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A Contribution in Stochastic Control Applied in Finance and Insurance

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Acknowledgments

Last thing to do :-)

Contents

Chapter 1

Introduction

Consider a claim g, sold at time $t \geq 0$ of maturity $T \geq t$, with underlying $X_{t,x}$ satisfying $X_{t,x}(t) = x$. In case of European option, the seller of the claim has to deliver the payoff $g(X_{t,x}(T))$ at terminal date T to the buyer. The natural question arising then is to determine a price π to be paid at time t to the seller which will satisfy both the seller and the buyer, so that the risk transfer may be done.

In the so-called *complete market* case of [Black 73, Ansel 92, Delbaen 94, Harrison 81], the seller may replicate the payoff of the claim by dynamically trading on the market. That is, under good integrability conditions on $\beta_T g(X_{t,x}(T))$, where β stands for the discount factor process, one can find a predictable process ν such that

$$\beta_T g\left(X_{t,x}(T)\right) = \mathbb{E}^{\mathbb{Q}}\left[\beta_T g\left(X_{t,x}(T)\right) \mid \mathcal{F}_t\right] + \int_t^T \nu_s \cdot dW_s^{\mathbb{Q}},$$

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where $\mathbb{Q}$ is the unique martingale measure, and $W^{\mathbb{Q}}$ is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion. The unique fair price is then $\mathbb{E}^{\mathbb{Q}}\left[\beta_T g\left(X_{t,x}(T)\right) \mid \mathcal{F}_t\right]$, since it would allow for arbitrage otherwise.

In the more realistic situation of *incomplete market*, when there are e.g. intrinsic, non traded sources of risk, both the valuation and the hedging problems may become highly non-trivial issues. Considering then the condition of *no-arbitrage* leads to an infinity of *viable* prices (see e.g. [Delbaen 94]). The risk taker needs thus to define the amount of money he has to invest at time t to be able to construct a financial portfolio that will reduce the risk in an acceptable way. On the other hand, the risk adverse agent has to be able to determine the amount of money he is willing to pay to accept the transfer. The pricing of contingent claims hence requires a description of preferences of buyers and sellers. Among the different approaches one could think of, we refer to [Broadie 98, Cvitanić 99, Cvitanić 96, Cvitanić 93, El Karoui 95, Karatzas 98] for the super-replication in incomplete markets, [Davis 97] for the marginal utility approach, [Bouleau 89], [Duffie 91], [Sondermann 85], [Schweizer 88], [Schweizer 91] or [Schweizer 99] for the quadratic methods, [Cvitanić 00], [Föllmer 99] or [Föllmer 00] for the quantile hedging and shortfall risk minimization.

The aim of this thesis is to contribute in this field.

1.1 The stochastic Target in Finance and Insurance

In a geometric form, a stochastic target problem can be formulated as follows. Let G be a Borel subset of a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$, and $Z^{\nu}_{t,z}$ a $\mathcal{Z}$-valued controlled process with initial conditions at time t $Z^{\nu}_{t,z}(t) = z \in \mathcal{Z}$. We are interested in the so-called *reachability set* $\Lambda(t)$ of initial conditions $z \in \mathcal{Z}$ such that $Z^{\nu}_{t,z}(T) \in G$ $\mathbb{P}$-a.s. for some $\nu \in \mathcal{U}$, with $\mathcal{U}$ the set of *admissible controls*:

$$\Lambda(t) := \left\{ z \in \mathcal{Z} : \text{ there exists } \nu \in \mathcal{U} \text{ s.t. } Z^{\nu}_{t,z}(T) \in G \ \mathbb{P}\text{-a.s.} \right\}. \tag{1.1.1}$$

This set was introduced in [Soner 02a], where they proved that it satisfies a dynamic programming principle, the so-called Geometric Dynamic Programming Principle (hereafter GDP). This GDP allows then to perform the derivation of the associated dynamic programming equation, as it usual in optimal control (see e.g. [Lions 82, Lions 83]).

As we shall see below, since the original treatment of this problem by Soner and Touzi, [Soner 02c, Soner 00, Soner 02a, Soner 03a], this theory seems to be now well established. Considering all the practical applications of this technology, this class of (non-standard) stochastic control problems may be seen as a part of the general tool box in optimal control. At first sight, relying on different dynamic programming principles, stochastic target problems and optimal control in standard form should have to be discussed separately. However, Bouchard and Dang have shown in [Bouchard 12a] that any optimal control problem in standard form admits a simple and natural representation in terms of a stochastic target problem.

quel rapport ?

1.1.1 The $\mathbb{P}$-almost sure criterion

Fix $\mathcal{Z} := \mathbb{R}^d \times \mathbb{R}$, $Z := (X, Y)$ and $G := \{z := (x, y) \in \mathbb{R}^d \times \mathbb{R} \text{ s.t. } \Psi(x, y) \geq 0\}$ for some Borel measurable map Ψ. Consider furthermore that both $y \mapsto \Psi(\cdot, y)$ and $y \mapsto Y^{\nu}_{t,x,y}(T)$ are non-decreasing, for all $\nu \in \mathcal{U}$. The set $\Lambda(t)$ can then be identified to $\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \geq \mathrm{y}(t, x)\}$, with

$$\mathrm{y}(t, x) := \inf \left\{ y \in \mathbb{R} : \text{ there exists } \nu \in \mathcal{U} \text{ s.t. } \Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T) \right) \geq 0 \ \mathbb{P}\text{-a.s.} \right\},$$

whenever the above infimum is achieved.

Soner and Touzi were the first to propose a treatment of this problem in its primal form. They were mainly motivated by applications to financial mathematics. Formulated as above, this problem may be seen as a generalization of the so-called super-replication problem, see e.g. [Broadie 98, Cvitanić 99, Cvitanić 96, Cvitanić 93, El Karoui 95, Karatzas 98].

In the literature, the super-replication problem is usually solved by convex duality. The idea is to start with applying the duality in order to retrieve a classical optimal control problem, see [Jouini 95, El Karoui 95, Cvitanić 93, Föllmer 97]. Hence, one may use classical dynamic programming to obtain the PDE characterization of the value function y. However, this dual approach does not allow to deal with a general framework, since it heavily relies on the fact that the wealth dynamics is linear in the control, and that the stock prices are not influenced by the trading strategy. In particular, it does not apply to large investor models or to more general dynamics or constraints, such as gamma constraint, where the primal approach of [Soner 02c, Soner 00, Soner 03b, Soner 02b, Cheridito 05, Soner 03b] does. The GDP of [Soner 02a] permits to obtain the PDE characterization directly from the initial formulation, without using the duality.

This approach was further exploited in [Touzi 00], Bouchard and Touzi [Bouchard 00], and extended to locally bounded jumps in [Bouchard 02], and to path dependent constraints in Bouchard and Vu [Bouchard 10c].

1.1.2 The moment constraint

The approach developed in Section 1.1.1 is very powerful to study a large family of non-standard stochastic control problems in which a target has to be reached with a probability one at time T. It was however limited to that case until Bouchard, Elie and Touzi [Bouchard 09], when the authors relaxed the $\mathbb{P}$-a.s. criterion $\Psi(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)) \geq 0$ into a moment constraint of the form $\mathbb{E}[\Psi(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T))] \geq p$, with $p \in \mathbb{R}$ a given threshold.

This new approach opened the door to a wide range of applications, especially in mathematical finance. Indeed, in most of the cases, the superhedging price leads to an unbearable cost, which is not reasonable in practice, where the expectation criterion allows to consider a large spectrum of risk criteria, as exemplified in Section 1.1.3.

The work of Bouchard, Elie and Touzi was extended to unbounded jumps in [Moreau 11], which is the object of Chapter 2.

We conclude this section with some references of recent advances in this field. In Bouchard and Dang [Bouchard 10a], the authors give a PDE characterization of a singular with state constraints version of the stochastic target problems. This work perfectly allows to treat the case of market models with proportional transaction costs, and more specifically to order book liquidation issues.

In [Bouchard 11c], Bouchard and Vu provide a PDE characterization of the minimal initial endowment required so that the terminal wealth of a financial agent can match a set of constraints in probability. Their original idea was to consider that the agent has a rough idea on the type of P&L he can afford, and has as a target. It was motivated by the fact that, if the attitude of the financial agent toward risk is usually described in academic literature in terms of utility or loss function, this is in practice not so trivial for an agent to characterize precisely their "utility function".

Having in mind a version of stochastic target robust to model uncertainty, we give in [Bouchard 12c] a weak formulation of a game version of stochastic target, in the same spirit of those of Bouchard and Touzi [Bouchard 11b] or Bouchard and Nutz [Bouchard 11a] for the standard optimal control. This work is developed in Chapter 4, see Sections 1.3, 1.4 and 1.4.2 for its introduction.

We finally refer to Bouchard, Elie and Reveillac [Bouchard 12b] for a BSDE formulation of this moment criterion, and to Bouchard, Elie and Imbert [Bouchard 10b] for an optimal stochastic control problem under stochastic target constraint.

1.1.3 Application of the moment criterion in Finance and Insurance

We shall briefly present in this section the possible applications of stochastic target in finance and insurance. This will be done within the framework of Bouchard, Elie and Touzi [Bouchard 09], but it can of course be extended into the mixed diffusion case of Chapter 2 or into the robust formulation of Chapter 4. We refer to chapters 3 and 4 for an example of explicit resolution in chosen applications.

Let X^ν be a process denoting roughly the risks in the portfolio of an agent (One might think of stocks, but also a fixed number of non-tradeable idiosyncratic sources of risks). Fix g, a map defined on $\mathbb{R}^d$ such that $g(X^\nu_{t,x}(T))$ has enough regularity. The quantity $g(X^\nu_{t,x}(T))$ may be seen as the random payoff of a European claim, given the initial condition $X^\nu_{t,x}(t) = x$. The process $Y^\nu_{t,x,y}$ shall represent the wealth of the agent, of initial value y at time t, where ν denotes his strategy in terms of X^ν. Fix finally $\kappa \in \mathbb{R}_+ \cup \{+\infty\}$, such that $-\kappa$ represent a finite credit line, and consider the value function

$$\mathrm{y}_\kappa(t,x,p) := \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] \geq p\right\}. \quad (1.1.2)$$

For $p = 1$ and

$$\Psi : (x,y) \longmapsto \mathbb{1}_{\{y \geq g(x)\}},$$

the value function (1.1.2) represents the super-replication price of the claim $g(X^\nu_{t,x}(T)$, as discussed above. If $p \in (0,1)$, the value function (1.1.2)

$$\mathrm{y}_\kappa(t,x,p) := \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}\left[Y^\nu_{t,x,y}(T) \geq g\left(X^\nu_{t,x}(T)\right)\right] \geq p\right\}, \quad (1.1.3)$$

allows for a treatment of the quantile hedging introduced in Föllmer and Leukert [Föllmer 99], in a more general framework. The formulation (1.1.3) describes perfectly the quantile hedging problem in a general framework where the strategy of the agent may influence the value of the risks (large investor). It also permits to deal with more general investment policy, where the original treatment of the problem by Fölmer and Leukert relies on the fact that this strategy is linear.

Consider now the case where $p \in \mathbb{R}$ and Ψ belongs to some general class of *utility functions*. More precisely, for an utility function $U : \mathbb{R} \to \mathbb{R}$ and

$$\Psi : (x, y) \in \mathbb{R}^d \times \mathbb{R} \longmapsto U(y - g(x)),$$

the problem (1.1.2) reads

$$\mathrm{y}_\kappa(t,x,p) := \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[U\left(Y^\nu_{t,x,y}(T) - g\left(X^\nu_{t,x}(T)\right)\right)\right] \geq p\right\}.$$

That is, finding the minimum amount of money the investor has to invest in some strategy ν in order to have its expected utility above a given threshold p. If p happens to be chosen as

$$p := \sup_{\nu' \in \mathcal{U}} \mathbb{E}\left[\Psi\left(0, Y^{\nu'}_{t,x,y_o}(T)\right)\right],$$

a straightforward reformulation of this problem defines the value function y_κ as the utility indifference price of the claim g:

$$\mathrm{y}_\kappa(t,x,p) = \inf\left\{y \geq -\kappa : \ \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y_o+y}(T)\right)\right] \geq p\right\}.$$

Finally, with some minor modifications in the previous reasoning allows to consider the case where $s\Psi$ belongs to some class of *risk functions*

$$\Psi : (x, y) \in \mathbb{R}^d \times \mathbb{R} \longmapsto -\rho(y - g(x)),$$

such as convex non-decreasing loss function $\rho : \mathbb{R} \to \mathbb{R}$, or the success ratio of [Föllmer 99]

$$\Psi : (x, y) \in \mathbb{R}^d \times \mathbb{R} \longmapsto \mathbb{1}_{\{g(x) \leq y\}}(x, y) + \frac{y}{g(x)} \mathbb{1}_{g(x) > y \wedge 0}.$$

The chapter 3 is dedicated to an example of such treatment in a mixed diffusion case.

Remark 1.1.1. As it was notified in the beginning of this section, one could consider (at least for the formulation of the problem) the game version stated in Chapter 4 in all the previous examples. The adverse control may be interpreted as a model uncertainty. If the agent is an insurance company, the adverse control has the natural interpretation of moral hazard. An application in this framework is developed in Section **??**.

1.2 The Geometric Dynamic Programming Principle

As it has been emphasized previously, the treatment of an optimal control problem consists in the characterization of the PDE satisfied by the value function. This characterization relies on the dynamic programming principle, and in the stochastic target problem, on the GDP of Soner and Touzi [Soner 02a]. The GDP stated in [Soner 02a] allows for a direct treatment of the problem in the $\mathbb{P}$-a.s. case. The use of this GDP in the controlled expected loss case requires to reformulate the problem into the class of standard stochastic target problem. This reformulation has been proposed in Bouchard, Elie and Touzi [Bouchard 09].

1.2.1 The Geometric Dynamic Programming in $\mathbb{P}$-a.s. criterion

Recall the geometric formulation (1.1.1) of a stochastic target problem of Section 1.1.1. For $\mathcal{Z} := \mathbb{R}^d$, The GDP allowed Soner and Touzi [Soner 02a] to derive the PDE satisfied, in the viscosity sense, by the characteristic function of this reachability set, by using similar methods as in [Barles 93, Chen 91, Evans 91, Soner 93].

The GDP reads as follows. Under good assumptions on the state process Z (e.g. if Z is Markovian) and on the set of controls $\mathcal{U}$ (stability under concatenation, and under measurable selection, which is the case if $\mathcal{U}$ is separable, cf. [Soner 02a, Lemma 2.1]), the reachability set

$$\Lambda(t) = \left\{ z \in \mathcal{Z} : Z^{\nu}_{t,z}(T) \in G \; \mathbb{P}\text{-a.s. for some admissible } \nu \right\}$$

coincides with the set $\bar{\Lambda}$

$$\bar{\Lambda}(t) := \left\{ z \in \mathcal{Z}, Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau) \; \mathbb{P}\text{-a.s. for some admissible } \nu \right\},$$

for all stopping times τ. This GDP has been extended, along the lines of [Soner 02a], to path dependent constraints. The scheme of the proof is to introduce, for each stopping time τ with values in $[t, T]$, the set

$$\bar{\Lambda}(t) := \left\{ Z \in \mathbb{R}^d; Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau)\mathbb{P}\text{-a.s. for some } \nu \in \mathcal{U} \right\},$$

and to show the double inclusion $\Lambda(t) \subseteq \bar{\Lambda}(t)$ and $\bar{\Lambda}(t) \subset \Lambda(t)$.

The first inclusion $\Lambda(t) \subseteq \bar{\Lambda}(t)$ is straightforward since, under a "Flow-like" Assumption and $(z, \nu) \in \mathbb{R}^d \times \mathcal{U}$ such that $Z^{\nu}_{t,z}(T) \in G$ $\mathbb{P}$-a.s., we have

$$Z^{\nu}_{\tau, Z^{\nu}_{t,z}(\tau)}(T) \in G \qquad \mathbb{P}\text{-a.s.}$$

The result is obtained using the pull-back of the probability $\mathbb{P}$ under the map $(\tau, Z^{\nu}_{t,z}(\tau))$.

The other inclusion is the "*tricky one*". For $(t, z) \in \bar{\Lambda}(t)$, there is $\nu \in \mathcal{U}$ such that

$$Z^{\nu}_{t,z}(\tau) \in \Lambda(\tau) \qquad \mathbb{P}\text{-a.s.}$$

Roughly speaking, the idea is that, depending on the realization of $(\tau, Z^{\nu}_{t,z}(\tau))$, we may pick an admissible control $\tilde{\nu}$ allowing to reach the target $\mathbb{P}$-a.s. at time T starting from time τ. Once this is achieved, the conclusion of the proof is straightforward considering the stability of controls under concatenation.

The existence of the control $\tilde{\nu}$ is performed with the Jankov-Von Neumann selection Theorem (see [Bertsekas 78, Proposition 7.49]), and under the Assumption of stability under measurable selection. We refer the interested reader to [Soner 02a] for the proof (see [Bouchard 10c] for the proof of the obstacle version), and go on with the idea, which is important to understand the need to state only a weak version in game settings, as exposed in Chapter 4.

Let $\mathcal{S}$ denote the initial conditions in both time and space. If $\mathcal{S}\times\mathcal{U}$ happens to be Borel, for every analytic subset B of $\mathcal{S}\times\mathcal{U}$, the Jankov-von Neumann Theorem (see [Bertsekas 78, Proposition 7.49]) states that there exists an analytically measurable map $\phi : \mathrm{Proj}_{\mathcal{S}}(B) \to \mathcal{U}$ such that $\mathrm{Gr}(\phi) \subset B$, with $\mathrm{Gr}(\phi) := \{(t', z', \phi(t', z')) \in \mathcal{S}\times\mathcal{U} | (t', z') \in \mathrm{Proj}_{\mathcal{S}}(B)\}$. If B is taken as the subset of elements $(t', z', \tilde{\nu}) \in \mathcal{S}\times\mathcal{U}$ allowing to reach the target:

tu veux dire bcp trop de chose et ca part dans tous les sens !

$$B := \left\{(t', z', \tilde{\nu}) \in \mathcal{S}\times\mathcal{U} \text{ s.t. } < Z^{\tilde{\nu}}_{t',z'}(T) \in G \qquad \mathbb{P}\text{-a.s.}\right\}, \tag{1.2.1}$$

and if B is an analytic subset of $\mathcal{S}\times\mathcal{U}$, then we can pick in an analytically measurable way from $(t', z') \in \mathrm{Proj}_{\mathcal{S}}(B)$ an admissible control allowing to steer the target G $\mathbb{P}$-a.s. at time T, if starting from (t', z').

At this point, we need however more measurability on the map ϕ. This is done in [Soner 02a] with the universal σ-algebra of [Bertsekas 78, Definition 7.18]. By assumption, $(\tau, Z^{\nu}_{t,z}(\tau)) \in \mathrm{Proj}_{\mathcal{S}}(B)$ $\mathbb{P}$-a.s. Then, with μ being the measure on $\mathcal{S}$ induced by $(\tau, Z^{\nu}_{t,z}(\tau))$, we need more specifically ϕ to be $\mathcal{B}_{\mathcal{S}}(\mu)$-measurable, where $\mathcal{B}_{\mathcal{S}}(\mu)$ is the completion of the Borel σ-field of $\mathcal{S}$ under μ. This is performed with [Bertsekas 78, Corollary 7.42.1], which essentially gives that ϕ is universally measurable, and hence $\mathcal{B}_{\mathcal{S}}(\tilde{\mu})$ for every probability measure on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$, and in particular for μ. Then $\phi = \phi_\mu$ μ-a.e., and the proof is concluded by stability under measurable selection.

Remark 1.2.1. As we tried to explain clearly here, for the measurable selection argument to hold , one need the set B defined in 1.2.1 to be an analytic subset of the Borel set $\mathcal{S}\times\mathcal{U}$. When extending the problem (1.1.1) to Differential Game settings, as it will be discussed in Section 1.4.2 and Chapter 4, this is not trivial how this can hold. Indeed, as we will introduce in Section 1.3, the player controlling ν will have to choose a strategy, that is a *function* of the adverse control, and cannot restrict to pick a control anymore. one need then a suitable topological structure on this set of strategy for the measurable selection argument to hold.

