow\ \mathrm{Int}(\mathrm{dom}(V^G(\cdot,\eta))) = \mathbb{R}^d\ .$$

Now, since, by the above argument, $V^G < 0$, (2.2) proves that $V^G(\cdot,\eta)$ is strictly increasing for $\succeq$. Finally, replacing $(x',\, \varepsilon,\, x)$ by $(x,\, \ell(x),\, 0)$ in (2.2) and sending $\ell(x)$ to $\infty$ proves that

$$\lim_{\ell(x)\to\infty} V^G(x,\eta) \quad = \quad 0 \qquad \forall\, x \in \mathbb{R}^d\ .$$

□

The duality result for the reservation price is a direct consequence of Theorem 2.1 and Proposition 2.2.

**Corollary 2.1** *Let assumptions* $\mathbf{M_1}$-$\mathbf{M_3}$*,* $\mathbf{C}$ *and* $\mathbf{D}$ *hold. Then, for all* $x \in \mathbb{R}^d$ *and* $\eta > 0$ *:*

$$\begin{aligned} p(x,\eta) &= -(U^\eta)^{-1}\left(V^G(x,\eta)\right) + (U^\eta)^{-1}\left(V^0(x,\eta)\right) \\ &= \sup_{(y,Y)\in\mathbf{Y}_e^*\,,\, y^1=1} E\left(YG - \frac{1}{\eta}Y^1 \ln Y^1 - yx\right) \\ &+ \inf_{(y,Y)\in\mathbf{Y}_e^*\,,\, y^1=1} E\left(\frac{1}{\eta}Y^1 \ln Y^1 + yx\right) \end{aligned}$$

*where the extrema are attained in* $\mathbf{Y}_e^*$.

**Proof.** Since $G \succeq 0$, it follows from Proposition III.2.1 (iv) that $V^G(x,\eta) \leq V^0(x,\eta)$. Hence, Proposition 2.2 proves that $p(x,\eta)$ exists and satisfies $V^G(x+p\mathbf{1}_1,\eta) = V^0(x,\eta)$. Since $V^G(x+p\mathbf{1}_1,\eta) = e^{-\eta p}\, V^G(x,\eta)$ by Remark 2.1, the result is a direct consequence of (i)-(iii) of Theorem 2.1. □

# 3 Application : Large risk aversion

## 3.1 The asymptotic result

In this section, we consider the asymptotic behaviour of the pricing function $p(x,\eta)$ as $\eta$ tends to $\infty$. Our main result is a consequence of the dual formulation obtained in Corollary 2.1 for the reservation price.

We first introduce the so-called super-replication problem of the contingent claim $G$ associated with our set of admissible trading strategies :

$$g(x) \;:=\; \inf\left\{w \in I\!R \;:\; X \;\succeq\; G \text{ for some } X \in \underline{\mathcal{X}}_e(x+w\mathbf{1}_1)\right\}\ .$$

**Remark 3.1** It follows from **C** and Proposition 2.1 (i) that $g(x) < \infty$ for all $x \in I\!R^d$ since $\bar{p}\mathbf{1}_1 \succeq G$ and $\bar{p}\mathbf{1}_1 \in \underline{\mathcal{X}}_e(x+\bar{p}\mathbf{1}_1-\ell(x)\mathbf{1}_1)$ by Proposition 2.1 (i) and Proposition III.2.1 (iii).

**Theorem 3.1** *Let assumptions* $\mathbf{M_1}$-$\mathbf{M_3}$, **C** *and* **D** *hold. Then, for all* $x \in I\!R^d$, $\eta > 0$ *:*

$$\text{(i)} \qquad \liminf_{\eta\to\infty} p(x,\eta) \geq \sup_{(y,Y)\in\mathbf{Y}_e^*\,,\; y^1=1} E(YG - yx) + \ell(x)\ .$$

$$\text{(ii)} \qquad \textit{If moreover } \mathbf{M_4} \textit{ holds,} \quad \limsup_{\eta\to\infty} p(x,\eta) \leq g(x) + \ell(x)\ .$$

**Remark 3.2** It will be clear in the proof that, in (i) of the above theorem, we may replace $\{(y,Y) \in \mathbf{Y}_e^*\,,\, y^1 = 1\}$ by $\{(y,Y) \in \mathbf{Y}_e\,,\, y^1 = 1\} = \{(y,Y) \in \mathbf{Y}_e \setminus \{0\}\,,\, y^1 = 1\}$ by definition of $\mathbf{Y}_e$ and Remark III.2.2.

Using the dual formulation of the hedging problem obtained by Kabanov (1999) and Kabanov and Last (1999) (see Section III.4), the above theorem provides a precise description of the limit behaviour of $p(x,\eta)$.

**Corollary 3.1** *Let assumptions* $\mathbf{M_1}$-$\mathbf{M_4}$*,* $\mathbf{C}$ *and* $\mathbf{D}$ *hold. Assume further that the stochastic basis is such that all martingales on it are continuous. Then, for all* $x \in \mathbb{R}^d$ *:*

$$\lim_{\eta \to \infty} p(x, \eta) \; = \; g(x) + \ell(x) \; .$$

**Remark 3.3** Recall that by Corollary 2.1

$$V^G(x + p(x,\eta)\mathbf{1}_1, \eta) \quad = \quad V^0(x, \eta) \; . \tag{3.1}$$

for all $x \in \mathbb{R}^d$ and $\eta > 0$. Let $x$ be equal to 0. Using the last equality, we can provide a very simple upper bound for the probability of missing the hedge by a constant $k$ when starting with an initial endowment equal to $p(0,\eta)\mathbf{1}_1$ :

$$\inf_{X \in \underline{\mathcal{X}}_e(p(0,\eta)\mathbf{1}_1)} P\left[\ell\left(X - G\right) \le -k\right] \quad \le \quad \exp\left[-\eta k\right] \quad .$$

The proof is obtained by Corollary 2.1, the relation $1\!\!1_{x \le 0} \le e^{-\eta x}$, $x \in \mathbb{R}$, and by arguing as in the proof of Theorem 3.2 in Barles and Soner (1996).

Notice that, for all positive $k$, this probability tends to zero as $\eta$ tends to $+\infty$. This is in agreement with Corollary 3.1 which states that $p(0,\eta)$ converges to the super-replication price $g(0)$. In fact this relation will hold in more general context as long as (3.1) is satisfied for $x = 0$. See Barles and Soner (1996) for a similar majoration.

## 3.2 Proof of Theorem 3.1

We start with the easy observation

**Lemma 3.1** *For all* $x$*,* $x' \in \mathbb{R}^d$ *and* $\eta > 0$ *:*

$$(U^\eta)^{-1}\left(V^G(x,\eta)\right) + g(x) - \ell(-x') \;\ge\; (U^\eta)^{-1}\left(V^0(x',\eta)\right) \;\ge\; \ell(x') \; .$$

**Proof.** Fix $x$ and $x' \in \mathbb{R}^d$. By Proposition 2.1 (i), $\ell(x')\mathbf{1}_1 \in \underline{\mathcal{X}}_e(x')$ and therefore $V^0(x',\eta) \ge U^\eta(\ell(x'))$. We now prove the left-hand side inequality. We claim that

$$V^G(x + (g(x) - \ell(-x'))\mathbf{1}_1, \eta) \quad \ge \quad V^0(x', \eta) \; . \tag{3.2}$$

The desired result is a direct consequence of (3.2) and Remark 2.1. We now prove (3.2). Let $X' \in \underline{\mathcal{X}}_e(x)$. Fix $\varepsilon > 0$. From Remark 3.1 and definition of $g(x)$, we can find some $\tilde{X} \in \underline{\mathcal{X}}_e(x + (g(x)+\varepsilon)\mathbf{1}_1)$ such that : $\tilde{X} - G \succeq 0$. Recall that $-\ell(-x')\mathbf{1}_1 \succeq x'$. Then, by Proposition 2.1 (iii), we can find some $\hat{X} \in \underline{\mathcal{X}}_e(-\ell(-x'))$ such that $\hat{X} \succeq X'$. Set $p_\varepsilon := -\ell(-x') + g(x) + \varepsilon$. By Proposition 2.1 (ii), $X = \tilde{X} + \hat{X} \in \underline{\mathcal{X}}_e(x + p_\varepsilon \mathbf{1}_1)$. Since $X - G \succeq X'$, $\ell(X - G) \geq \ell(X')$ by Proposition III.2.1 (iv), and therefore

$$V^G(x + p_\varepsilon \mathbf{1}_1, \eta) \;\; \geq \;\; EU^\eta(\ell(X-G)) \geq EU^\eta(\ell(X')) \; .$$

(3.2) is then obtained by sending $\varepsilon$ to 0 and using Proposition 2.2. □

**Lemma 3.2** *Under* $\mathbf{M_4}$, *for all* $x \in I\!R^d$ *:*

$$\lim_{\eta\to\infty} (U^\eta)^{-1}\left(V^0(x,\eta)\right) \;\; = \;\; \ell(x) \; ,$$

*and then by Theorem 2.1 (i)-(iii)*

$$\lim_{\eta\to\infty} \inf_{(y,Y)\in\mathbf{Y}_e^*\, ,\, y^1=1} E\left(yx + \frac{1}{\eta}Y^1 \ln Y^1\right) \;\; = \;\; \ell(x) \; .$$

**Proof.** By Lemma 3.1, $(U^\eta)^{-1}\left(V^0(x,\eta)\right) \geq \ell(x)$. So it suffices to prove that

$$\limsup_{\eta\to\infty} (U^\eta)^{-1}\left(V^0(x,\eta)\right) \;\; \leq \;\; \ell(x) \; .$$

By Theorem 2.1 (i)-(iii) with $G = 0$,

$$(U^\eta)^{-1}\left(V^0(x,\eta)\right) \;\; \leq \;\; yx + \frac{1}{\eta} EY^1 \ln Y^1 \; .$$

for all $(y,Y) \in \mathbf{Y}_e^*$ with $y^1 = 1$. This proves that

$$\limsup_{\eta\to\infty} (U^\eta)^{-1}\left(V^0(x,\eta)\right) \;\; \leq \;\; \inf_{(y,Y)\in\mathbf{Y}_e^*,\, y^1=1} yx \; .$$

The desired result is then obtained by Proposition III.2.1 (i) and by noticing that, under $\mathbf{M_4}$, for all $y \in K^*$ with $y^1 \neq 0$, $(y, yH) \in \mathbf{Y}_e^*$. □

**Proof of Theorem 3.1.** By Corollary 2.1, for all $\eta > 0$ and $(y,Y) \in \mathbf{Y}_e^*$ with $y^1 = 1$

$$p(x,\eta) \quad \geq \quad E\left(YG - \frac{1}{\eta}Y^1 \ln Y^1 - yx\right) + \inf_{(y,Y)\in\mathbf{Y}_e^*\,,\, y^1=1} E\left(\frac{1}{\eta}Y^1 \ln Y^1 + yx\right)$$

and then by Lemma 3.2

$$\liminf_{\eta\to\infty} p(x,\eta) \quad \geq \quad E\left(YG - yx\right) + \ell(x)$$

for all $(y,Y) \in \mathbf{Y}_e^*$ with $y^1 = 1$, which leads to the first inequality of Theorem 3.1.

We now prove the second inequality. By Lemma 3.2 and the easy observation that $\ell(\ell(x)\mathbf{1}_1) = \ell(x)$ (see Proposition III.2.1 (iii)), we obtain that

$$\lim_{\eta\to\infty} (U^\eta)^{-1}\left(V^0(x,\eta)\right) \quad = \quad \ell(x) = \lim_{\eta\to\infty} (U^\eta)^{-1}\left(V^0(\ell(x)\mathbf{1}_1,\eta)\right) \ .$$

Since by Corollary 2.1

$$p(x,\eta) \quad = \quad -(U^\eta)^{-1}\left(V^G(x,\eta)\right) + (U^\eta)^{-1}\left(V^0(x,\eta)\right)$$

this proves that

$$\limsup_{\eta\to\infty} p(x,\eta) \quad = \quad \limsup_{\eta\to\infty} -(U^\eta)^{-1}\left(V^G(x,\eta)\right) + (U^\eta)^{-1}\left(V^0(\ell(x),\eta)\right)$$

and the result follows from Lemma 3.1 (with $x' = \ell(x)\mathbf{1}_1$) and Proposition III.2.1 (iii). □

## 3.3 Proof of Corollary 3.1

The result is obtained by combining (3.3), Remark 3.4, Lemma 3.3 below with Theorem 3.1.

Recall from Chapter III the definition of

$$\mathcal{D} \quad := \quad \left\{Z \in L^0(K^*, I\!\!F) \ : \ Z_0^1 = 1\ ,\ (Z^iS^i)_{i\leq d} \in \mathcal{M}(P)\right\} \ .$$

We also define

$$\mathcal{D}_e \ := \ \left\{ Z \in \mathcal{D} \ : \ Z_T^1 \neq 0 \ , \ Z_T^1 \ln Z_T^1 \ \in \ L^1 \right\} \ .$$

$\mathcal{D}$ (resp. $\mathcal{D}_e$) plays the same role than the set of equivalent martingale measures (resp. with finite relative entropy) in frictionless financial markets and it is not difficult to see that

$$\{(Z_0, Z_T) \ : \ Z \in \mathcal{D}_e\} \ \subset \ \mathbf{Y}_e^* \ . \tag{3.3}$$

(see Section 3 in Kabanov and Last 1999).

Now, set $H_t := E(H \mid \mathcal{F}_t)$ and notice that

$$\left\{(y, (yH_t)_{t\geq 0}) \ : \ y \in K^*, \ y^1 = 1\right\} \subset \mathcal{D}_e \ . \tag{3.4}$$

**Remark 3.4** The super-replication price is given by

$$g(x) \ = \ \sup_{Z\in\mathcal{D}} E(Z_T G - Z_0 x) \ .$$

To see this, let $X_T^{x,L} \in \underline{\mathcal{X}}_e(x)$ be such that $X_T^{x,L} \succeq G$. Then, $X_T^{x,L} \succeq 0$ and by Proposition III.4.1, for all $t \in [0,T]$ and $Z \in \mathcal{D}$, $Z_t X_t^{x,L} \geq E(Z_T X_T^{x,L} \mid \mathcal{F}_t)$ $\geq 0$ and therefore $X_t^{x,L} \succeq 0$ (see (III.2.2) and (3.4)). Since $\underline{\mathcal{X}}_e(x) \cap \mathcal{X}_+(x)$ is equal to the restriction to $\mathcal{X}_+(x)$ of the set of admissible strategies $\underline{\mathcal{X}}(x)$ used in Kabanov and Last (1999), we may apply their Theorem 3.2 - see Section III.4.

**Lemma 3.3** *Suppose that* $\mathbf{M_1}$–$\mathbf{M_4}$ *hold. Assume further that the stochastic basis is such that all martingales on it are continuous. Let* $G \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ *be such that* $G \succeq -cS_T$ *for some real constant c. Then for all* $x \in \mathbb{R}^d$

$$\gamma(x) \ := \ \sup_{Z\in\mathcal{D}} E(Z_T G - Z_0 x) \ = \ \sup_{Z\in\mathcal{D}_e} E(Z_T G - Z_0 x) \ .$$

**Proof.** Let $x \in \mathbb{R}^d$ be fixed. Denote by $\gamma_e(x)$ the right-hand side term of the equality of the lemma. Since $\mathcal{D}_e \subset \mathcal{D}$, we clearly have $\gamma(x) \geq \gamma_e(x)$. To see that the reverse inequality holds fix $Z \in \mathcal{D}$ and, for all and $n \in I\!N$, define the stopping time :

$$\tau^n \;:=\; \inf\Big\{t \;:\; |\ln(Z^1_t) - \ln(H_t)| \;\geq\; n\Big\} \wedge T$$

and

$$(Z^n_t)^i \;:=\; Z^i_t 1\!\!1_{t\leq\tau^n} + (1/H_{\tau^n}) Z^i_{\tau^n} H_t 1\!\!1_{t>\tau^n} \quad t \leq T \quad i = 1, \ldots, d\;.$$

It is easily checked that for all $n$, $Z^n \in \mathcal{D}_e$. Also notice that $Z^n_T \longrightarrow Z_T$ $P-$a.s. as $n \to \infty$, and $Z^n_0 = Z_0$. Now recall that $G \succeq -cS_T$ for some constant $c$. Then, by (III.2.2), $Z^n_T(G + cS_T) \geq 0$, and we get by the martingale property of $Z^n S$ and $ZS$ together with Fatou's lemma :

$$\begin{aligned}
\liminf_{n\to\infty} E\left[Z^n_T G - Z^n_0 x\right] &= \liminf_{n\to\infty} E\left[Z^n_T(G + cS_T) - Z^n_0 x - c(Z^n_T S_T)\right] \\
&= \liminf_{n\to\infty} E\left[Z^n_T(G + cS_T)\right] - Z_0 x - c(Z_0 S_0) \\
&\geq E\left[Z_T(G + cS_T)\right] - Z_0 x - c(Z_0 S_0) \;=\; E\left[Z_T G\right] - Z_0 x\;.
\end{aligned}$$

From the arbitrariness of $Z \in \mathcal{D}$, this proves that $\gamma_e(x) \geq \gamma(x)$. □

# 4 Proof of Theorem 2.1

We make the proof only for some $x \in K$ such that $\ell(x) > \bar{p} > 0$ (see Remark 2.1).

We split the proof of Theorem 2.1 in different lemmas. In this section, we will always assume that $\eta = 1$ and we will omit the dependence of the involved functions with respect to $\eta$ and $G$. For instance, we will write $V(x)$ for $V^G(x, \eta)$.

We shall approximate our optimization problem $V(x)$ by a sequence of optimization problems defined on the family of utility functions

$$U_n(x) \;:=\; -\left(1 + \frac{x}{n}\right)^{-n} \quad \text{for} \;\; n \geq 1\;.$$

Clearly, $(U_n)_n$ is an increasing sequence of functions that converges uniformly on compact subsets to $U$. Defining $x_n := x + (n/2)\mathbf{1}_1$ and $G_n := G + (n/2)\mathbf{1}_1$, we will show in particular that the sequence of functions

$$V_n(x) := \sup_{X \in \underline{\mathcal{X}}_+(x_n)} EU_n\left(\ell(X - G_n)\right) \ ,$$

converges to $V(x)$, where $\underline{\mathcal{X}}_+(x_n)$ is a subset of $\underline{\mathcal{X}}_e(x_n)$ defined accordingly with the enlarged set of dual variables $\mathbf{Y}_+$ that contains $\mathbf{Y}_e$

$$\underline{\mathcal{X}}_+(x_n) := \{X \in \underline{\mathcal{X}}(x_n) \ : \ EYX \leq yx_n \quad \forall\, (y, Y) \in \mathbf{Y}_+\} \ .$$

Existence of the above optimal control problem $V_n(x)$, will be obtained after proving existence in the dual problem

$$W_n(x) := \inf_{(y,Y) \in \mathbf{Y}_+ \setminus \{0\}} E\left(\tilde{U}_n(Y^1) + yx_n - YG_n\right) \ ,$$

where $\tilde{U}_n$ is the Fenchel transform of $-U_n(-\cdot)$ :

$$\tilde{U}_n(y) \ := \ \sup_{x \in \mathbb{R}} \left(U_n(x) - yx\right) = U_n(I_n(y)) - yI_n(y) = -(n+1)y^{\frac{n}{n+1}} + ny$$

with $I_n := (U_n')^{-1}$ and $y \geq 0$.

Finally, it will turn out that the optimum $V(x) = EU(\ell(X_* - G))$ is the limit of $EU(\ell(X_n - G))$ where, for each $n$, $X_n$ is related to some optimal terminal wealth for the problem $V_n(x)$.

**Remark 4.1** The idea of approximating $U$ by the sequence $(U_n)_n$ is similar to that used in Schachemayer (2000). The difference is that we also replace $x$ by $x_n$ and $G$ by $G_n$. The reason is the following. We know that the infimum in $W_n$ may not be attained under the condition $EY^1 = y^1$ (which partly explain why we use $\mathbf{Y}_+$ instead of $\mathbf{Y}_e$ in the definition of $W_n$). Since we want that the sequence of extrema $(y_n, Y_n)$ for $W_n$ converges to some extremum for $W$, we need to find a way to "force" the sequence $(EY_n^1 - y_n^1)_n$ to converge to 0. Since we cannot manage as in Schachermayer (2000), this role is played by $x_n$ and $G_n$ (see Step 2 of the proof of Lemma 4.4). This approximation argument will work, basically, because, by Remark 2.1, $V^{G_n}(x_n) = V^G(x)$ (see Step 6 of Lemma 4.4).