1.2.2 The Geometric Dynamic Programming in moment criterion

When dealing with stochastic target problems with controlled expected loss as introduced in Section 1.1.2, the underlying reachability set (although it was not introduced explicitly in Bouchard, Elie and Touzi [Bouchard 09]) is now

$$\Lambda(t) := \left\{(z,p) \in \mathbb{R}^d \times \mathbb{R} : \text{ there exists } \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\Psi\left(Z^{\nu}_{t,z}(T)\right)\right] \geq p\right\}.$$

When trying to apply directly the GDP of Soner and Touzi described in the previous section, one might think to establish a GDP of the following form

$$\Lambda(t) = \left\{\begin{array}{c}(z,p) \in \mathbb{R}^d \times \mathbb{R} : \text{ there exists } \nu \in \mathcal{U} \text{ and some process } M \\ \text{s.t. } \left(Z^{\nu}_{t,z}(\tau), M_{t,p}(\tau)\right) \in \Lambda(\tau) \ \mathbb{P}\text{-a.s.}\end{array}\right\},$$

with $M_{t,p}$ being some process with initial value p at time t. The original idea in [Bouchard 09, Proposition 3.1] (extended to the mix diffusion case in Proposition

2.3.2) was to determine this process with the martingale representation Theorem. We may hence write the reachability as

$$\Lambda(t) := \left\{(z,p) \in \mathbb{R}^d \times \mathbb{R} : \text{ there exists } \nu,\alpha,\chi \in \hat{\mathcal{U}} \text{ s.t. } \tilde{\Psi}\left(Z^{\nu}_{t,z}(T), M^{\alpha,\chi}_{t,p}(T)\right) \geq 0\right\},$$

where $\tilde{\Psi} : (z,p) \in \mathbb{R}^d \times \mathbb{R} \longmapsto \Psi(z) - p$, and we refer to Section 2.2.1 for the definition of the set $\hat{\mathcal{U}}$.

The dynamic programming PDE is then derived from the GDP of [Soner 02a] up to non-trivial difficulties. In a Brownian filtration, (where the only additional control is α), the major difficulties comes from the fact that this additional control has no a priori boundedness properties. The usual HJB operator fails then to have the required semicontinuity. This is handled in [Bouchard 09] with a local relaxation of this operator.

1.2.3 The mixed diffusion case

In Chapter 2, we extend the results of [Bouchard 09] presented in the previous section into the mixed diffusions case. Namely, for $0 \leq t \leq T$, we are given two controlled diffusion processes $\left\{X^{\nu}_{t,x}(s), t \leq s \leq T\right\}$ and $\{Y^{\nu}_{t,x,y}(s), t \leq s \leq T\}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}$. These processes satisfy the initial condition $\left(X^{\nu}_{t,x}(t), Y^{\nu}_{t,x,y}(t)\right) = (x,y)$, and are $\mathbb{R}^d \times \mathbb{R}$-valued strong solutions of the stochastic differential equations

$$\begin{aligned} dX(s) &= \mu_X\left(X(s),\nu_s\right) ds + \sigma_X\left(X(s),\nu_s\right) dW_s \\ &\quad + \int_E \beta_X\left(X(s-),\nu^1_s,\nu^2_s(e),e\right) J(de,ds) \\ dY(s) &= \mu_Y\left(Z(s),\nu_s\right) ds + \sigma_Y\left(Z(s),\nu_s\right) dW_s \\ &\quad + \int_E \beta_Y\left(Z(s-),\nu^1_s,\nu^2_s(e),e\right) J(de,ds). \end{aligned}$$

We consider in Chapter 2 a filtration $\mathbb{F}$ generated by a Brownian motion W and a E-marked right continuous point process J, compared to [Bouchard 09], where the filtration $\mathbb{F}$ is generated by the Brownian motion W, and $\beta_X \equiv \beta_Y \equiv 0$. We shall see briefly below that this has non-trivial impacts on both the formulation and the derivation of the associated partial differential equations.

Let Ψ be some measurable map. For a given threshold p, the aim of the controller is to determine the minimal initial condition y for which it is possible to find a control ν satisfying $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p$. Namely, he wants to compute:

$$\mathrm{y}(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p \text{ for some control } \nu\right\} \quad (1.2.2)$$

where $\kappa \in \mathbb{R}_+$. As explained in previous section, increasing the dimension of both the state and the control processes allows Bouchard, Elie and Touzi [Bouchard 09] to reduce this problem into a standard problem of super-replication. It implies in the Brownian controlled SDEs case to deal with a possibly unbounded control α. In the mixed diffusion case, the martingale representation Theorem gives birth to an additional control (α, χ), where α is as in [Bouchard 09], but χ is a process taking values in a set of unbounded measurable maps on E.

In order to clarify these issues, we consider the following SDEs:

$$\begin{aligned}
\begin{pmatrix} X^{\nu}_{t,x}(s) \\ M^{\alpha,\chi}_{t,p}(s) \end{pmatrix} &= \begin{pmatrix} x \\ p \end{pmatrix} + \int_t^s \begin{pmatrix} \mu\left(X^{\nu_r}_{t,x}(r), \nu_r\right) \\ -\int_E \chi(e,r)\lambda(de) \end{pmatrix} dr + \int_t^s \begin{pmatrix} \sigma\left(X^{\nu_r}_{t,x}(r), \nu_r\right) \\ \alpha_r^T \end{pmatrix} dW_r \\
&\quad + \int_t^s \int_E \begin{pmatrix} \beta\left(X^{\nu_r}_{t,x}(r), \nu_r, e\right) \\ \chi(e,r) \end{pmatrix} J(de,dr) \\
Y^{\nu}_{t,x,y}(s) &= y + \int_t^s \mu_Y\left(X^{\nu}_{t,x}(r), Y^{\nu}_{t,x,y}(r), \nu_r\right) dr + \int_t^s \sigma_Y\left(X^{\nu}_{t,x}(r), Y^{\nu}_{t,x,y}(r), \nu_r\right) dW_r \\
&\quad + \int_t^s \int_E \beta_Y\left(X^{\nu}_{t,x}(r), Y^{\nu}_{t,x,y}(r), \nu_r, e\right) J(de,dr),
\end{aligned}$$

where λ is the predictable intensity kernel of J, and $(X^{\nu}_{t,x}(s), M^{\alpha,\chi}_{t,p}(s))$ stands for the augmented state process, and (ν, α, χ) for the augmented control. The possible unboundedness of the additional control α can be handled as in Bouchard, Elie and Touzi [Bouchard 09] with a local relaxation of the usual HJB operator. The viscosity supersolution (resp. subsolution) is then stated in terms of upper semicontinuous envelop H^* (resp. lower semicontinuous envelop H_*) of the HJB operator H. In the mixed diffusions case, this relaxation is however not enough to guarantee this semicontinuity. This can be seen in the proof of the supersolution property, where we may lose the local properties with an unbounded jump. The non-local relaxation of test function will allow us "*to control the distance between the value function and the test function*". For a test function $\varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, the relaxed operators are then

$$H^*(\Theta, \varphi) = \limsup_{\substack{\varepsilon \searrow 0, \Theta' \to \Theta \\ \eta \to 0, \psi \xrightarrow[u.c.]{} \varphi}} H_{\varepsilon,\eta}(\Theta', \psi) \qquad H_*(\Theta, \varphi) = \liminf_{\substack{\varepsilon \searrow 0, \Theta' \to \Theta \\ \eta \to 0, \psi \xrightarrow[u.c.]{} \varphi}} H_{\varepsilon,\eta}(\Theta', \psi)$$

where, in a very informal way, $\Theta' := (t', x', p', y', q_t, q_x, q_p, q_{xx}, q_{xp}, q_{pp})$ converges toward (t, x, p, y) and the local partial derivatives of φ at this point (the local relaxation), $\varepsilon > 0$ and $\eta \in [-1, 1]$ are numbers controlling the volatility and the size of the jump, and the non-local relaxation stands in the convergence of a "relaxed" test function ψ toward φ uniformly on compact subset.

We are then able to derive the associated PDE as well as the terminal condition, see Theorems 2.2.5, 2.2.9 and Corollaries 2.3.7, 2.3.17.

1.3 The Differential Games

Being interested in Chapter 4 in the statement of a game version of the GDP of Soner and Touzi, we were confronted as already mentioned earlier to non-trivial measurable issues. The aim of the present section is to introduce the theory of differential games.

The theory of two-controller, zero sum differential games was initiated by Isaacs [Isaacs 65]. (In the game theory literature, the term *player* rather than *controller* is frequently used) Early rigorous theory of differential game value made use of time discretization, see for example [Friedman 71]. They were superseded by the more convenient Elliott-Kalton differential Game values [Elliott 72a], affected by two controllers with opposite goals.

1.3.1 The deterministic case and non-anticipative strategies

The intuitive idea is that there are two players I and II. Each of these players have opposite goals. One aims at maximizing some objective function, where the other aims at minimizing it, no matter which action is chosen by the other. The controls of the players are denoted by ν and ϑ.

The main difficulty lies in the fact that the players play in continuous time, and observe each other continuously, so that instantaneous switches of ν and ϑ are possible. To overcome this problem in the deterministic case ($\sigma \equiv 0$) or in the special stochastic cases ($\sigma \equiv \sigma(Z)$), Fleming [Fleming 64, Fleming 61], Friedman [Friedman 71], Elliott and Kalton [Elliott 74a, Elliott 74b] or Subbotina, Subbotin and Tret'Jakov [N.N. 85] introduced two approximate games, namely, a *lower* and an *upper* one.

In each of these games, one player has an *instantaneous information advantage*. In the lower game, player II is allowed to know ν_s before choosing ϑ_s, while in the upper game, player I chooses ν_s knowing ϑ_s.

Using arguments from the theory of viscosity solutions of Hamilton-Jacobi equations introduced by Crandall and Lions [Crandall 83] (see also [Crandall 92]) Barron, Evans and Jensen [Barron 84], Evans and Souganidis [Evans 84] and Souganidis [Souganidis 85a, Souganidis 85b] established the existence of the *lower* value and the *upper* value for the deterministic case, and in the case where the Isaacs' condition holds, they showed that the game has a *value*.

In the Elliott-Kalton definition of upper value (resp. lower value), the minimizing player (resp. maximizing player) chooses a control, and the other player chooses a strategy. However, since we wish to formalize the fact that no player can guess in advance the future behavior of the other player, we have to require that such a map is non-anticipative. Non anticipative strategies were introduced by [Varaiya 67], [Roxin 69], Elliott and Kalton [Elliott 72a, Elliott 72b], and were extensively used in the viscosity solution approach of differential games and, in particular, in the former work of Evans and Souganidis [Evans 84].

In order to put the game into the so-called *normal form*, one should be able to say that, for any pair of non-anticipating strategies $(\mathfrak{u}, \mathfrak{v})$, there is a unique pair of controls (ν, ϑ) such that

$$\mathfrak{u}(\vartheta) = \nu \qquad \mathfrak{v}(\nu) = \vartheta. \tag{1.3.1}$$

The pair (ν, ϑ) would be the natural answer of the players to the strategies $(\mathfrak{u}, \mathfrak{v})$. Unfortunately, this is not possible, as we may find either an infinite number of such controls, or none of them would.

The asymmetric roles of maximizing and minimizing in this definition have also sometimes been criticized within the game theory community. We may then find different notions of strategies to overcome some of these issues, such as the subclass of Elliott-Kalton strategies consisting in the strictly progressive strategies (see e.g. [Fleming 06, Chapter XI, Section 9]), Delay strategies, or Random strategies.

1.3.2 The stochastic case

Fleming and Souganidis were the first to study in a rigorous manner two-player zero-sum stochastic differential games, in the so-called *Bolza form*. Their work has translated the results from the deterministic into the stochastic framework. We refer to [Buckdahn 05, Rainer 07, Tang 07, Mataramvura 08, Cardaliaguet 09, Buckdahn 11] for advance researches in the field of stochastic differential games. We also refer to Biswas [Biswas 10], who has extended the results of Fleming and Souganidis, stated on the Wiener space $C_0([t, T]; \mathbb{R}^d)$, into the Poisson-Wiener space.

Once the notion of upper and lower value functions has been introduced, the main step is to establish that these functions are viscosity solutions of the associated Bellman-Isaacs' equations. As usual, this would immediately follow from the fact that these functions satisfy the dynamic programming principle. However, as we have in Chapter 4, they have encountered some non-trivial measurability issues, and hence were able to establish only half of the desired inequalities for some restrictions of the value functions (with the so-called *r-strategies*). In fact, combining these inequalities with a discretization argument (see the definition of *π-controls*), and the uniqueness result of Ishii [Ishii 89], they were able to show that the lower and the upper value functions are the unique viscosity solutions of the HJBI equations, and that they hence satisfy the DPP.

1.4 Weak formulations of dynamic programming principles

As underlined in Section 1.1, the link between an optimal control problem (either in deterministic or in the stochastic cases) and the HJB equation heavily relies on Dynamic Programming Principle (hereafter DPP), which relates problem at time t with the same problem stated at a later time τ. The proof of the DPP requires in general some regularity property on the value function of the problem. The aim of Bouchard and Touzi [Bouchard 11b] and Bouchard and Nutz [Bouchard 11a] in the state constraint case was to provide a *weak* formulation of this DPP when the value fails to have the needed regularity, which avoids the use of measurable selection.

When considering a game version of stochastic target problems, the lack of topological structure on the set of non-anticipative strategies makes the use of a measurable selection impossible. We have instead stated a weaker formulation of the GDP, similar to those of [Bouchard 11b, Bouchard 11a]. This formulation turned out to be enough for the derivation of the dynamic programming equation in the Markov-diffusion case, when considering stochastic target problem in controlled expected loss.

1.4.1 Weak formulations of dynamic programming principle in optimal control

Given a set of admissible controls $\nu \in \mathcal{U}$, a controlled state process $Z^\nu_{t,z}$ with values in $\mathbb{R}^d$ of initial value $Z^\nu_{t,z}(t) = z$, a time horizon $T > 0$, and g a function defined on $\mathbb{R}^d$. An optimal control problem usually takes the following form

$$v(t,z) := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[g\left(Z^\nu_{t,z}(T)\right)\right]. \tag{1.4.1}$$

A formal statement of the DPP would be, for any stopping time τ with values in $[t,T]$:

$$v(t,z) = \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[v\left(\tau, Z^\nu_{t,z}(\tau)\right)\right]. \tag{1.4.2}$$

If we consider the problem (1.4.1) from a financial point of view, one could think of an agent who wishes to maximize a reward function by dynamically trading on the market, when starting from initial wealth z at time t.

The proof of (1.4.2) usually requires some measurability on the value function v. In [Bouchard 11b], Bouchard and Touzi avoid this assumption by stating a weaker DPP. Namely, they show that, for any stopping time τ with values in $[t,T]$, one has

$$\sup_{\nu \in \mathcal{U}} \mathbb{E}\left[v^*\left(\tau, Z^\nu_{t,z}(\tau)\right)\right] \geq v(t,z) \geq \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\varphi\left(\tau, Z^\nu_{t,z}(\tau)\right)\right]$$
$$\text{for every upper-semicontinuous minorant } \varphi \text{ of } v,$$

where v^* is the upper semicontinuous envelope of v. Although this DPP is weaker than (1.4.2), thinking of the viscosity solutions, and the fact that they are stated in terms of semicontinuous envelope of a locally bounded value function, this formulation turns out to be enough for the derivation of the corresponding dynamic programming equations in Markov settings. The proof relies on an appropriate covering argument, and avoids the measurable selection argument, which will be of important use for us in Chapter 4.

In the same spirit, Bouchard and Nutz [Bouchard 11a] give a weak formulation of DPP for a stochastic optimal control problem under generalized state constraint. In a classical form, the problem of optimal control with state constraint is a problem of the form (1.4.1), with the additional constraint that the state process has to remain in a given subset $\mathcal{O}$ of the state space. For the corresponding PDEs, we refer to [Ishii 96, Soner 86a, Soner 86b] for the first order, and to [Katsoulakis 94, Lasry 89] for the second order.

However, the weak DPP of Bouchard and Touzi [Bouchard 11b] does not apply directly in that case. Intuitively, the idea is based on the fact that a control ν may be optimal from an initial condition (t, z) (i.e. the state process $Z^{\nu}_{t,z}$ remains in $\mathcal{O}$), but it may violate this state constraint for another position z' of z. The key idea in order to tackle this issue is to introduce some continuity assumption. If the set $\mathcal{O}$ is open, for an optimal control ν for (t, z) and z' sufficiently close to z, the probability that $Z^{\nu}_{t,z'}$ remains in $\mathcal{O}$ is small enough. This idea leads Bouchard and Nutz to consider state constraints in expectation form $\mathbb{E}[f(Z^{\nu}_{t,z}(T)] \leq m$ which, with the same idea of those of [Bouchard 09], can be formulated dynamically with an appropriate family of martingales.

Similar arguments are used in the weak version of the GDP, see Chapter 4.

1.4.2 The game version of the Geometric Dynamic Programming Principle

In Chapter 4, we are interested in a stochastic game problem. Since the GDP of Soner and Touzi does not apply in this particular case, we need its *game version*.

Consider still a finite time horizon $T > 0$, a Borel subset of a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$ and a $\mathcal{Z}$-valued process $Z^{\nu,\vartheta}_{t,z}$ with initial conditions $Z^{\nu,\vartheta}_{t,z}(t) = z$, controlled now by two players. Our aim is to investigate a game version of the reachability set:

$$\Lambda(t) := \left\{ z \in \mathcal{Z} : \text{there exists } \mathfrak{u} \text{ s.t. } Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,z}(T) \in G \ \mathbb{P}\text{-a.s. for all } \vartheta \right\}.$$

The aim of the player controlling $\mathfrak{u}$ is to reach the target G $\mathbb{P}$-a.s., whatever could the player controlling ϑ do to prevent it. We introduce, as in Fleming and Souganidis [Fleming 89], the notion of non-anticipating strategies. For each $\vartheta \in \mathcal{V}$, an admissible strategy for player I associates, in a non-anticipating way, a control $\mathfrak{u}[\vartheta] \in \mathcal{U}$. The set of admissible strategies for player I is denoted by $\mathfrak{U}$.

We introduce a continuity Assumption on the target, through the consideration of a continuous function ℓ, and we allow the controls to depend on information occurring before the beginning of the game. The reachability set then becomes

$$\Lambda(t) := \left\{ \begin{array}{l} (z, p) \in \mathcal{Z} \times \mathbb{R} : \text{there exists } \nu \in \mathfrak{U} \text{ s.t.} \\ \operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z^{\nu[\vartheta],\vartheta}_{t,z}(T) \right) | \mathcal{F}_t \right] \geq p \ \mathbb{P}\text{-a.s.} \end{array} \right\}.$$

By analogy with [Bouchard 09], an informal version of the GDP should be that $\Lambda(t)$ coincides with the set of elements $(z, p) \in \Lambda(t)$ for which there exists an admissible strategy and a family $\{M^{\vartheta}, \vartheta \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ such that $(Z^{\nu[\vartheta],\vartheta}_{t,z}(\tau), M^{\vartheta}(\tau)) \in \Lambda(\tau)$ $\mathbb{P}$-a.s. for all $\vartheta \in \mathcal{V}$ and stopping times τ. In the above, $\mathcal{M}_{t,p}$ is a suitable set of martingales starting from p at time t.

We hence provide a weak version of this assertion, divided in two parts, usually called (GDP1) and (GDP2). More precisely, our GDP is stated in terms of the sets

$$\bar{\Lambda}(t) := \left\{ \begin{array}{l} (z, p) \in \mathcal{Z} \times \mathbb{R} : \text{ there exist } (t_n, z_n, p_n) \to (t, z, p) \\ \text{such that } (z_n, p_n) \in \Lambda(t_n) \text{ and } t_n \geq t \text{ for all } n \geq 1 \end{array} \right\}$$

and

$$\mathring{\Lambda}_\iota(t) := \left\{(z,p) \in \mathcal{Z} \times \mathbb{R} :\, (t',z',p') \in B_\iota(t,z,p) \text{ implies } (z',p') \in \Lambda(t')\right\},$$

where $B_\iota(t,z,p)$ denotes the open ball of $[0,T] \times \mathcal{Z} \times \mathbb{R}$ with center (t,z,p) and radius $\iota > 0$. The proof of this GDP heavily relies on a result from [Buckdahn 08], where it is stated that, in a Brownian framework, and with $\mathcal{Z} = \mathbb{R}^d$, the map

$$(t,z) \in [0,T] \times \mathcal{Z} \longmapsto \operatorname*{ess\,sup}_{\nu \in \mathfrak{U}} \operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\nu[\vartheta],\vartheta}(T)\right)|\mathcal{F}_t\right]$$

is deterministic. We then begin to state the GDP, in an abstract framework, with strong regularity assumptions on this map K, and finally relax these assumptions, using the fact that the GDP is stated in a weak version.