We now list some properties concerning the newly introduced control problems and their relations with the initial one.

(R1) $(U_n)_n$ *is an increasing sequence bounded from above by* $U$ *and* $\underline{\mathcal{X}}_+(x) \subset \mathcal{X}_U(x)$. *In particular, Remark 2.1 shows that* $(V_n(x))_n$ *is bounded from above by* $V(x)$.

(R2) $(\tilde{U}_n)_n$ *is increasing and bounded from above by* $\tilde{U}$. *Since* $\mathbf{Y}_e \subset \mathbf{Y}_+$ *and* $EY^1 = y^1$ *for all* $(y, Y) \in \mathbf{Y}_e$, *it follows that* $(W_n(x))_n$ *is bounded from above by* $W(x)$. *In particular, by Remark 2.7,* $\sup_n W_n(x) \leq W(x) < \infty$.

(R3) $\tilde{U}_n$ *converges to* $\tilde{U}$ *uniformly on compact subsets.*

(R4) $I_n(y) \to \infty$ *as* $y \to 0$.

(R5) *For all* $n \geq 1$ *and* $y \geq 0$, $-1 \leq \tilde{U}_n(y) \leq ny$.

Before turning to the proofs, we shall notice that some of the convergence results used in the proofs may hold only up to a subsequence. It will be clear that we will always be able to (and we will always do) assume that the original sequence converges.

## 4.1 Duality for finite $n$'s

The proof is close to that of Deelstra, Pham and Touzi (2000) but some of the conditions of their main Theorem do not hold in our setting.

We prove the duality all $n \geq n_0 \geq 1$, where $n_0$ is such that $\ell(-G) > -n_0/2$. This is possible since $G$ is assumed to be bounded for the partial ordering $\succeq$. The above properties will be very useful in the proofs.

**Proposition 4.1** *For all* $n \geq n_0$ *and* $(y, Y) \in \mathbf{Y}_+$ *:*

(i) $-C \leq V_n(x) \leq W_n(x) \leq W(x) \leq C$, *for some scalar* $C$ *independent of* $n$.

(ii) $\tilde{U}_n(Y^1) - YG_n > -C_n$, *for some scalar* $C_n$.

(iii) $\tilde{U}_n(Y^1) - YG > -C$, *for some scalar* $C$ *independent of* $n$.

(iv) $(y, Y) \in \mathbf{Y}_+ \longmapsto \tilde{U}_n(Y^1) + yx - YG$ *is convex.*

(v) *The following inequalities hold*

$$\begin{aligned} E(\tilde{U}_n(Y^1) + yx_n - YG_n) &\geq E(\tilde{U}_n(Y^1) + y^1\ell(x_n) - YG_n) \\ &\geq E\tilde{U}_n(Y^1) + y^1(\ell(x) - \bar{p}) + (n/2)(y^1 - EY^1) \\ &\geq E\tilde{U}_n(Y^1) + y^1(\ell(x) - \bar{p}) \geq E\tilde{U}_n(Y^1) \\ &= E(U_n(I_n(Y^1)) - Y^1 I_n(Y^1)) \\ &\geq E(-1 - Y^1 - Y^1 I_n(Y^1)) \\ &\geq -1 - y^1 - EY^1 I_n(Y^1) \,. \end{aligned}$$

**Proof.** (i) Notice that for $x \in K$, $\ell(x)\mathbf{1}_1 \in \underline{\mathcal{X}}_+(x)$ (by arguing like in Proposition 2.1 (i)), and therefore, by (R1) and Proposition III.2.1 (iii)-(iv), we get that for all $n \geq n_0 : -\infty < U_{n_0}(-(n_0)/2) \leq EU_n(\ell(\ell(x_n)\mathbf{1}_1 - G_n)) \leq V_n(x)$. To see that the second inequality of (i) holds, fix $X \in \underline{\mathcal{X}}_+(x_n)$, $(y, Y) \in \mathbf{Y}_+$, and notice that by definition of $\tilde{U}_n$

$$U_n(\ell(X - G_n)) \;\leq\; \tilde{U}_n(Y^1) + Y^1\ell(X - G_n) \leq \tilde{U}_n(Y^1) + Y(X - G_n) \tag{4.1}$$

where the last inequality is obtained by definition of $\mathbf{Y}_+$ and Proposition III.2.1 (i). The result follows by definition of $\underline{\mathcal{X}}_+(x_n)$. The rest of the statement is a direct consequence of (R2). By (4.1), $\tilde{U}_n(Y^1) - YG_n \geq U_n(\ell(-G_n))$. Since $\ell(-G) > -n_0/2$ implies that $\ell(-G_n) > -n/2$, this proves (ii). (iii) is obtained similarly by using (R2). (iv) follows from the convexity of $\tilde{U}_n$. (v) follows from $EY^1 \leq y^1$ and $0 \leq EYG \leq y^1\bar{p}$ , Proposition III.2.1 (i)-(iii), $\ell(x) > \bar{p}$ and the fact that, for all $y \geq 0$, $\tilde{U}_n(y) = U_n(I_n(y)) - yI_n(y)$ where $U_n(I_n(y)) = -y^{\frac{n}{n+1}}$ $\geq -1 - y$. □

**Lemma 4.1** *There exists some* $(y_n, Y_n) \in \mathbf{Y}_+$ *such that :*

$$W_n(x) \;\geq\; E\left(\tilde{U}_n(Y_n^1) + y_n x_n - Y_n G_n\right) \,.$$

**Proof.** Let $(y_k, Y_k)$ be a minimizing sequence and notice that by Proposition 4.1 (i)-(v), for all sufficiently large $k$,

$$\infty > E\left(\tilde{U}_n(Y_k^1) + y_k^1(\ell(x) - \bar{p})\right) \,.$$

Since $\ell(x)-\bar{p}>0$, (R5) implies that $y^1_k \to y^1_n$ as $k\to\infty$ (possibly along some subsequence). From the compactness of $\{y\in K^*: y^1=1\}$ (see (III.2.1)) this proves that $y_k \to y_n$ (possibly along some subsequence).

Recall that $0\le EY^1_k \le y^1_k$. Since $(y^1_k)_k$ is bounded, by Komlòs theorem and the compactness of $\{y\in K^*: y^1=1\}$, we obtain that there exists a sequence $\tilde{Y}_k \in \mathrm{conv}(Y_j, j\ge k)$ that converges $P-$a.s. to some $Y_n\in L^0(I\!R^d,\mathcal{F}_T)$. Taking the same convex combination to define $\tilde{y}_k$ and using the convexity of $\mathbf{Y}_+$, it follows that $(\tilde{y}_k,\tilde{Y}_k)\in\mathbf{Y}_+$. Finally, by closedness of $K^*$ and the fact that $E\tilde{Y}_kX\le \tilde{y}_kx$ for all $x\in K$, $X\in\mathcal{X}_+(x)$ and $k$, we obtain that $(y_n,Y_n)\in\mathbf{Y}_+$ by Fatou's Lemma.

By Proposition 4.1 (iv), $(\tilde{y}_k,\tilde{Y}_k)$ is also a minimizing sequence. Now, by Proposition 4.1 (ii) and Fatou's Lemma

$$\begin{aligned}\liminf_{k\to\infty} \tilde{y}_kx_n + E\left(\tilde{U}_n(\tilde{Y}^1_k)-\tilde{Y}_kG_n\right) &\ge y_nx_n + E\left(\liminf_{k\to\infty}\tilde{U}_n(\tilde{Y}^1_k)-\tilde{Y}_kG_n\right)\\ &= E\left(\tilde{U}_n(Y^1_n)+y_nx_n-Y_nG_n\right)\ .\end{aligned}$$

Since $(\tilde{y}_k,\tilde{Y}_k)$ is a minimizing sequence, this proves that

$$W_n(x) \ \ge\ E\left(\tilde{U}_n(Y^1_n)+y_nx_n-Y_nG_n)\right)\ .$$

□

**Lemma 4.2** *Let $(y_n,Y_n)\in\mathbf{Y}_+$ be as in Lemma 4.1. Then $(y_n,Y_n)\in\mathbf{Y}^*_+$, is optimal for $W_n(x)$ and*

$$0 = EY_n(I_n(Y^1_n)\mathbf{1}_1+G_n)-y_nx_n = \sup_{(y,Y)\in\mathbf{Y}_+} EY(I_n(Y^1_n)\mathbf{1}_1+G_n)-yx_n\ . \quad (4.2)$$

**Proof.** Let $(y_n,Y_n)$ be as in Lemma 4.1 and fix $(y,Y)\in\mathbf{Y}_+\setminus\{0\}$. Fix $\varepsilon\in(0,1/2)$. By convexity of $\mathbf{Y}_+$, $(y_\varepsilon,Y_\varepsilon):=\varepsilon(y,Y)+(1-\varepsilon)(y_n,Y_n)\in\mathbf{Y}_+\setminus\{0\}$. By Lemma 4.1, the convexity of $\tilde{U}_n$ and the fact that $\tilde{U}'_n=-I_n$, we have :

$$\begin{aligned}0 &\ge \frac{1}{\varepsilon}\left[E\left(\tilde{U}_n(Y^1_n)+y_nx_n-Y_nG_n\right)-E\left(\tilde{U}_n(Y^1_\varepsilon)+y_\varepsilon x_n-Y_\varepsilon G_n\right)\right]\\ &\ge E(Y-Y_n)\left(I_n(Y^1_\varepsilon)\mathbf{1}_1+G_n\right)+(y_n-y)x_n\ . \end{aligned} \quad (4.3)$$

We shall prove later that for all $\varepsilon \in (0,1/2)$ :

$$(Y^1 - Y_n^1) I_n(Y_\varepsilon^1) \ \geq \ -4(1+n(Y^1+Y_n^1)) - n(Y^1 - Y_n^1) \,. \qquad (4.4)$$

We now take the limit as $\varepsilon \to 0$ in (4.3). Since the right hand side term of (4.4) is in $L^1$, it follows from Fatou's Lemma that

$$0 \ \geq \ E(Y - Y_n)\left(I_n(Y_n^1)\mathbf{1}_1 + G_n\right) + (y_n - y)x_n \,.$$

Moreover, since $\mathbf{Y}_+$ is a cone, replacing $(y,Y)$ by $1/2(y_n,Y_n)$ and then by $2(y_n,Y_n)$ proves that $EY_n(I_n(Y_n^1)\mathbf{1}_1 + G_n) - y_n x_n = 0$.

Now, assume that $(y,Y) \in \mathbf{Y}_+^*$ (which is possible by Assumption **D** and $\mathbf{Y}_e \subset \mathbf{Y}_+$). Now assume that $P(A) > 0$ where $A := \{Y_n^1 = 0\}$. By the same argument as above :

$$\begin{aligned} 0 \ &\geq \ \liminf_{\varepsilon\to 0} E(Y - Y_n)\left(I_n(Y_\varepsilon^1)\mathbf{1}_1 + G_n\right) + (y_n - y)x_n \\ &\geq \ E \liminf_{\varepsilon\to 0} (Y - Y_n)\left(I_n(Y_\varepsilon^1)\mathbf{1}_1 + G_n\right) + (y_n - y)x_n \\ &= \ \infty \end{aligned}$$

since $\lim_{y\to 0} I_n(y) = \infty$ and $\lim_{y\to 0} 0 I_n(y) = 0$. This proves that $P(A) = 0$.

We now prove (4.4). Since $\varepsilon < \frac{1}{2}$, $Y_\varepsilon^1 \geq \frac{1}{2}Y_n^1$, $\frac{1}{4}(Y^1 - Y_n^1) + Y_\varepsilon^1 \geq \frac{1}{4}Y_n^1 \geq 0$, by convexity of $\tilde{U}_n$, $\tilde{U}_n' = -I_n$ and by (R5)

$$\begin{aligned} n\left(\frac{1}{4}(Y^1 - Y_n^1) + Y_\varepsilon^1\right) \ &\geq \ \tilde{U}_n(\frac{1}{4}(Y^1 - Y_n^1) + Y_\varepsilon^1) \\ &\geq \ \tilde{U}_n(Y_\varepsilon^1) - \frac{1}{4} I_n(Y_\varepsilon^1)(Y^1 - Y_n^1) \\ &\geq \ -1 - \frac{1}{4} I_n(Y_\varepsilon^1)(Y^1 - Y_n^1) \end{aligned}$$

and the result is obtained by observing that $Y_\varepsilon^1 \leq Y^1 + Y_n^1$. □

**Lemma 4.3** *Let $(y_n, Y_n) \in \mathbf{Y}_+$ be as in Lemma 4.1. Then,*

$$X_n \ := \ I_n(Y_n^1)\mathbf{1}_1 + G_n \in \underline{\mathcal{X}}_+(x_n) \,.$$

**Proof.** Since $I_n(Y_n^1)\mathbf{1}_1 + G_n \succeq -(n/2)\mathbf{1}_1$, this is a direct consequence of Lemma 4.2 and Lemma 8.4 in Deelstra, Pham and Touzi (2000). □

**Corollary 4.1** *Let* $(y_n, Y_n)$ *and* $X_n$ *be as in Lemma 4.3, then :*

$$EU(\ell(X_n - G_n)) = V_n(x) \quad = \quad W_n(x) = E\left(\tilde{U}_n(Y_n^1) + y_n x_n - Y_n G_n\right) .$$

**Proof.** This follows from Lemmas 4.2 and 4.3 together with Proposition 4.1 (i). □

**Remark 4.2** Notice that, for all $k \geq n$ and $(y, Y) \in \mathbf{Y}_+$, $yx_k - EYG_k = yx_n - EYG_n + (1/2)(k-n)(y^1 - EY^1) \geq yx_n - EYG_n$ since $EY^1 \leq y^1$. Using (R2), this proves that $(W_n(x))_n$ is increasing. Hence, Corollary 4.1 shows that $(V_n(x))_n$ is also increasing.

## 4.2 Existence in the initial problem

**Lemma 4.4** *(i), (ii) and (iii) of Theorem 2.1 hold. Moreover,*

$$W_n(x) = V_n(x) \quad \longrightarrow V(x) = W(x),$$

*and there exists a sequence of finite convex combinations* $(\tilde{y}_n, \tilde{Y}_n) \in conv((y_k, Y_k)\ ,\ k \geq n) \cap \mathbf{Y}_+^*$ *that converges to some* $(y_*, Y_*) \in \mathbf{Y}_+$ *with* $E\tilde{Y}_n^1 \to EY_*^1$*, where* $(y_n, Y_n)$ *is given by Lemma 4.1.*

**Proof. Step 1.** $V(x) \leq W(x)$.

Fix $X \in \underline{\mathcal{X}}_e(x)$. By the same argument as in Proposition 4.1 (i), $EU(\ell(X - G)) \leq E\tilde{U}(Y^1) + yx - YG$ for all $(y, Y) \in \mathbf{Y}_e$, hence the desired result.

**Step 2.** *There exists a sequence of finite convex combinations* $(\tilde{y}_n, \tilde{Y}_n) \in conv((y_k, Y_k)\ ,\ k \geq n) \cap \mathbf{Y}_+^*$ *that converges to some* $(y_*, Y_*) \in \mathbf{Y}_+$*. Moreover,* $\lim_n E\tilde{Y}_n^1 = \lim_n \tilde{y}_n^1 = y_*^1$.

By (R2)-(R5), Lemma 4.1 and Proposition 4.1 (v), there exists some real $C$ such that, for all $n \geq n_0$,

$$C \quad > \quad E\left(\tilde{U}_n(Y_n^1) + y_n x_n - Y_n G_n\right) \geq -1 + y_n^1 \ell(x) - y_n^1 \bar{p} + \frac{n}{2}(y_n^1 - EY_n^1) .$$

Recalling that $\ell(x) - \bar{p} > 0$ and $EY_n^1 \le y_n^1$ proves that $y_n^1$ converges (after possibly passing to a subsequence) and that $\lim_n EY_n^1 - y_n^1 = 0$. The rest of the statement is obtained by arguing as in the proof of Lemma 4.1.

**Step 3.** *The sequence $(\tilde{Y}_n^1)_n$ is uniformly integrable. Hence, from Step 2, $E\tilde{Y}_n^1 \to EY_*^1 = y_*^1$.*

Notice that it suffices to prove that $(Y_n^1)_n$ is uniformly integrable. Suppose to the contrary that $(Y_n^1)_n$ is not uniformly integrable. Then, there exists some $\alpha > 0$ such that for all $C > 0$,

$$\limsup_{n\to\infty} E\left(Y_n^1 1\!\mathrm{I}_{Y_n^1 \ge C}\right) \quad > \quad \alpha \ .$$

Now notice that for all $n \ge 1$,

$$\liminf_{y\to\infty} \tilde{U}_n(y)/y \quad \ge \quad (n-1) \ , \tag{4.5}$$

since $\tilde{U}_n(y) \ge U_n(1-n) + (n-1)y$ and $U_n(1-n) > -\infty$. Now, fix $C^n$ and some large $n$ and $m > n$ such that

$$\tilde{U}_n(y) \ge (n-2)y \quad \forall\, y \ge C^n \quad , \quad E\left(Y_m^1 1\!\mathrm{I}_{Y_m^1 \ge C^n}\right) > \alpha \ .$$

By (R2)-(R5), Corollary 4.1 and Proposition 4.1 (v), we get

$$\begin{aligned} C &\ge W_m(x) \ge E\tilde{U}_m(Y_m^1) \ge E\tilde{U}_n(Y_m^1) \ge E\tilde{U}_n(Y_m^1) 1\!\mathrm{I}_{Y_m^1 \ge C^n} - 1 \\ &\ge E(n-2) Y_m^1 1\!\mathrm{I}_{Y_m^1 \ge C^n} - 1 \ge (n-2)\alpha - 1 \ , \end{aligned}$$

for some real $C$ independent of $n$ and we get a contradiction by sending $n$ to $\infty$.

**Step 4.** $(y_*, Y_*) \in \mathbf{Y}_e^*$, *is optimal for $W(x)$ and*

$$\sup_{(y,Y)\in\mathbf{Y}_e} EY(I(Y_*^1)\mathbf{1}_1 + G) - yx \quad = \quad EY_*(I(Y_*^1)\mathbf{1}_1 + G) - y_* x = 0 \ . \tag{4.6}$$

Let $\zeta^{k,n}$ be the coefficients corresponding to the definition of $\tilde{Y}_n$. Then, by convexity of $\tilde{U}_n$, (R2) and Lemma 4.2

$$\begin{aligned}
& E\left(\tilde{U}_n(\tilde{Y}_n^1) + \tilde{y}_n x_n - \tilde{Y}_n G_n\right) \\
& \leq E \sum_{k \geq n} \zeta^{k,n} \left(\tilde{U}_n(Y_k^1) + y_k x_n - Y_k G_n\right) \\
& \leq E \sum_{k \geq n} \zeta^{k,n} \left(\tilde{U}_k(Y_k^1) + y_k x_k - Y_k G_k + \frac{1}{2}(y_k^1 - EY_k^1)(n-k)\right) \\
& \leq \sum_{k \geq n} \zeta^{k,n} W_k(x) \leq \sum_{k \geq n} \zeta^{k,n} W(x) = W(x) \ . \qquad (4.7)
\end{aligned}$$

since $EY_k^1 \leq y_k^1$ and $k \geq n$. By (R3), Proposition 4.1 (iii) and $E\tilde{Y}_n^1 \leq \tilde{y}_n^1$, it follows from Fatou's Lemma that

$$\begin{aligned}
E\left(\tilde{U}(Y_*^1) + y_* x - Y_* G\right) \ & \leq \ E\left(\liminf_{n\to\infty} \tilde{U}_n(\tilde{Y}_n^1) - \tilde{Y}_n G\right) + \lim_{n\to\infty} \tilde{y}_n x \\
& \quad + \liminf_{n\to\infty} \frac{n}{2}(\tilde{y}_n^1 - E\tilde{Y}_n^1) \\
& \leq \ \liminf_{n\to\infty} E\left(\tilde{U}_n(\tilde{Y}_n^1) + \tilde{y}_n x_n - \tilde{Y}_n G_n\right) \\
& \leq \ \liminf_{n\to\infty} \sum_{k\geq n} \zeta^{k,n} W_k(x) \leq W(x) \ . \qquad (4.8)
\end{aligned}$$

Since $|y_* x| < \infty$, the last inequality together with Remarks 2.5 and 2.7 proves in particular that $Y_*^1 \ln Y_*^1 \in L^1$.

The fact that $P(Y_*^1 = 0) = 0$ and (4.6) are obtained as in the proof of Lemma 4.2 by using $\mathbf{D}$ and the inequalities $-y_1 \ln((1-\varepsilon)y_1)^- \leq y_1 \ln(\varepsilon y_2 + (1-\varepsilon)y_1) \leq y_1 \ln(y_1)^+ + y_2 \ln(y_2)^+$ for all $y_1$, $y_2 > 0$ and $\varepsilon \in (0,1)$.

Since $(y_*, Y_*) \in \mathbf{Y}_+$ is such that $Y_*^1 \ln Y_*^1 \in L^1$, $EY_*^1 = y_*^1$ (by Step 3) and $P(Y_*^1 = 0) = 0$, $(y_*, Y_*)$ is in $\mathbf{Y}_e^*$. By (4.8), this proves that $(y_*, Y_*)$ is optimal for $W(x)$.

This concludes the proof of (i) of Theorem 2.1.

**Step 5.** $W_n(x) = V_n(x) \to V(x) = W(x)$.

By Proposition 4.1 (i) and Remark 4.2, $W_n(x)$ converges to some $W_\infty(x)$

$\leq W(x)$. By Step 4 and (4.8), we have

$$W(x) = E\left(\tilde{U}(Y_*^1) + y_* x - Y_* G\right) \leq \liminf_{n\to\infty} \sum_{k\geq n} \zeta^{k,n} W_k(x) = W_\infty \leq W(x) .$$

Since, $W_n(x) = V_n(x) \leq V(x) \leq W(x)$ by Corollary 4.1, (R1) and Step 1, this concludes the proof.

**Step 6.** $X_* := I(Y_*^1)\mathbf{1}_1 + G \in \mathcal{X}_U$ *attains the supremum in* $V(x)$.

Define $X'_n := X_n - (n/2)\mathbf{1}_1$ where $X_n$ is defined as in Corollary 4.1 and notice that

$$X'_n - G = X_n - G_n , \quad n \geq 1 . \tag{4.9}$$

By Proposition 2.1 (iv) and $\underline{\mathcal{X}}_+(x_n) \subset \underline{\mathcal{X}}_e(x_n)$, $X'_n \in \underline{\mathcal{X}}_e(x_n - (n/2)\mathbf{1}_1) = \underline{\mathcal{X}}_e(x)$. Then, by Step 5, (R1) and (4.9),

$$\begin{aligned} V(x) &= \lim_{n\to\infty} EU_n(\ell(X_n - G_n)) \leq \lim_{n\to\infty} EU(\ell(X_n - G_n)) \\ &= \lim_{n\to\infty} EU(\ell(X'_n - G)) \leq V(x) = W(x) . \end{aligned}$$

The result is then obtained by noticing that $EU(I(Y_*^1)) = EU(\ell(X_* - G))$ and that, by Step 4,

$$W(x) = E\tilde{U}(Y_*^1) + y_* x - EY_* G = EU(I(Y_*^1)) .$$

**Step 7.** $W(x) = \inf_{(y,Y)\in\mathbf{Y}_e^*,\, y^1=1} U\left(E\left(yx - YG + Y^1 \ln Y^1\right)\right)$ .

Clearly, $\mathbf{Y}_e$ is a cone and since $y_*^1 \neq 0$, $\zeta/y_*^1(y_*, Y_*) \in \mathbf{Y}_e^*$, for all $\zeta > 0$. Then, it follows from Step 4 that

$$W(x) = \inf_{(y,Y)\in\mathbf{Y}_e^*,\, y^1=1} \inf_{\zeta>0} E\left(\tilde{U}(\zeta Y^1) + \zeta(yx - YG)\right) .$$

Now, fix $(y, Y) \in \mathbf{Y}_e$ such that $y^1 = 1$. Then, direct computation shows that :

$$\inf_{\zeta>0} E\left(\tilde{U}(\zeta Y^1) + \zeta(yx - YG)\right) = \inf_{\zeta>0} E\left(-\zeta Y^1 + \zeta Y^1 \ln(\zeta Y^1) + \zeta(yx - YG)\right)$$

$$
\begin{aligned}
&= \inf_{\zeta>0}(-\zeta+\zeta\ln\zeta)EY^1 \\
&\quad +\zeta E(yx - YG + Y^1\ln(Y^1)) \\
&= \inf_{\zeta>0}\tilde{U}(\zeta)+\zeta E(yx - YG + Y^1\ln Y^1) \\
&= U\left(E\left(yx - YG + Y^1\ln Y^1\right)\right) ,
\end{aligned}
$$

where the last equality is obtained by noticing that for all $c \in I\!R$, $\inf_{\zeta>0}\tilde{U}(\zeta) + \zeta c = U(c)$. □

**Lemma 4.5** *(iv) of Theorem 2.1 holds.*

**Proof. Step 1.** $\sup_n EHI_n(Y_n^1)^- < \infty$ *where* $(y_n, Y_n)$ *is as in Lemma 4.1.*

Notice that, for all $n$, $I_n(\cdot)^-$ is increasing, and therefore $HI_n(Y_n^1)^- \leq Y_n^1 I_n(Y_n^1)^- + HI_n(H)^-$. Also notice that for all $y \geq 0$, $yI_n(y) = n(y^{\frac{n}{n+1}} - y)$ and that direct computation shows that $\sup_n \sup_{y\geq 0} yI_n(y) \leq 1$. By Proposition 4.1 (i)-(v) and the previous argument, there exists some real $C$ such that, for all $n$,

$$C \geq -1 - y_n^1 - 1 + EY_n^1 I_n(Y_n^1)^- .$$

This proves that $\sup_n EY_n^1 I_n(Y_n^1)^- < \infty$ since $(y_n^1)_n$ is bounded. So it suffices to prove that $\sup_n EHI_n(H)^- < \infty$. Notice that for all $n$, $E\tilde{U}_n(H)^+ < \infty \Leftrightarrow EHI_n(H)^- < \infty$ and that $E\tilde{U}(H)^+ < \infty \Leftrightarrow EHI(H)^- < \infty$. Hence the result follows from $\mathbf{M_4}$ and (R2).

**Step 2.** Let $X_n$ and $X'_n$ be defined as in Step 6 of Lemma 4.4. Recall that since $X_n \in \underline{\mathcal{X}}_+(x_n) \subset \underline{\mathcal{X}}_e(x_n)$, it follows from Proposition 2.1 (iv) that $X'_n \in \underline{\mathcal{X}}_e(x)$. Let $L^n \in \mathcal{A}$ be such that $X'_n = X_T^{x,L^n}$. Fix $w \in \mathrm{Int}(K^*)$. By the same arguments as in Kabanov and Last (1999) Lemma 3.3, $EHwX'_n \leq wx - cEH|L_T^n|$ for some $c > 0$ and $(HwX'_n)^- \leq w^1 HI_n(Y_n^1)^-$ where the last term is uniformly bounded in $L^1$ (by Step 1) and $w^1 > 0$. Hence, $EH|L_T^n| \leq C$, for some $C \in I\!R$. Then, from Kabanov and Last (1999) Lemma 3.4, there exists a subsequence of $L^n$ which is Cesaro convergent and, using the linear structure of the dynamic of $X_T^{x,L^n} = X'_n$, there exists a convex combination $\tilde{X}'_n \in \mathrm{conv}(X'_k,\ k \geq n) \cap \underline{\mathcal{X}}_e(x)$ that converges to some $X_* \in \mathcal{X}(x)$.

**Step 3.** $EU\left(\ell(\tilde{X}'_n - G)\right) \leq V(x)$, since $\tilde{X}'_n \in \underline{\mathcal{X}}_e(x)$ . We shall prove in Step 4 that

$$\lim_{n\to\infty} EU\left(\ell(\tilde{X}'_n - G)\right) \;=\; EU\left(\ell(X_* - G)\right) \; . \tag{4.10}$$

Using the previous inequality, this proves that

$$EU\left(\ell(X_* - G)\right) \;\leq\; V(x) \; .$$

On the other hand, by Lemma 4.4, concavity of $U(\ell(\cdot))$, (R1), (4.9) and (4.10)

$$\begin{aligned} V(x) &= \lim_{n\to\infty} \sum_{k\geq n} \zeta^{k,n} EU_k\left(\ell(X_k - G_k)\right) = \lim_{n\to\infty} \sum_{k\geq n} \zeta^{k,n} EU_k\left(\ell(X'_k - G)\right) \\ &\leq \limsup_{n\to\infty} \sum_{k\geq n} \zeta^{k,n} EU\left(\ell(X'_k - G)\right) \leq \limsup_{n\to\infty} EU\left(\ell(\tilde{X}'_n - G)\right) \\ &= EU\left(\ell(X_* - G)\right) \; , \end{aligned}$$

where $\zeta^{k,n}$ are the coefficients of the convex combination defining $\tilde{X}'_n$. Combining the two last inequalities proves

$$V(x) \;=\; EU\left(\ell(X_* - G)\right) = \lim_{n\to\infty} E\left(U(\ell(\tilde{X}'_n - G))\right) \; ,$$

which implies that $X_*$ is in $\mathcal{X}_U(x)$ and is optimal for $V(x)$.

**Step 4.** It remains to prove (4.10). Let $(\tilde{y}_n, \tilde{Y}_n)$ and $(y_*, Y_*)$ be as in Lemma 4.4. By combining the convex combinations, we may always assume that the coefficients $\zeta^{k,n}$ defining $\tilde{X}'_n$ and $\tilde{Y}^1_n$ are the same. Fix some large $n$ and notice that, by concavity of $U(\ell(\cdot))$, (R1), Corollary 4.1 and (4.9)

$$\begin{aligned} U\left(\ell(\tilde{X}'_n - G)\right) &\geq \sum_{k\geq n} \zeta^{k,n} U\left(\ell(X'_k - G)\right) = \sum_{k\geq n} \zeta^{k,n} U\left(\ell(X_k - G_k)\right) \\ &\geq \sum_{k\geq n} \zeta^{k,n} U_k\left(\ell(X_k - G_k)\right) = \sum_{k\geq n} \zeta^{k,n}\left(-(Y^1_k)^{\frac{k}{k+1}}\right) \geq -1 - \tilde{Y}^1_n \; , \end{aligned}$$

where the last equality is obtained by direct computation. Since $\tilde{Y}_n^1 \to Y_*^1$ $P-$ a.s. and $EY_*^1 = \lim_{n\to\infty} E\tilde{Y}_n^1$ by Lemma 4.4, we get from the previous inequalities and Fatou's lemma

$$\begin{aligned}\liminf_{n\to\infty} EU\left(\ell(\tilde{X}_n' - G)\right) + (1 + EY_*^1) &= \liminf_{n\to\infty} E\left(U\left(\ell(\tilde{X}_n' - G)\right) + 1 + \tilde{Y}_n^1\right) \\ &\geq EU\left(\ell(X_* - G)\right) + (1 + EY_*^1)\ .\end{aligned}$$

Now, since $U$ is bounded from above,

$$\limsup_{n\to\infty} EU\left(\ell(\tilde{X}_n' - G)\right) \ \leq \ EU\left(\ell(X_* - G)\right)\ .$$

Combining the two last results proves (4.10). □

# Chapter V.

# Large risk aversion exponential pricing under proportional transaction costs

We[1] generalize the convergence result obtained in the previous chapter for the *reservation price*. An important feature of our analysis is that we do not make use of the dual formulation for the utility optimization problem as in the previous Chapter. This enable us to enlarge the set of admissible strategies and to relax some assumptions on the contingent claim $G$. We exploit the exponential structure of the function to simplify the proofs. In the next chapter, we provide an other demonstration that works for a more general class of functions.

[1]This part is based on Bouchard, Kabanov and Touzi (2000).

# 1 The problem formulation

In this chapter, we enlarge the set of *admissible* strategies of the previous chapter by considering all the strategies in $\underline{\mathcal{A}}$ (see Chapter III), i.e. we will say that a trading strategy $L$ is *admissible* if there is a constant $\delta_L \geq 0$ such that

$$X_t^{x,L} \succeq -\delta_L S_t, \quad t \leq T. \tag{1.1}$$

Recall that the set of admissible strategies $\underline{\mathcal{A}}$ does not depend on $x$ and that we denoted by $\underline{\mathcal{X}}(x)$ is the convex set of terminal values of portfolio processes starting from the initial endowment $x$, i.e.

$$\underline{\mathcal{X}}(x) := \{X_T^{x,L} : \ L \in \underline{\mathcal{A}}\}.$$

Our optimization problems are now defined on $\underline{\mathcal{X}}(x)$, i.e., for $\eta > 0$,

$$\begin{aligned} V^0(x,\eta) &:= \sup_{X \in \underline{\mathcal{X}}(x)} EU^\eta\left(\ell(X)\right) , \\ V^G(x,\eta) &:= \sup_{X \in \underline{\mathcal{X}}(x)} EU^\eta\left(\ell(X-G)\right) \end{aligned}$$

and the *reservation price* is defined as

$$p(x,\eta) := \inf\left\{w \in I\!R : \ V^G(x + w\mathbf{1}_1, \eta) \geq V^0(x,\eta)\right\}$$

with

$$U^\eta(y) := -e^{-\eta y}, \quad y \in I\!R.$$

**Remark 1.1** By Proposition III.2.1 (ii), $\ell(x) \in \underline{\mathcal{X}}(x)$ and therefore, since $U^\eta$ is bounded from above by 0, $V^0(x,\eta)$ is well defined and finite. $V^G(x,\eta)$ is also well defined and bounded from above.

There is no need of any dual formulation to see that the particular form of the exponential utility function implies that, if $V^0(x,\eta) < 0$ and $V^G(x,\eta) < 0$,

$$p(x,\eta) = \frac{1}{\eta} \ln \frac{V^G(x,\eta)}{V^0(x,\eta)} . \tag{1.2}$$

In fact, this identity follows from a slightly more general assertion.

**Lemma 1.1** *Fix* $(x, y) \in I\!R^{2d}$ *and* $\eta > 0$. *Let*

$$p(x, y, \eta) := \inf \left\{ w \in I\!R : \ V^G(x + w\mathbf{1}_1, \eta) \geq V^0(y, \eta) \right\}.$$

*Then, if* $V^0(y, \eta) < 0$ *and* $V^G(x, \eta) < 0$,

$$p(x, y, \eta) = \frac{1}{\eta} \ln \frac{V^G(x, \eta)}{V^0(y, \eta)} .$$

**Proof.** Set $p := p(x, y, \eta)$. Assume that $V^G(x, \eta) > -\infty$. It is easy to see that the function $V^G(x, \eta)$ is strictly increasing and continuous in the $x^1$ variable. Thus, by $V^0 < 0$, $V^G < 0$ and $V^G$, $V^0$ finite, $p$ is finite and

$$V^0(y, \eta) = V^G(x + p\mathbf{1}_1, \eta) .$$

By Remark III.2.1 and Proposition III.2.1 (iii)

$$V^G(x + p\mathbf{1}_1, \eta) = V^{G - p\mathbf{1}_1, \eta}(x) = e^{-\eta p} V^G(x, \eta)$$

and the result follows.

If $V^G(x, \eta) = -\infty$, then, by the previous argument, for all $w \in I\!R$,

$$V^G(x + w\mathbf{1}_1, \eta) = -\infty$$

and therefore, since $V^0$ is finite, $p(x, \eta) = \inf \emptyset = +\infty$. Since $V^0 < 0$, the desired result holds. □

Finally, recall from Chapter III the definitions of the set of hedging endowments

$$\Gamma := \{x \in I\!R^d : \ X \succeq G \text{ for some } X \in \underline{\mathcal{X}}(x)\}.$$

and of the *super-replication price* of the contingent claim

$$g(x) \ := \ \inf \{ w \in I\!R : \ x + w\mathbf{1}_1 \in \Gamma \} \ . \tag{1.3}$$

**Lemma 1.2** *Let* $x, y \in I\!R^d$ *and* $\eta > 0$. *Then,*

$$\frac{1}{\eta} \ln[-V^0(y, \eta)] \ \leq \ -\ell(y).$$

*and, if* $g(x) < \infty$,

$$\frac{1}{\eta} \ln[-V^G(x, \eta)] - g(x) + \ell(-y) \ \leq \ \frac{1}{\eta} \ln[-V^0(y, \eta)]$$

This is the counterpart of Lemma IV.3.1.

**Proof.** The proof is very similar and we skip it. It suffices to not notice that $g(x) < \infty \Rightarrow V^G(x,\eta) > -\infty$ by using the definition of $g(x)$ and recalling that $V^G(x,\eta) = e^{\eta g(x)} V^G(x + (g(x))\mathbf{1}_1, \eta) > -\infty$. □

# 2 The asymptotic result

Recall the definition of

$$\begin{aligned} \mathcal{D} &:= \left\{ Z \in L^0(K^*, I\!\!F) \; : \; Z_0^1 = 1 \; , \; (Z^i S^i)_{i \le d} \in \mathcal{M}(P) \right\} \; , \\ \mathcal{D}_e &:= \left\{ Z \in \mathcal{D} \; : \; Z_T^1 \neq 0 \; , \; Z_T^1 \ln Z_T^1 \; \in \; L^1 \right\} \end{aligned}$$

introduced in Chapter III and IV. It is easy to see that the elements of $\mathcal{D}_e$ have all components strictly positive. Recall from Chapter III that (1.1) ensures that for every admissible strategy $L$ the process $ZX^{x,L}$ is a supermartingale and therefore $EZ_T X_T^{x,L} \le Z_0 x$.

We introduce the following hypotheses:

$\mathbf{M_1}$. The process $S$ is continuous.

$\mathbf{M_2}$. There exists a probability measure $Q \sim P$ such that $S \in \mathcal{M}(Q)$.

$\mathbf{M_3}$. The cone $K$ is proper (i.e. $K \cap (-K) = \{0\}$).

$\mathbf{M_4}$. The set $\mathcal{D}_e \neq \emptyset$ and $\mathcal{D}_e^T$ is dense in $\mathcal{D}^T$ in $L^1$ where $\mathcal{D}^T := \{Z_T, \, Z \in \mathcal{D}\}$ and $\mathcal{D}_e^T := \{Z_T, \, Z \in \mathcal{D}_e\}$.

Our main result is

**Theorem 2.1** *Assume that* $\mathbf{M_1}$ – $\mathbf{M_4}$ *hold. Then*

$$\lim_{\eta \to \infty} p(x,\eta) = g(x) + \ell(x).$$

It is a simple corollary of the hedging theorem and the following more technical assertion which we prove in Section 3.

**Theorem 2.2** *Assume that* $\mathbf{M_2}$ *and* $\mathbf{M_4}$ *hold. Then*

$$\begin{aligned}\limsup_{\eta\to\infty} p(x,\eta) &\leq g(x)+\ell(x),\\ \liminf_{\eta\to\infty} p(x,\eta) &\geq \gamma(x)+\ell(x),\end{aligned}$$

*where*

$$\gamma(x) := \sup_{Z\in\mathcal{D}} E(Z_T G - Z_0 x).$$

**Proof of Theorem 2.1.** In view of Theorem 2.2 it remains to check that $g(x) \leq \gamma(x)$ when $\gamma(x) < \infty$. According to Kabanov and Last (1999), if $\mathbf{M_1}$ – $\mathbf{M_3}$ hold, then $\Gamma = \{x : \ \gamma(x) \leq 0\}$. Since $\gamma(x+\gamma(x)\mathbf{1}_1) = 0$, the point $x+\gamma(x)\mathbf{1}_1$ is in $\Gamma$ and $\gamma(x) \geq g(x)$ by (1.3). □

Theorem 2.2 is the counterpart of Theorem IV.3.1 that was obtained by using the dual formulation for the utility maximization problem (see Theorem IV.2.1 and Corollary IV.2.1). In this Chapter, we show how it can be obtained directly.

# 3 Proof of Theorem 2.2

**Lemma 3.1** *For arbitrary* $Z \in \mathcal{D}_e$ *the following inequality holds:*

$$\frac{1}{\eta}\ln[-V^G(x,\eta)] \geq E(Z_T G - Z_0 x) - \frac{1}{\eta} E Z_T^1 \ln Z_T^1.$$

**Proof.** Let $X \in \underline{\mathcal{X}}(x)$. Since $Z_T \in K^*$ and $Z_T^1 > 0$ we have by Proposition III.2.1 $(i)$ that

$$\ell(X-G) \leq \frac{Z_T}{Z_T^1}(\xi - G).$$

Thus,

$$\begin{aligned}EU^\eta(\ell(X-G)) &\leq EU^\eta\left(\frac{Z_T X - Z_T G}{Z_T^1}\right)\\ &= EZ_T^1 U^\eta\left(\frac{Z_T X - Z_T G}{Z_T^1} + \frac{1}{\eta}\ln Z_T^1\right)\\ &\leq U^\eta\left(E(Z_T X - Z_T G) + \frac{1}{\eta} E Z_T^1 \ln Z_T^1\right)\end{aligned}$$

due to the Jensen inequality applied with the measure $P^1 := Z^1_T P$. Since $EZ_T X \le Z_0 x$, it follows that

$$V^G(x,\eta) \le U^\eta \left( E(Z_0 x - Z_T G) + \frac{1}{\eta} E Z^1_T \ln Z^1_T \right) .$$

Obviously, this bound is equivalent to the assertion of the lemma. □

**Remark 3.1** In view of the last inequality, it follows from $\mathbf{M_4}$ and the existence of some real $c$ such that $G \succeq -cS_T$ that $V^G<0$. Similarly, $V^0 < 0$.