For $(z,p) \in \Lambda(t)$, the key idea is to construct a family of càdlàg martingales $\{M^\vartheta, \vartheta \in \mathcal{V}\}$ such that, for all $\vartheta \in \mathcal{V}$ and stopping times τ,

$$\left(Z_{t,z}^{\nu[\vartheta],\vartheta}(\tau), M^\vartheta(\tau)\right) \in \bar{\Lambda}(\tau).$$

Given the strategy $\nu \in \mathfrak{U}$ induced by $(z,p) \in \Lambda(t)$, we start with the construction of a family of martingales $\{M^\vartheta, \vartheta \in \mathcal{V}\}$ which bounds from below $\{\operatorname{ess\,inf}_{\bar{\vartheta} \in \mathcal{V}} \mathbb{E}[\ell(Z_{t,z}^{\nu,\vartheta \oplus_r \bar{\vartheta}}(T)|\mathcal{F}_r)], \vartheta \in \mathcal{V}\}$. Fix now, for each $n \in \mathbb{N}$ a discrete stopping time τ_n taking values in $\{t_i^n, i \leq n\} \subset (t,T]$ such that $\tau_n \downarrow \tau$ as $n \to \infty$, and a countable covering $(B_j)_{j \in \mathbb{N}}$ of $\mathcal{Z}$. We construct thus an admissible ε-optimal strategy starting from (t_i^n, z_j) on the event $\{\tau_n = t_i^n, Z_{t,z}^{\nu,\vartheta \oplus_{\tau_n} \bar{\vartheta}}(\tau_n) \in B_j\}$, where z_j is a given representant of B_j for each $j \in \mathbb{N}$. The end of the proof is then performed by continuity of the map K, with a suitable concatenation of the ε-optimal strategies, and using the relaxed form of $\bar{\Lambda}(\tau)$. Observe that in this proof, the fact that K is deterministic is crucial, since it allows us to avoid the non-trivial issue of $\mathbb{P}$-null sets encountered when constructing the ε-optimal strategy.

As for the second part of the GDP, starting from

$$\left(Z_{t,z}^{\nu[\vartheta],\vartheta}(\tau^\vartheta), M^\vartheta(\tau^\vartheta)\right) \in \mathring{\Lambda}_\iota(\tau^\vartheta)$$

allows to construct a strategy which is ε-optimal when starting in $B_\iota(\tau^\vartheta, Z_{t,z}^{\nu[\vartheta],\vartheta}(\tau^\vartheta), M^\vartheta(\tau^\vartheta))$. Considering then an appropriate sequence of stopping times τ_n^ϑ, good estimates on (Z,M) uniformly in $\vartheta \in \mathcal{V}$ (recall the abstract settings), and continuity of K, the required result is obtained by controlling the probability that

$$\left(\tau_n^\vartheta, Z_{t,z}^{\nu,\vartheta}(\tau_n^\vartheta), M^\vartheta(\tau_n^\vartheta)\right) \in B_\iota\left(\tau^\vartheta, Z_{t,z}^{\nu,\vartheta}(\tau^\vartheta), M^\vartheta(\tau^\vartheta)\right).$$

Remark 1.4.1. We shall point out that, when allowing the controls to depend on information occurring before the beginning of the game, the value functions are defined as combination of essential supremum and essential infimum, see Chapter 4, and are *a priori* random variables. Our results in Chapter 4 however crucially relies on the fact that they are deterministic. To address this issue, one might refer to the method developed in Peng [Peng 97] (see also [Li 09, Theorem A.1]). However, this method introduced for value functions involving only control processes, does not apply very well for value functions involving strategies, since strategies do not have, in general, any continuity property.

In [Buckdahn 08, Proposition 4.1, Lemma 4.1], the authors handle this issue for the stochastic differential games in the framework of Brownian controlled SDEs. In [Buckdahn 10], the authors pointed out that this method does not apply in a more general mixed diffusion framework. They hence provide a new argument for the framework of Stochastic Differential Games driven by Brownian motion and Poisson random measure, see [Buckdahn 10, Lemma 3.1, Lemma 3.2].

1.4.3 Derivation of the Hamilton-Jacobi-Bellman-Isaacs' equation

The game version of the GDP allows to derive the PDEs associated to two given problems introduced in [Soner 02c, Soner 02a], in a Brownian framework, for controls taking their values in bounded subsets of $\mathbb{R}^d$. The first is a game version of the characteristic function of the complement of the reachability set Λ, whereas the second is a game version of the problem (1.2.2). In these two particular cases, the weak version of the GDP seems tailor-made for the derivation of the PDEs in the viscosity sense.

Chapter 4 is concluded with an interesting application. We indeed consider the example of controlling the hedging loss of an investor having sold an European claim of payoff $g(X^{\vartheta}_{t,x}(T)$, provided that he has a utility function Ψ. The adverse control stands here for the realized drift and volatility (μ, ϑ) of the underlying X^{ϑ}. Define $\ell(x,y) := \Psi(y - g(x))$, and for a finite credit line $-\kappa \leq 0$:

$$\mathrm{y}_\kappa(t,x,p) := \inf \left\{ \begin{array}{c} y \geq -\kappa : \;\exists\, \nu \in \mathfrak{U} \text{ s.t. } \; \text{for all } \vartheta \in \mathcal{V} \\ \mathbb{E}\left[\left(X^{\vartheta}_{t,x}(T), Y^{\nu,\vartheta}_{t,x,y}(T) \right) | \mathcal{F}_t \right] \geq p \ \mathbb{P}\text{-a.s.} \end{array} \right\}.$$

Under relatively mild assumptions, this problem ends up in a "relaxed" superhedging price of the claim g.

Namely, we prove that

$$\mathrm{y}_\kappa(t,x,p) = \max\left(\sup_{\vartheta \in \mathcal{V}^0} \mathbb{E}\left[g\left(X^{\vartheta}_{t,x}(T) \right) | \mathcal{F}_t \right] + \Psi^{-1}(p), -\kappa \right), \tag{1.4.3}$$

where $\mathcal{V}^0$ denotes the subset of adverse controls such that $\mu \equiv 0$. This essentially coincides with a degenerate super-replication price of the claim g.

1.5 Hybrid claims : Between Finance and Insurance

During the last years, insurance products being a combination of both insurance risk (e.g. mortality or longevity, or yield of crop) and financial risk (such as the value of a portfolio or the price of a given good) have appeared. These insurance liabilities cannot be priced neither using the usual actuarial principles, nor by no-arbitrage argument only. The interested reader could find a review of the interplay between the two fields in Embrechts [Embrechts 00].

Any insurer is well aware of the definitions of *fair insurance premium* as well as the necessary *loading* (see e.g. [Buhlmann 70], [Gerber 79]) or [Bowers 86]. Motivated by the use of the strong law of large number, a *premium principle* prescribes charging the *fair* or so-called *actuarial* value of a claim G, equal to $\mathbb{E}[G]$, with $\mathbb{E}$ standing for the expectation with respect to the historical measure $\mathbb{P}$, by some *safety loading* $SL(G)$, so that the premium $\pi(G)$ is $\pi(G) := \mathbb{E}[G] + SL(G)$. Some of the most usual principle are

- the *principle of equivalence*, $SL(G) = 0$;
- the *expected value principle*, $SL(G) = a\mathbb{E}[G]$;
- the *standard deviation principle*, $SL(G) = a\sqrt{\mathrm{Var}[G]}$;
- the *variance principle*, $SL(G) = a\mathrm{Var}[G]$.

There are also the *exponential principle*, the *Esscher principle* or the *generalized $(1-\alpha)$-percentile principle*, which states that the premium should be calculated respectively as

- $\pi(G) = \frac{1}{\eta}\log\mathbb{E}\left[e^{\eta G}\right]$;
- $\pi(G) = \frac{\mathbb{E}\left[Ge^{\eta G}\right]}{\mathbb{E}\left[e^{\eta G}\right]}$;
- $\pi(G) = \varepsilon\mathbb{E}[G] + (1-\varepsilon)F^{\leftarrow}(1-\alpha)$,

with $F^{\leftarrow}$ being the generalized inverse of the distribution function of G.

Utility theory enters as a natural (though perhaps academic) tool to provide insight into decision making in the face of uncertainty. Considering the decision maker's preference for various distributions of outcomes, one could use a utility based pricing rule, such as

$$U(x) = \mathbb{E}[U(x + \pi - G)]$$

where x stands for the initial wealth of the insurer, and U is an increasing concave twice differentiable function satisfying the Inada conditions. By Jensen's inequality and the concavity of U,

$$\pi(G) \geq \mathbb{E}[G],$$

and in the case where U is linear, we have the equality.

Let us step now from the insurance to the financial framework. As in the insurance framework, we might use the actuarial premium principle $\mathbb{E}[G]$ in order to price a claim G. However, in the finance context, the whole argument against using $\mathbb{E}[G]$ as a premium is based on the notion of no arbitrage (see [Delbaen 94]). In a complete financial market, the problem of pricing and hedging a given contingent claim has a clear and unique solution (see e.g. [Ansel 92, Delbaen 94, Harrison 81]): the prices of a claim is the expectation of its discounted payoff under the martingale measure, which is unique. A risk neutral probability measure $\mathbb{Q}$ changes the original measure $\mathbb{P}$, and from an insurance point of view, it could be seen as a way to give more weight to unfavorable events in a risk averse environment. In insurance mathematics, it should explain the safety loading.

Examples of *hybrid products* such as unit-linked life insurance contracts, catastrophe insurance futures and bonds, integrated risk-management solution or even agricultural revenue coverage introduced these last few years in both life or non-life insurance justifies the interest of both insurance and financial mathematics. Let the stochastic process $(S_t)_{t\geq 0}$, defined on a given filtered probability space, denote the value at time t of a stock. Typically, the agents are interested in pricing claims of the following form

$$G_n := \sum_{i=1}^{N} f\left(S, G^i\right), \tag{1.5.1}$$

where for $i \in \{1, \cdots, N\}$, the G^i are independent and identically distributed random variables, N denotes the number of unit claims $f(S, G^i)$ sold, and f being some measurable function. In 1.5.1, one could think e.g. of unit-linked contract,

$$f(S, G^i) = \mathbf{1}_{\{G^i > T\}} S_T,$$

unit-linked with guarantee,

$$f(S, G^i) = \mathbf{1}_{\{G^i > T\}} \max(S_T, K),$$

guaranteed annual return

$$f(S, G^i) = K \sum_{j=1}^{T} \mathbf{1}_{\{G^i > j\}} \max\left(1 + \frac{S_j - S_{j-1}}{S_{j-1}}, 1 + \delta_j\right),$$

with

$$\frac{S_j - S_{j-1}}{S_{j-1}}$$

being the return in year j on the asset S, and δ_j the guaranteed return in year j, provided that the customer i is still alive, or even a revenue guarantee

$$f(S, G^i) := \left(\max(S_T, K_S) K_G - S_T G^i\right)^+,$$

which is a guarantee that the customer i would have the expected revenue from his expected production K_G sold at the better price $\max(S_T, K_S)$.

Such contracts, and especially unit-linked contracts have been studied by actuaries since the late sixties, and Brennan and Schwartz [Brennan 79a, Brennan 79b] proposed valuation principles consisting in combining law of large number with financial valuation. They first replaced the insured risk by their expected value, so that the modified claim only contains financial uncertainty. Namely, it remains to the insurer to price and hedge the following modified claim

$$G_n := \sum_{i=1}^{N} \mathbb{E}\left[f\left(S, G^1\right) \middle| \mathcal{F}_T^S \right], \tag{1.5.2}$$

where $\mathcal{F}^S$ denotes the filtration of the market. In most cases, the hedging strategy of this kind of claims only consists in buy and hold strategy which consist in buying a number of shares of stocks or liquid options depending on the structure of the payoff f. This strategy has been widely used in practice, see e.g. [Boyle 03, Milevsky 00, Milevsky 06].

We are interested in the sequel in the pricing of such claims, and the establishments of sufficient conditions for this *trivial pricing rule* to fail, or to hold.

1.6 Utility indifference Pricing

1.6.1 Introduction

As introduced in the very beginning of this manuscript, in the incomplete markets case, the agents have to define their attitude toward risk.

We shall focus in this Section on the framework where the preference of the agents are described by a concave utility function U. As discussed in Section 1.1.3, the stochastic target problem in controlled expected loss allows to treat this situation in some particular settings. However, when using more complex probabilistic models of financial assets such as non-Markovian diffusion or semimartingale models, direct methods from stochastic optimal control discussed previously become increasingly difficult to handle.

1.6.2 Maximizing utility of terminal wealth

Consider an agent whose goal is to trade dynamically in a financial market up to horizon T, in order to achieve maximum expected utility.

The preferences of the investor are represented by a Von Neumann-Morgenstern utility function $U : \mathbb{R} \to [-\infty, \infty)$ which must be not identical to $-\infty$, increasing and concave. No consumption occurs before time T. The agent has the initial endowment x and can invest in the financial market. The resulting optimization problem is

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}\left[U(X_T)\right] \tag{1.6.1}$$

where $\mathfrak{X}(x)$ is the set of random wealths which can be obtained at time T with initial wealth x. The formulation of the problem with random endowment, namely when the agent receives at T an additional cashflow G is the following

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}\left[U(X_T + G)\right]. \tag{1.6.2}$$

The study of 1.6.1 requires a specification of

1. the financial market model and the admissible terminal wealths
2. the technical assumption on U
3. some joint condition on the market model and the utility function.

1.6.2.1 Financial market model

The financial market model considered is frictionless and consists of N risky assets globally indicated with $S := (S^1, \cdots, S^N)$, and one risk free asset (money market account) assumed equal to 1 (i.e. the prices are discounted). The trading can occur continuously in $[0,T]$, S is in fact a $\mathbb{R}^N$-valued, continuous time process, defined on z filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$. Since the wealth from an investment in this market is a (stochastic) integral, S is assumed to be a semimartingale, so that the object "integral with respect to S" is mathematically well defined. For expository reasons, S is a *locally bounded* semimartingale. This class of models is already very general, as all the diffusions are locally bounded semimartingales, as well as any jump-diffusion processes with bounded jumps.

There are no restrictions on the quantities the agent can buy, sell or sell short. H_t is the random vector with the number of shares the agent holds in the infinitesimal interval $[t, t+dt]$. To be technically precise, H must be a predictable process. As there is no consumption and no infusion of money in the trading period, the wealth from the strategy H is the process X that solves

$$X_t = x + \int_0^t H_u \cdot dS_u.$$

As usual in continuous time trading, to avoid phenomena like doubling strategies, not every self-financing H is allowed. A self-financing strategy H is said *admissible* only if during the trading the losses don't exceed a finite credit line, i.e. H is admissible if there exists some constant $\kappa > 0$ such that

$$\text{for all } t \in [0,T], \quad \int_0^t H_u \cdot dS_u \ge -\kappa \ \mathbb{P}\text{-a.s.}, \tag{1.6.3}$$

so that for any x, the wealth process is also bounded from below. Maximizing expected utility from terminal wealth means in fact *maximizing expected utility from the set $\mathfrak{X}(x)$ of those random variables X_T that can be represented as $X_T = x + \int_0^T H_t \cdot dS_t$ with H admissible in the sense of 1.6.3.*

As shown by [Delbaen 94], a financially relevant set of probabilities is $\mathcal{M}^e$. Under each probability $\mathbb{Q} \in \mathcal{M}^e, S$ is a (local) martingale, and thus $\mathbb{Q}$ is a risk-neutral probability. This is the theoretical justification of the use os each of these $\mathbb{Q}$s as a pricing measure for any derivative claim B.

But we need the less restrictive set $\mathcal{M}^a$ of the absolutely continuous (local) martingale probabilities $\mathbb{Q}$ for S as this is the set which will show up in the dual problem. The set $\mathcal{M}^a$ can be characterized in the following way

$$\mathcal{M}^a := \left\{ \mathbb{Q} \ll \mathbb{P} : \ \mathbb{E}^{\mathbb{Q}} \left[\int_0^T H_u \cdot dS_u \right] \leq 0 \text{ for all admissible } H \right\}. \tag{1.6.4}$$

Therefore, given any $X_T \in \mathfrak{X}(x)$ and any $\mathbb{Q} \in \mathcal{M}^a$,

$$\mathbb{E}^{\mathbb{Q}}[X_T] = \mathbb{E}^{\mathbb{Q}} \left[x + \int_0^T H_u \cdot dS_u \right] \leq x.$$

1.6.2.2 Hypothesis on U

Regarding U, it is here required that:

- U is strictly concave, strictly increasing and differentiable over $\mathbb{R}$
- U satisfies the Inada conditions.

Also, U must satisfy the *reasonable asymptotic elasticity condition* introduced in [Kramkov 99] and [Schachermayer 01]. In the cited references, it is also shown that this condition is necessary and sufficient for the duality to work properly if U is fixed and one considers all possible financial markets.

1.6.2.3 The joint condition

The convex conjugate V of U is the function

$$V(y) = \sup_x U(x) - xy,$$

and, apart from some minus signs, it coincids with the Fenchel conjugate of U. So V is a convex function, which is identically equal to $+\infty$ when $y < 0$. It is also differentiable on $(0, \infty)$ and its derivative is $-I = (U')^{-1}$. Note that

$$U(x) = \inf_{y>0} xy + V(y). \tag{1.6.5}$$

Let us recall that a probability $\mathbb{Q}$ absolutely continuous w.r.t. $\mathbb{P}$ is said to have finite entropy (or, also, finite divergence) if its density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is integrable when composed with V

$$\mathbb{E}\left[V\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] < \infty.$$

The joint condition required between preferences and the market is actually a condition between V and the set of probabilities $\mathcal{M}^a$, that is

$$\exists\ \mathbb{Q}^0 \in \mathcal{M}^a \text{ s.t. } \mathbb{E}\left[V\left(\frac{d\mathbb{Q}^0}{d\mathbb{P}} \right) \right] < \infty. \tag{1.6.6}$$

1.6.3 Duality in incomplete market models

Suppose the market is arbitrage free. Let us recall the *primal* problem (1.6.1):

$$u(x) := \sup_{X \in \mathfrak{X}(x)} \mathbb{E}\left[U(X_T)\right],$$

where $u(x)$ denotes the optimal level of the expected utility.

It has been stated in [Kramkov 99] in case of utility function defined on $\mathbb{R}_+$ (extended in [Schachermayer 01] in the case of utility functions defined on the whole real line, and in [Owen 02] for the case where the terminal wealth is affected by a random endowment) that, if the utility function has *reasonable asymptotic elasticity*, then the primal and the dual problem coincide. Namely,

$$u(x) = \inf_{y>0} \{v(y) + y(x + G)\},$$

whith

$$v(y) := \begin{cases} \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y\dfrac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \text{if } y > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

1.6.4 Utility Asymptotics - Pricing of hybrid claims

In Chapter 6, we investigate the problem of the price of a claim combining both insurancial risks and pure financial risks as introduced in Section 1.5. In principle, these liabilities cannot be priced neither by applying the usual actuarial principles of diversification, nor by arbitrage-free replication arguments. Still, it has been often proposed in the literature (and widely used in practice) to combine these two approaches by suggesting to hedge a pure financial payoff computed by taking the mean under the historical/objective probability on the part of the risk that can be diversified.

Consider an insurance company selling to the client i a claim with discounted payoff g^i, paid at maturity T, whose value depends on the evolution of some tradable financial assets $S = (S_t)_{t\geq 0}$ and some additional idiosyncratic risk. The g^i's are usually not unconditionally independent, but still independent conditionally to S. In such a situation, and if the financial market formed by the assets S is complete, it is tempting to play on the ability to diversify the conditionally idiosyncratic risks and cover the systemic pure financial risk by dynamically trading on the market. If the g^i's are independent and identically distributed given S, then the price of each of these contingent claims could be defined as $\bar{p} := \mathbb{E}^{\mathbb{Q}}\left[\bar{g}(S)\right]$ where $\bar{g}(S) := \mathbb{E}\left[g^i | S\right]$ does not depend on i, and $\mathbb{Q}$ denotes the unique martingale measure on the pure financial market (i.e. restricted to S). The rationality behind this is the following: by an informal application of the law of large numbers conditionally to S, we obtain the convergence $G_n/n := \sum_{i=1}^n g^i/n \to \bar{g}(S)$ a.s. for a large number n of sold contracts. In the above, the payoff $\bar{g}(S)$ only depends on S and can thus be hedged dynamically by trading on the (complete) pure financial market. Hence, by replicating the mean payoff $\bar{g}(S)$, we end up with a zero net position in mean (under the initial probability measure $\mathbb{P}$).

This solution seems to ignore the fact that playing with the law of large numbers on the diversifiable part of the risk requires selling a large number of contracts, and therefore may lead to huge positions on the financial market. If the law of large numbers does not operate well enough, then the losses may be leveraged by an unfavorable evolution of the financial market.

On some complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with a locally bounded càdlàg semi-martingale S, we denote as usual the set $\mathcal{M}$ of $\mathbb{P}$-equivalent local martingale measures such that S is a $(\mathbb{F}, \mathbb{Q})$-local martingale. We are interested in this paper in the *pure incomplete market* case, but intend to study the so-called *half-complete market* defined as follows, for some fixed $\mathbb{Q}^* \in \mathcal{M}$, and with $\mathbb{F}^S$ the filtration generated by S (we do not impose that $\mathcal{F}_T^S = \mathcal{F}$).

Definition 1.6.1. *We say that the pure financial market is complete, in short* **(HCM)** *holds, if*

$$\mathbb{E}^{\mathbb{Q}^*}[\xi] = \mathbb{E}^{\mathbb{Q}}[\xi] \text{ for all } \mathbb{Q} \in \mathcal{M} \text{ and } \xi \in L^\infty(\mathcal{F}_T^S),$$

where $L^\infty(\mathcal{F}_T^S)$ denotes the set of essentially bounded $\mathcal{F}_T^S$-measurable random variables.