**Lemma 3.2** *Under* $\mathbf{M_2}$ *and* $\mathbf{M_4}$

$$\begin{aligned} \ell(x) &= \inf_{Z\in\mathcal{D}_e} Z_0 x, \\ \gamma(x) &= \sup_{Z\in\mathcal{D}_e} E(Z_T G - Z_0 x). \end{aligned}$$

**Proof.** For any $y \in K^*$ the process $Z$ with $Z_t = (dQ_t/dP_t)y$ is in $\mathcal{D}$. Thus, $\mathbf{M_2}$ ensures that $K^* \cap \{y : \ y^1 = 1\}$ coincides with the set $\{Z_0 \ : \ Z \in \mathcal{D}\}$. But by $\mathbf{M_4}$ the set $\{Z_0 : \ Z \in \mathcal{D}_e\}$ is dense in the latter (if the terminal values of martingales converge in $L^1$ then the initial values also converge). The first identity follows now from Proposition III.2.1 $(i)$.

Let $Z \in \mathcal{D}$ and $Z^n \in \mathcal{D}_e$ are such that $Z^n_T$ converges to $Z_T$ in $L^1$-sense and also a.s. (this is possible by $\mathbf{M_4}$ again). Since $G \succeq -cS_T$ we have the bound

$$Z^n_T G \ge -c Z^n_T S_T.$$

Recalling that $Z^n S$ and $ZS$ are martingales, we get by Fatou's lemma

$$EZ_T G - Z_T x \le \liminf_n (EZ^n_T G - Z^n_0 x) \le \sup_{Z\in\mathcal{D}_e} E(Z_T G - Z_0 x)$$

and the second identity holds since $\mathcal{D}_e \subset \mathcal{D}$. □

**Proof of Theorem 2.2.** At first, we check that

$$\lim_{\eta\to\infty} \frac{1}{\eta} \ln[-V^0(x,\eta)] = -\ell(x). \tag{3.1}$$

Indeed, applying Lemma 3.1 with $G = 0$ we get:

$$\liminf_{\eta\to\infty} \frac{1}{\eta} \ln[-V^0(x,\eta)] \ \ge \ - \inf_{Z\in\mathcal{D}_e} Z_0 x \ = \ -\ell(x)$$

in virtue of Lemma 3.2. The converse inequality follows from the first inequality of Lemma 1.2.

As a corollary of (3.1), we have

$$\lim_{\eta\to\infty} \frac{1}{\eta} \ln[-V^0(x,\eta)] = \lim_{\eta\to\infty} \frac{1}{\eta} \ln[-V^0(\ell(x)\mathbf{1}_1,\eta)].$$

If $g(x) < \infty$, it follows from the representation (1.2), Remark 3.1 and this identity that

$$\begin{aligned}\limsup_{\eta\to\infty} p(x,\eta) &= \limsup_{\eta\to\infty} \frac{1}{\eta}\Big(\ln[-V^G(x,\eta)] - \ln[-V^0(\ell(x)\mathbf{1}_1,\eta)]\Big)\\ &\leq g(x) - \ell(-\ell(x)\mathbf{1}_1) = g(x) + \ell(x),\end{aligned}$$

where we used the second inequality of Lemma 1.2 with $y = \ell(x)\mathbf{1}_1$. Clearly, the result also holds if $g(x) = \infty$.

At last, in virtue of Lemmas 3.1 and 3.2

$$\liminf_{\eta\to\infty} \frac{1}{\eta} \ln[-V^G(x,\eta)] \geq \sup_{Z\in\mathcal{D}_e} E(Z_T G - Z_0 x) = \gamma(x).$$

The second inequality of Theorem 2.2 follows from (1.2), (3.1) and Remark 3.1. □

# 4 Final comments

An inspection of the proof shows that in Theorem 2.1 the hypotheses $\mathbf{M_1}$ – $\mathbf{M_3}$ can be replaced by a single one: the closure of $\Gamma$ coincides with the set $\{x : \gamma(x) \leq 0\}$. Thus, the assertion holds for a discrete-time model (where all strategies are admissible) under $\mathbf{M_2}$ and $\mathbf{M_4}$ only. Indeed, the hedging theorem in Delbaen *et al.* (1999) ensures that $\Gamma = \{x : \gamma(x) \leq 0\}$ if $\mathbf{M_2}$ is fulfilled. One may expect that the hedging theorems are true under weaker assumptions which automatically will substitute $\mathbf{M_1}$ – $\mathbf{M_3}$ in our formulation.

The large risk-averse asymptotic in Delbaen *et al.* (2000) is obtained assuming that all martingales on the stochastic basis are continuous. In fact, one needs only the property that the set $\mathcal{Q}_e$ of martingale measures with finite entropy is dense in the set of all local martingale measures $\mathcal{Q}$. It was shown in Kabanov and Stricker (2000) that this is always the case if $\mathcal{Q}_e \neq \emptyset$. The corresponding result for the sets $\mathcal{D}$ and $\mathcal{D}_e$ is not available yet but, if all martingales on stochastic basis are continuous and $\mathcal{Q}_e \neq \emptyset$, $\mathbf{M_4}$ is fulfilled, see the proof of Theorem 3.7 in Kabanov (1999).

We worked here with the traditional exponential utility function (to simplify formulae, shifted by a constant which does not matter). We show in the next chapter that the result can be extended to the case $(U^\eta)' = (U')^\eta$ where $U$ satisfies some mild assumptions.

# Chapter VI.

# Large risk aversion pricing under proportional transaction costs for a class of utility functions including CRRA and CARA functions

As in the previous chapter, we study the asymptotic of the *reservation price* as the absolute risk aversion tends to infinity but we extend the previous result to a general class of utility functions including CARA[1] and CRRA[2] utility functions. Hence, we prove that the convergence not only holds for constant absolute risk aversion but also for constant relative risk aversion.

[1] Constant Absolute Risk Aversion : $-U''(x)/U'(x) = \text{constant}$.
[2] Constant Relative Risk Aversion : $-xU''(x)/U'(x) = \text{constant}$.

# 1 The problem formulation

Let $U$ be a $C^1$ increasing and concave function defined on $(-k, \infty)$ for some $k \in I\!R_+ \cup \{\infty\}$ such that

$$\lim_{y \downarrow -k} U'(y) = \infty \qquad \text{and} \qquad \lim_{y \uparrow \infty} U'(y) = 0 .$$

In this chapter, we consider a family $(U^\eta)_{\eta > 0}$ of $C^1$ increasing and concave functions defined on $(-k, \infty)$ such that there exists some $a > 0$ and $b \in I\!R$ satisfying :

$$(U^\eta)' \;=\; (U')^{a\eta + b} \qquad \text{for all} \quad \eta \;>\; 0 . \tag{1.1}$$

The reason for choosing such a parameterization is that

$$-(U^\eta)''/(U^\eta)' = -(a\eta + b)U''/U'$$

and therefore

$$-(U^\eta)''/(U^\eta)' \to \infty \;\text{ as }\; \eta \to \infty ,$$

i.e. the absolute risk aversion tends to $\infty$ as $\eta$ tends to $\infty$.

Fix $c > 0$ and let $\eta \geq c$. We denote by $I^\eta$ the inverse of $U^{\eta\prime}$ and by $\tilde{U}^\eta$ the Fenchel transform of $-U^\eta(-\cdot)$. We assume that

**$\mathbf{U_1}$**. There exists $\alpha$ and $\beta \in I\!R$ such that $yI^c(y) \leq \alpha y + \beta$, for all $y > 0$.

**$\mathbf{U_2}$**. There exists $\alpha$ and $\beta \in I\!R$ such that $U^c(I^c(y)) \geq \alpha y + \beta$, for all $y > 0$.

As it will be clear in the proofs, the important point is that $a\eta + b$ tends to $\infty$ as $\eta$ tends to $\infty$. Therefore, in order to alleviate the notations, we will assume that $a = 1$ and $b = 0$. The general case is obtained by the same arguments.

We now fix $k \in I\!R_+ \cup \{\infty\}$. Given $x \succ -k\mathbf{1}_1$, we shall denote by $\mathcal{A}_k(x; G)$ the set of processes in $\underline{\mathcal{A}}$ (see Chapter III) satisfying the additional condition

$$X_T^{x,L} - G \succ -k$$

and we set

$$\mathcal{X}_k(x;G) := \left\{X \in L^0(\mathcal{F}_T) \,:\, X = X_T^{x,L} \text{ for some } L \in \mathcal{A}_k(x;G)\right\} .$$

We define similarly $\mathcal{A}_k(x;0)$ and $\mathcal{X}_k(x;0)$ by replacing $G$ by 0. Clearly, $\mathcal{A}_\infty = \underline{\mathcal{A}}$ and $\mathcal{X}_\infty = \underline{\mathcal{X}}$.

**Remark 1.1** Fix $x$, $x' \in I\!R^d$ and $k \in I\!R_+ \cup \{\infty\}$.
(i) If $x' \succeq x \succ -k\mathbf{1}_1$ then $\mathcal{A}_k(x;G) \subset \mathcal{A}_k(x';G)$.
(ii) For all $X \in \mathcal{X}_k(x;G)$, $X' \in \mathcal{X}_k(x';G)$ and $\beta \in [0,1]$, $\beta X + (1-\beta)\, X' \in \mathcal{X}_k(\,\beta x + (1-\beta)\, x';\, G)$.
(iii) For all $y \in I\!R$ and for all trading strategy $L$ in $\mathcal{A}_k(x;G) \cap \mathcal{A}_k(x+y\mathbf{1}_1;G)$ : $X_x^L + y\mathbf{1}_1 \;=\; X_{x+y\mathbf{1}_1}^L$ . This follows from the normalization of the non-risky asset $S^1$ to unity.
(iv) For all $x \succ -k\mathbf{1}_1$, $\ell(x)\mathbf{1}_1 \in \mathcal{X}_k(x;0)$.

Our optimization problems are defined by

$$V^0(x,\eta) := \sup_{X \in \mathcal{X}_k(x;0)} EU^\eta\left(\ell(X)\right) \quad , \quad V^G(x,\eta) := \sup_{X \in \mathcal{X}_k(x;G)} EU^\eta\left(\ell(X-G)\right) \ ,$$

and the pricing function is given by

$$p(x,\eta) \;:=\; \inf\left\{w \in I\!R \,:\, V^G(x+w\mathbf{1}_1,\eta) \geq V^0(x,\eta)\right\} .$$

**Remark 1.2** Fix $x \succ -k$ and assume that $p(x,\eta)$ is finite. We claim that, under Assumption **D** and $\mathbf{U_3}$ below, $V^G(x+p(x,\eta)\mathbf{1}_1,\eta) < \infty$ and $V^0(x,\eta) < \infty$ (see Remark 3.2 below). By Remark 1.1 (iv), this proves that $|V^0(x,\eta)| < \infty$. Moreover, by Remark 1.1 (i) and (ii), $V^0$ and $V^G$ are concave and increasing in the $x^1$ variable. This implies that $V^G(x+p(x,\eta)\mathbf{1}_1,\eta) = V^0(x,\eta)$ whenever $p(x,\eta)$ is finite.

We recall the definition of the super-replication price for the contingent claim $G$ :

$$g(x) \;:=\; \inf\left\{w \in I\!R \,:\, X \succeq G \text{ for some } X \in \underline{\mathcal{X}}(x+w\mathbf{1}_1)\right\} .$$

and we assume that

$$g(0) < \infty \ .$$

This implies that

$$g(x) < \infty \qquad \text{for all } x \in I\!R^d \ .$$

Indeed, it is easily checked that for all $x \in I\!R^d$, $g(x) \leq g(0) - \ell(x)$. Also notice that we can rewrite $g$ as

$$g(x) \;=\; \inf\left\{w \in I\!R \;:\; X \succeq G \text{ for some } X \in \mathcal{X}_k(x + w\mathbf{1}_1; G)\right\} \ .$$

We conclude this section with the easy observation,

**Lemma 1.1** *For all $x_1 \in I\!R^d$, $x_2 \succ -k\mathbf{1}_1$ and $\eta > 0$ :*

$$V^G(x_1 + (g(x_1) - \ell(-x_2))\mathbf{1}_1, \eta) \;\geq\; V^0(x_2, \eta) \;\geq\; U^\eta\left(\ell(x_2)\right) \ .$$

**Proof.** The proof is similar to that of Lemma IV.3.1. □

**Remark 1.3** Lemma 1.1 implies that for all $x \in I\!R^d$ and $\eta \geq c$,

$$p(x, \eta) \leq g(x) - \ell(-x) < \infty \ .$$

# 2 The asymptotic result

Recall the definition of

$$\mathcal{D} \;:=\; \left\{Z \in L^0(K^*, I\!F) \;:\; Z_0^1 = 1 \ , \ (Z^i S^i)_{i \leq d} \in \mathcal{M}(P)\right\}$$

introduced in Chapter III. Also recall that the definition of $\underline{\mathcal{A}}$ ensures that, for every admissible strategy $L$, the process $ZX^{x,L}$ is a supermartingale and therefore $EZ_T X_T^{x,L} \leq Z_0 x$.

As in Chapter V, we introduce the set

$$\mathcal{D}^T \;:=\; \left\{Y \in L^1 \;:\; Y = Z_T^1, \text{ for some } Z \in \mathcal{D}\right\} \ .$$

Notice that, if $Y \in \mathcal{D}^T$, then $EY = 1$ and that $Z_0^1 > 0$ for all $Z \in \mathcal{D}$. This follows from the normalization of $S^1$ and the martingale property of $ZS$.

We assume that

**D**. There exists some $Y \in \mathcal{D}^T$ such that $Y > 0$ $P-$a.s. and for all $\zeta > 0$,

$$E\tilde{U}^c(\zeta Y) < \infty \, .$$

$\mathbf{U_3}$. There exists a function $\psi_1$ mapping $\mathbb{R}^2$ into $(-k, \infty)$ and some family of functions $(\psi_{2,\eta})_\eta$ mapping $L^1((0,\infty))$ into $(0,\infty)$ such that for all $Y \in \mathcal{D}^T$ satisfying **D** and $\eta \geq c$

$$EU^\eta\left(I(xY^{\frac{1}{\eta}})\right) = U^\eta\left(\psi_1\left(I(x), \psi_{2,\eta}(Y)\right)\right) \, , \tag{2.1}$$

satisfying for all $y > 0$

$$\lim_{x\to\infty} \psi_1(I(x), y) = -k \, , \tag{2.2}$$

and such that for all sequence $x_\eta > 0$,

$$\limsup_{\eta\to\infty} \psi_1\left(I(x_\eta), \psi_{2,\eta}(Y)\right) \leq \limsup_{\eta\to\infty} I(x_\eta) \tag{2.3}$$

where $I$ is the inverse of $U'$.

These assumptions will be satisfied in particular for CRRA and CARA functions (see our Example Section).

Let $\mathcal{D}_e^T$ be the set of random variables $Y \in \mathcal{D}^T$ such that **D** holds and $\mathcal{D}_e$ be the set of process $Z \in \mathcal{D}$ such that $Z_T^1 \in \mathcal{D}_e^T$. Notice that, if $Z \in \mathcal{D}_e$, then $Z_T^1 > 0$ $P-$a.s.

**Remark 2.1** It follows from $\mathcal{D}_e \subset \mathcal{D}$ and Remark III.4.2 that

$$\sup_{Z\in\mathcal{D}_e} E(Z_TG - Z_0x) \leq g(x) \, .$$

In particular, $g(x) < \infty$ implies that

$$\sup_{Z\in\mathcal{D}_e} E(Z_TG - Z_0x) < \infty \, .$$

Finally, we assume that
**H**. For all $x \in I\!R^d$, $\inf_{Z\in\mathcal{D}_e} Z_0 x = \ell(x)$.

**Remark 2.2** Recall from Chapter V that Remark III.2.1 (i) implies **H** if there exists a probability measure $Q \sim P$ such that $S \in \mathcal{M}(Q)$ and if $\mathcal{D}_e^T$ is dense in $\mathcal{D}^T$ in $L^1$.

We can now state the generalization of Theorem V.2.2.

**Theorem 2.1** *For all $x \succ -k\mathbf{1}_1$ :*

$$\limsup_{\eta\to\infty} p(x,\eta) \le g(x) + \ell(x) \qquad \textit{and} \qquad \liminf_{\eta\to\infty} p(x,\eta) \ge \gamma_e(x) + \ell(x)\ .$$

*where* $\gamma_e(x) := \sup_{Z\in\mathcal{D}_e} E(Z_T G - Z_0 x)$.

**Remark 2.3** Clearly, we could deduce from Theorem 2.1 the counterpart of Theorem V.2.1 under assumptions similar to $\mathbf{M_1}$ and $\mathbf{M_3}$ of Chapter III, i.e. prove that : $\gamma_e(x) = g(x)$ and therefore that $\lim_{\eta\to\infty} p(x,\eta) = g(x) + \ell(x)$.

# 3 Proof of Theorem 2.1

**Lemma 3.1** *For all $Y \in \mathcal{D}_e^T$ and $\zeta > 0$*

$$E\sup_{\eta\ge c} Y[I^\eta(\zeta Y)]^- \ < \ \infty\ , \tag{3.1}$$
$$E\sup_{\eta\ge c} Y[I^\eta(\zeta Y)]^+ \ < \ \infty\ . \tag{3.2}$$

**Proof.** We first prove (3.2). Notice that $I$ is decreasing. Hence, the result is obtained by using $\mathbf{U_1}$, $EY = 1$, $I^\eta(y) = I(y^{\frac{1}{\eta}})$ and by noticing that

$$\begin{aligned} I^\eta(\zeta Y) &\le [I(\zeta^{\frac{1}{\eta}})]^+ 1\!\mathrm{I}_{Y\ge 1} + [I(\zeta^{\frac{1}{\eta}} Y^{\frac{1}{c}})]^+ 1\!\mathrm{I}_{Y<1} \\ &= [I(\zeta^{\frac{1}{\eta}})]^+ 1\!\mathrm{I}_{Y\ge 1} + [I^c(\zeta^{\frac{c}{\eta}} Y)]^+ 1\!\mathrm{I}_{Y<1}\ . \end{aligned}$$

We now prove (3.1). Direct computation shows that, for all $\zeta > 0$

$$U^c\left(I^c(\zeta Y)\right) - \zeta Y I^c(\zeta Y) = \tilde{U}^c(\zeta Y)\ .$$

Therefore, by $\mathbf{U_2}$ and $\mathbf{D}$, there exists some $\alpha$ and $\beta \in I\!R$ such that for all $\zeta > 0$

$$E\alpha Y + \beta - E\zeta Y I^c(\zeta Y) < \infty \; .$$

Using the fact that $EY = 1$ again and the fact that $EY[I^c(\zeta Y)]^+ < \infty$, by (3.2), this proves that for all $\zeta > 0$

$$EY[I^c(\zeta Y)]^- \;\; < \;\; \infty \; . \tag{3.3}$$

Now, fix $\eta \geq c$. Recall that $I^\eta(y) = I(y^{\frac{1}{\eta}})$ and that $I$ is decreasing. Hence,

$$[YI^\eta(\zeta Y)]^- \;\; \leq \;\; [I(\zeta^{\frac{1}{\eta}})]^- 1\!\!1_{Y \leq 1} + [YI(\zeta^{\frac{1}{\eta}} Y^{\frac{1}{c}})]^- 1\!\!1_{Y>1}$$

and (3.1) holds by (3.3). □

**Lemma 3.2** *Fix $x \in I\!R^d$ such that $\ell(x) > -k$. Set $p_\eta := p(x, \eta)$ and let $(\eta_n)_n$ be a sequence such that $\lim_{n\to\infty} p_{\eta_n} = p^*$ exists in $I\!R \cup \{-\infty\}$. Then, for all $Z \in \mathcal{D}_e$,*

$$\limsup_{n\to\infty} (U^{\eta_n})^{-1} \left( V^G(x + p_{\eta_n}\mathbf{1}_1, \eta_n) \right) \;\; \leq \;\; E(Z_0(x + p^*\mathbf{1}_1) - Z_T G) \; .$$

**Remark 3.1** For ease of notations we only write $\eta$ for $\eta_n$. We split the proof of the last Lemma in 5 parts.

We first explain the basic argument of the proof. Fix $x \in I\!R^d$ and $Z \in \mathcal{D}_e$. Assume that, for all $\eta \geq c$, we can find some $\zeta_\eta > 0$ such that

$$V^G(x + p_\eta \mathbf{1}_1, \eta) \;\; \leq \;\; EU^\eta \left( I \left( \zeta_\eta (Z_T^1)^{\frac{1}{\eta}} \right) \right) \; . \tag{3.4}$$

By assumption, we can find some functions $\psi_1$ and $\psi_{2,\eta}$ such that

$$EU^\eta \left( I(\zeta_\eta (Z_T^1)^{\frac{1}{\eta}}) \right) \;\; = \;\; U^\eta \left( \psi_1 \left( I(\zeta_\eta), \psi_{2,\eta}(Z_T^1) \right) \right) \; .$$

Now, assume that

$$\limsup_{\eta\to\infty} \psi_1 \left( I(\zeta_\eta), \psi_{2,\eta}(Z_T^1) \right) \;\; \leq \;\; E(Z_0(x + p^*\mathbf{1}_1) - Z_T G) \; , \tag{3.5}$$

then the desired result holds. (3.4) is obtained in Step 3 after some preparation in Step 1 and Step 2. (3.5) is obtained in Step 4 under $\mathbf{U_3}$.

**Proof.** Fix $Z \in \mathcal{D}_e$ and set $Y = Z_T^1$. From the assumptions on $U'$, it is easily checked that $I$ is a continuous function mapping $(0,\infty)$ into $(-k,\infty)$ satisfying :

$$\lim_{y\uparrow\infty} I(y) = -k \qquad \text{and} \qquad \lim_{y\downarrow 0} I(y) = \infty \; . \tag{3.6}$$

Also recall that, for all $y > 0$,

$$I^\eta(y) := (U^{\eta\prime})^{-1}(y) = I(y^{\frac{1}{\eta}}) \; . \tag{3.7}$$

For all $\eta \geq c$ and $\zeta > 0$, we define $F(\zeta;\eta) = EYI\left(\zeta Y^{\frac{1}{\eta}}\right)$.

The aim of Step 1 and Step 2 is to show that, for all $\eta \geq c$, we can find some $\zeta_\eta > 0$ such that $F(\zeta_\eta;\eta) = E(Z_0(x+p_\eta \mathbf{1}_1) - Z_T G)$. In Step 1 (resp. Step 2), we study the function $F(\cdot;\eta)$ (resp. the quantity $E(Z_0(x+p_\eta\mathbf{1}_1) - Z_T G)$). The proof is concluded in Step 3.

**Step 1 :** We first prove that : *for all* $\eta \geq c$ , $F(\cdot;\eta)$ *is a decreasing continuous function defined on* $(0,\infty)$ *satisfying* $\lim_{\zeta\uparrow\infty} F(\zeta;\eta) = -k$ *and* $\lim_{\zeta\downarrow 0} F(\zeta;\eta) = \infty$ .

Fix $\eta \geq c$. Clearly, $F$ is decreasing and defined on $(0,\infty)$ by assumptions on $I$ and Lemma 3.1.

For all $0 < \zeta_0 < \zeta$, -$Y\ I\left(\zeta Y^{\frac{1}{\eta}}\right) \geq$ -$Y\ I\left(\zeta_0 Y^{\frac{1}{\eta}}\right) \in L^1$ by Lemma 3.1 again. Notice that by (3.6), $\lim_{\zeta\uparrow\infty} I\left(\zeta Y^{\frac{1}{\eta}}\right) = -k$ $P-$a.s. Then, by Fatou's Lemma :

$$\liminf_{\zeta\uparrow\infty} -EY\left(I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) \;\geq\; EYk \;=\; k$$

since by construction $EY$=1. Hence, $\limsup_{\zeta\uparrow\infty} EYI\left(\zeta Y^{\frac{1}{\eta}}\right) \leq -k$ . Since $I$ is bounded from below by $-k$, this proves that

$$\lim_{\zeta\uparrow\infty} F(\zeta;\eta) = \lim_{\zeta\uparrow\infty} EYI\left(\zeta Y^{\frac{1}{\eta}}\right) = -k \; .$$

Now, since $I$ is decreasing, for all $0 < \zeta < \zeta_0 < \infty$, $YI\left(\zeta Y^{\frac{1}{\eta}}\right)$ is bounded from below by $YI\left(\zeta_0 Y^{\frac{1}{\eta}}\right) \in L^1$. Then, by Fatou's Lemma and (3.6) again,

$$\liminf_{\zeta\downarrow 0} F(\zeta;\eta) \geq EY \liminf_{\zeta\downarrow 0} I\left(\zeta Y^{\frac{1}{\eta}}\right) = \infty \; .$$

Finally, the continuity of $F(\cdot,\eta)$ is obtained from the continuity of $I$ by using the same kind of arguments.

**Step 2 :** We now prove that : *for all* $\eta \geq c$, $-k < E(Z_0(x+p_\eta \mathbf{1}_1) - Z_T G) < \infty$ .

The right-hand side inequality is trivially satisfied. Indeed, $p_\eta < \infty$, by Remark 1.3, and there exists some $c > 0$ such that $G \succeq -cS_T$ and therefore, by the martingale property of $(Z^i S^i)_{i\leq d}$, $EZ_T G \geq -cZ_0 S_0 > -\infty$.

If $k = \infty$, the left-hand side inequality holds by Remark 2.1.

We now consider the case $k < \infty$. We suppose that

$$E(Z_0(x+p_\eta \mathbf{1}_1) - Z_T G) \leq -k = \inf_{y>0} I(y) \tag{3.8}$$

and we work towards a contradiction of the definition of $p(x,\eta)$. Notice that by Proposition III.2.1 (i),

$$(Z_T^1)^{-1} Z_T(X-G) \geq \ell(X-G).$$

By concavity of $U^\eta$, for all $\zeta > 0$ :

$$\begin{aligned} U^\eta(\ell(X-G)) &\leq U^\eta\left(I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) \\ &+ (U^\eta)'(I(\zeta Y^{\frac{1}{\eta}}))\left((Z_T^1)^{-1}(Z_T X - Z_T G) - I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) \\ &= U^\eta\left(I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) \\ &+ \zeta^\eta Y\left((Z_T^1)^{-1}(Z_T X - Z_T G) - I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) \end{aligned}$$

since $I(y) = (U'(y))^{-1} = (U^{\eta\prime})^{-1}(y^\eta)$. By taking expectation and using the supermartingale property of $ZX$, we get by definition of $Y = Z_T^1$ :

$$\begin{aligned} EU^\eta(\ell(X-G)) &\leq EU^\eta\left(I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) + \zeta^\eta E(Z_0(x+p_\eta \mathbf{1}_1) - Z_T G) \\ &- \zeta^\eta EYI\left(\zeta Y^{\frac{1}{\eta}}\right) . \end{aligned} \tag{3.9}$$

By (3.8), $EY = 1$ and arbitrariness of $X \in \mathcal{X}_k(x+p_\eta \mathbf{1}_1; G)$, this proves that :

$$\begin{aligned} V^G(x+p_\eta \mathbf{1}_1, \eta) &\leq EU^\eta\left(I\left(\zeta Y^{\frac{1}{\eta}}\right)\right) \\ &\leq U^\eta\left(\psi_1\left(I(\zeta), \psi_{2,\eta}(Y)\right)\right) \end{aligned}$$

where $\psi_1$ and $\psi_{2,\eta}$ are defined as in $\mathbf{U_3}$. Then, sending $\zeta$ to $\infty$ and using (2.2), we get

$$V^G(x + p_\eta \mathbf{1}_1, \eta) \;\le\; \lim_{y \to -k} U^\eta(y) \; .$$

On the other hand, since $-k < \ell(x) \in \mathcal{X}_k(x; 0)$, we clearly have

$$V^0(x, \eta) \;\ge\; U^\eta(\ell(x)) \;>\; \lim_{y \to -k} U^\eta(y) \; ,$$

since by assumption on $(U^\eta)'$, $\lim_{y \to -k} (U^\eta)'(y) = \infty$. Hence,

$$V^0(x, \eta) \;>\; V^G(x + p_\eta \mathbf{1}_1, \eta)$$

and we get a contradiction with the definition of $p_\eta = p(x, \eta)$.

**Step 3 :** For all $\eta \ge c$, there exists some $\zeta_\eta > 0$ such that $F(\zeta_\eta; \eta) = y_\eta := E(Z_0(x + p_\eta \mathbf{1}_1) - Z_T G)$. Moreover, $V^G(x + p_\eta \mathbf{1}_1, \eta) \;\le\; EU^\eta\left(I\left(\zeta_\eta Y^{\frac{1}{\eta}}\right)\right)$ .

Fix $\eta \ge c$. Notice that, by Step 2, $-k < y_\eta < \infty$. Then, existence of $\zeta_\eta$ follows from Step 1. Replacing $\zeta$ by $\zeta_\eta$ in (3.9), we get for all $X \in \mathcal{X}_k(x + p_\eta \mathbf{1}_1; G)$ :

$$\begin{aligned} EU^\eta(\ell(X - G)) \;&\le\; EU^\eta\left(I\left(\zeta_\eta Y^{\frac{1}{\eta}}\right)\right) \\ &+\; E(\zeta_\eta)^\eta\left[\left((Z_0(x + p_\eta \mathbf{1}_1) - Z_T G)\right) - YI\left(\zeta_\eta Y^{\frac{1}{\eta}}\right)\right] \\ &=\; EU^\eta\left(\left(I\left(\zeta_\eta Y^{\frac{1}{\eta}}\right)\right)\right) \; . \end{aligned}$$

The result follows from the arbitrariness of $X \in \mathcal{X}_k(x + p_\eta \mathbf{1}_1; G)$.

**Step 4 :** $\limsup_{\eta \to \infty} I(\zeta_\eta) = y^* := \lim_\eta y_\eta \;=\; E(Z_0(x + p^* \mathbf{1}_1) - Z_T G)/Z_0^1$.

First assume that $y^* > -k$ . Then, by Step 1, there exists some $a > 0$ such that for all sufficiently large $\eta$, $\zeta_\eta \le a$. Denote by $\eta_n$ a subsequence such that $\lim_n I(\zeta_{\eta_n}) = \limsup_\eta I(\zeta_\eta)$. Then, for all sufficiently large $n$,

$$[YI\left(\zeta_{\eta_n} Y^{\frac{1}{\eta_n}}\right)]^- \;\le\; \sup_{\eta \ge c}[YI\left(aY^{\frac{1}{\eta}}\right)]^- \;\in\; L^1$$

by Lemma 3.1. Using the fact that for all $\eta_n \geq c$, $F(\zeta_{\eta_n};\eta_n) = y_{\eta_n}$ as well as Fatou's Lemma, we get :

$$\begin{aligned} y^* &= \liminf_{n\to\infty} EYI\left(\zeta_{\eta_n} Y^{\frac{1}{\eta_n}}\right) \geq E \liminf_{n\to\infty} YI\left(\zeta_{\eta_n} Y^{\frac{1}{\eta_n}}\right) \\ &= \liminf_{n\to\infty} I\left(\zeta_{\eta_n}\right) = \limsup_{\eta\to\infty} I\left(\zeta_\eta\right) \ . \end{aligned}$$

We now consider the case $\lim_{\eta\uparrow\infty} y_\eta = -k$. By Step 1, we can find some $b > 0$ such that for all sufficiently large $\eta$, $\zeta_\eta \geq b$ and therefore

$$[YI\left(\zeta_\eta Y^{\frac{1}{\eta}}\right)]^+ \ \leq \ \sup_{\eta\geq c}[YI\left(bY^{\frac{1}{\eta}}\right)]^+ \ \in \ L^1$$

by Lemma 3.1. Then by Fatou's Lemma, it is easily checked that $\limsup_{\eta\uparrow\infty} I\left(\zeta_\eta\right) = -k = y^*$.

**Step 5 :** We can now conclude the proof of the Lemma. By Step 3 :

$$V^G(x + p_\eta \mathbf{1}_1, \eta) \ \leq \ EU^\eta\left(I\left(\zeta_\eta Y^{\frac{1}{\eta}}\right)\right) \tag{3.10}$$

for $\zeta_\eta$ defined as in Step 3. By $\mathbf{U_3}$, there exists a function $\psi_1$ mapping $R^2$ into $(-k,\infty)$ and some function $\psi_{2,\eta}$ mapping $L^1((0,\infty))$ into $(0,\infty)$ such that for all $Y \in \mathcal{D}_e^T$ and $\eta \geq c$

$$EU^\eta\left(I(\zeta_\eta Y^{\frac{1}{\eta}})\right) \ = \ U^\eta\left(\psi_1(I(\zeta_\eta),\psi_{2,\eta}(Y))\right) \tag{3.11}$$

and such that,

$$\limsup_{\eta\to\infty} \psi_1\left(I(\zeta_\eta),\psi_{2,\eta}(Y)\right) \ \leq \ \limsup_{\eta\to\infty} I(\zeta_\eta)\ . \tag{3.12}$$

Hence, from (3.10), we get

$$(U^\eta)^{-1}\left(V^G(x + p_\eta\mathbf{1}_1,\eta)\right) \ \leq \ \psi_1\left(I(\zeta_\eta),\psi_{2,\eta}(Y)\right)$$

and the desired result is obtained by taking limsup in the previous inequality and using (3.12) as well as Step 4. □

**Remark 3.2** By (3.11) and (3.10),

$$V^G(x + p(x,\eta)\mathbf{1}_1,\eta) \ \leq \ U^\eta\left(\psi_1(I(\zeta_\eta),\psi_{2,\eta}(Y))\right) < \infty\ .$$

Replacing $G$ and $p(x,\eta)$ by 0, we also get

$$V^0(x,\eta) \;<\; \infty \,.$$

**Proof of Theorem 2.1**

Apply Lemma 3.2, with $G=0$, to get

$$\limsup_{\eta\to\infty}(U^\eta)^{-1}\left(V^0(x,\eta)\right) \le \inf_{Z\in\mathcal{D}_e} Z_0x = \ell(x)$$

by **H**. Then, it follows from Lemma 1.1 that

$$\lim_{\eta\to\infty}(U^\eta)^{-1}V^0(x,\eta) \;=\; \ell(x) \quad \text{for all} \quad x \succ -k\mathbf{1}_1 \,. \tag{3.13}$$

**Proof of the first inequality.** Set $p_\eta := p(x,\eta)$ and $p^* := \limsup_{\eta\to\infty} p_\eta$. If $p^* = -\infty$ the result is trivial. If $p^* > -\infty$, there exists a subsequence of $p_{\eta_n}$ such that $\lim_{n\to\infty} p_{\eta_n} = p^*$ and $p_{\eta_n} > -\infty$, for all $n$. Hence, by Remark 1.3, $(p_{\eta_n})_n$ if finite, and by Remark 1.2 :

$$(U^{\eta_n})^{-1}\left(V^G(x+p(x,\eta_n)\mathbf{1}_1,\eta_n)\right) \;=\; (U^{\eta_n})^{-1}\left(V^0(x,\eta_n)\right) \,. \tag{3.14}$$

Hence,

$$\lim_{n\to\infty}(U^{\eta_n})^{-1}\left(V^G(x+p_{\eta_n}\mathbf{1}_1,\eta_n)\right) - (U^{\eta_n})^{-1}\left(V^0(x,\eta_n)\right) \;=\; 0 \,. \tag{3.15}$$

Now assume that $p^* > g(x)+\ell(x)$. Then, we can find some $\varepsilon>0$ such that for all sufficiently large $n$, $p_{\eta_n} \ge g(x)+\ell(x)+\varepsilon = g(x)+\ell(x+\varepsilon\mathbf{1}_1)$ by Proposition III.2.1 (iii). Applying Lemma 1.1 with $x_1 = x$ and $x_2 = \ell(x+\varepsilon\mathbf{1}_1)$ and using Proposition III.2.1 (iii) again, we obtain that for all $n$ :

$$(U^{\eta_n})^{-1}\left(V^G\left(x+g(x)\mathbf{1}_1+\ell(x+\varepsilon\mathbf{1}_1)\mathbf{1}_1\right)\right) \ge \ell(x)+\varepsilon \,.$$

By increase of $V^G$ with respect to $x^1$ and $p_{\eta_n} \ge g(x)+\ell(x)+\varepsilon$, this proves that

$$(U^{\eta_n})^{-1}\left(V^G(x+p_{\eta_n}\mathbf{1}_1,\eta_n)\right) \ge \ell(x)+\varepsilon.$$

Using the last inequality together with (3.13), we get :

$$\liminf_{n\to\infty}(U^{\eta_n})^{-1}\left(V^G(x+p_{\eta_n}\mathbf{1}_1,\eta_n)\right)-(U^{\eta_n})^{-1}\left(V^0(x,\eta_n)\right) \;\geq\; \ell(x)+\varepsilon-\ell(x) \;>\; 0\,,$$

and we end up with a contradiction to (3.15). This proves that $\limsup_{\eta\to\infty} p(x,\eta) \leq g(x)+\ell(x)$.

**Proof of the second inequality.** Set $p_* := \liminf_{\eta\to\infty} p_\eta$. Let $\eta_n$ be a subsequence such that $\lim_{n\to\infty} p_{\eta_n} = p_*$. By definition of $p_{\eta_n}$ and the right-hand side inequality of Lemma 1.1 :

$$(U^{\eta_n})^{-1}\left(V^G(x+p_{\eta_n}\mathbf{1}_1,\eta_n)\right) \;\geq\; (U^{\eta_n})^{-1}\left(V^0(x,\eta_n)\right) \;\geq\; \ell(x)\;. \quad (3.16)$$

Sending $n$ to $\infty$, we get, by Lemma 3.2 :

$$E((p_*\mathbf{1}_1+x)Z_0-Z_TG) \;\geq\; \ell(x)\,,$$

for all $Z\in\mathcal{D}_e$. Since, by Remark 2.1, $\sup_{Z\in\mathcal{D}_e} E(Z_TG-xZ_0)<\infty$, and $Z_0p_*\mathbf{1}_1 = p_*$ for all $Z\in\mathcal{D}_e$, this proves that :

$$\liminf_{\eta\to\infty} p(x,\eta) \;\geq\; \sup_{Z\in\mathcal{D}_e} E(Z_TG-xZ_0)+\ell(x)\;.$$

□

# 4 Examples

## 4.1 CARA Utility

The class of CARA[3] utility functions is characterized by a constant absolute risk aversion (i.e independent of the wealth). It coincides with the exponential family $U^\eta(x) := -\eta^{-1}\exp(-\eta x)$.

[3] see e.g. Blanchard and Fischer (1989).

We fix the constant $c$ to be equal to 1. Clearly, $U^{\eta\prime} = (U^{1\prime})^\eta$ where $U^1$ is a $C^1$ increasing and concave function defined on $I\!R$ such that

$$\lim_{y\downarrow-\infty} U^{1\prime}(y) = \infty \qquad \text{and} \qquad \lim_{y\uparrow\infty} U^{1\prime}(y) = 0\ .$$

Clearly, $U^1$ and $I^1$ satisfy $\mathbf{U_1}$-$\mathbf{U_2}$ where $I^\eta(y) = -\eta^{-1}\ln(y)$ for all $y > 0$ and $\eta \geq 1$. We now prove that $\mathbf{U_3}$ holds. Notice that for all $x > 0$ and $y > 0$, $I^1(xy) = I^1(x) - \ln(y)$. Hence, for all $\eta \geq 1$, $Y \in \mathcal{D}_e^T$ and for all $x > 0$

$$\begin{aligned} EU^\eta\left(I^1(xY^{\frac{1}{\eta}})\right) &= E - \eta^{-1}\exp\left(-\eta\left(I^1(x) - \eta^{-1}\ln(Y)\right)\right) \\ &= E - \eta^{-1}\exp\left(-\eta I^1(x) + \eta\eta^{-1}\ln(Y)\right) \\ &= -EY\eta^{-1}\exp\left(-\eta I^1(x)\right) = U^\eta\left(I^1(x)\right)\ . \end{aligned}$$

Hence, we can choose $\psi_1(x, y) = x$ and $\psi_{2,\eta} = 1$ and clearly $\mathbf{U_3}$ holds.

## 4.2 CRRA

The class of CRRA utility functions is characterized by a constant relative risk aversion (i.e independent of the wealth). Any CRRA function is of the form $U^\eta(x) = -\eta^{-1}\, x^{-\eta}$ for $\eta > -1$ and $\eta \neq 0$ or $U(x) = \ln(x)$ (limit case for $\eta \to 0$).