We first give simple counterexamples where the trivial pricing rule defined as above does not apply. Consider for example an aggregated claim

$$G_n := \sum_{i=1}^{n} g^i, \quad n \geq 1,$$

where the g^i's have the same law and are independent conditionally to $\mathcal{F}_T^S$ under $\mathbb{P}$, and **(HCM)** holds. Then the trivial pricing rule does not apply neither for an utility function with bounded from below domain, or for the exponential utility. In the latter, the trivial pricing rule under **(HCM)** has been established in [Becherer 03] for a sequence of exponential utility functions with vanishing risk aversion. These two counterexamples lead us to consider a sequence of utility functions defined on the whole real line, with absolute risk aversion converging to 0 at infinity.

We are hence interested in establishing conditions for this pricing rule to hold. Consider a sequence $(U_n)_{n\in\mathbb{N}}$ of utility function defined on the whole real line and satisfying the usual assumptions (Inada, reasonable asymptotic elasticity, see [Schachermayer 01]). Assume furthermore that $\mathcal{M} \neq \emptyset$ and that for each $n \in \mathbb{N}$, the corresponding dual problem (see e.g. [Schachermayer 01]) is finite. The unit utility indifference prices $p_n(G_n, U_n)$ given by

$$p_n(G_n, U) := \inf\{p \in \mathbb{R} : \sup_X \mathbb{E}\left[U(X + np - G_n)\right] \geq \sup_X \mathbb{E}\left[U(X)\right]\},$$

are well defined for any $n \geq 1$ and existence for the optimal *dual probability and multiplier*, given by

$$(y_n^0, \mathbb{Q}_n^0) := \arg\min \left\{ \mathbb{E}\left[V_n \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right],\ (y, \mathbb{Q}) \in (0, \infty) \times \mathcal{M} \right\},$$

is guaranteed. In the above, for each n, the function V_n is the usual convex conjugate of U_n. Under the additional assumptions

$$\sup_{n \geq 1} |G_n/n|_{L^\infty} < \infty,$$

and

$$n|r_n|_\infty \underset{n\to\infty}{\longrightarrow} 0\,, \qquad \text{with} \quad r_n : x \mapsto -\frac{U_n''(x)}{U_n'(x)}\,,$$

with $|r_n|_\infty := \sup_{x\in\mathbb{R}} |r_n(x)|$, we state that,

$$\lim_{n\to\infty} p_n(G_n, U_n) = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right],$$

where the limit has to be understood as liminf and limsup. As a byproduct, under the weaker condition $\|r_n\|_\infty \to 0$, and whenever the sequence $(G_n)_{n\geq 1}$ is assumed to be uniformly bounded in L^∞, this provides a general convergence result for bounded sequences of contingent claims when the absolute risk aversion vanishes in the sup norm, which is of own interest.

Focus now on $\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right]$, and the sequence of optimizers $(\mathbb{Q}_n^0)_{n\in\mathbb{N}}$. Under **(HCM)**, and if $G_n/n \to \bar{g}$ as $n \to \infty$, we prove that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right] = \mathbb{E}^{\mathbb{Q}^*}[\bar{g}].$$

In order to deal with the general incomplete market case, we introduce the following Assumption.

Assumption 1.6.1. *There exist two sequences of strictly positive numbers* $\left(\eta_n^1\right)_{n\geq 1}$ *and* $\left(\eta_n^2\right)_{n\geq 1}$ *converging toward* 0 *such that*

$$0 < \eta_n^2 \leq r_n(x) \leq \eta_n^1 \text{ for all } x \in \mathbb{R} \text{ and } n \geq 1,$$
$$\lim_{n\to\infty} \eta_n^2/\eta_n^1 = 1.$$

With Q^e the element of $\mathcal{M}$ that minimizes the relative entropy $E[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}]$, and under Assumption 1.6.1, we proved that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n^0}\left[G_n/n\right] = \mathbb{E}^{\mathbb{Q}^e}[\bar{g}],$$

Assumption 1.6.1 may seem to be quite strong, since it basically says that the sequence of utility functions behave asymptotically as an exponential type utility function. It however gives a good insight on the asymptotical behavior of an utility function satisfying $r_n \to 0$ as $n \to \infty$ uniformly in x.

1.7 Organization of this manuscript

Part I

Stochastic target in finance and insurance

Chapter 2

Stochastic Target With Controlled Loss in Jump Diffusion Models

Contents

The work presented in this chapter is taken from [Moreau 11], and has been accepted for publication in **SIAM**, *Journal on Control and Optimization*.

2.1 Introduction

We are interested in this chapter in the stochastic target problem with expected loss discussed in Sections 1.1.2, 1.1.3, and 1.2.2. We focus now on the framework introduced in Section 1.2.3.

For $0 \le t \le T$, and given two controlled diffusion processes $\{X^\nu_{t,x}(s), t \le s \le T\}$ and $\{Y^\nu_{t,x,y}(s), t \le s \le T\}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}$, satisfying the initial condition $\big(X^\nu_{t,x}(t), Y^\nu_{t,x,y}(t)\big) = (x, y)$. We are interested in finding the minimal initial condition y for which it is possible to find a control ν satisfying $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p$ for some given Borel measurable map Ψ, non-decreasing in the y-variable, and for a threshold p. Namely, we want to characterize the value function:

$$\hat{v}(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu\right\}, \qquad (2.1.1)$$

in the mixed diffusion case. If $\Psi(x,y) := \mathbb{1}_{\{y-g(x)\geq 0\}}$ and $p \in (0,1)$,

$$\widehat{v}(t,x,p) = \inf\left\{y \geq -\kappa : \mathbb{P}\left[V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0\right] \geq p \text{ for some } \nu\right\}, \quad (2.1.2)$$

this problem coincides with the quantile hedging problem discussed in [Föllmer 99], in the context of financial mathematics. In this paper, the process X represents the prices of some given securities. The process Y models the wealth of an investor, based on some portfolio strategy ν. Importantly, the coefficients of the diffusion Y are linear in the control variable and the process X is not affected by the control ν. In this context, Föllmer and Leukert [Föllmer 99] used a duality argument to convert this problem into a classical test problem in mathematical statistics.

In order to deal with the problem (2.1.2) in a more general case, Bouchard, Elie and Touzi [Bouchard 09] introduced an additional controlled diffusion process $P^{\alpha}_{t,p}$, which appears to (essentially) correspond to the conditional probability of reaching the target $V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0$. This allowed them to rewrite the problem 2.1.2 in the form

$$\hat{v}(t,x,p) = \inf\left\{y \geq -\kappa : \mathbf{1}_{\{V(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T))\geq 0\}} \geq P^{\alpha}_{t,p}(T) \text{ for some } (\nu,\alpha)\right\},$$

where α is a predictable square integrable process coming from the martingale representation of $\mathbb{P}\left[V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \mid \mathcal{F}_{\cdot}\right] = P^{\alpha}_{t,p_0} := p_0 + \int_t^{\cdot} \alpha_s \cdot dW_s$, for some $p_0 \geq p$. The key point is that this reformulation reduces the original problem 2.1.2 into a classical stochastic target problem of the form

$$\hat{v}(t,x,p) := \inf\left\{y \geq -\kappa : \hat{V}\left(X^{\nu}_{t,x}(T), P^{\alpha}_{t,p}(T), Y^{\nu}_{tx,y,}(T)\right) \geq 0 \text{ for some } \nu,\alpha\right\},$$

as studied in [Soner 02a, Soner 02c], for an augmented system (X,Y,P) and an augmented control (ν,α). The major difference being that the new control α can no longer be assumed to take values in a compact set, as it is given by the martingale representation theorem.

Up to a non-trivial relaxation, Bouchard, Elie and Touzi [Bouchard 09] were able to provide a PDE characterization for the value function $\hat{v}$ in the sense of discontinuous viscosity solutions, for a discontinuous operator which corresponds to the one used in [Soner 02a] and [Soner 02c].

The aim of this chapter is to extend the work of Bouchard, Elie and Touzi [Bouchard 09] to the setting of jump diffusions, in its more general form 2.1.1.

Diffusing the conditional expectation $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\middle| \mathcal{F}_s\right]$ for $s \in [t,T]$, and considering it as an additional controlled state variable $P^{\alpha,\chi}_{t,p}$ will allow us to convert this problem into a singular stochastic target problem. Here, the additional control χ comes from the jump part of the martingale representation.

This leads to technical difficulties, mainly because of this new control χ. The first one was already handled in [Bouchard 02], and consists in the consideration of an additional (non-local) term in the PDE characterization. Secondly, part of the control now takes values in an unbounded set of measurable maps, as opposed to a compact subset of $\mathbb{R}^d$. The local relaxation of the associated HJB operator introduced in [Bouchard 09] will not be sufficient to ensure the semicontinuity needed, and we shall have to introduce a new (non-trivial) relaxation of the non-local part of the associated operator. Furthermore, this non-local operator complicates significantly the discussion of the boundary conditions at $p = m$ and $p = M$ when the map Ψ takes values in $[m, M]$.

Compared to [Bouchard 09], where they discuss general problem of the form (2.1.1), but state their results for the problem (2.1.2), we aim to state our results for the the problem (2.1.1). In particular, we shall see that the convex face-lifting phenomenon in the p-variable observed in [Bouchard 09] for (2.1.2) extends to a much more general context.

This chapter is organized as follows. In Section 2.2, we present the general formulation of stochastic target problem with unbounded measurable map controls, in mixed diffusion case. It contains the statement of the corresponding dynamic programming equation. In Section 2.3, we give the arguments allowing us to translate the problem of expected controlled loss into the case of singular stochastic target problem of the previous section. The boundary conditions for the stochastic target problem with controlled expected loss.

In all this paper, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript ^{T} stands for transposition, $\cdot$ denotes the scalar product on $\mathbb{R}^n$, $|\cdot|$ the Euclidean norm, and $\mathbb{M}^n$ denotes the set of n-dimensional square matrices. We denote by $\mathbb{S}^n$ the subset of elements of $\mathbb{M}^n$ which are symmetric. For a subset $\mathcal{O}$ of $\mathbb{R}^n$, $n \geq 1$, we denote by $\overline{\mathcal{O}}$ its closure, by $\mathrm{Int}(\mathcal{O})$ its interior and by $\mathrm{dist}(x, \mathcal{O})$ the Euclidean distance from x to $\mathcal{O}$ with the convention $\mathrm{dist}(x, \emptyset) = \infty$. Finally, we denote by $B_r(x)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. Given a locally bounded map v on a subset B of $\mathbb{R}^n$, we define the lower and upper semicontinuous envelopes

$$v_*(b) := \liminf_{B \ni b' \to b} v(b') \qquad v^*(b) := \limsup_{B \ni b' \to b} v(b'), b \in \overline{B}.$$

The convex hull of a function f will be denoted $\circledcirc(f)$, and we recall that it is the greatest convex function lower or equal to f. We will use the same notation for the convex hull of a subset, i.e. $\circledcirc(A)$ is the convex hull of the subset A, and we recall that it is the smallest convex subset containing A, in the sense of inclusion.
In this paper, inequalities between random variable have to be understood in the a.s. sense.

2.2 Singular stochastic target problems

2.2.1 Problem formulation

Let $T > 0$ be a fixed time, E a borel subset of $\mathbb{R}_+$, equipped with its Borel σ-field $\mathcal{E}$, $J(de, dt) = \sum_{i=1}^{d} J^i(de, dt)$ be a E-marked right-continuous point process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let W be a $\mathbb{R}^d$-Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that W and J are independent. We denote by $\mathbb{G} := \{\mathcal{G}_t, 0 \leq t \leq T\}$ the $\mathbb{P}$-completed filtration generated by $(W_., J(de, \cdot))$. We assume that $\mathcal{G}_0$ is trivial. The random measure $J(de, dt)$ is assumed to have a predictable $(\mathbb{P}, \mathbb{G})-$intensity kernel $\lambda(de)dt$ such that $\lambda(E) < \infty$, and we denote by $\tilde{J}(de, dt) := J(de, dt) - \lambda(de)dt$ the associated compensated random measure. By $\mathbb{H}^2_\lambda$, we denote the set of maps $\chi : \Omega \times [0, T] \times E \to \mathbb{R}$ which are $\mathcal{P} \bigotimes \mathcal{E}$ measurable [1] and such that

augmentée plutot

$$\|\chi\|_{\mathbb{H}^2_\lambda} := \left(\mathbb{E} \left[\int_0^T \int_E \chi_t(e)^2 \lambda(de)dt \right] \right)^{\frac{1}{2}} < \infty.$$

We can always assume that $\mathbb{P}[J(E \setminus \text{supp}(\lambda), [0, T]) > 0] = 0$, and therefore that $E = \text{supp}(\lambda)$. Let $\mathcal{U}_0 = \mathcal{U}_0^1 \times \mathcal{U}_0^2$ be the collection of predictable processes $\nu = (\nu^1, \nu^2)$ with $\nu^1 \in L^2([0, T])$ and $\nu^2 \in \mathbb{H}^2_\lambda$ $\mathbb{P}$-a.s., and with values in a given closed subset $U = U^1 \times \mathbb{L}^2_\lambda$ of $\mathbb{R}^d \times \mathbb{L}^2_\lambda$. Here $\mathbb{L}^2_\lambda$ denotes the set of measurable functions $\pi : E \to \mathbb{R}$ such that $\|\pi\|^2_\lambda < \infty$, with

$$\|\pi\|^2_\lambda := \int_E |\pi(e)|^2 \lambda(de).$$

For $t \in [0, T]$, $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$ and $\nu := (\nu^1, \nu^2) \in \mathcal{U}_0$, we define $Z^\nu_{t,z} := (X^\nu_{t,x}, Y^\nu_{t,x,y})$ as the $\mathbb{R}^d \times \mathbb{R}$-valued solution of the stochastic differential equation

$$\begin{aligned} X(s) =& \mu_X(X(s), \nu_s) ds + \sigma_X(X(s), \nu_s) dW_s \\ &+ \int_E \beta_X\left(X(s-), \nu^1_s, \nu^2_s(e), e\right) J(de, ds) \\ dY(s) =& \mu_Y(Z(s), \nu_s) ds + \sigma_Y(Z(s), \nu_s) dW_s \\ &+ \int_E \beta_Y\left(Z(s-), \nu^1_s, \nu^2_s(e), e\right) J(de, ds) \end{aligned} \tag{2.2.1}$$

satisfying the initial condition $Z(t) = (x, y)$. Here,

$$\begin{aligned} (\mu_X, \sigma_X) &: \mathbb{R}^d \times U \to \mathbb{R}^d \times \mathbb{M}^d \\ (\mu_Y, \sigma_Y) &: \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R} \times \mathbb{R}^d \end{aligned}$$

are locally Lipschitz, and are assumed to satisfy, for $u := (u^1, u^2) \in U$,

$$|\mu_Y(x, y, u)| + |\mu_X(x, u)| + |\sigma_Y(x, y, u)| + |\sigma_X(x, u)| \leq K(x, y)\left(1 + |u^1| + \left\|u^2\right\|_\lambda\right)$$

[1] $\mathcal{P}$ denotes the σ-algebra of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$.

where K is a locally bounded map. Moreover

$$\beta_X : \mathbb{R}^d \times U \times E \to \mathbb{R}^d$$
$$\beta_Y : \mathbb{R}^d \times \mathbb{R} \times U \times E \to \mathbb{R}$$

are continuous and are assumed to satisfy, for some $M \geq 0$,

$$\begin{aligned} \int_E \left(|\beta_X(x, u(e), e)|^2 + |\beta_Y(z, u(e), e)|^2\right) \lambda(de) \leq M \left(1 + |z|^2 + |u|^2\right) \\ \int_E |\beta_X(x, u(e), e) - \beta_X(x', u(e), e)|^2 \lambda(de) \leq M \left|x - x'\right|^2 \\ \int_E |\beta_Y(z, u(e), e) - \beta_Y(z', u(e), e)|^2 \lambda(de) \leq M \left|z - z'\right|^2, \end{aligned} \tag{2.2.2}$$

where we have used the notation $u(e) = (u^1, u^2(e))$ and $|u|^2 := |u^1|^2 + \left\|u^2\right\|_\lambda^2$. We denote by $\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2$ a subset of elements of $\mathcal{U}_0$ for which (2.2.1) admits an unique strong solution for all given initial data. We assume furthermore that any constant controls with values in U belongs to $\mathcal{U}$. We also allow for state constraints and we denote by $\mathbf{X}$ the interior of the support of the controlled process X.

Let V be a measurable map from $\mathbb{R}^{d+1}$ to $\mathbb{R}$ such that, for every fixed x, the function

$$y \longmapsto V(x, y) \text{ is non-decreasing and right continuous.}$$

We then define the stochastic target problem as follows

$$v(t, x) := \inf \left\{y \geq -\kappa : V\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\}, \tag{2.2.3}$$

with $\kappa \in \mathbb{R}_+ \cup \{+\infty\}$. At this point, the set U may not be bounded, and we will see later that dealing with unbounded controls will be required in the analysis of Section 2.3.

In order to be consistent and avoid the process Y to cross the level $-\kappa$, we shall assume all over this paper that

$$\begin{gathered} \mu_Y(x, -\kappa, u) \geq 0, \quad \sigma_Y(x, -\kappa, u) = 0 \quad \text{and} \quad \beta_Y(x, y, u, e) \geq -(y + \kappa) \\ \text{for all } (x, y, u, e) \in \mathbf{X} \times \mathbb{R} \times U \times E. \end{gathered} \tag{2.2.4}$$

As usual in this kind of problem, our analysis requires that

$$y' \geq y \text{ and } y \in \Gamma(t, x) \Rightarrow y' \in \Gamma(t, x) \quad \text{for all} \quad (t, x, y, y') \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$$

where

$$\Gamma(t, x) := \left\{y \geq -\kappa : V\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\}.$$

This allows to characterize the closure of $\Gamma(t, x)$ as $[v(t, x), +\infty)$, which will be of important use in the following. Indeed, let us assume that the infimum in the definition of v is attained, and let $y = v(t,x)$. Then we can find some $\nu \in \mathcal{U}$ such that $V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}\right) \geq 0$. Hence, if we start with $y' > y$, we should be able to find some $\nu' \in \mathcal{U}$ such that $V\left(X^{\nu'}_{t,x}(T), Y^{\nu'}_{t,x,y'}\right) \geq 0$. If this property does not hold (which can be the case in a jump diffusion model), it is not possible to characterize the set $\Gamma(t,x)$ by its lower bound $v(t,x)$.

Remark 2.2.1. Let us observe that this problem can be formulated equivalently as

$$v(t,x) := \inf \left\{ y \geq -\kappa : Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right) \text{ for some } \nu \in \mathcal{U} \right\},$$

where g is the generalized inverse of V at 0:

$$g(x) := \inf \left\{ y \geq -\kappa : V(x,y) \geq 0 \right\}, \tag{2.2.5}$$

recall (2.2.4).

Example 2.2.1. Consider the case where $\mathbf{X} = (0,\infty)^d$ and X is defined by the stochastic differential equation

$$dX_{t,x}(s) = \mu\left(X_{t,x}(s)\right) ds + \sigma\left(X_{t,x}(s)\right) dW_s + \int_E \beta\left(X_{t,x}(s-), e\right) J(de, ds)$$
$$X_{t,x}(t) = x \in (0,\infty)^d,$$

with $Y^{\nu}_{t,x,y}$ given by

$$Y^{\nu}_{t,x,y}(s) = y + \int_t^s \nu^1_r \cdot dX_{t,x}(r), \quad \text{for} \quad s \geq t \quad \text{and} \quad \nu = \left(\nu^1, \nu^2\right) \in \mathcal{U}.$$

This corresponds to the situation where the process X is not affected by the control:

$$\begin{matrix} \mu_X(x,u) = \mu(x), \quad \sigma_X(x,u) = \sigma(x) \\ \text{and} \quad \beta_X(x,u(e),e) = \beta(x,e) \end{matrix} \qquad \text{are independent of } u$$

and

$$\mu_Y(x,y,u) := u^1 \cdot \mu(x), \quad \sigma_Y(x,y,u) := \sigma^T(x) u^1, \quad \beta_Y(x,y,u(e),e) := u^1 \cdot \beta(x,e).$$

In financial mathematics, the process X should be interpreted as the price of d risky securities. Because of the jump diffusions, we are in an incomplete market, so that the uniqueness of a $\mathbb{P}$-equivalent martingale measure is not satisfied. The process Y represents the wealth process induced by the trading strategy ν, where ν^1_s indicates the number of units of the assets in the portfolio at time s.

Finally, for some Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}_+$ and

$$V(x,y) := y - g(x),$$

$v(t,x)$ coincides with the usual superhedging price of the contingent claim $g\left(X_{t,x}(T)\right)$.

2.2.2 Main results

The main result of this section is the derivation of the dynamic programming equation corresponding to the stochastic target problem (2.2.3), in the present context of possibly unbounded controls and jumps.