Here we consider a more general class of functions, i.e., for all $\eta \geq 1$, we define on $I\!R$

$$U^\eta(x) := -\eta^{-1}(x + k)^{-\eta}$$

for some real positive constant $k$. Then, for all $\eta \geq 1$,

$$(U^\eta)'(x) := (x + k)^{-\eta-1} \quad \text{and} \quad I^\eta(x) := x^{-\frac{1}{\eta+1}} - k$$

Clearly, $(U^\eta)' = (U^{1\prime})^{\frac{\eta+1}{2}}$ and $I^\eta(y) = I^1(y^{\frac{2}{\eta+1}})$ (for $y > 0$) where $U^1$ is a $C^1$ increasing and concave function defined on $(-k, \infty)$ such that

$$\lim_{y\downarrow-k} U^{1\prime}(y) = \infty \qquad \text{and} \qquad \lim_{y\uparrow\infty} U^{1\prime}(y) = 0\ .$$

Direct computation shows that for all $y > 0$

$$\begin{aligned} yI^1(y) &= y^{\frac{1}{2}} - yk \;\le\; 1 + y \;, \\ U^1(I^1(y)) &= -y^{\frac{1}{2}} \;\ge\; -1 - y \;. \end{aligned}$$

Therefore, $U^1$ and $I^1$ satisfy $\mathbf{U_1}$-$\mathbf{U_2}$. We now prove that $\mathbf{U_3}$ holds. Notice that we need to adapt the condition (2.1) since in our setting $I^\eta(y)$ is equal to $I^1(y^{\frac{2}{\eta+1}})$ and not to $I^1(y^{\frac{1}{\eta}})$. As mentioned in Section 1, the result of this paper also holds under this modification. By direct computation for all $\eta \ge 1$, $Y \in \mathcal{D}_e^T$ and for all $\zeta > 0$

$$\begin{aligned} & EU^\eta\left(I^1(\zeta Y^{\frac{2}{\eta+1}})\right) \\ = \;& E - \frac{1}{\eta}\left((\zeta Y^{\frac{2}{\eta+1}})^{-\frac{1}{2}}\right)^{-\eta} = E - \frac{1}{\eta}\left(\zeta^{-\frac{1}{2}} Y^{\frac{-1}{\eta+1}}\right)^{-\eta} \\ = \;& E - \frac{1}{\eta}\left(\zeta^{-\frac{1}{2}}\right)^{-\eta} Y^{\frac{\eta}{\eta+1}} = -\frac{1}{\eta}\left(\zeta^{-\frac{1}{2}}\right)^{-\eta} EY^{\frac{\eta}{\eta+1}} \\ = \;& -\frac{1}{\eta}\left((I^1(\zeta)+k)\left(EY^{\frac{\eta}{\eta+1}}\right)^{-\frac{1}{\eta}}\right)^{-\eta} = U^\eta\left((I^1(\zeta)+k)\left(EY^{\frac{\eta}{\eta+1}}\right)^{-\frac{1}{\eta}} - k\right) \;. \end{aligned}$$

Hence, we can choose $\psi_1(x,y) = (x+k)y - k$ and $\psi_{2,\eta}(Y) = (EY^{\frac{\eta}{\eta+1}})^{-\frac{1}{\eta}}$. Now, notice that $E\sup_{\eta\ge 1}|Y^{\frac{\eta}{\eta+1}}| \le 1 + EY < \infty$. Then, by Lebesgue's dominated convergence theorem, $EY^{\frac{\eta}{\eta+1}}$ converges to 1 as $\eta$ tends to $\infty$. This proves that $\mathbf{U_3}$ holds.

# Part C.

# Résolution des problèmes de contrôle sous forme standard par Monte-Carlo : premiers résultats

# Chapter VII.

# Résolution des problèmes de contrôle sous forme standard par Monte-Carlo : premiers résultats

Dans cette partie, on étudie un problème de contrôle général sous forme standard associé à un processus de Markov. Suivant l'idée introduite par Carriere (1996), on propose une méthode de simulation permettant à la fois d'estimer la fonction valeur et les contrôles optimaux. Cette méthode est très générale. On utilise la programmation dynamique pour se ramener à un calcul d'espérances conditionnelles qui sont estimées par régression non-paramétrique.

# 1 Introduction

Soit $Z^\nu = (X^\nu, Y)$ un processus markovien contrôlé, à valeurs dans $\mathbb{R}^d$. On s'intéresse au problème sous forme standard

$$V(0,z) := \sup_\nu E\left[f(Z^\nu(T)) \mid Z_0^\nu = z\right] \tag{1.1}$$

pour une certaine fonction mesurable $f$.

Dans cette partie, on propose une méthode simple permettant d'estimer, en même temps, la fonction valeur $V$ et les contrôles optimaux.

Ce type de problème est très souvent rencontré en finance. Par exemple, on peut supposer que $Y$ correspond au processus de prix d'actifs financiers et que $X^\nu$ est un processus de portefeuille induit par une dotation initiale $x$ et une stratégie financière $\nu$. Ainsi, (1.1) correspond à la maximisation d'un critère dépendant de la valeur terminale du portefeuille et de celle des actifs risqués. Par exemple, si $f(x,y) = U(x)$ où $U$ est concave, croissante, il s'agit d'un problème de gestion optimale de portefeuille par fonction d'utilité. Si $f(x,y) = U(x - g(y))$, il s'agit d'un problème de couverture optimale, par fonction d'utilité, d'un actif contigent $g(Y_T)$ (par exemple un call européen).

L'approche classique, pour estimer $V$, consiste à dériver des équations aux dérivées partielles (EDP), satisfaites par $V$, puis à les résoudre numériquement. En général, cela conduit à des équations non linéaires (voir par exemple Fleming et Soner 1992). Si on s'intéresse à un problème de maximisation d'utilité dans un modèle avec coûts de transaction proportionnels, on obtient une équation de type variationnelle (voir par exemple Davis, Panas et Zariphopoulou 1993). Si $Z$ a des sauts, on doit résoudre un terme integrodifférentiel (voir par exemple Bensoussan et Lions 1982). En général, la résolution numérique de telles équations devient difficile, voir même impossible, lorsque $d$ devient trop grand.

D'un autre côté, il est bien connu que les méthodes de Monte-Carlo donnent de bons résultats en matière d'évaluation d'option (voir par exemple Boyle, Broadie and Glasserman 1997 pour une présentation des techniques les plus répendues). Pourtant, elles n'ont été utilisées que très récemment pour résoudre les problèmes de gestion optimale de portefeuille. Detemple, Garcia et Rindisbacher (1999) combinent le calcul de Malliavin avec des techniques de simulation. Cvitanić, Goukasian et Zapatero (2000) proposent une méthode de Monte-Carlo pure. Malheureusement, ces deux approches sont très restrictives et contournent complètement le problème d'optimisation en utilisant des caractérisations particulières (formulation duale), dépendant du modèle étudié, de la richesse optimale terminale. Ceci n'est pas toujours possible. Par exemple, ces deux approches ne peuvent être appliquées à des modèles avec coûts de transaction.

Pour toutes ces raisons, on propose une nouvelle approche par Monte-Carlo, très simple, qui permet d'estimer, en même temps, la fonction valeur $V$ et les contrôles optimaux. Elle est applicable dès que le processus $Z^\nu$ est markovien et dès que l'on peut le simuler. Elle s'applique dans des contextes très généraux, y compris aux problèmes avec coûts de transaction proportionnels.

Elle repose sur une idée utilisée par Carriere (1996) pour calculer le prix d'options américaines (voir également Longstaff et Schwartz 1998). Elle n'utilise que la programmation dynamique associée au problème.

Dans la prochaine section, on introduit le modèle général en temps discret et le problème de contrôle. Ensuite, après une brève introduction aux techniques de régression par noyaux, on présente l'algorithme. Dans la Section 4, on propose une extension naturelle aux modèles en temps continu. Enfin, on présente des résultats numériques obtenus sur un exemple simple.

Pour le moment, nous n'avons aucun résultat théorique de convergence, mais cette étude est un premier pas encourageant.

## 2 Le modèle

On fixe un horizon de temps $T$ et on se place dans un espace de probabilité filtré complet $(\Omega, \mathcal{F}, P)$. On considère un processus de Markov à temps discret $Y$, à valeurs dans un sous-espace $\mathcal{Y}$ de $I\!R^d$. On considère également un processus d'innovation $\varepsilon$, à valeurs dans $I\!R^d$. On suppose que la filtration engendrée par $(Y,\varepsilon)$, $\mathcal{F}_t := \sigma\{(Y_s,\varepsilon_s),\ \ s \leq t\}$, vérifie $\mathcal{F}_0 = \{\Omega,\emptyset\}$ et $\mathcal{F}_T = \mathcal{F}$. Ici, le terme innovation signifie que, pour tout $t \in [0,T-1]$, $\varepsilon_{t+1}$ est indépendant de $\mathcal{F}_t$.

On considère un sous-ensemble, compact pour simplifier, $U$ de $I\!R^d$ et on définit $\mathcal{U}$ comme l'ensemble des processus $\nu$ adaptés à la filtration $I\!F := \{\mathcal{F}_t;\ t = 1,\ldots,T\}$ et à valeurs dans $U$. On introduit le processus contrôlé $X^\nu$ à valeurs dans un sous-espace $\mathcal{X}$ de $I\!R^d$ vérifiant :

$$\begin{aligned} X^\nu_{t+1} &= r\left(t+1, X^\nu_t, Y_t, \varepsilon_{t+1}, \nu_t\right) \quad 0 \leq t \leq T-1\ , \\ X^\nu_0 &= x_0\ , \end{aligned}$$

où $r$ est une fonction mesurable de $[0,T] \times \mathcal{X} \times \mathcal{Y} \times I\!R^d \times U$ dans $\mathcal{X}$. On s'intéresse donc simplement à un processus (partiellement) contrôlé $Z^\nu := (X^\nu, Y)$. On verra par la suite pourquoi cette décomposition s'avère très intéressante dans certains cas tout en restant très générale.

L'objectif de cette partie est de résoudre numériquement le problème de contrôle suivant :

$$V(t,x,y) = \sup_{\nu\in\mathcal{U}} E\left[f(X^\nu_T, Y_T)\ |X^\nu_t = x, Y_t = y\right] \quad , \quad (t,x,y) \in [0,T] \times \mathcal{X} \times \mathcal{Y}\ ,$$

pour une fonction mesurable $f$ de $\mathcal{X} \times \mathcal{Y}$ dans $I\!R$.

La formulation retenue est en fait très générale. Voici quelques exemples d'applications possibles.

*Premier exemple : Assurance optimale et auto-protection.*

On suppose que le patrimoine d'un agent est soumis à des chocs aléatoires assurables. Il peut se couvrir auprès d'une compagnie d'assurance, en payant une prime par période $\Pi$, ou s'auto-protéger en investissant dans un actif sans risque lui rapportant un rendement par période $\rho > 0$. La dynamique de sa richesse s'écrit alors

$$X^\nu_{t+1} = (1+\rho)\left(X^\nu_t - \nu_t \Pi\right) - (1-\nu_t)\varepsilon_{t+1}$$

où $\varepsilon$ est un processus de Poisson univarié et $U = [0,1]$. On suppose que $f$ est une fonction croissante et concave de $x$. Le problème de contrôle correspond alors à la maximisation de l'utilité espérée de la richesse terminale

$$V(t,x) = \sup_{\nu\in\mathcal{U}} E\left[f(X_T^\nu) \mid X_t^\nu = x\right] \ , \ (t,x) \in [0,T] \times I\!R \ ,$$

Ici, le processus $Z^\nu$ est entièrement contrôlé ($Y = 0$). Il sera clair par la suite que notre algorithme est en partie dégénéré dans une telle situation. Il reste cependant valable et l'application numérique de la Section 4 montre même qu'il donne de bons résultats.

*Deuxième exemple : Assurance optimale, auto-protection et actif risqué.*
On reprend l'exemple précédent mais on suppose maintenant que l'agent peut également investir dans un actif risqué dont le processus de prix est modélisé par $Y$ à valeurs dans $(0,\infty)$. L'innovation $\varepsilon$ est maintenant un processus à valeurs dans $I\!R^2$ où $\varepsilon^1$ et $\varepsilon^2$ sont respectivement un processus de Poisson et un processus gaussien centré de matrice de variance-covariance diag$[\sigma^2\mathbf{1}]$. On pose $U = [0,1]^2$. $\nu^1$ représente le niveau de couverture par assurance et $\nu^2$ la proportion de richesse investie dans l'actif risqué. Sous la condition d'autofinancement, la dynamique de la richesse s'écrit maintenant

$$X_{t+1}^\nu = (1+\rho)\left(X_t^\nu(1-\nu_t^2) - \nu_t^1\Pi\right) + \nu_t^2 X_t^\nu \frac{Y_{t+1}}{Y_t} - (1-\nu_t^1)\varepsilon_{t+1}$$

où l'on suppose que $Y$ a pour dynamique

$$Y_{t+1} = Y_t e^{\mu - \frac{1}{2}\sigma^2 + \varepsilon_{t+1}^2} \ .$$

et $\mu > 0$. Le problème s'écrit alors

$$V(t,x,y) = \sup_{\nu\in\mathcal{U}} E\left[f(X_T^\nu) \mid X_t^\nu = x, \ Y_t = y\right] \ , \ (t,x,y) \in [0,T] \times I\!R \times (0,\infty) \ .$$

Cette fois-ci, $Z^\nu$ est vraiment composé d'un partie contrôlée et d'une autre qui ne l'est pas. C'est le cas le plus intéressant dans le sens où notre approche se justifie pleinement.

*Troisième exemple : Evaluation d'un call européen sur moyenne arithmétique.*
On suppose que la dynamique de $Y$ est la même que dans l'exemple précédent avec $\mu$ =0, que $X_T$ s'écrit

$$X_T = \frac{1}{T}\left(\sum_{t=1}^{T} Y_t\right)$$

et que $f(x) = [x - K]^+$ $(K > 0)$. Dans ce contexte,

$$V(t,x,y) = E\left[f(X_T) \mid X_t = x,\ Y_t = y\right]\ ,\ (t,x,y) \in [0,T] \times \mathcal{X} \times \mathcal{Y}\ ,$$

correspond au prix d'arbitrage, à la date $t$, d'un call européen sur moyenne arithmétique, portant sur l'actif $Y$. Ici, la partie contrôlée de $Z^\nu$ est dégénérée puisque l'espace des contrôles $\mathcal{U}$ est réduit à un singleton. Notre algorithme fonctionne également mais l'on revient en fait au cas déjà étudié par Carriere (1996).

On pourrait multiplier les exemples d'applications : marchés financiers avec grands investisseurs dans lesquels la composition du portefeuille à un impact sur la dynamique des cours des actifs financiers, problème de couverture d'option par fonction d'utilité ou par approche par quantile, problème d'investissement optimal dans l'auto-protection contre un risque assurable comme dans la Section I.8.1 de cette thèse, etc. Tant que $(X^\nu, Y)$ est markovien, il est possible de se ramener à la formulation générale proposée.

# 3 Idée générale et algorithme

On revient maintenant à notre problème de contrôle initial :

$$V(0,x_0,y_0) = \sup_{\nu \in \mathcal{U}} E\left[f(X_T^\nu, Y_T)\ |X_0^\nu = x_0, Y_0 = y_0\right]\quad ,\quad (x_0,y_0) \in \mathcal{X} \times \mathcal{Y}\ .$$

En général, puisque $(X^\nu, Y)$ est Markov et $\varepsilon_{t+1}$ est par hypothèse indépendant de $\mathcal{F}_t$, $V$ vérifie l'égalité de programmation dynamique :

$$V(t,x,y) = \sup_{\nu \in \mathcal{U}} E\left[V(t+1, X_{t+1}^\nu, Y_{t+1}) \mid X_t^\nu = x,\ Y_t = y\right]\quad ,$$

pour tout $(t,x,y) \in [0, T-1] \times \mathcal{X} \times \mathcal{Y}$. Supposons maintenant que l'on connaisse $V(t+1,\cdot,\cdot)$, d'après l'équation précédente, le calcul de $V(t,\cdot,\cdot)$ se ramène à la maximisation, sur le compact $U$, d'une espérance conditionnelle :

$$J(t,x,y,\nu) := E\left[V\left(t+1, X_{t+1}^\nu, Y_{t+1}\right)\ \mid X_t^\nu = x,\ Y_t = y\right]\ .$$

Etant donnée la dynamique de $X^\nu$, on obtient en fait

$$J(t,x,y,\nu) = E\left[V\left(t+1, r\left(t+1, x, Y_t, \varepsilon_{t+1}, \nu\right), Y_{t+1}\right) \mid Y_t = y\right] . \qquad (3.1)$$

Si l'on peut calculer cette espérance, pour $\nu \in U$ fixé, on peut déduire $J(t,\cdot,\cdot,\cdot)$ à partir de $V(t+1,\cdot,\cdot)$. En prenant le maximum de $J(t,\cdot,\cdot,\cdot)$ sur $U$, on obtient ensuite $V(t,\cdot,\cdot)$ et, par récurrence, on peut finalement obtenir $V(0,x_0,y_0)$. Le problème réside donc dans le calcul de (3.1). Dans cette partie, on propose de l'estimer en combinant l'approche par Monte-Carlo aux régressions non-paramétriques par noyaux (on pourrait bien évidemment utiliser d'autres types de régressions non-paramétriques, l'objectif étant avant tout d'approximer l'espérance). Avant d'aller plus loin dans l'exposé de cette méthode, on commence par décrire brièvement la méthode de régression par noyaux.

## 3.1 Introduction préliminaire aux méthodes de régression par noyaux de convolution

Pour simplifier, on considère dans cette sous-section un couple de variables aléatoires $(X,Y)$ à valeurs dans $\mathbb{R} \times \mathbb{R}^d$ et on suppose que $X$ peut s'écrire comme une fonction aléatoire de $Y$. On cherche à estimer

$$m(y) := E[X \mid Y = y]$$

de manière non-paramétrique (on peut remarquer que (3.1) est obtenue en remplaçant $X$ par $V\left(t+1, r\left(t+1, x, Y_t, \varepsilon_{t+1}, \nu\right), Y_{t+1}\right)$ et $Y$ par $Y_t$).

Supposons que l'on dispose de $n$ observations $(X_i, Y_i)_i$. Pour estimer $E[X]$, on calcule généralement la moyenne empirique :

$$\frac{1}{n}\sum_{i=1}^{n} X_i .$$

Ici, comme l'on cherche à estimer l'espérance de $X$ conditionnellement à $Y = y$, on pourrait simplement calculer une moyenne empirique conditionnelle

$$\frac{\sum_{i=1}^{n} X_i \mathbb{1}_{Y_i=y}}{\sum_{i=1}^{n} \mathbb{1}_{Y_i=y}} .$$

Cependant, comme $Y$ est à valeurs dans $I\!R^d$, la probabilité d'avoir $Y_i = y$ peut être très faible, voire nulle, ce qui signifie que, d'un point de vue pratique, cet estimateur a peu de chance de donner de bons résultats.

L'idée de la régression par noyaux consiste à considérer, non pas, seulement, les observations $i$ telles que $Y_i = y$, mais toutes les observations telles que $Y_i$ "n'est pas trop loin de $y$". Eventuellement, on peut pondérer la contribution de $X_i$ dans le calcul de l'espérance par la distance entre $Y_i$ et $y$. La technique la plus simple consiste à retenir toutes les observations telles que $|Y_i - y| \leq h$ pour un certain $h > 0$. On approxime alors l'espérance conditionnelle par

$$\frac{\sum_{i=1}^n X_i 1\!\!1_{|Y_i - y| \leq h}}{\sum_{i=1}^n 1\!\!1_{|Y_i - y| \leq h}} .$$

Si l'on veut pondérer les contributions par la distance avec $y$, on peut remplacer $1\!\!1_{|Y_i - y| \leq h}$ par, par exemple, $(1 - |Y_i - y|/h) 1\!\!1_{|Y_i - y| \leq h}$. On fait donc une moyenne empirique, éventuellement pondérée, des observations $X_i$ telles que $Y_i$ n'est pas trop loin de $y$. Si l'application $m(\cdot)$ est suffisamment lisse, on peut espérer en avoir une bonne approximation lorsque $n$ tend vers l'infini et $h$ tend vers 0, c'est-à-dire lorsque l'on a de plus en plus d'observations et que l'on se restreint, de plus en plus, à ne considérer que celles pour lesquelles $Y_i$ est très proche de $y$.