Before stating our main results, we need to introduce additional notations. Given a smooth function φ, $u \in U$ and $e \in E$, we now define the operators

$$\mathcal{L}^u\varphi(t,x) := \partial_t\varphi(t,x) + \mu_X(x,u)\cdot D\varphi(t,x) + \frac{1}{2}\mathrm{Trace}\left(\sigma_X\sigma_X^T(x,u)D^2\varphi(t,x)\right)$$
$$\mathcal{G}^{u,e}\varphi(t,x) := \beta_Y\left(x,\varphi(t,x),u(e),e\right) - \varphi\left(t,x+\beta_X(x,u(e),e)\right) + \varphi(t,x),$$

where $\partial_t\varphi$ stands for the partial derivative with respect to t, $D\varphi$ and $D^2\varphi$ denote the gradient vector and the Hessian matrix with respect to the x variable. We then define the following relaxed semi-limits

$$\begin{aligned} H^*(\Theta,\varphi) &:= \limsup_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta \\ \eta\to 0,\psi\underset{u.}{\longrightarrow}\varphi}} H_{\varepsilon,\eta}\left(\Theta',\psi\right) \\ H_*(\Theta,\varphi) &:= \liminf_{\substack{\varepsilon\searrow 0,\Theta'\to\Theta \\ \eta\to 0,\psi\underset{u.}{\longrightarrow}\varphi}} H_{\varepsilon,\eta}\left(\Theta',\psi\right), \end{aligned} \tag{2.2.6}$$

where, for $\Theta = (t,x,y,k,q,q',A) \in \mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^d, \psi \in \mathcal{C}^{1,2}\left([0,T]\times\mathbb{R}^d\right)$, $\varepsilon\geq 0$ and $\eta\in[-1,1]$,

$$H_{\varepsilon,\eta}(\Theta,\psi) := \sup_{u\in\mathcal{N}_{\varepsilon,\eta}(t,x,y,q',\psi)} \mathbf{A}^u(\Theta),$$

with

$$\mathbf{A}^u(\Theta) := \mu_Y(x,y,u) - k - \mu_X(x,u)\cdot q - \frac{1}{2}\mathrm{Trace}\left[\sigma_X\sigma_X^T(x,u)A\right],$$
$$\mathcal{N}_{\varepsilon,\eta}(t,x,y,q',\psi) := \left\{\begin{array}{c} u\in U \text{ s.t. } |N^u(x,y,q')|\leq\varepsilon \text{ and} \\ \Delta^{u,e}(t,x,y,\psi)\geq\eta \quad \text{for } \lambda\text{-a.e. } e\in E \end{array}\right\},$$
$$N^u(x,y,q') := \sigma_Y(x,y,u) - \sigma_X(x,u)^Tq',$$
$$\Delta^{u,e}(t,x,y,\psi) := \beta_Y(x,y,u(e),e) - \psi\left(t,x+\beta_X(x,u(e),e)\right) + y$$

and the convergence $\psi \underset{u.}{\longrightarrow} \varphi$ in (2.2.6) has to be understood in the sense that ψ converges uniformly towards φ.

Also notice that, given $\eta\in[-1,1]$, $\left(\mathcal{N}_{\varepsilon,\eta}\right)_{\varepsilon\geq 0}$ is non-decreasing in ε so that

$$H_*(\Theta,\varphi) := \liminf_{\substack{\eta\to 0,\Theta'\to\Theta \\ \psi\underset{u.}{\longrightarrow}\varphi}} H_{0,\eta}\left(\Theta',\psi\right).$$

For ease of notations, we shall often simply write $Hv(t,x)$ in place of $H(t,\, x,\, v(t,x)$, $\partial_t v(t,x)$, $Dv(t,x)$, $Dv(t,x)$, $D^2v(t,x)$, $v)$. We shall similarly use the notations H^*v

and H_*v.

In order to handle the possible unboundedness of the jumps in Section 2.2.3.1, we shall need the following definition of viscosity super solution.

Definition 2.2.2. *We say that a l.s.c. (resp. u.s.c.) function U (resp. V) is a viscosity supersolution of $H^*U \geq 0$ (resp. subsolution of $H_*V \leq 0$) on $[0,T) \times \mathbb{R}^d$ if for every smooth function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ of linear growth and $(t_o, x_o) \in [0,T) \times \mathbb{R}^d$ such that $\min_{[0,T]\times\mathbb{R}^d}(U - \varphi) = (U-\varphi)(t_o,x_o) = 0$ (resp. $\max_{[0,T]\times\mathbb{R}^d}(V - \varphi) = (V-\varphi)(t_o,x_o) = 0$), we have*

$$H^*\varphi(t_o,x_o) \geq 0 \qquad (\textit{resp. } H_*\varphi(t_o,x_o) \leq 0).$$

We will need for the proof of the supersolution property on $[0,T] \times \mathbb{R}^d$ (see Sections 2.2.3.1 and 2.2.3.2) the following technical Assumption. Define for sake of clarity, for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R})$, $u \in U$ and $(t,x,z_1,z_2) \in [0,T] \times \mathbb{R}^{2d+1}$

$$\mathcal{L}^u_{X,Z}\bar{\varphi}(t,x,z) := \mathcal{L}^u\varphi(t,x) - \mu_X(x,u)\cdot z_1 - \mu_Y(x,y,u)z_2, \tag{2.2.7}$$

y pas introduit

where $z =: (z_1,z_2) \in \mathbb{R}^d \times \mathbb{R}$ and $\bar{\varphi}(t,x,z) := \varphi(t,x) - |z|^2$.

Assumption 2.2.3. *For all $\varepsilon > 0, \eta \in [-1,1], (t_o,x_o) \in [0,T]\times\mathbb{R}^d, \varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and finite C_1 satisfying*

$$\sup_{u \in \mathcal{N}_{\varepsilon,\eta}(t,x,y,D\varphi,\varphi)} \{\mu_Y(x,y,u) - \mathcal{L}^u\varphi(t,x)\} \leq 2C_1$$
$$\textit{for all } (t,x) \in B_\varepsilon(t_o,x_o) \textit{ and } y \in \mathbb{R} \textit{ s.t. } |y - \varphi(t,x)| \leq \varepsilon,$$

there exists $\varepsilon' > 0, \eta' \in [-1,1]$ and a finite C_2 such that

$$\begin{aligned}&\sup_{u \in \mathcal{N}_{\varepsilon',\eta'}(t,x,y,D\varphi,\varphi)} \left\{\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}(t,x,z)\right\} \leq 2C_1 + |C_1| \\ &\textit{for all } (t,x,y,z) \in [0,T]\times\mathbb{R}^{2d+2} \textit{ s.t. } \begin{cases} (t,x,z) \in B_{\varepsilon'}(t_o,x_o,0) \\ y \in \mathbb{R} \textit{ s.t. } |y - \bar{\varphi}(t,x)| \leq \varepsilon', \end{cases}\end{aligned} \tag{2.2.8}$$

and

$$\begin{aligned}&\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}(t,x,z)\right]^+}{1 + |N^u(x,y,D\varphi)|} \leq C_2\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(x,u)\right|\right) \\ &\textit{for all } (t,x,y,z) \in [0,T]\times\mathbb{R}^{2d+2} \textit{ s.t. } \begin{cases} (t,x,z_1,z_2) \in B_{\varepsilon'}(t_o,x_o,0) \\ y \in \mathbb{R} \textit{ s.t. } |y - \bar{\varphi}(t,x)| \leq \varepsilon', \end{cases} \\ &\textit{and } u \in U \textit{ such that } \Delta^{u,\cdot}(t,x,y,\varphi) \geq \eta \ \lambda\textit{-a.e.}\end{aligned} \tag{2.2.9}$$

As in [Bouchard 09, Soner 02a, Soner 02c], the proof of the subsolution property requires an additional regularity assumption on the set valued map $\mathcal{N}_{0,\eta}(\cdot, f)$.

Assumption 2.2.4. (*Continuity of* $\mathcal{N}_{0,\eta}(t,x,y,q,f)$) *For* $f \in \mathcal{C}^0\left([0,T]\times\mathbb{R}^d\right)$, $\eta > 0$, *let* B *be a subset of* $[0,T)\times\boldsymbol{X}\times\mathbb{R}\times\mathbb{R}^d$ *such that* $\mathcal{N}_{0,2\eta}(\cdot,f)\neq\emptyset$ *on* B. *Then, for every* $\varepsilon>0, (t_0,x_0,y_0,q_0)\in Int(B)$, *and* $u_0\in\mathcal{N}_{0,2\eta}(t_0,x_0,y_0,q_0,f)$, *there exists an open neighborhood* B' *of* (t_0,x_0,y_0,q_0) *and a locally Lipschitz map* $\overline{\nu}$ *defined on* B' *such that* $|\overline{\nu}(t_0,x_0,y_0,q_0)-u_0|\leq\varepsilon$ *and* $\overline{\nu}(t,x,y,q)\in\mathcal{N}_{0,\eta}(t,x,y,q,f)$ *on* B'.

We also assume that v is locally bounded, so that v_* and v^* are finite. Our first result characterizes v as a discontinuous viscosity solution of the variational inequation (2.2.17) in the following sense.

Theorem 2.2.5. *Under Assumption 2.2.3, the function* v_* *is a viscosity supersolution on* $[0,T)\times\boldsymbol{X}$ *of*

$$H^* v_* \geq 0. \tag{2.2.10}$$

If in addition Assumption 2.2.4 holds, then the function v^* *is a viscosity subsolution on* $[0,T)\times\boldsymbol{X}$ *of*

$$\min\left\{H_* v^*, v^*+\kappa\right\}\leq 0 \tag{2.2.11}$$

The proof of this result is reported in Section 2.2.3.

Remark 2.2.6.

1. Note that the operator H^* would not be upper-semicontinuous in φ, for the uniform convergence, without the relaxation in the test function on the non-local part. This is the counterpart of the local relaxation introduced in [Bouchard 09] on the derivatives of the test function.

deja dit de virer 2.

2. Assumption 2.2.4 is the counterpart of [Bouchard 09, Assumption 2.1] in the case of mixed diffusion. It can also be related to the definition of N_{Lip} and K_{Lip} in Chapter 4. The main idea is the existence of an ε-optimal Lipschitz-continuous selector for the set-valued map $\mathcal{N}$, see Chapter 5.

3. Notice that we impose the Definition 2.2.2 of viscosity solution for integrability issue. This heavily relies on the relaxation of the operator in its test function parameter, in terms of uniform convergence. Indeed, consider the case where the relaxation is stated in terms of uniform convergence on compact sets. Then for every $(t_o,x_o)\in[0,T]\times\mathbb{R}^d$ and test function φ, the family of auxiliary test functions $(\varphi_\iota)_\iota$ defined for each $\iota>0$ as $\varphi_\iota(t,x):=\varphi(t,x)\pm\iota|x-x_o|^n$, for some $n>0$. This family converges uniformly on compact subsets towards φ as $\iota\to 0$. However, the presence of the jumps may imply that $\varphi_\iota(\cdot,X)$ may fail to be integrable for n large enough.

4. Assumption 2.2.3 is of technical nature, and is needed in the proof of (2.2.10) for integrability issues. It was missing in [Bouchard 09, Theorems 2.1 and 2.2, Corollaries 3.1 and 3.2], although it is satisfied in their Section 4. This condition enable us to control the drift $\mu_Y-\mathcal{L}^u\varphi$ in terms of BMO martingales, and thus to define a change of measure with uniformly integrable martingale, see Section 2.2.3.1. Equation (2.2.8) essentially stands in an additional relaxation

of the operator. The relaxation in terms of z_1 in (2.2.8) is obvious by definition of H^*, whereas the relaxation in z_2 is new. Equation (2.2.9) is also new, and consists essentially in constraints on the partial derivatives of the test function, see the proof of Corollary 3.2.1 in the particular case of stochastic target under controlled loss.

5. In the context of [Bouchard 09, Section 4], where the process X is not influenced by the control, Equation (2.2.9) is too strong. Indeed, one would have in that case a control in terms of σ_X. This remark would also hold whenever the jumps are locally bounded, as in [Bouchard 02].

ce n'est pas une equation, plutot assertion. Pas compris pourquoi too strong

Example 2.2.2. In the context of Example 2.2.1, first notice that the process X is not influenced by the control ν. Hence, Assumption 2.2.3 reduces in this context in a control of $\frac{|\mu_Y(u)|}{|\sigma_Y(u)|}$. It is thus trivially satisfied since these coefficients are linear in u. Then, direct computations show that v_* is a viscosity supersolution on $[0,T)\times(0,\infty)^d$ of

$$0 \le \min\left\{-\partial_t\varphi - \frac{1}{2}\sigma^2 D^2\varphi, D\varphi\cdot\beta(\cdot,e) - \varphi\left(\cdot+\beta(\cdot,e)\right)+\varphi\right\}, \quad \text{for } \lambda\text{-a.e. } e\in E$$

and that v^* is a viscosity subsolution of

$$0 \ge \min\left\{-\partial_t\varphi - \frac{1}{2}\sigma^2 D^2\varphi, D\varphi\cdot\beta(\cdot,e) - \varphi\left(\cdot+\beta(\cdot,e)\right)+\varphi\right\} \quad \text{for } e\in E'\in\mathcal{E} \text{ s.t. } \lambda(E')>0.$$

We next discuss the terminal conditions on $\{T\}\times\mathbf{X}$. By the definition of the stochastic target problem, we have

$$v(T,x) = g(x) \text{ for every } x\in\mathbb{R}^d,$$

where g is defined in (2.2.5). However, the possible discontinuities of v might imply that the limits $v_*(T,\cdot)$ and $v^*(T,\cdot)$ do not agree with this boundary condition. We then need to introduce, as in [Bouchard 09], the set-valued map

$$\mathbf{N}(t,x,y,q,\psi) := \left\{\begin{array}{l}(r,s)\in\mathbb{R}^d\times\mathbb{R} : \exists\, u\in U \text{ s.t. } r = N^u(x,y,q) \\ \quad \text{and } s\le \Delta^{u,e}(t,x,y,\psi) \quad \text{for } \lambda\text{-a.e. } e\in E\end{array}\right\},$$

together with the signed distance function from its complement $\mathbf{N}^c$ to the origin:

$$\delta := \mathrm{dist}(0,\mathbf{N}^c) - \mathrm{dist}(0,\mathbf{N}),$$

where we recall that dist stands for the (unsigned) Euclidean distance. Then,

$$0\in \mathrm{int}\left(\mathbf{N}(t,x,y,q,\psi)\right) \text{ iff } \delta(t,x,y,q,\psi)>0. \tag{2.2.12}$$

The upper and lower-semicontinuous envelopes of δ are respectively denoted by δ^* and δ_*, and we will abuse notation by writing $\delta_*v(t,x) = \delta_*\left(t,x,v(t,x),Dv(t,x),v\right)$ and $\delta^*v(t,x) = \delta^*\left(t,x,v(t,x),Dv(t,x),v\right)$. For $\varphi\in\mathcal{C}^2\left(\mathbb{R}^d\right)$, we similarly define $\delta_*\varphi(x) = \delta_*\left(T,x,\varphi(x),D\varphi(x),\varphi\right)$ and the same definition holds for $\delta^*\varphi(x)$.

Remark 2.2.7. From the convention $\sup\emptyset = -\infty$ and the supersolution property (2.2.10) in Theorem 2.2.5, it follows that

$$\delta^* v_* \geq 0 \text{ on } [0,T) \times \mathbb{R}^d$$

in the viscosity sense. Then, if $\mathbf{N}^c \neq \emptyset$, this means that v is subject to a gradient constraint.

Remark 2.2.8. 1. Assume that for every (x,y,q) and $r \in \mathbb{R}^d$, there is an unique solution $\bar{u}(x,y,q,r)$ to the equation $N^u(x,y,q) = r$, i.e.

$$N^u(x,y,q) = r \quad \text{iff} \quad u = \bar{u}(x,y,q,r).$$

Assume further that $\bar{u}$ is locally Lipschitz continuous, so that Assumption 2.2.4 trivially holds. For ease of notations, we set $\bar{u}_0(x,y,q) := \bar{u}(x,y,q,0)$. For a bounded set of controls U, it follows that, for any smooth function φ, $H^*\varphi(t,x) \geq 0$ implies that

$$\bar{u}_0\left(x,\varphi(t,x),D\varphi(t,x)\right) \in U, \quad \mathbf{A}^{\bar{u}_0}(\cdot,\varphi,\partial_t\varphi,D\varphi,D^2\varphi)(t,x) \geq 0$$
$$\text{and } \Delta^{\bar{u}_0,e}(t,x,\varphi(t,x),\varphi) \geq 0 \quad \text{for } \lambda\text{-a.e. } e \in E.$$

Similarly, $H_*\varphi(t,x) \leq 0$ implies that

$$\text{either } \bar{u}_0\left(x,\varphi(t,x),D\varphi(t,x)\right) \notin \text{int}U,$$
$$\text{or} \quad \mathbf{A}^{\bar{u}_0}(\cdot,\varphi,\partial_t\varphi,D\varphi,D^2\varphi)(t,x) \leq 0$$
$$\text{or } \Delta^{\bar{u}_0,e}(t,x,\varphi(t,x),\varphi) < 0 \quad \text{for} \quad e \in E' \in \mathcal{E} \quad \text{s.t.} \quad \lambda(E') > 0.$$

The following result states that the constraint discussed in Remark 2.2.7 propagates up to the boundary. Here, the main difficulty is due to the unboundedness of the set U and the presence of jumps in the diffusions. As discussed in Section 2.3.4 (see Corollary 2.3.17), the unboundedness of the controls may imply that the condition $\{H^*v_*(T,\cdot) < \infty\}$ is not satisfied. Notice that in the framework of Chapter 4, The same kind of condition would ne needed for the terminal condition on the subsolution property if the adverse control were possibly unbounded.

Theorem 2.2.9. *Under Assumption 2.2.3, the function $x \mapsto v_*(T,x)$ is a viscosity supersolution of*

$$\min\left\{(v_*(T,\cdot) - g_*)\,\mathbf{1}_{\{H^*v_*(T,\cdot)<\infty\}}, \delta^*v_*(T,\cdot)\right\} \geq 0 \text{ on } \boldsymbol{X}, \tag{2.2.13}$$

and, under Assumption (2.2.4), $x \in \boldsymbol{X} \mapsto v^*(T,x)$ is a viscosity subsolution of

$$\min\left\{v^*(T,\cdot) - g^*, \delta_*v^*(T,\cdot)\right\} \leq 0 \text{ on } \boldsymbol{X}. \tag{2.2.14}$$

We conclude this section by some remarks. Remark 2.2.11 establishes the link between this work and those of [Soner 02c], [Bouchard 02] and [Bouchard 09]. Remark 2.2.12 was already in [Bouchard 09], and Remark 2.2.10 will be of important use in the proofs of Section 2.3.5 below.

Remark 2.2.10. Assume that

$$\operatorname*{ess\,sup}_{u\in\mathcal{N}_{\cdot},e\in E}\{|\beta_X(\cdot,u(e),e)|+|\beta_Y(\cdot,u(e),e)|\}\text{ is locally bounded,}\\ \text{and } E \text{ is compact.} \tag{2.2.15}$$

Then, the operator H is continuous for the uniform convergence in its $\psi \in \mathcal{C}^{1,2}$ parameter. In this case, the test function ψ appearing in the form $\psi(t, x + \beta_X(x, u(e), e))$ in the definition of H^* can be replaced by v_* itself. To see this, note that for any $\varepsilon > 0, (t_0, x_0)$ and $\varphi \in \mathcal{C}^{1,2}$ such that $(v_* - \varphi)$ achieves a strict minimum at (t_0, x_0), one can find a sequence of smooth function φ_n^ε such that $\varphi_n^\varepsilon = \varphi$ on $B_\varepsilon(t_0, x_0)$, $\varphi_n^\varepsilon \le v_*$, and $\varphi_n^\varepsilon \uparrow v_*$ uniformly on compact sets of $(B_{2\varepsilon}(t_0, x_0))^c$. This allows to replace the original test function φ by v_* on $(B_{2\varepsilon}(t_0, x_0))^c$. It then suffices to send $\varepsilon \to 0$ and use the continuity induced by (2.2.15).

The same remark holds for the subsolution property.

Remark 2.2.11. Note that $\delta(x, y, q) \le 0$ whenever $\operatorname{int}(N(x, y, q)) \neq \emptyset$, so that the subsolution property does not carry any information. This would be the case when the control set U has empty interior.

Remark 2.2.12. When the set U is bounded, and $\beta_X \equiv \beta_Y \equiv 0$, i.e. there is no jumps, it was proved in Soner and Touzi [Soner 02c] that the value function v is a discontinuous viscosity solution of

$$\sup_{u\in\mathcal{N}_0(\cdot,v,Dv)(t,x)}\{\mu_Y(x,v(t,x),u)-\mathcal{L}^u v(t,x)\}=0, \tag{2.2.16}$$

where

$$\mathcal{N}_0(x,y,q) := \{u\in U : N^u(x,y,q)=0\}\\ \text{and } N^u(x,y,q) := \sigma_Y(x,y,u)-\sigma_X(x,u)^T q,$$

with the standard convention $\sup\emptyset = -\infty$. In the case of a convex compact set U, with jumps and $\mathbb{R}^d$-valued controls, i.e. $\mathcal{U}^2 = \{0\}$, Bouchard [Bouchard 02] showed that v is a viscosity solution of an equation of the form

$$\sup_{u\in\mathcal{N}_0(\cdot,v,Dv)(t,x)}\left\{\min\left\{\mathcal{L}^u\varphi(t,x),\inf_{e\in E}\mathcal{G}^{u,e}\varphi(t,x)\right\}\right\}=0. \tag{2.2.17}$$

Finally the case of unbounded set U with no jumps was considered by Bouchard, Elie and Touzi [Bouchard 09]. In this paper, the authors introduced a relaxation on the operator (2.2.16), in order to deal with this unboundedness. This relaxation applies to the space variable x, the function φ, its gradient and its Hessian matrix, at the local point (t, x). Such a relaxation is required in order to ensure that the subsolution (resp. super-solution) property is stated in terms of a lower semi-continuous (resp. upper semi-continuous) operator. In our jump-diffusion framework, a similar relaxation is required, but it should involve the additional non-local term $\mathcal{G}^{u,e}$ in (2.2.17). One shall note that this relaxation is introduced in the Kernel $\mathcal{N}_\varepsilon$ with $\varepsilon \ge 0$, so that our PDEs do not take the form of (2.2.17). This is however a pure technical consideration, since we recover the same inequalities when considering particular frameworks, see e.g. Example 2.2.2.

2.2.3 Derivation of the PDE for singular stochastic target problems

This section is dedicated to the proof of Theorems 2.2.5 and 2.2.9. We first recall the geometric dynamic programming principle of Soner and Touzi [Soner 02a], see also Bouchard and Vu [Bouchard 10c]. We next report the proof of the supersolution properties in Sections 2.2.3.1 and 2.2.3.2, and the proof of the subsolution properties in Sections 2.2.3.3 and 2.2.3.4.

Theorem 2.2.13. ***(Geometric Dynamic Programming Principle)*** *Fix* $(t,x) \in [0,T) \times \boldsymbol{X}$ *and let* $\{\theta^\nu, \nu \in \mathcal{U}\}$ *be a family of* $[t,T]-$*valued stopping times. Then,*

(GDPj1) *If* $y > v(t,x)$*, then there exists* $\nu \in \mathcal{U}$

$$Y^\nu_{t,x,y}(\theta^\nu) \geq v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right).$$

(GDPj2) *For every* $-\kappa \leq y < v(t,x), \nu \in \mathcal{U}$,

$$\mathbb{P}\left[Y^\nu_{t,x,y}(\theta^\nu) > v\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)\right] < 1.$$

2.2.3.1 The supersolution property on $[0,T) \times \mathbf{X}$

We follow the arguments of [Bouchard 09] up to non trivial modifications due to the presence of the jumps, and the consideration of Assumption 2.2.3.

Step 1: Let $(t_0, x_0) \in [0,T) \times \mathbf{X}$ and φ be a smooth function of linear growth such that

$$\min_{[0,T)\times\mathbf{X}} (\text{strict})\,(v_* - \varphi) = (v_* - \varphi)(t_0, x_0) = 0.$$

Assume that $H^*\varphi(t_0,x_0) =: -4\eta < 0$ for some $\eta > 0$, and let us work towards a contradiction. We define the family $\{f_\iota, \iota > 0\}$ of real valued functions defined on $\mathbb{R}^d$ for all $\iota > 0$ by

$$f_\iota : x \in \mathbb{R}^d \longmapsto \frac{2\iota}{\pi}\int_0^{\pi|x-x_0|} \sin^2 u du \mathbb{1}_{\{|x-x_0|\leq 1\}} + \iota\mathbb{1}_{\{|x-x_0|>1\}}. \tag{2.2.18}$$

Observe that for each $\iota > 0$,

$$\begin{gathered} f_\iota \in C^2(\mathbb{R}^d;\mathbb{R}) \text{ is of linear growth,} \\ 0 = f_\iota(x_0) = \min_{x\in\mathbb{R}^d} f_\iota(x), \\ (f_\iota)_{\iota>0} \text{ converges uniformly towards } 0 \text{ as } \iota \to 0. \end{gathered} \tag{2.2.19}$$

We also notice for later use that for all $\iota > 0$, we have

$$\begin{gathered} f_\iota(x) \geq \gamma_{\varepsilon,\iota} := \iota\left(\left(\varepsilon - \frac{\sin(2\pi\varepsilon)}{2\pi}\right)\mathbb{1}_{\{|x-x_0|\leq 1\}} + \mathbb{1}_{\{|x-x_0|>1\}}\right) > 0 \\ \text{for all } \varepsilon > 0 \text{ and } x \in \mathbb{R}^d \text{ such that } |x - x_0| \geq \varepsilon. \end{gathered} \tag{2.2.20}$$

Set $\varphi_\iota(t,x) := \varphi(t,x) - f_\iota(x)$ for $\iota > 0$. By definition of H^* and the fact that $\varphi_\iota \underset{u.}{\longrightarrow} \varphi$ as $\iota \to 0$, we may find $\varepsilon, \iota > 0$ small enough, such that, after possibly changing $\eta > 0$

$$\mu_Y(x,y,u) - \mathcal{L}^u \varphi_\iota(t,x) \leq -2\eta$$
$$\text{for all } (t,x,y) \in [0,T) \times \mathbf{X} \times \mathbb{R} \text{ s.t. } \begin{cases} (t,x) \in B_\varepsilon(t_0,x_0) \\ |y - \varphi_\iota(t,x)| \leq \frac{\eta}{2}, \end{cases}$$
$$\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}\left(t,x,y,D\varphi_\iota(t,x),\varphi_\iota\right),$$

where we recall that $B_\varepsilon(t_o,x_o)$ denotes the ball of center (t_o,x_o) and radius ε. Define now for all $z := (z_1,z_2) \in \mathbf{X} \times \mathbb{R}$ and $(t,x) \in [0,T] \times \mathbf{X}$ the function $\bar\varphi_\iota(t,x,z) := \varphi_\iota(t,x) - |z|^2$, and observe that, since the partial derivatives in (t,x) of $\bar\varphi_\iota$ and φ_ι coincide, we have for every $u \in U, (t,x,y,z) \in [0,T] \times \mathbf{X} \times \mathbb{R} \times \mathbf{X} \times \mathbb{R}$:

$$\mathcal{L}^u \bar\varphi_\iota(t,x,z) = \mathcal{L}^u \varphi_\iota(t,x).$$

We recall from (2.3.7), for every $u \in U$, $(t,x,z) \in [0,T] \times \mathbf{X}^2 \times \mathbb{R}$ and $y \in \mathbb{R}$ the definition of the operator

$$\mathcal{L}^u_{X,Z} \bar\varphi_\iota(t,x,z) = \mathcal{L}^u \varphi_\iota(t,x) - \mu_X(x,u) \cdot z_1 - \mu_Y(x,y,u) z_2.$$

By Assumption 2.2.3, there exists then a finite constant $C > 0$ such that, after possibly changing ε and $\eta > 0$, we have

$$\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z} \bar\varphi_\iota(t,x,z) \leq -\eta$$
$$\text{for all } (t,x,z,y) \in [0,T) \times \mathbf{X}^2 \times \mathbb{R}^2 \text{ s.t. } \begin{cases} (t,x,z) \in B_\varepsilon(t_0,x_0,0) \\ |y - \bar\varphi_\iota(t,x,z)| \leq \frac{\eta}{4}, \end{cases} \tag{2.2.21}$$
$$\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}\left(t,x,y,D\varphi_\iota(t,x),\varphi_\iota\right)$$

and

$$\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar\varphi_\iota(t,x,z)\right]^+}{1 + |N^u(x,y,D\varphi_\iota|} \leq C\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(x,u)\right|\right)$$
$$\text{for all } (t,x,z) \in B_\varepsilon(t_o,x_o,0) \text{ and } y \in \mathbb{R} \text{ s.t. } |y - \bar\varphi_\iota(t,x,z)| \leq \frac{\eta}{4}$$
$$\text{and for all } u \in U \text{ s.t. } \Delta^{u,\cdot}(t,x,y,\varphi_\iota) \geq -\eta \ \lambda\text{-a.e.,} \tag{2.2.22}$$

Notice that we still have

$$0 = v_*(t_0,x_0) - \bar\varphi_\iota(t_0,x_0,0) = \min_{[0,T)\times\mathbf{X}^2\times\mathbb{R}} (\text{strict})\left(v_* - \bar\varphi_\iota\right).$$

Let $\partial_p B_\varepsilon(t_0,x_0,0) := \{t_0+\varepsilon\} \times \overline{B}_\varepsilon(t_0,x_0,0) \cup [t_0,t_0+\varepsilon) \times \partial B_\varepsilon(x_0,0)$ denote the parabolic boundary of $B_\varepsilon(t_0,x_0,0)$. Set

$$\zeta := \min_{\partial_p B_\varepsilon(t_0,x_0,0)} \left(v_* - \bar\varphi_\iota\right),$$

and observe that $\zeta > 0$ since the above minimum is strict. We now define $\mathcal{V}_\varepsilon(t_0,x_0,0) := \partial_p B_\varepsilon(t_0,x_0,0) \cup [t_0,t_0+\varepsilon) \times B^c_\varepsilon(x_0) \times B_\varepsilon(0)$, and with $\gamma_{\varepsilon,\iota}$ defined as in (2.2.20), we observe that

$$(v_* - \bar\varphi_\iota)(t,x,z) \geq \zeta \wedge \gamma_{\varepsilon,\iota} =: \xi > 0 \text{ for } (t,x,z) \in \mathcal{V}_\varepsilon(t_0,x_0,0)$$

since $(t_0,x_0,0)$ is a strict minimizer, and $|x-x_0| \geq \varepsilon$ on $B^c_\varepsilon(x_0)$, recall (2.2.20).

step 2: Let $(t_n,x_n,z_n)_{n\geq 1}$ be a sequence in $[0,T) \times \mathbf{X}^2 \times \mathbb{R}$ which converges to $(t_0,x_0,0)$ and such that $v(t_n,x_n) \to v_*(t_0,x_0)$. Set $y_n := v(t_n,x_n) + n^{-1}$ and observe that

$$\gamma_n := y_n - \bar\varphi_\iota(t_n,x_n,z_n) \to 0. \tag{2.2.23}$$

For each $n \geq 1$, we have $y_n > v(t_n,x_n)$. Thus, it follows from **(GDPj1)** that there exists some $\nu^n \in \mathcal{U}$ such that

$$Y^n(t\wedge\theta_n) \geq v\left(t\wedge\theta_n, X^n(t\wedge\theta_n)\right), \quad t \geq t_n, \tag{2.2.24}$$

where

$$\begin{aligned}\theta_n^o &:= \{s \geq t_n : (s, X^n(s), Z^n(s)) \notin B_\varepsilon(t_0,x_0,0)\}\\ \theta_n &:= \left\{s \geq t_n : |Y^n(s) - \bar\varphi_\iota\left(s, X^n(s), Z^n(s)\right)| \geq \frac{\eta}{4}\right\} \wedge \theta_n^o,\end{aligned} \tag{2.2.25}$$

and

$$(X^n,Y^n,Z^n) := \left(X^{\nu^n}_{t_n,x_n}, Y^{\nu^n}_{t_n,x_n,y_n}, Z^{\nu^n}_{t_n,x_n,z_n}\right),$$

$$Z^{\nu^n}_{t_n,x_n,z_n}(s) := z_n + \frac{1}{2}\int_{t_n}^s \begin{pmatrix} \mu_Y\left(X^n(u),Y^n(u),\nu^n_u\right) \\ \mu_X\left(X^n(u),\nu^n_u\right)\end{pmatrix} du.$$

z_n=0 ?

By the inequalities $v \geq v_* \geq \varphi_\iota \geq \bar\varphi_\iota$, this implies that

$$\begin{aligned}& Y^n(t\wedge\theta_n) - \bar\varphi_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\\ &\geq \mathbf{1}_{\{t\geq\theta_n\}}\left[Y^n(t\wedge\theta_n) - \bar\varphi_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\right]\\ &\geq \mathbf{1}_{\{t\geq\theta_n\}}\Big[\left(Y^n(t\wedge\theta_n) - \bar\varphi_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\right)\mathbf{1}_{\{\theta_n<\theta_n^o\}}\\ &\qquad + \left(v_*\left(t\wedge\theta_n, X^n(t\wedge\theta_n)\right) - \bar\varphi_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right)\right)\mathbf{1}_{\{\theta_n=\theta_n^o\}}\Big]\\ &\geq \left[\frac{\eta}{4}\mathbf{1}_{\{\theta_n<\theta_n^o\}} + \xi\mathbf{1}_{\{\theta_n=\theta_n^o\}}\right]\mathbf{1}_{\{t\geq\theta_n\}}\end{aligned}$$

and therefore

$$Y^n(t\wedge\theta_n) - \bar\varphi_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right) \geq \left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t\geq\theta_n\}} \geq 0. \tag{2.2.26}$$

step 3: Since $\bar\varphi_\iota$ is smooth, recall (2.2.19), it follows from Itô's lemma, (2.2.23), the definition of Y^n and (2.2.26), that

$$\begin{aligned}a_n + \int_{t_n}^{t\wedge\theta_n}(b^n_s + d^n_s)\,ds + \int_{t_n}^{t\wedge\theta_n}\psi^n_s dW_s + \int_{t_n}^{t\wedge\theta_n}\int_E c^{n,e}_s \tilde J(de,ds)\\ \geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t<\theta_n\}},\end{aligned} \tag{2.2.27}$$

ensuite tu les remets ensemble en ecrivant en terme de J et non \tilde J. Le faire ici aussi where we recall that $\tilde{J}$ is the compensated jump measure and

$$a_n := -\left(\frac{\eta}{4} \wedge \xi\right) + \gamma_n, \qquad b_s^n := \mu_Y(X^n(s), Y^n(s), \nu_s^n) - \mathcal{L}_{X,Z}^{\nu_s^n} \bar{\varphi}_\iota(s, X_s^n, Z^n(s))$$
$$c_s^{n,e} := \varphi_\iota(s, X_{s-}^n) - \varphi_\iota\left(s, X_{s-}^n + \beta_X(X_{s-}^n, \nu_s^n(e), e)\right)$$
$$+\beta_Y\left(X_{s-}^n, Y_{s-}^n, \nu_s^n(e), e\right)$$
$$\psi_s^n := N^{\nu^n}\left(Z_s^n, D\varphi_\iota(s, X_s^n)\right), \qquad d_s^n := \int_E c_s^{n,e}\lambda(de). \tag{2.2.28}$$

In view of (2.2.23), we have

$$a_n \to -\left(\frac{\eta}{4} \wedge \xi\right) < 0 \text{ for } n \to \infty. \tag{2.2.29}$$

Observe now that, for every $n \geq 1$, the definition of θ_n implies that for all $s \in [t_n, \theta_n)$, we have

$$|Y^n(s) - \bar{\varphi}_\iota\left(s, X_s^n, Z^n(s)\right)| \leq \frac{\eta}{4}.$$

Hence, we have

$$c_s^{n,e} \geq -\eta \quad \text{for} \quad \lambda\text{-a.e.} e \in E \text{ and } s \in [t_n, \theta_n], \tag{2.2.30}$$

since otherwise we would have

$$Y^n(\theta_n) - \bar{\varphi}_\iota\left(\theta_n, X^n(\theta_n), Z^n(\theta_n)\right) \leq \frac{-3\eta}{4},$$

which is in contradiction with (2.2.24). Hence, by (2.2.21) and the definition of the Kernel $\mathcal{N}_{\varepsilon,-\eta}$, for all n$\geq 1$, $s \in [t_n, \theta_n]$, we have

$$|\psi_s^n| \leq \varepsilon \qquad \Longrightarrow \qquad b_s^n \leq -\eta. \tag{2.2.31}$$

step 4: We now introduce, for each $n \geq 1$, the set

$$A_n := \left\{s \in [t_n, \theta_n] : b_s^n > -\eta\right\}.$$

Observe that, for all $n \geq 1$, (2.2.31) implies that the process ψ^n satisfies

$$|\psi_s^n| > \varepsilon \qquad \text{for all} \qquad s \in A_n, \tag{2.2.32}$$

so that we can define the process α^n as

$$\alpha_s^n := \frac{-b_s^n}{|\psi_s^n|^2}\psi_s^n \mathbb{1}_{A_n}(s).$$

By definition of θ_n, (2.2.32), (2.2.22), (2.2.30), we have,

$$|\alpha_s^n| \leq C\left(1 + |\sigma_Y(X^n(s), Y^n(s), \nu_s^n)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(X^n(s), \nu_s^n)\right|\right),$$

for all $s \in [t_n, \theta_n)$. We now claim that processes $\int_{t_n}^{\theta_n \wedge \cdot} \sigma_X(X^n(s), \nu_s^n) dW_s$ and $\int_{t_n}^{\theta_n \wedge \cdot} \sigma_Y(X^n(s), Y^n(s), \nu_s^n) dW_s$ are BMO-martingales, so that $\int_{t_n}^{\theta_n \wedge \cdot} \alpha_s^n dW_s$ is itself a BMO-martingale. By [Kazamaki 94, Theorem 2.3], we may thus define the uniformly integrable exponential martingale

$$L_\cdot^n := \mathcal{E}\left(\int_{t_n}^{\cdot} \alpha_s^n dW_s\right)_{\cdot \wedge \theta_n}$$

where $\mathcal{E}$ denotes the Doleans-Dade exponential. Hence, by Girsanov's Theorem,

$$\hat{W}_\cdot^n := W_\cdot - \int_{t_n}^{\cdot \wedge \theta_n} \alpha_s^n ds$$

is a $\hat{\mathbb{Q}}^n$-Brownian motion, with $\hat{\mathbb{Q}}^n$ the equivalent probability defined by its density $\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}}\Big|_{\mathcal{G}_\cdot} := L_\cdot^n$. Recalling (2.2.27), we have

$$\begin{aligned} a_n + \int_{t_n}^{t \wedge \theta_n} b_s^n \mathbb{1}_{A_n^c} ds + \int_{t_n}^{t \wedge \theta_n} \psi_s^n d\hat{W}_s^n + \int_{t_n}^{t \wedge \theta_n} \int_E c_s^{n,e} J(de, ds) \\ \geq -\left(\frac{\eta}{4} \wedge \xi\right) \mathbf{1}_{\{t < \theta_n\}}. \end{aligned} \tag{2.2.33}$$

Define now for each $n \geq 1$ the process

$$M_\cdot^n := \mathcal{E}\left(\int_{t_n}^{\cdot} \int_E \left(\frac{1}{nT\left(|d_s^n| + 1\right)} - 1\right) \tilde{J}(de, ds)\right)_{\cdot \wedge \theta_n}.$$

Since

$$\int_{t_n}^{\cdot} \int_E \left(\frac{1}{nT\left(|d_s^n| + 1\right)} - 1\right) \tilde{J}(de, ds) \geq -1,$$

M^n is a non-negative local-martingale (see e.g. [Brémaud 81, Theorem T10]), and from the fact that

$$\frac{1}{nT\left(|d_s^n| + 1\right)} \leq \frac{1}{nT}, \tag{2.2.34}$$

together with $\int_E \lambda(de) < +\infty$, we deduce from [Brémaud 81, Theorems T10 and T11] that M^n is uniformly integrable. We may hence define the equivalent martingale measure $\frac{d\tilde{\mathbb{Q}}^n}{d\hat{\mathbb{Q}}^n}\Big|_{\mathcal{G}_\cdot} := M_\cdot^n$, and by Girsanov's Theorem again, we have

$$\int_{t_n}^{\cdot} \int_E \tilde{J}^n(de, ds) := \int_{t_n}^{\cdot} \int_E J(de, ds) - \int_{t_n}^{\cdot} \int_E \frac{1}{nT\left(|d_s^n| + 1\right)} \lambda(de) ds$$

is a $\tilde{\mathbb{Q}}^n$-martingale. Notice that $\hat{W}^n$ is a $\tilde{\mathbb{Q}}^n$-Brownian motion. Hence, (2.2.33) leads to

$$\begin{aligned} a_n + \int_{t_n}^{t \wedge \theta_n} b_s^n \mathbb{1}_{A_n^c} + \frac{1}{nT} \frac{d_s^n}{\left(|d_s^n| + 1\right)} ds + \int_{t_n}^{t \wedge \theta_n} \psi_s^n d\hat{W}_s^n + \int_{t_n}^{t \wedge \theta_n} \int_E c_s^{n,e} \tilde{J}^n(de, ds) \\ \geq -\left(\frac{\eta}{4} \wedge \xi\right) \mathbf{1}_{\{t < \theta_n\}}. \end{aligned}$$

Recall from the definition of θ_n that $\theta_n \leq T$, which combined with (2.2.34) gives

$$
\begin{aligned}
S_t^n &:= a_n + \frac{1}{n} + \int_{t_n}^{t\wedge\theta_n} b_s^n \mathbb{1}_{A_n^c} + \int_{t_n}^{t\wedge\theta_n} \psi_s^n d\hat{W}_s^n + \int_{t_n}^{t\wedge\theta_n}\int_E c_s^{n,e} \tilde{J}^n(de,ds) \\
&\geq a_n + \int_{t_n}^{t\wedge\theta_n} b_s^n \mathbb{1}_{A_n^c} + \frac{1}{nT}\frac{d_s^n}{(|d_s^n|+1)} ds + \int_{t_n}^{t\wedge\theta_n} \psi_s^n d\hat{W}_s^n + \int_{t_n}^{t\wedge\theta_n}\int_E c_s^{n,e} \tilde{J}^n(de,ds) \\
&\geq S_t^n \geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbf{1}_{\{t<\theta_n\}},
\end{aligned}
$$

parentheses

virer le 2eme

and from definition of A_n, (2.2.2) and the fact that $\bar{\varphi}_\iota$ is a linear growth in its x variable, S^n is local supermartingale, bounded by below, and hence a supermartingale. It follows then that

$$
a_n + \frac{1}{n} = S_{t_n}^n \geq \mathbb{E}^{\tilde{\mathbb{Q}}^n}\left[S_{\theta_n}^n | \mathcal{F}_{t_n}\right] \geq -\left(\frac{\eta}{4}\wedge\xi\right)\mathbb{E}^{\tilde{\mathbb{Q}}^n}\left[\mathbf{1}_{\{\theta_n<\theta_n\}}\right] = 0
$$

which contradicts (2.2.29) for n large enough.

step 5: In order to conclude , it remains to prove that $\int_{t_n}^{\cdot}\sigma_X(X^n(s),\nu_s^n)dW_s$ and $\int_{t_n}^{\cdot}\sigma_Y(X^n(s),Y^n(s),\nu_s^n)dW_s$ are BMO-martingales. We shall focus on $\int_{t_n}^{\cdot}\sigma_X(X^n(s),\nu_s^n)dW_s$, the result for $\int_{t_n}^{\cdot}\sigma_Y(X^n(s),Y^n(s),\nu_s^n)dW_s$ following the exact same argument.