D'une manière générale, on peut définir un noyau de convolution, $\mathcal{K}$, comme une application de $I\!R^d$ dans $I\!R_+$, bornée, intégrable par rapport à la mesure de Lebesgue et telle que

$$\int_{I\!R^d} \mathcal{K}(r) dr = 1 .$$

On supposera en outre que $\mathcal{K}(r)$ décroît lorsque $|r|$ croît et que

$$\lim_{|r| \to \infty} |r|^d \mathcal{K}(r) = 0 .$$

Lorsque cette propriété est vérifiée, on parle de noyau de Parzen-Rosenblatt (voir, e.g., Bosq et Lecoutre 1987 pour plus de détails). Par exemple, $\frac{1}{2} 1\!\!1_{|\cdot| \leq 1}$ est bien un noyau de Parzen-Rosenblatt (il s'agit du noyau uniforme), $(1 - |\cdot|) 1\!\!1_{|\cdot| \leq 1}$ correspond au noyau triangulaire, $(1/\sqrt{2\pi}) e^{-\frac{|\cdot|^2}{2}}$ correspond au noyau exponentiel.

Comme dans l'exemple ci-dessus, on a besoin d'un paramètre $h$ qui permette de contrôler la contribution des observations en fonction de la distance avec $y$. On définit donc

$$K_h(\cdot) = \frac{1}{h^d}\mathcal{K}\left(\frac{\cdot}{h}\right) \quad ,$$

où $h > 0$ est appelé la fenêtre. On peut remarquer que, pour tout $h$, $K_h$ est encore un noyau de convolution. L'estimateur par noyaux s'écrit alors

$$\bar{m}_h(y) := \sum_{i=1}^{n} \frac{K_h(Y_i - y)X_i}{\sum_{i=1}^{n} K_h(Y_i - y)} \ .$$

Ici, $h$ joue le même rôle que dans l'exemple ci-dessus. Comme $\mathcal{K}(r)$ décroît lorsque $|r|$ croît, plus $h$ est petit et plus le poids des observations telles que $Y_i$ est loin de $y$ est faible dans la moyenne pondérée. A la limite, lorsque $h$ tend vers 0, on ne retient plus que les $(X_i, Y_i)$ tels que $Y_i = y$.

Sous certaines hypothèses, sur le type de noyau utilisé et sur les variables $(X, Y)$, on peut montrer que l'estimateur par noyaux est convergent en loi :

$$\sqrt{nh_n^d}\left(\frac{\bar{m}_{h_n}(y) - m(y)}{[v_{h_n}(y)/f_{h_n}(y)\int \mathcal{K}^2(y)dy]^{\frac{1}{2}}}\right) \longrightarrow_{\text{(loi)}} N(0,1) \quad \text{quand} \quad nh_n^{d+4} \longrightarrow 0 \ ,$$

où $f_{h_n}$ (resp. $v_{h_n}$) est un estimateur convergent de la densité de $Y$ (resp. de la variance de $X$ conditionnellement à $Y = y$ ).

On a donc une convergence en loi à la vitesse $\sqrt{nh_n^d}$ lorsque $h_n$ tend suffisamment vite vers 0. Il faut cependant noter qu'il ne s'agit pas d'une convergence en $\sqrt{n}$ car $h_n$ doit tendre vers 0 lorsque $n$ tends vers l'infini. Ainsi, la vitesse maximale que l'on puisse obtenir est légèrement inférieure à $n^{\frac{2}{d+4}}$. En effet, pour s'assurer que $nh_n^{d+4} \longrightarrow 0$, il faut que $h_n$ soit de la forme $n^{-\frac{1+\varepsilon}{d+4}}$, pour un $\varepsilon > 0$. Ceci nous donne une vitesse de $\sqrt{n^{\frac{4-\varepsilon d}{d+4}}}$ soit légèrement inférieure à $h^{\frac{2}{d+4}}$, pour $\varepsilon$ très petit. La vitesse décroît donc avec la dimension de l'espace dans lequel $Y$ prend ses valeurs. En prenant par exemple $d = 4$, on obtient une vitesse légèrement inférieure à $n^{\frac{1}{4}}$. Pour une même précision, il faut en fait que $n$ croîsse en puissance de $d/2$ ($n^{\frac{2}{d+4}} = M$ implique $n = M^{\frac{d+4}{2}}$).

Il existe bien d'autres résultats de convergence (presque sure, en norme $L^2$, etc) mais la convergence en loi est souvent la plus utilisée car elle permet de construire des intervalles de confiance. On peut, par exemple, se référer à Bosq (1996) ou à Bosq et Lecoutre (1987) pour plus de détails sur ces techniques.

## 3.2 L'algorithme

On montre maintenant comment utiliser les techniques de régression par noyaux pour résoudre numériquement notre problème de contrôle :

$$V(0,x_0,y_0) = \sup_{\nu\in\mathcal{U}} E\left[f(X_T^\nu, Y_T)\ |X_0^\nu = x_0, Y_0 = y_0\right] \quad , \quad (x_0,y_0)\in\mathcal{X}\times\mathcal{Y} \, .$$

D'après (3.1), on doit estimer, à $(t,x,\nu)\in[0,T-1]\times\mathcal{X}\times U$ fixé, une espérance conditionnelle qui ne dépend plus que de $y$ :

$$J(t,x,y,\nu) = E\left[V\left(t+1, r\left(t+1,x,Y_t,\varepsilon_{t+1},\nu\right), Y_{t+1}\right)\ \mid Y_t = y\right] \, .$$

Imaginons que l'on ait un grand nombre, $n$, d'observations $(Y_t^i)_i$ de $Y_t$, et $(\varepsilon_{t+1}^i)_i$ de $\varepsilon_{t+1}$. D'après les résultats exposés dans la sous-section précédente, on peut l'estimer en régressant par noyaux

$$\left(V\left(t+1, X_{t+1}^{x,\nu,i}, Y_{t+1}^i\right)\right)_i \quad \text{sur} \quad (Y_t^i)_i \, ,$$

où

$$X_{t+1}^{x,\nu,i} = r\left(t+1, x, Y_t^i, \varepsilon_{t+1}^i, \nu\right) \, .$$

Etant donné un noyau $K_h$, l'estimateur correspondant s'écrira alors

$$\tilde{J}\left(t,x,y,\nu\right) := \sum_{i=1}^n \frac{K_h(Y_t^i - y) V\left(t+1, X_{t+1}^{x,\nu,i}, Y_{t+1}^i\right)}{\sum_{i=1}^n K_h(Y_t^i - y)} \, .$$

Evidemment, on ne dispose pas, dans notre contexte, de telles observation. Toutefois, connaissant la dynamique de $Y$ et la loi des innovations, il est possible de simuler des réalisations de $(Y,\varepsilon)$. On peut ensuite utiliser ces simulations comme s'il s'agissait d'observations. Comme par définition,

$$V(T,x,y) = f(x,y) \quad (x,y)\in\mathcal{X}\times\mathcal{Y} \, ,$$

on peut implémenter l'algorithme suivant pour estimer $V$ :

Pas préliminaires :

1/ On simule tout d'abord $n$ réplications indépendantes de $(Y, \varepsilon)$, notées $(Y^i, \varepsilon^i)_{i \in I}$, $I = \{1, \ldots, n\}$, sous la condition initiale $Y_0^i = y$.

2/ Pour tout $(i, j) \in I \times I$ et $t \in \{1, \ldots, T\}$, on calcule

$$K_h^{ij}(t) \; := \; \frac{K_h\left[Y_t^j - Y_t^i\right]}{\sum_{j \in I} K_h\left[Y_t^j - Y_t^i\right]}$$

et

$$X_t^{x,\nu,i} := r(t-1, x, Y_{t-1}^i, \varepsilon_t^i, \nu) \;\; , \quad \nu \in U \; .$$

3/ Ensuite, on construit une suite de maillages $\{\Sigma_t\}_{t=0,\ldots,T-1}$ de $\mathcal{X}$ telle que $\Sigma_0 = \{x_0\}$. Pour tout $t \in \{0, \ldots, T-1\}$, on n'approximera $V(t, \cdot, \cdot)$ que sur $\Sigma_t \times \{Y_t^i\}_{i \in I}$.

Premier pas :

1/ On calcule

$$f\left(X_T^{x,\nu,i}, Y_T^i\right) \;\; , \;\; x \in \Sigma_{T-1} \; , \; i \in I \; , \; \nu \in U \; .$$

2/ *Régression.* Pour tout $x \in \Sigma_{T-1}$ et $\nu \in U$, on calcule

$$\tilde{J}\left(T-1, x, Y_{T-1}^i, \nu\right) \;\; = \;\; \sum_{j \in I} K_h^{ij}(T-1) f\left(X_T^{x,\nu,j}, Y_T^j\right) \; .$$

3/ *Optimization.* Pour tout $x \in \Sigma_{T-1}$ et $i \in I$, on calcule

$$\tilde{V}\left(T-1, x, Y_{T-1}^i\right) \;\; = \;\; \max_{\nu \in U} \tilde{J}\left(T-1, x, Y_{T-1}^i, \nu\right) \; .$$

et on enregistre l'argument maximal $\tilde{\nu}(T-1, x, Y_{T-1}^i)$.

Pas intermédiaires : On répète la procédure suivante pour $t = T-2$ à 1. Connaissant $\tilde{V}(t+1, x, Y_{t+1}^i)$ pour tout $x \in \Sigma_{t+1}$ et $i \in I$ :

1/ *Interpolation.* Pour tout $x \in \Sigma_t$ et $\nu \in U$, on calcule pour tout $i \in I$

$$\tilde{V}\left(t+1, X_{t+1}^{x,\nu,i}, Y_{t+1}^i\right) \;\; ,$$

en interpolant $\tilde{V}(t+1,\cdot,Y^i_{t+1})$.

2/ *Régression.* Pour tout $x \in \Sigma_t$ et $\nu \in U$, on calcule

$$\tilde{J}(t,x,Y^i_t,\nu) \;=\; \sum_{j\in I} K^{ij}_h(t)\tilde{V}\left(t+1, X^{x,\nu,j}_{t+1}, Y^j_{t+1}\right)\ ,\ \ i \in I\ .$$

3/ *Optimisation.* Pour tout $x \in \Sigma_t$, $i \in I$, on calcule

$$\tilde{V}(t,x,Y^i_t) \;=\; \max_{\nu\in U} \tilde{J}(t,x,Y^i_t,\nu)$$

et on enregistre l'argument maximal $\tilde{\nu}(t,x,Y^i_t)$.

<u>Pas final</u> :

1/ *Interpolation.* On calcule pour tout $i \in I$ et $\nu \in U$

$$\tilde{V}\left(1, X^{x_0,\nu,i}_1, Y^i_1\right)\ ,$$

en interpolant $\tilde{V}(1,\cdot,i)$.

2/ *Régression.* Pour tout $\nu \in U$, on calcule

$$\tilde{J}(0,x_0,y_0,\nu) \;=\; \frac{1}{n}\sum_{i\in I} \tilde{V}\left(1, X^{x_0,\nu,i}_1, Y^i_1\right)\ .$$

3/ *Optimisation.* Finalement, on calcule

$$\tilde{V}(0,x_0,y_0) \;=\; \max_{\nu\in U} \tilde{J}(0,x_0,y_0,\nu)$$

et on enregistre l'argument maximal $\tilde{\nu}(0,x_0,y_0)$.

On peut noter que, lorsque $Z^\nu = (X^\nu, 0)$, le pas de régression est dégénéré. On fait en fait du Monte-Carlo pur. Pour $Z^\nu = (0, Y)$, le pas d'optimisation disparaît, c'est le cas étudié par Carriere (1996).

On peut remarquer que l'estimateur de $V$, $\tilde{V}$, ne dépend des simulations $(Y^i)_i$ qu'à travers l'indice $i$ ce qui simplifie largement l'implémentation de l'algorithme et réduit la dimension des données à conserver en mémoire.

Par ailleurs, d'après la sous-section précédente, on peut espérer avoir une vitesse de convergence de $\tilde{J}(T-1,\cdot,\cdot,\cdot)$ de l'ordre de $n^{\frac{2}{d+4}}$. Ceci devrait nous permettre d'approximer correctement $\hat{\nu}(T-1,\cdot,\cdot)$ et donc $V(T-1,\cdot,\cdot)$. En ce qui concerne les pas suivants $T-2$, $T-3$, etc, le problème est plus délicat car les erreurs d'estimation s'accumulent. Si elles s'additionnent simplement, l'estimateur de $V(0,x_0,y_0)$ devrait être lui aussi convergent mais, notamment à cause du pas d'optimisation, ce n'est pas du tout évident. La convergence théorique de $\tilde{V}(0,x_0,y_0)$ est donc, a priori, délicate à démontrer et devrait faire l'objet de futures recherches.

Par ailleurs, on peut remarquer, que dès le premier pas, l'utilisation de la régression par noyaux engendre une perte de vitesse de convergence par rapport aux techniques de Monte-Carlo pures, pour lesquelles elle est de l'ordre de $\sqrt{n}$ au lieu de $n^{\frac{2}{4+d}}$ (voir par exemple Lapeyre et Pardoux 1998). D'un autre côté, elle permet de réduire le nombre de simulations à réaliser puisque l'on tire une fois pour toutes les trajectoires au lieu de faire du Monte-Carlo à chaque pas. En outre, il faut bien noter que $d$ correspond à la dimension de l'espace des variables sur lesquelles on régresse et non à la dimension de l'espace des variables simulées. Typiquement, dans l'exemple numérique de la Section 4, l'espace des simulations est de dimension 2 mais on n'effectue la régression que sur une variable réelle ($y$) et donc $d=1$.

## 3.3 Choix du maillage

En théorie, on peut calculer les trajectoires simulées $(X_t^{\nu,i})$ vérifiant

$$\begin{aligned} X_{t+1}^{\nu,i} &= r\left(t+1, X_t^{\nu,i}, Y_t^i, \varepsilon_{t+1}^i, \nu\right) \quad , \ t \leq T-1 \\ X_0^{\nu,i} &= x \end{aligned}$$

pour tout $\nu \in U$ si $U$ est fini. Dans ce cas, il n'est pas utile d'introduire un maillage sur $\mathcal{X}$. Dans, la pratique, $U$ n'est pas forcément fini, et, même si c'est le cas, il en général impossible numériquement de conserver en mémoire toutes les trajectoires si $n$ et $\mathrm{card}(U)$ sont grands. C'est pourquoi on a introduit la sequence de maillages $(\Sigma_t)_t$.

Evidemment, on peut toujours choisir un maillage arbitraire, suffisamment large. Dans la pratique, il semble plus judicieux de tenir compte des simulations $(Y^i, \varepsilon^i)$ pour le construire. Par exemple, si on a une idée des contrôles optimaux, on peut calculer $X_t^{\nu,i}$ en fonction des simulations $(Y^i, \varepsilon^i)$ et du contrôle optimal estimé, puis construire le maillage en fonction des simulations $X_t^{\nu,i}$. On ne calcule donc les trajectoires que pour un contrôle et non pour tout les contrôles appartenant à $\mathcal{U}$. Par exemple, dans notre application numérique, on construit $\Sigma_t$ en fonction des valeurs moyennes, maximales et minimales des simulations $X_t^{\nu,i}$ pour un contrôle $\nu$ que l'on suppose être proche du contrôle optimal.

Si l'on n'a aucune idée sur le contrôle optimal, on peut commencer par appliquer l'algorithme sur un maillage large, pour en obtenir une première approximation, puis construire un nouveau maillage en fonction de ses estimations et relancer la procédure sur le nouveau maillage. Lors de la deuxième étape, on peut en même temps chercher à accroître la précision, par exemple passer d'une précision à 1 près à une précision au dixième autour de la première estimation du contrôle.

En théorie, on n'a pas besoin de maillage sur $\mathcal{Y}$ puisque l'on ne calcule la fonction valeur qu'aux points simulés. En pratique, la convergence reposant sur un grand nombre de simulations, il paraît difficile de calculer, par optimisation, la fonction $\tilde{V}$ pour toutes les trajectoires simulées $Y^i$ (évidemment si $\mathcal{U}$ est réduit à un singleton comme dans Carriere 1986, ce problème ne se pose pas). On propose donc de n'effectuer l'optimisation que pour un sous-ensemble $I'$ de $I$. On peut, par exemple, se restreindre aux premières simulations. On estime ensuite $\tilde{V}$ sur $I \setminus I'$ par interpolation par noyaux. C'est ce que l'on fait dans l'application numérique ci-dessous.

Dans ce cas, il faut s'assurer que pour tous les $i \in I'$, l'estimation de $V(\cdot, \cdot, Y_t^i)$ est correcte. On peut par exemple vérifier qu'il y a suffisamment de $Y_t^j$ proches de $Y_t^i$ de manière à ce que la régression par noyaux donne de bons résultats. Par ailleurs, il faut s'assurer que le sous-semble $I'$ corresponde à un bon échantillon des simulations de $I$ tout entier, c'est-à-dire, que pour tout $i \in I$ on puisse trouver un $j \in I'$ tel que $Y_t^j$ soit proche de $Y_t^i$. Si ce n'est pas le cas, l'interpolation par noyaux risque de donner de mauvais résultats et, in fine, on aura une mauvaise estimation de $V(t,\cdot,Y_t^i)$. Comme l'algorithme est basé sur une récurrence, l'erreur risque de se propager d'un pas à l'autre et d'engendrer une mauvaise estimation de $V(0,x_0,y_0)$.

# 4 Extension aux modèles en temps continu

Supposons que l'on s'intéresse au même type de problème de contrôle mais cette fois-ci pour un modèle en temps continu.

Par exemple, on considère un problème de gestion de portefeuille par fonction d'utilité. On suppose que $f$ est une fonction de $x$, concave, croissante, définie sur $\mathbb{R}$, et que $(X^\nu, Y)$ sont solutions sur $[0,T]$ de

$$\begin{aligned} dY_t^1 &= Y_t^1\left(\mu dt + Y_t^2 dB_t^1\right) \\ dY_t^2 &= \kappa(\bar\sigma - Y_t^2)dt + \sigma dB_t^2 \\ dX_t^\nu &= \nu_t X_t^\nu \frac{dY_t^1}{Y_t^1} \ , \\ (Y_0, X_0^\nu) &= (y,x) \in (0,\infty)^3 \end{aligned}$$

où $\mu$, $\bar\sigma$, $\kappa$ sont des réels strictement positifs à définir, et $B$ est un mouvement Brownien à valeurs dans $\mathbb{R}^2$. $Y^1$ correspond au processus de prix d'un actif risqué dont la volatilité stochastique est modélisée par $Y^2$. $X^\nu$ correspond à l'évolution d'un portefeuille dont une proportion $\nu_t$ est investie, en $t$, dans l'actif risqué (ici, on a supposé qu'il existe un actif sans risque, de rendement nul pour simplifier). On suppose que le processus $\nu$ appartient à une certaine classe de stratègies admissibles $\mathcal{U}$, à définir, et on s'intéresse au problème d'allocation optimale :

$$V(t,x,y) = \sup_{\nu\in\mathcal{U}} E\left[f(X_T^\nu) \mid X_t^\nu = x,\ Y_t = y\right] \ , \ (t,x,y) \in [0,T]\times\mathbb{R}\times(0,\infty)^2 \ .$$

Pour utiliser notre algorithme, il suffit de se ramener à un problème en temps discret, proche du problème initial. On peut ainsi utiliser, par exemple, les schémas d'Euler pour discrétiser les processus $(X^\nu, Y)$. En appliquant notre algorithme au problème discrétisé pour un pas de temps $\Delta t$ petit, on peut espérer avoir une bonne approximation de $V$ et $\hat\nu$.

Cette technique est déjà largement employée dans l'estimation de prix d'options par méthode de Monte-Carlo et donne de bons résultats (voir par exemple Boyle, Broadie et Glasserman 1997 ou Lapeyre et Pardoux 1998).

La difficulté additionnelle, par rapport au cas discret, vient de l'erreur engendrée par la discrétisation en temps. Elle est généralement de l'ordre de $\Delta t/T$ pour le calcul de l'espérance.

On peut également noter que notre approche est beaucoup plus simple que celle par EDP lorsque que la dynamique de $Y$ est compliquée. Par exemple, si $Y$ est un processus de diffusion mixte, on aura, typiquement, un terme intégro-différentiel à résoudre dans l'EDP, ce qui est coûteux numériquement. Par contre, il n'est pas tellement compliqué de simuler un processus à saut en plus d'un brownien discrétisé surtout si les trajectoires ne sont simulées qu'une fois.

Enfin, l'approche par EDP fonctionne généralement mal lorsque le nombre de variable dépasse 2 ou 3.

## 5 Application numérique sur un exemple simple

L'objectif de cette section est de montrer, à travers un exemple très simple, que notre algorithme permet d'estimer correctement, à la fois, la fonction valeur et les contrôles optimaux.