Denote for all $n \geq 1$ and $s \in [t_n, \theta_n]$

$$
\Delta X^n(s) := X^n(s) - X^n(s-),
$$

with $X^n(\cdot-)$ being the left limit of $X^n(\cdot)$. By smoothness of $\bar{\varphi}_\iota$ together with the definition (2.2.25) of θ_n, definition of Z^n, (2.2.22) and (2.2.2), for each $n \geq 1$, there exists a constant K_n such that for all $s < \theta_n$

$$
\max\left(|X^n(s)|_\infty\,;\left|\int_{t_n}^{s}\mu_X\left(s, X^n(s),\nu_s^n\right)ds\right|_\infty;\;|\Delta X^n(s)|_\infty\right) \leq K_n. \tag{2.2.35}
$$

pas besoin des \infty

Being interested in the process $\int_{t_n}^{\theta_n\wedge\cdot}\sigma_X(X^n(s),\nu^n(s))dW_s$, we may restrict ourselves to stopping times τ_n taking their values $\mathbb{P}$-a.s. in $[t_n,\theta_n]$. By continuity of the path: $r \in [t_n,\theta_n] \mapsto \int_{t_n}^{r}\sigma_X(X^n(s),\nu^n(s))dW_s$, we have, for every $\tau_n \in [t_n,\theta_n)$

pas de i

$$
\begin{aligned}
\int_{\tau_n}^{\theta_n}\sigma_X\left(X^n(s),\nu^n(s)\right)dW_s =& X^n(\theta_n-) - X^n(\tau_n) - \int_{\tau_n}^{\theta_n}\mu_X^i\left(X^n(s),\nu^n(s)\right)ds \\
&- \sum_{s<\theta_n}\Delta X^n(s)
\end{aligned}
$$

\tau_n<s

By (2.2.35) together with Jensen's inequality, we thus have

$$
\begin{aligned}
&\left|\mathbb{E}\left[\left\langle \int_{t_n}^{\theta_n} \sigma_X\left(X^n(s), \nu^n(s)\right) dW_s - \int_{t_n}^{\tau_n} \sigma_X\left(X^n(s), \nu^n(s)\right) dW_s \right\rangle |\mathcal{F}_{\tau_n}\right]\right|_\infty \\
=&\left|\mathbb{E}\left[\left\langle \int_{\tau_n}^{\theta_n} \sigma_X\left(X^n(s), \nu^n(s)\right) dW_s \right\rangle |\mathcal{F}_{\tau_n}\right]\right|_\infty \\
=&\left|\mathbb{E}\left[\left| \int_{\tau_n}^{\theta_n} \sigma_X\left(X^n(s), \nu^n(s)\right) dW_s \right|^2 |\mathcal{F}_{\tau_n}\right]\right|_\infty \\
\leq& 4\left|\mathbb{E}\left[|X^n(\theta_n-)|^2 + |X^n(\tau_n)|^2 + \left|\int_{\tau_n}^{\theta_n} \mu_X\left(X^n(s), \nu^n_s\right) ds\right|^2 + \left|\sum_{s<\theta_n} \Delta X^n(s)\right|^2 |\mathcal{F}_{\tau_n}\right]\right|_\infty \\
\leq& C_n\left(1 + \left|\mathbb{E}\left[J(E, [\tau_n, \theta_n)) \sum_{s<\theta_n} |\Delta X^n(s)|^2 \,|\mathcal{F}_{\tau_n}\right]\right|_\infty\right) \\
\leq& C_n\left(1 + \left|\mathbb{E}\left[J(E, [\tau_n, \theta_n))^2 K_n^2 |\mathcal{F}_{\tau_n}\right]\right|_\infty\right) < \infty,
\end{aligned}
$$

C_n ?

s>\tau_n

since $\lambda(E) < \infty$, and so follows the result.

□

Remark 2.2.14. Note that, in the above proof, the relaxation of the non-local part of the operator in term of uniform convergence is required in order to pass from the initial test function φ to the penalized one φ_ι. It allows to obtain the inequality $v_* \geq \varphi_\iota + \xi$ outside of the ball $B_\varepsilon(x_0)$, which is crucial in our proof. This is not required in [Bouchard 09] where processes are continuous. It is neither required in [Bouchard 02], where the non-local operator is already continuous and the size of the jump is locally bounded.

2.2.3.2 The supersolution property on $\{T\} \times \mathbf{X}$

We split the proof in different lemmas.

Lemma 2.2.1. *Let $x_0 \in \boldsymbol{X}$ and $\varphi \in \mathcal{C}^2(\boldsymbol{X})$ be such that*

$$0 = \left(v_*(T, \cdot) - \varphi\right)(x_0) = \min_{\boldsymbol{X}}(strict)\left(v_*(T, \cdot) - \varphi\right)$$

then

$$\delta^*\varphi(x_0) \geq 0.$$

The proof relies on the upper semi-continuity of δ^*, and follows the exact same idea as in [Soner 02c, Lemma 5.2]. We may however give the main steps of this proof for sake of completeness. As in [Soner 02c], the key idea is to consider an auxiliary test function φ_n, penalized in both space and time, and to consider local minimizers (t_n, x_n) of $(v_* - \varphi_n)$. After having proved that $(t_n, x_n) \to (T, x_0)$, we prove that $\lim_{n\to\infty} v_*(t_n, x_n) = v_*(T, x_0)$, and then conclude that the viscosity property of v_* holds in (t_n, x_n). We conclude by using the upper semi-continuity of δ^* and the supersolution property of Theorem (2.2.5) and (2.2.12) on $[0, T) \times \mathbf{X}$.

ajouter l'hypothese !

Lemma 2.2.2. *v_* is a viscosity supersolution of*

$$(v_*(T,\cdot) - g_*)\,\mathbf{1}_{\{H^*v_*(T,\cdot)<\infty\}} \geq 0 \text{ on } \boldsymbol{X}. \tag{2.2.36}$$

Proof. Let $x_o \in \mathbf{X}$ and φ be a smooth function of linear growth such that

$$\min_{\mathbf{X}}(\text{strict})\,(v_*(T,\cdot) - \varphi) = (v_*(T,\cdot) - \varphi)\,(x_o).$$

step 1: Assume that $H^*v_*(T,x_o) < \infty$, $\varphi(x_o) = v_*(T,x_o) < g_*(x_o)$, and let us work towards a contradiction. Since $v(T,\cdot) = g$ by the definition of the problem and $g \geq g_*$, there is a constant $\eta > 0$ such that $\varphi - v(T,\cdot) \leq \varphi - g_* \leq -\eta$ on $B_\varepsilon(x_o)$ for some $\varepsilon > 0$. Since x_o is a strict minimizer, we have

$$2\zeta := \min_{x\in\partial B_\varepsilon(x_o)} v_*(T,x) - \varphi(x) > 0,$$

and it follows from the lower semi-continuity of v_* that there exists $r > 0$ such that

$$v(t,x) - \varphi(x) \geq v_*(t,x) - \varphi(x) \geq \zeta > 0$$
$$\text{for all } (t,x) \in [T-r,T] \times \partial B_\varepsilon(x_o),$$

and hence

$$v(t,x) - \varphi(x) \geq \zeta \wedge \eta > 0$$
$$\text{for } (t,x) \in ([T-r,T) \times \partial B_\varepsilon(x_o)) \cup (\{T\} \times B_\varepsilon(x_o)) =: \mathcal{V}_{\varepsilon,r}(T,x_o).$$

Define $\varphi_\iota(x) := \varphi(x) - f_\iota(x)$, for $\iota > 0$ and f_ι as in (2.2.18). With similar arguments as those of Section 2.2.3.1 and by (2.2.20), we have

$$v(t,x) - \varphi_\iota(x) \geq \zeta \wedge \eta \wedge \gamma_{\varepsilon,\iota} =: 4\xi > 0$$
$$\text{for } (t,x) \in \left([T-r,T] \times \bar{B}^c_\varepsilon(x_o)\right) \cup (\{T\} \times B_\varepsilon(x_o))\,.$$

We now use the fact that $H^*\varphi(x_o) =: \frac{C}{2} < \infty$. Set

$$\tilde{\varphi}_\iota(t,x) := \varphi_\iota(x) + (C+2\eta)(t-T) \leq \varphi_\iota(x).$$

Then, by (2.2.19), for $r, \iota > 0$ sufficiently small and after possibly changing $\varepsilon, \eta > 0$, we have

$$v(t,x) - \tilde{\varphi}_\iota(t,x) \geq 2\xi > 0 \text{ for } (t,x) \in \mathcal{V}_{\varepsilon,r}(T,x_o) \cup [T-r,T] \times \bar{B}^c_\varepsilon(x_o),$$
$$\mu_Y(x,y,u) - \mathcal{L}^u\tilde{\varphi}_\iota(t,x) \leq -2\eta \text{ for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\widetilde{\varphi}_\iota(t,x),\widetilde{\varphi}_\iota)$$
$$\text{and } (t,x,y) \in [T-r,T] \times \mathbf{X} \times \mathbb{R} \text{ s.t. } x \in B_\varepsilon(x_o) \text{ and } |y - \widetilde{\varphi}_\iota(t,x)| \leq \frac{\eta}{2}.$$

Indeed, $\mu_Y(x,y,u) - \mathcal{L}^u\tilde{\varphi}_\iota(t,x) = \mu_Y(x,y,u) - \mathcal{L}^u\varphi_\iota(x) - C - 2\eta \leq -2\eta$ as soon as $\mu_Y(x,y,u) - \mathcal{L}^u\varphi_\iota(x) \leq C$, and we have $\mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\widetilde{\varphi}_\iota(t,x),\varphi_\iota) \subset \mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\bar{\varphi}_\iota(t,x),\widetilde{\varphi}_\iota)$.
We now define for every $(t,x,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d+1}$ and $\iota > 0$ the function

$\bar{\varphi}_\iota(t,x,z) := \tilde{\varphi}_\iota(t,x) - |z|^2$. By Assumption 2.2.3, and after possibly changing $\varepsilon, \eta > 0$, there is $C' > 0$ such that

$$v(t,x) - \bar{\varphi}_\iota(t,x,z) \geq \xi > 0$$
$$\text{for } (t,x,z) \in \bar{\mathcal{V}}_{\varepsilon,r}(T,x_o,0) \cup [T-r,T] \times \bar{B}^c_\varepsilon(x_o) \times B_\varepsilon(0),$$
$$\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}_\iota(t,x,z) \leq -\eta \text{ for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t,x,y,D\widetilde{\varphi}_\iota(t,x),\widetilde{\varphi}_\iota)$$
$$\text{and } (t,x,y,z) \in [T-r,T] \times \mathbb{R}^{2d+2} \times \mathbb{R} \text{ s.t. } \begin{cases} (x,z) \in B_\varepsilon(x_o,0) \\ |y - \widetilde{\varphi}_\iota(t,x)| \leq \dfrac{\eta}{4} \end{cases}$$

and

$$\frac{\left[\mu_Y(x,y,u) - \mathcal{L}^u_{X,Z}\bar{\varphi}_\iota(t,x,z)\right]^+}{|N^u(x,y,D\tilde{\varphi}_\iota|} \leq C_2\left(1 + |\sigma_Y(x,y,u)| + \sum_{i=1}^d \left|\sigma_X^{i,\cdot}(x,u)\right|\right)$$
$$\text{for all } (t,x,z) \in B_\varepsilon(t_o,x_o,0) \quad \text{and} \quad y \in \mathbb{R} \text{ s.t. } |y - \tilde{\varphi}_\iota(t,x)| \leq \frac{\eta}{4}$$
$$\text{and for all } u \in U \text{ s.t. } \Delta_{u,\cdot}(t,x,y,\tilde{\varphi}_\iota) \geq -\eta \ \lambda\text{-a.e.},$$

where $\bar{\mathcal{V}}_{\varepsilon,r}(T,x_o,0)$ is constructed around $(T,x_o,0)$ as $\mathcal{V}_{\varepsilon,r}(T,x_o)$.
step 2: Let $(t_n,x_n,z_n)_{n\geq 1}$ be a sequence in $[T-r,T] \times \mathbf{X} \times \mathbf{X} \times \mathbb{R}$ which converges to $(T,x_o,0)$ and such that $v(t_n,x_n) \to v_*(T,x_o)$. Set $y_n := v(t_n,x_n) + n^{-1}$, and observe that

$$\gamma_n := y_n - \bar{\varphi}(t_n,x_n,z_n) \to 0.$$

For each $n \geq 1$, we have $y_n > v(t_n,x_n)$. Then, by **(GDPj1)**, there exists some $\nu^n \in \mathcal{U}$ such that

$$Y^n(t \wedge \theta_n) \geq v\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right), \quad t \geq t_n,$$

where

$$\theta^o_n := \{s \geq t_n : (s, X^n(s), Z^n(s)) \notin \mathcal{V}_{\varepsilon,r}(T,x_o,0)\}$$
$$\theta_n := \left\{s \geq t_n : |Y^n(s) - \bar{\varphi}_\iota\left(s, X^n(s), Z^n(s)\right)| \geq \frac{\eta}{4}\right\} \wedge \theta^o_n,$$

and

$$(X^n, Y^n, Z^n) := \left(X^{\nu^n}_{t_n,x_n}, Y^{\nu^n}_{t_n,x_n,y_n}, Z^{\nu^n}_{t_n,x_n,z_n}\right),$$

z_n=0

$$Z^{\nu^n}_{t_n,x_n,z_n} = z_n + \frac{1}{2}\int_{t_n}^s \begin{pmatrix} \mu_Y\left(X^n(u), Y^n(u), \nu^n_u\right) \\ \mu_X\left(X^n(u), \nu^n_u\right) \end{pmatrix} ds.$$

Using the inequalities $v \geq v_* \geq \tilde{\varphi}_\iota \geq \bar{\varphi}_\iota$, this implies that

$$\begin{aligned}
&Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right) \\
&\geq \left[Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right)\right] \mathbf{1}_{\{t \geq \theta_n\}} \\
&\geq \mathbf{1}_{\{t \geq \theta_n\}} \left[\left(Y^n(t \wedge \theta_n) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right)\right) \mathbf{1}_{\{\theta_n < \theta^o_n\}}\right. \\
&\quad \left. + \left(v\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) - \bar{\varphi}_\iota\left(t \wedge \theta_n, X^n(t \wedge \theta_n), Z^n(t \wedge \theta_n)\right)\right) \mathbf{1}_{\{\theta_n = \theta^o_n\}}\right] \\
&\geq \left[\varepsilon \mathbf{1}_{\{\theta_n < \theta^o_n\}} + \xi \mathbf{1}_{\{\theta_n = \theta^o_n\}}\right] \mathbf{1}_{\{t \geq \theta_n\}}
\end{aligned}$$

and therefore

$$Y^n(t\wedge\theta_n) - \bar{\varphi}_\iota\left(t\wedge\theta_n, X^n(t\wedge\theta_n), Z^n(t\wedge\theta_n)\right) \geq (\varepsilon\wedge\xi)\,\mathbf{1}_{\{t\geq\theta_n\}} \geq 0.$$

By repeating the arguments of steps 3 and 4 of Section 2.2.3.1, we end up to a contradiction.

□

2.2.3.3 The subsolution property on $[0,T)\times\mathbf{X}$

The proof of the subsolution property is a straightforward combination of the arguments of [Bouchard 02] and [Bouchard 09]. We provide it for completeness.

step 1: Let $(t_0,x_0)\in[0,T)\times\mathbf{X}$ and φ be a smooth function of linear growth such that

$$0=(v^*-\varphi)(t_0,x_0) > (v^*-\varphi)(t,x) \text{ for } (t_0,x_0)\neq(t,x)\in[0,T)\times\mathbf{X}.$$

We assume that $v^*(t_0,x_0)>-\kappa$ and we show that

$$H_*\varphi(t_0,x_0)\leq 0.$$

Assume to the contrary that

$$4\eta := H_*\varphi(t_0,x_0) > 0.$$

By (2.2.6), and after possibly changing $\eta>0$, we may find $\varepsilon>0$ and $\iota>0$ sufficiently small such that

$$\mu_Y(x,y,u)-\mathcal{L}^u\varphi_\iota(t,x)\geq 2\eta$$

for some $u\in\mathcal{N}_{0,\eta}(t,x,y,D\varphi_\iota(t,x),\varphi_\iota)$, for all $(t,x,y)\in[0,T)\times\mathbf{X}\times\mathbb{R}$ such that $(t,x)\in B_\varepsilon(t_0,x_0)$ and $|y-\varphi_\iota(t,x)|\leq\frac{\eta}{4}$, where $\varphi_\iota(t,x):=\varphi(t,x)+f_\iota(x)$, recall (2.2.18) and (2.2.19). Observe that we still have

$$0=(v^*-\varphi_\iota)(t_0,x_0)=\max_{[0,T)\times\mathbf{X}}(\text{strict})\,(v^*-\varphi_\iota)\,. \tag{2.2.37}$$

For ε sufficiently small, and after possibly changing $\eta>0$, Assumption 2.2.4 then implies that

$$\min\left\{\begin{array}{c}\mu_Y\left(x,y,\hat{\nu}\left(t,x,y,D\varphi_\iota(t,x)\right)\right)-\mathcal{L}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x))}\varphi_\iota(t,x),\\ \mathcal{G}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x)),e}\varphi_\iota(t,x)\end{array}\right\}\geq\eta \tag{2.2.38}$$
$$\text{for } \lambda\text{-a.e. } e\in E \text{ and for all } (t,x,y)\in[0,T)\times\mathbf{X}\times\mathbb{R}$$
$$\text{s.t. } (t,x)\in B_\varepsilon(t_0,x_0) \text{ and } |y-\varphi_\iota(t,x)|\leq\frac{\eta}{4},$$

where $\hat{\nu}$ is a locally Lipschitz map satisfying

$$\hat{\nu}\left(t,x,y,D\varphi_\iota(t,x)\right)\in\mathcal{N}_{0,\eta}\left(t,x,y,D\varphi_\iota(t,x),\varphi_\iota\right) \text{ on } B_\varepsilon(t_0,x_0). \tag{2.2.39}$$

Observe that, since (t_0, x_0) is a strict maximizer in (2.2.37), we have

$$-\zeta := \max_{\partial_p B_\varepsilon(t_0,x_0)} (v^* - \varphi_\iota) < 0$$

where $\partial_p B_\varepsilon(t_0, x_0)$ denotes the parabolic boundary of $B_\varepsilon(t_0, x_0)$. As in the previous sections, by (2.2.20), we have for all $(t, x) \in [0, T) \times B_\varepsilon^c(x_0)$

$$(v^* - \varphi_\iota)(t, x) \leq -\gamma_{\varepsilon,\iota}.$$

Thus, for all $(t, x) \in ([t_0, t_0 + \varepsilon) \times B_\epsilon^c(x_0)) \cup \left(\{t_0 + \varepsilon\} \times \overline{B}_\varepsilon(x_0)\right)$,

$$(v^* - \varphi_\iota)(t, x) \leq -(\gamma_{\varepsilon,\iota} \wedge \zeta) =: -\xi < 0. \tag{2.2.40}$$

step 2: We now show that (2.2.38), (2.2.39) and (2.2.40) lead to a contradiction of **(GDPj2)**.