On considère les dynamiques suivantes

$$\begin{aligned} Y_{t+1} &= Y_t + \varepsilon^1_{t+1} \\ X^\nu_{t+1} &= X^\nu_t + \nu_t \left(\mu + Y_t \varepsilon^2_{t+1}\right) \\ (X^\nu_0, Y_0) &= (x_0, y_0) \ , \end{aligned}$$

où $\mu$ et $y_0$ sont des réels strictement positifs et $\varepsilon$ est un processus d'innovation. Pour le moment, on suppose simplement que $\varepsilon^2$ est processus gaussien, centré réduit.

On cherche à estimer

$$V(0,x_0,y_0) \;:=\; \sup_{\nu\in\mathcal{U}} E\left(-e^{-X_T^\nu}\right)$$

pour $U = I\!R_+$.

Les $\varepsilon_t$ étant i.i.d., il n'est pas difficile de calculer directement les contrôles optimaux en utilisant simplement la formule de la transformée de Laplace pour les variables aléatoires normales. On obtient :

$$\hat{\nu}(Y_t) \;=\; \frac{\mu}{Y_t^2}\;.$$

Dans la suite, on comparera toujours les contrôles estimés à la valeur (arrondie au dixième) de ce contrôle optimal.

Il serait facile de simplifier ce problème en remarquant que, pour tout $(t,x,y)$, $V(t,x,y) = e^{-x}V(t,0,y)$. Il suffirait donc d'estimer $V(t,0,y)$. Dans nos calculs, on n'utilise ni cette simplification, ni la formule explicite des contrôles optimaux.

Dans un premier temps, on restreint $U$ au singleton $\{\mu/y_0^2\}$ et on suppose que le processus $\varepsilon^1$ est nul. Ceci nous permet de tester la procédure d'interpolation sur $x$ et de vérifier que l'utilisation de la programmation dynamique n'engendre pas trop d'erreurs. Les résultats (Tableau 1 de l'Appendice) montrent que l'on sur-estime $V(0,x_0,y_0)$ seulement de 0.61%, ce qui est très satisfaisant.

Puisque, pour un contrôle fixé, on a une bonne estimation de la fonction de coût correspondante, on peut maintenant passer à l'optimisation. On se restreint tout d'abord à $U = \{3,4,\ldots,7\}$. Les résultats du Tableau 2, montrent que l'on obtient, très rapidement, une bonne approximation des contrôles optimaux.

On peut maintenant tenter d'augmenter la précision en optimisant sur $U = \{4.7, 4.8, \ldots, 5.5\}$. Là encore, le Tableau 3 montre que l'approximation est très bonne (au dixième). Par contre, le temps de calcul est relativement long.

Puisque pour le moment on a supposé que $\varepsilon^1$ était nul (et donc $Y$ constant et égal à $y_0$), on n'a pas encore testé la procédure de régression par noyaux.

On suppose maintenant que les $\varepsilon_t^1$ sont i.i.d. et uniformément distribués sur $[-y_0/(2T),\ y_0/(2T)]$. On optimise sur $U = \{3, 4, \dots, 7\}$ et on utilise 36.000 trajectoires simulées (voir Tableau 4). On refait le même calcul pour des $\varepsilon_t^1$ i.i.d., uniformément distribués sur $[-y_0/(4T),\ y_0/(4T)]$, et $U = \{3.5, 4, \dots, 7\}$ (voir Tableau 5). Dans les deux cas, les estimations sont satisfaisantes. La procédure de régression par noyaux fonctionne donc bien.

Globalement, l'algorithme converge bien et permet d'obtenir rapidement une idée assez précise des contrôles optimaux même lorsque la volatilité $Y$ est stochastique.

Ces résultats sont très encourageants, même si, pour des raisons techniques, il ne nous a pas été possible, pour le moment, de pousser cette étude plus avant. La prochaine étape consiste à démontrer, théoriquement, la convergence des estimateurs.

# 6 Appendice

## Tableau 1

$n = 20.000$, $T = 4$, $y_0 = 0.14/\sqrt{4}$, $\mu = 0.1/4$, $\varepsilon^1 = 0$, $U = \{\mu/y_0^2\}$. Temps de calcul = 6 min.

| t=0 | $x$ | $\tilde{V}$ | $V$ | $(\tilde{V} - V)/\lvert V\rvert$ |
|---|---|---|---|---|
| | 0,00 | -7,75E-01 | -7,80E-01 | 0,61 % |

| t=1 | $x$ | $\tilde{V}$ | $V$ | $(\tilde{V} - V)/\lvert V\rvert$ |
|---|---|---|---|---|
| | -1,28 | -2,96E+00 | -2,98E+00 | 0,46 % |
| | -0,81 | -1,86E+00 | -1,86E+00 | 0,46 % |
| | -0,34 | -1,16E+00 | -1,17E+00 | 0,46 % |
| | 0,13 | -7,28E-01 | -7,32E-01 | 0,46 % |
| | 0,58 | -4,61E-01 | -4,64E-01 | 0,46 % |
| | 1,04 | -2,92E-01 | -2,94E-01 | 0,46 % |
| | 1,49 | -1,85E-01 | -1,86E-01 | 0,47 % |

| t=2 | $x$ | $\tilde{V}$ | $V$ | $(\tilde{V} - V)/\lvert V\rvert$ |
|---|---|---|---|---|
| | -1,85 | -5,58E+00 | -5,60E+00 | 0,30 % |
| | -1,43 | -3,67E+00 | -3,68E+00 | 0,31 % |
| | -1,01 | -2,41E+00 | -2,42E+00 | 0,31 % |
| | -0,59 | -1,59E+00 | -1,59E+00 | 0,31 % |
| | -0,17 | -1,04E+00 | -1,05E+00 | 0,31 % |
| | 0,25 | -6,85E-01 | -6,88E-01 | 0,31 % |
| | 0,61 | -4,80E-01 | -4,81E-01 | 0,31 % |
| | 0,96 | -3,36E-01 | -3,37E-01 | 0,31 % |
| | 1,32 | -2,35E-01 | -2,36E-01 | 0,31 % |
| | 1,68 | -1,65E-01 | -1,65E-01 | 0,31 % |
| | 2.04 | -1,56E-01 | -1,19E-01 | 0,31 % |

| t=3 | $x$ | $\tilde{V}$ | $V$ | $(\tilde{V}-V)/\lvert V\rvert$ |
|---|---|---|---|---|
| | -2,07 | -7,40E+00 | -7,41E+00 | 0,13 % |
| | -1,72 | -5,22E+00 | -5,22E+00 | 0,13 % |
| | -1,37 | -3,68E+00 | -3,68E+00 | 0,13 % |
| | -1,02 | -2,59E+00 | -2,60E+00 | 0,13 % |
| | -0,67 | -1,83E+00 | -1,83E+00 | 0,13 % |
| | -0,32 | -1,29E+00 | -1,29E+00 | 0,13 % |
| | 0,03 | -9,08E-01 | -9,10E-01 | 0,13 % |
| | 0,38 | -6,40E-01 | -6,41E-01 | 0,13 % |
| | 0,73 | -4,52E-01 | -4,53E-01 | 0,13 % |
| | 1,08 | -3,20E-01 | -3,20E-01 | 0,13 % |
| | 1,42 | -2,26E-01 | -2,26E-01 | 0,13 % |
| | 1,77 | -1,60E-01 | -1,60E-01 | 0,13 % |
| | 2,12 | -1,13E-01 | -1,13E-01 | 0,13 % |
| | 2,47 | -7,96E-02 | -7,98E-02 | 0,13 % |
| | 2,81 | -5,63E-02 | -5,63E-02 | 0,13 % |

# Tableau 2

$n$=10.000, $T = 4$, $U = \{3,4,5,6,7\}$, $\varepsilon^1 = 0$, $y_0 = 0.14/\sqrt{4}$, $\mu = 0.1/2$. Temps de calcul = 7 min.

| t=0 | $x$ | $\tilde{V}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $V$ | $(\tilde{V}-V)/\lvert V\rvert$ |
|---|---|---|---|---|---|---|
| | 0,00 | -7.78E-01 | 5,0 | 5,1 | -7,75E-01 | -1,29 % |

| t=1 | | | t=2 | | | t=3 | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $\tilde{\nu}$ | $\hat{\nu}$ | $x$ | $\tilde{\nu}$ | $\hat{\nu}$ | $x$ | $\tilde{\nu}$ | $\hat{\nu}$ |
| - 1,18 | 5,0 | 5,1 | - 1,54 | 5,0 | 5,1 | - 1,92 | 5,0 | 5,1 |
| - 0,74 | 5,0 | 5,1 | - 1,18 | 5,0 | 5,1 | - 1,59 | 5,0 | 5,1 |
| - 0,30 | 5,0 | 5,1 | - 0,81 | 5,0 | 5,1 | - 1,26 | 5,0 | 5,1 |
| 0,14 | 5,0 | 5,1 | - 0,45 | 5,0 | 5,1 | - 0,93 | 5,0 | 5,1 |
| 0,61 | 5,0 | 5,1 | - 0,09 | 5,0 | 5,1 | - 0,60 | 5,0 | 5,1 |
| 1,08 | 5,0 | 5,1 | 0,27 | 5,0 | 5,1 | - 0,27 | 5,0 | 5,1 |
| 1,55 | 5,0 | 5,1 | 0,69 | 5,0 | 5,1 | 0,06 | 5,0 | 5,1 |
| | | | 1,11 | 5,0 | 5,1 | 0,39 | 5,0 | 5,1 |
| | | | 1,53 | 5,0 | 5,1 | 0,72 | 5,0 | 5,1 |
| | | | 1,95 | 5,0 | 5,1 | 1,06 | 5,0 | 5,1 |
| | | | 2,36 | 5,0 | 5,1 | 1,39 | 5,0 | 5,1 |
| | | | | | | 1,72 | 5,0 | 5,1 |
| | | | | | | 2,06 | 5,0 | 5,1 |
| | | | | | | 2,39 | 5,0 | 5,1 |
| | | | | | | 2,72 | 5,0 | 5,1 |

On répète l'estimation de $V(0,0)$ 500 fois et on obtient pour les estimateurs $\tilde{V}(0,0)$ :

| Valeur moyenne | Ecart-Type | Variance | Valeur min | Valeur max |
|---|---|---|---|---|
| -0,7750 | 0,0037 | 0,0000 | -0,7850 | -0,7618 |

# Tableau 3

$n = 100.000$, $T = 4$, $U = \{4.7, 4.8, \ldots, 5.5\}$, $\varepsilon^1 = 0$, $y_0 = 0.14/\sqrt{4}$, $\mu = 0.1/2$. Temps de calcul = 3 heures.

| t=0 | $x$ | $\tilde{V}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $V$ | $(\tilde{V}-V)/\|V\|$ |
|---|---|---|---|---|---|---|
| | 0,00 | -7,76E-01 | 5,2 | 5,1 | -7,75E-01 | -0,13 % |

| t=1 | $x$ | $\tilde{V}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $V$ | $(\tilde{V}-V)/\|V\|$ |
|---|---|---|---|---|---|---|
| | - 1,42 | -3,44E+00 | 5,1 | 5,1 | -3,43E+00 | -0,26 % |
| | - 0,91 | -2,05E+00 | 5,1 | 5,1 | -2,04E+00 | -0,26 % |
| | - 0,39 | -1,22E+00 | 5,1 | 5,1 | -1,22E+00 | -0,26 % |
| | 0,13 | -7,28E-01 | 5,1 | 5,1 | -7,26E-01 | -0,26 % |
| | 0,66 | -4,28E-01 | 5,1 | 5,1 | -4,27E-01 | -0,26 % |
| | 1,19 | -2,52E-01 | 5,1 | 5,1 | -2,51E-01 | -0,26 % |
| | 1,72 | -1,48E-01 | 5,1 | 5,1 | -1,48E-01 | -0,26 % |

| t=2 | $x$ | $\tilde{V}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $V$ | $(\tilde{V}-V)/\|V\|$ |
|---|---|---|---|---|---|---|
| | - 1,72 | -4,91E+00 | 5,2 | 5,1 | -4,90E+00 | - 0,16 % |
| | - 1,32 | -3,31E+00 | 5,2 | 5,1 | -3,30E+00 | -0,17 % |
| | - 0,93 | -2,23E+00 | 5,2 | 5,1 | -2,23E+00 | -0,17 % |
| | - 0,53 | -1,50E+00 | 5,2 | 5,1 | -1,50E+00 | -0,17 % |
| | - 0,14 | -1,01E+00 | 5,2 | 5,1 | -1,01E+00 | -0,17 % |
| | 0,26 | -6,83E-01 | 5,2 | 5,1 | -6,82E-01 | -0,17 % |
| | 0,73 | -4,24E-01 | 5,2 | 5,1 | -4,23E-01 | -0,17 % |
| | 1,21 | -2,63E-01 | 5,2 | 5,1 | -2,63E-01 | -0,17 % |
| | 1,69 | -1,63E-01 | 5,2 | 5,1 | -1,63E-01 | -0,17 % |
| | 2,16 | -1,01E-01 | 5,2 | 5,1 | -1,01E-01 | -0,17 % |
| | 2,64 | -6,29E-02 | 5,2 | 5,1 | -6,27E-02 | - 0,21 % |

| t=3 | $x$ | $\tilde{V}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $V$ | $(\tilde{V}-V)/\lvert V\rvert$ |
|---|---|---|---|---|---|---|
| | - 2,24 | -8,87E+00 | 5,0 | 5,1 | -8,84E+00 | - 0,26 % |
| | - 1,87 | -6,09E+00 | 5,0 | 5,1 | -6,08E+00 | - 0,26 % |
| | - 1,49 | -4,19E+00 | 5,0 | 5,1 | -4,17E+00 | - 0,26 % |
| | - 1,12 | -2,88E+00 | 5,0 | 5,1 | -2,87E+00 | - 0,26 % |
| | - 0,74 | -1,98E+00 | 5,0 | 5,1 | -1,97E+00 | - 0,26 % |
| | - 0,37 | -1,36E+00 | 5,0 | 5,1 | -1,35E+00 | - 0,26 % |
| | 0,01 | -9,33E-01 | 5,0 | 5,1 | -9,31E-01 | - 0,26 % |
| | 0,38 | -6,41E-01 | 5,0 | 5,1 | -6,40E-01 | - 0,26 % |
| | 0,75 | -4,42E-01 | 5,0 | 5,1 | -4,41E-01 | - 0,26 % |
| | 1,13 | -3,05E-01 | 5,0 | 5,1 | -3,04E-01 | - 0,26 % |
| | 1,50 | -2,10E-01 | 5,0 | 5,1 | -2,10E-01 | - 0,26 % |
| | 1,87 | -1,45E-01 | 5,0 | 5,1 | -1,45E-01 | - 0,26 % |
| | 2,24 | -1,00E-01 | 5,0 | 5,1 | -9,97E-02 | - 0,26 % |
| | 2,61 | -6,90E-02 | 5,0 | 5,1 | -6,88E-02 | - 0,26 % |
| | 2,98 | -4,76E-02 | 5,0 | 5,1 | -4,74E-02 | - 0,26 % |

# Tableau 4

$n = 36.000$, $I' = \{1, \ldots, 10\}$, $T = 4$, les $\varepsilon_t^1$ sont i.i.d. et uniformément distribués sur $[-y_0/(2T), y_0/(2T)]$, $y_0 = 0.14/\sqrt{4}$, $\mu = 0.1/2$, $U = \{3, 4, 5, 6, 7\}$. Temps de calcul = 33 min.

| | t=0 | | t=1 | | t=2 | | t=3 | |
|---|---|---|---|---|---|---|---|---|
| $i$ | $\tilde{\nu}$ | $\hat{\nu}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $\tilde{\nu}$ | $\hat{\nu}$ |
| 1 | 5 | 5,1 | 5 | 4,5 | 5 | 4,5 | 5 | 4,9 |
| 2 | 5 | 5,1 | 6 | 5,9 | 6 | 5,8 | 5 | 5,3 |
| 3 | 5 | 5,1 | 6 | 6,4 | 6 | 5,6 | 7 | 6,4 |
| 4 | 5 | 5,1 | 4 | 4,2 | 5 | 4,7 | 4 | 4,1 |
| 5 | 5 | 5,1 | 6 | 5,7 | 6 | 6,0 | 7 | 7,3 |
| 6 | 5 | 5,1 | 5 | 4,6 | 4 | 4,4 | 4 | 3,8 |
| 7 | 5 | 5,1 | 4 | 4,1 | 5 | 5,0 | 5 | 4,7 |
| 8 | 5 | 5,1 | 6 | 6,6 | 5 | 5,2 | 5 | 5,5 |
| 9 | 5 | 5,1 | 5 | 4,9 | 6 | 6,1 | 7 | 7,1 |
| 10 | 5 | 5,1 | 5 | 5,3 | 4 | 4,3 | 4 | 3,8 |

On répète l'estimation de $V(0,0)$ 100 fois et on obtient pour les estimateurs $\tilde{V}(0,0)$ :

| Valeur moyenne | Ecart-Type | Variance | Valeur min | Valeur max |
|---|---|---|---|---|
| -0,7704 | 0,0022 | 0,0000 | 0,7748 | -0,7648 |

On estime maintenant $V(0,0)$ en utilisant la forme du contrôle optimal pour $n = 5.000.000$ et obtient comme estimation : $-0,7705$.

# Tableau 5

$n = 100.000$, $I' = \{1, \ldots, 20\}$, $T = 4$, les $\varepsilon_t^1$ sont i.i.d. et uniformément distribués sur $[-y_0/(4T),\, y_0/(4T)]$, $y_0 = 0.14/\sqrt{4}$, $\mu = 0.1/2$, $U = \{3, 3.5, \ldots, 7\}$. Temps de calcul = 3 H 13 min.

| | t=0 | | t=1 | | t=2 | | t=3 | |
|---|---|---|---|---|---|---|---|---|
| $i$ | $\tilde{\nu}$ | $\hat{\nu}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $\tilde{\nu}$ | $\hat{\nu}$ | $\tilde{\nu}$ | $\hat{\nu}$ |
| 1 | 5,0 | 5,1 | 5,0 | 4,9 | 4,0 | 4,3 | 4,5 | 4,6 |
| 2 | 5,0 | 5,1 | 5,5 | 5,4 | 6,0 | 6,2 | 5,5 | 5,7 |
| 3 | 5,0 | 5,1 | 5,5 | 5,5 | 6,0 | 6,2 | 7,0 | 7,0 |
| 4 | 5,0 | 5,1 | 4,5 | 4,7 | 4,0 | 4,3 | 4,5 | 3,9 |
| 5 | 5,0 | 5,1 | 5,0 | 5,0 | 5,5 | 5,4 | 5,5 | 5,4 |
| 6 | 5,0 | 5,1 | 5,5 | 5,2 | 5,0 | 4,8 | 5,0 | 4,8 |
| 7 | 5,0 | 5,1 | 5,0 | 4,8 | 4,0 | 4,3 | 5,0 | 4,8 |
| 8 | 5,0 | 5,1 | 5,5 | 5,4 | 6,0 | 6,1 | 5,5 | 5,5 |
| 9 | 5,0 | 5,1 | 5,0 | 4,5 | 4,5 | 4,5 | 4,5 | 4,4 |
| 10 | 5,0 | 5,1 | 6,0 | 5,8 | 5,5 | 5,8 | 6,0 | 6,0 |
| 11 | 5,0 | 5,1 | 5,0 | 4,8 | 5,0 | 5,1 | 5,5 | 5,2 |
| 12 | 5,0 | 5,1 | 5,5 | 5,5 | 5,0 | 5,1 | 5,0 | 5,0 |
| 13 | 5,0 | 5,1 | 5,5 | 5,6 | 5,5 | 5,2 | 5,0 | 5,0 |
| 14 | 5,0 | 5,1 | 4,5 | 4,7 | 5,0 | 5,0 | 5,0 | 5,3 |
| 15 | 5,0 | 5,1 | 4,5 | 4,7 | 5,0 | 5,0 | 5,5 | 5,5 |
| 16 | 5,0 | 5,1 | 5,5 | 5,6 | 5,0 | 5,2 | 5,0 | 4,8 |
| 17 | 5,0 | 5,1 | 5,0 | 5,0 | 5,0 | 5,0 | 4,5 | 4,6 |
| 18 | 5,0 | 5,1 | 5,5 | 5,2 | 5,5 | 5,2 | 5,5 | 5,7 |
| 19 | 5,0 | 5,1 | 5,5 | 5,5 | 5,0 | 5,2 | 5,0 | 5,3 |
| 20 | 5,0 | 5,1 | 4,5 | 4,7 | 5,0 | 5,0 | 5,0 | 4,9 |

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