Let $(t_n, x_n)_{n\geq 1}$ be a sequence in $[0, T) \times \mathbf{X}$ which converges to (t_0, x_0) and such that $v(t_n, x_n) \to v^*(t_0, x_0)$. Set $y_n := v(t_n, x_n) - n^{-1}$, and observe that

$$\gamma_n := y_n - \varphi_\iota(t_n, x_n) \to 0. \tag{2.2.41}$$

Also notice that $y_n \geq -\kappa$ for n large enough.

Let $Z^n := (X^n, Y^n)$ denote the solution of (2.2.1) associated to the Markovian control $\hat{\nu}^n := \hat{\nu}(\cdot, X^n, Y^n, D\varphi_\iota(\cdot, X^n))$ and the initial condition $Z^n(t_n) = (x_n, y_n)$. Since $\hat{\nu}$ is locally Lipschitz, this solution is well defined up to the stopping time

$$\theta_n := \inf\left\{s \geq t_n : |Y^n(s) - \varphi_\iota(s, X^n(s))| \geq \frac{\eta}{4}\right\} \wedge \theta_n^o, \tag{2.2.42}$$

with

$$\theta_n^o := \inf\{s \geq t_n : (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\}. \tag{2.2.43}$$

Note that (2.2.38), (2.2.41), and a standard comparison theorem implies that

$$Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n)) \geq \frac{\eta}{4} \quad \text{on} \quad \left\{|Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n))| \geq \frac{\eta}{4}\right\}$$

for n large enough. Indeed, $Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n)) \geq \gamma_n > -\varepsilon$ for n large enough. Since $-v \geq -v^* \geq -\varphi_\iota$, we then deduce from (2.2.40), (2.2.42) and (2.2.43) that

$$\begin{aligned}
&Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) \\
&\quad \geq \mathbf{1}_{\{\theta_n < \theta_n^o\}}(Y^n(\theta_n) - \varphi_\iota(\theta_n, X^n(\theta_n))) \\
&\qquad + \mathbf{1}_{\{\theta_n = \theta_n^o\}}(Y^n(\theta_n^o) - v^*(\theta_n^o, X^n(\theta_n^o))) \\
&\quad \geq \frac{\eta}{4}\mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}}(Y^n(\theta_n^o) - v^*(\theta_n^o, X^n(\theta_n^o))) \\
&\quad \geq \frac{\eta}{4}\mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}}(Y^n(\theta_n^o) + \xi - \varphi_\iota(\theta_n^o, X^n(\theta_n^o))) \\
&\quad \geq \frac{\eta}{4} \wedge \xi + \mathbf{1}_{\{\theta_n = \theta_n^o\}}(Y^n(\theta_n^o) - \varphi_\iota(\theta_n^o, X^n(\theta_n^o))).
\end{aligned} \tag{2.2.44}$$

We may continue by using Itô's formula:

$$Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) \geq \frac{\eta}{4} \wedge \xi + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \left(\gamma_n + \int_{t_n}^{\theta_n} \alpha(s, X_s^n, Y_s^n)\, ds \right.$$
$$\left. + \int_{t_n}^{\theta_n} \int_E \delta(s, X_s^n, Y_s^n, e)\, J(de, ds) \right)$$

where

$$\begin{aligned} \alpha(t,x,y) &:= \mu_Y(x, y, \hat{\nu}(t,x,y,D\varphi_\iota(t,x))) - \mathcal{L}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x))}\varphi_\iota(t,x) \\ \delta(t,x,y,e) &:= \beta_Y(x, y, \hat{\nu}(t,x,y,D\varphi_\iota(t,x))(e), e) \\ &\quad - \varphi_\iota(t, x + \beta_X(x, \hat{\nu}(t,x,y,D\varphi_\iota(t,x))(e), e)) + \varphi_\iota(t,x) \end{aligned}$$

and the diffusion coefficient vanishes by (2.2.39). Recalling (2.2.38), the fact that $\gamma_n \to 0$, and that $\eta, \xi > 0$, this implies that

$$Y^n(\theta_n) > v(\theta_n, X^n(\theta_n)) \text{ for sufficiently large } n.$$

Since the initial position of the process Y^n is $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$, this is clearly in contradiction with **(GDPj2)**.

□

2.2.3.4 The subsolution property on $\{T\} \times \mathbf{X}$

The proof combines arguments used in the two previous sections (2.2.3.2) and (2.2.3.3). The only difference between this proof and the one in [Bouchard 09] relies on the presence of the jumps. However, it can be handled by following [Bouchard 02]. We then only explain the main steps. Let $x_0 \in \mathbf{X}$ and φ be a smooth function of linear growth such that

$$\max_{\mathbf{X}} (\text{strict})\, (v^*(T,\cdot) - \varphi) = (v^*(T,\cdot) - \varphi)(x_0) = 0.$$

Assume that, for some $\eta > 0$,

$$\begin{aligned} &0 < \delta_* \varphi(x_0) \\ &0 < 4\eta < \varphi(x_0) - g^*(x_0) = v^*(T, x_0) - g^*(x_0) \end{aligned}$$

Set $\varphi_\iota(t,x) = \varphi(x) + f_\iota(x) + \iota\sqrt{T-t}$, recall (2.2.18). Since the partial derivatives in x of φ and φ_ι are the same for $x = x_0$, by (2.2.12) and Assumption 2.2.4, together with (2.2.19), using the fact that $\varphi_\iota \geq \varphi$, for $\iota > 0$ small enough, after possibly changing $\eta > 0$, we can find $r, \varepsilon > 0$ and a locally Lipschitz map $\hat{\nu}$ satisfying,

$$\hat{\nu}(t,x,y,D\varphi_\iota(t,x)) \in \mathcal{N}_{0,\eta}(t,x,y,D\varphi_\iota(x), \varphi_\iota). \tag{2.2.45}$$

such that

$$\begin{aligned} &0 < \delta_* \varphi_\iota(t,x) \\ &0 < 4\eta < \varphi_\iota(T, x_0) - g^*(x_0) = v^*(T, x_0) - g^*(x_0) \end{aligned} \tag{2.2.46}$$

for all $(t,x,y) \in [T-r,T) \times \mathbf{X} \times \mathbb{R}$ s.t. $x \in B_r(x_0)$ and $|y - \varphi_\iota(t,x)| \leq \varepsilon$. Since $\partial_t \varphi_\iota \to -\infty$ as $t \to T$, we deduce that, for $r > 0$ small enough,

$$\mu_Y\left(x, y, \hat{\nu}\left(t, x, y, D\varphi_\iota(t,x)\right)\right) - \mathcal{L}^{\hat{\nu}(t,x,y,D\varphi_\iota(t,x))}\varphi_\iota(t,x) \geq \eta \tag{2.2.47}$$

for all $(t,x,y) \in [T-r,T) \times \mathbf{X} \times \mathbb{R}$ s.t. $x \in B_r(x_0)$ and $|y - \varphi_\iota(t,x)| \leq \frac{\eta}{4}$. Also observe that, since $v^* - \varphi_\iota$ is upper-semicontinuous and $(v^* - \varphi_\iota)(T, x_0) = 0$, we can choose $r > 0$ such that

$$v(t,x) \leq \varphi_\iota(t,x) + \frac{\varepsilon}{2} \text{ for all } (t,x) \in [T-r,T) \times B_r(x_0). \tag{2.2.48}$$

Moreover, combining the identity $v(T,x_0) = g(x_0)$, (2.2.20), (2.2.46), (2.2.47), (2.2.48), the fact that x_0 achieves a strict maximum, and using similar arguments as those of Section 2.2.3.2 above, recall 2.2.20, we see that

$$v(t,x) - \varphi_\iota \leq -\left(\zeta \wedge \gamma_{\varepsilon,\iota}\right) =: -\xi \tag{2.2.49}$$

for all $(t,x) \in \left([T-r,T] \times \overline{B_r^c}(x_0)\right) \cup \left(\{T\} \times B_r(x_0)\right)$ and for some $r, \varepsilon > 0$ small enough, but so that the above inequalities still hold. By following the arguments in step 2 of Section 2.2.3.3, we see that (2.2.46), (2.2.45), (2.2.48) and (2.2.49) lead to a contradiction of **(GDPj2)**.

□

2.3 Target reachability with controlled expected loss

2.3.1 Problem reduction

We now turn to the main motivation for the above analysis: the stochastic target problem with controlled expected loss. Let Ψ be a measurable map from $\mathbb{R}^{d+1}$ to $\mathbb{R}$ such that, for every fixed x, the function

$$y \longmapsto \Psi(x,y) \text{ is non-decreasing and right continuous.}$$

We define L as the closed convex hull of the image of Ψ

$$L := \overline{\circledcirc}\left(\Psi\left(\mathbf{X} \times [-\kappa, \infty)\right)\right) = [m, M],$$

with $m < M$, $m, M \in [-\infty, +\infty]$. For $p \in L$, we define the stochastic target problem with controlled expected loss as follows:

$$\hat{v}(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right)\right] \geq p \text{ for some } \nu \in \mathcal{U}\right\}, \tag{2.3.1}$$

with $\kappa \in \mathbb{R}_+ \cup \{+\infty\}$.

The aim of this section is to convert the problem (2.3.1) into the class of standard stochastic target problems as defined in Section 2.2. The dynamic programming equation for the target reachability with controlled expected loss will then be deduced directly from Theorem 2.2.5 above.

Following [Bouchard 09], we introduce an additional controlled state variable

$$P_{t,p}^{\alpha,\chi}(s) := p + \int_t^s \alpha_r \cdot dW_r + \int_t^s \int_E \chi_{s,e} \tilde{J}(de, ds), \quad s \in [t, T],$$

where the additional controls α, χ are $\mathbb{F}$-predictable measurable processes, with $\chi \in \mathbb{H}_\lambda^2$ and α is $\mathbb{R}^d$-valued and such that $\mathbb{E}\left[\int_0^T |\alpha_s|^2 ds\right] < \infty$. We denote by $\mathcal{A}$ the collection of such processes (α, χ). For $\hat{\nu} := (\nu, \alpha, \chi)$, we then set $\hat{X}^{\hat{\nu}} := (X^\nu, P^{\alpha,\chi})$. We also define $\hat{\mathbf{X}} := \mathbf{X} \times L, \hat{U} := U \times \mathbb{R}^d \times \mathbb{L}_\lambda^2$, and denote by $\hat{\mathcal{U}} = \mathcal{U} \times \mathcal{A}$ the corresponding set of admissible controls. Abusing notations, we also set $Y^{\hat{\nu}} = Y^\nu$. Finally, we introduce the function

$$\hat{V}(\hat{x}, y) := \Psi(x, y) - p, \quad \text{for} \quad y \geq -\kappa \quad \text{and} \quad \hat{x} = (x, p) \in \overline{(\mathbf{X} \times L)}.$$

We make the following assumption, which allows us to use the stochastic integral representation theorem.

Assumption 2.3.1. *$\Psi\left(X_{t,x}^\nu(T), Y_{t,x,y}^\nu(T)\right)$ is square integrable for all initial conditions $(t, x, y) \in [0, T] \times \boldsymbol{X} \times [-\kappa, +\infty)$ and controls $\nu \in \mathcal{U}$.*

Following the arguments of [Bouchard 09], we can now relate $\hat{v}$ to a stochastic target problem with unbounded controls, and controls taking the form of measurable functions on E.

Proposition 2.3.2. *For all $(t, \hat{x}) := (t, x, p) \in [0, T] \times \hat{\boldsymbol{X}}$, we have*

$$\hat{v}(t, \hat{x}) = u(t, \hat{x}) = w(t, \hat{x}),$$

where

$$u(t, x, p) := \inf\left\{y \geq -\kappa : \hat{V}\left(\hat{X}_{t,\hat{x}}^{\hat{\nu}}(T), Y_{t,x,y}^{\hat{\nu}}(T)\right) \geq 0 \textit{ for some } \hat{\nu} \in \hat{\mathcal{U}}\right\} \quad (2.3.2)$$

$$w(t, x, p) := \inf\left\{\begin{array}{l} y \geq -\kappa : \widehat{V}\left(\hat{X}_{t,\hat{x}}^{\hat{\nu}}(T), Y_{t,x,y}^{\hat{\nu}}(T)\right) \geq 0 \\ \textit{and } P_{t,p}^{\alpha,\chi} \in L \textit{ for some } \hat{\nu} \in \hat{\mathcal{U}} \end{array}\right\}. \quad (2.3.3)$$

Proof. <u>step 1:</u> We first show that $\hat{v} \geq u$. For $y > \hat{v}(t, x, p)$, we may find $\nu \in \mathcal{U}$ such that $p_0 := \mathbb{E}\left[\Psi\left(X_{t,x}^\nu(T), Y_{t,x,y}^\nu(T)\right)\right] \geq p$. By the stochastic integral representation theorem, recall Assumption (2.3.1), there exists $(\alpha, \chi) \in \mathcal{A}$ such that

$$\Psi\left(X_{t,x}^\nu(T), Y_{t,x,y}^\nu(T)\right) = p_0 + \int_t^T \alpha_s \cdot dW_s + \int_t^T \int_E \chi_{s,e} \tilde{J}(de, ds) = P_{t,p_0}^{\alpha,\chi}(T).$$

Since $p_0 \geq p$, it follows that $\Psi\left(X_{t,x}^\nu(T), Y_{t,x,y}^\nu(T)\right) \geq P_{t,p}^{\alpha,\chi}(T)$, and therefore $y \geq u(t, x, p)$ from the definition of the problem u.

step 2: We next show that $u \geq \hat{v}$. For $y > u(t,x,p)$, we have $\widehat{V}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0$ for some $\hat{\nu} = (\nu, \alpha, \chi) \in \hat{\mathcal{U}}$. It follows that

$$\mathbb{E}\left[\widehat{V}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T), Y^{\hat{\nu}}_{t,x,y}(T)\right)\right] = \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) - P^{\alpha,\chi}_{t,p}(T)\right] \geq 0,$$

and since $P^{\alpha,\chi}_{t,p}$ is a martingale

$$\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p = \mathbb{E}\left[P^{\alpha,\chi}_{t,p}(T)\right],$$

so that $y \geq \hat{v}(t,x,p)$ by the definition of $\hat{v}$.

step 3: The inequality $u \leq w$ is obvious. To see that the converse inequality holds, consider some $y > u(t,x,p)$. Then there exists some $\hat{\nu} = (\nu, \alpha, \chi) \in \hat{\mathcal{U}}$ such that

$$\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq P^{\alpha,\chi}_{t,p}(T). \tag{2.3.4}$$

Define

$$\tau := T \wedge \inf\left\{s > t : P^{\alpha,\chi}_{t,p}(s) \leq m\right\} \text{ and}$$
$$\tilde{\alpha}_s := \alpha_s \mathbf{1}_{\{s \leq \tau\}},$$
$$\tilde{\chi}_{s,e} := \left[-\left(\chi_{s,e} \vee \left(m - P^{\alpha,\chi}_{t,p}(s-)\right)\right)^{-} + (\chi_{s,e})^{+}\right] \mathbf{1}_{\{s \leq \tau\}} \text{ for } s \in [t,T].$$

Clearly, $P^{\alpha,\chi}_{t,p}(T) = P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T)$ on the event $\{\tau = T\}$. Since $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T) = m$ on the event $\{\tau < T\}$, it follows from (2.3.4) that

$$\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T).$$

We finally observe that $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T) \geq m$ by the definition of $\tilde{\alpha}$ and $\tilde{\chi}$, and that the last inequality implies that $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}(T) \leq M$. By the martingale property of the process $P^{\tilde{\alpha},\tilde{\chi}}_{t,p}$, we conclude that it is valued in the interval $[m, M] = L$. Hence, $y \geq w(t,x,p)$. □

Let us observe that the problem (2.3.2) can be alternatively formulated as

$$\hat{v}(t,x,p) = \inf\left\{y \geq -\kappa : Y^{\hat{\nu}}_{t,x,y}(T) \geq \hat{g}\left(\hat{X}^{\hat{\nu}}_{t,\hat{x}}(T)\right) \text{ for some } \hat{\nu} = (\nu, \alpha, \chi) \in \hat{\mathcal{U}}\right\}$$

where $\hat{g}$ is the generalized inverse of $\hat{V}$ at 0

$$\hat{g}(\hat{x}) := \inf\left\{y : \widehat{V}(\hat{x}, y) \geq 0\right\}.$$

Remark 2.3.3. 1. In the case where the infimum in the definition of $\hat{v}(t,x,p)$ is achieved and there exists a control $\nu \in \mathcal{U}$ satisfying

$$\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] = p$$

with $y = \hat{v}(t,x,p)$, the above argument shows that the corresponding process $P^{\alpha,\chi}_{t,p}$ coincides with the conditional expectation of $\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)$, i.e.

$$P^{\alpha,\chi}_{t,p}(s) = \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \middle| \mathcal{G}_s\right] \quad \text{for } s \in [t,T].$$

2. Equation (2.3.3) shows that one can restrict to controls α and χ such that $P_{t,p}^{\alpha,\chi}$ takes values in L. This is rather natural since the latter should be interpreted as a conditional expectation of Ψ, which convex hull is L, and this corresponds to the natural domain $[m, M]$ of the variable p. Notice also that the value function $\hat{v}(\cdot, p)$ is constant for $p < m$, and equal ∞ for $p > M$. In both cases, the natural domain of $\hat{v}$ is therefore $[0, T] \times \overline{\mathbf{X}} \times [m, M]$.

3. Moreover, in the special case where m and/or M are finite, the fact that $P_{t,p}^{\alpha,\chi}$ takes values in L allows us to consider that the jump coefficient χ is bounded. This will be useful in the proofs of Section 2.3.5. Indeed we may write in that particular case
$$m - P_{t,p}^{\alpha,\chi}(s-) \leq \chi_s \leq M - P_{t,p}^{\alpha,\chi}(s-),$$
with $P_{t,p}^{\alpha,\chi}(s-) \in [m, M]$.

Example 2.3.1. Given a non-negative function h, let us consider the case where $\check{\Psi}(x, y) = \frac{y}{h(x)} \wedge 1$, with the convention $\frac{y}{0} = +\infty$ for $y \in \mathbb{R}_+$. For $\kappa = 0$, we then obtain

$$\check{v}(t, x, p) = \inf\left\{y \in \mathbb{R}_+ : \mathbb{E}\left[\frac{Y_{t,x,y}^{\nu}(T)}{g\left(X_{t,x}^{\nu}(T)\right)} \wedge 1\right] \geq p \text{ for some } \nu \in \mathcal{U}\right\},$$

which is the problem of the expected success ratio studied in [Föllmer 99]. Using (2.3.2), we see that the above problem reduces to

$$\check{v}(t, x, p) = \inf\left\{y \in \mathbb{R}_+ : \check{V}\left(\hat{X}_{t,x,p}^{\hat{\nu}}(T), Y_{t,x,y}^{\hat{\nu}}(T)\right) \geq 0 \text{ for some } \hat{\nu} = (\nu, \alpha, \chi) \in \hat{\mathcal{U}}\right\},$$

where $\check{V}(x, p, y) = \check{\Psi}(x, y) - p$.

Example 2.3.2. One can similarly recover the problem of stochastic target under controlled probability of success studied in [Bouchard 09] and [Föllmer 99]:

$$\tilde{v}(t, x, p) := \inf\left\{y \in \mathbb{R}_+ : \mathbb{P}\left[\tilde{\Psi}\left(X_{t,x}^{\nu}(T), Y_{t,x,y}^{\nu}\right) \geq 0\right] \geq p \text{ for some } \nu \in \mathcal{U}\right\},$$

for some measurable map $\tilde{\Psi}$ from $\mathbb{R}^{d+1}$ into $\mathbb{R}$ such that, for every fixed $x \in \mathbb{R}^d$, the function $y \mapsto \tilde{\Psi}(x, y)$ is non-decreasing and right-continuous. The reduction of the problem (2.3.2) leads to

$$\tilde{v}(t, x, p) := \inf\left\{y \in \mathbb{R}_+ : \tilde{V}\left(\hat{X}_{t,x,p}^{\hat{\nu}}(T), Y_{t,x,y}^{\hat{\nu}}\right) \geq 0 \text{ for some } \hat{\nu} \in \hat{\mathcal{U}}\right\},$$

where $\tilde{V}(x, p, y) = \mathbf{1}_{\{\tilde{\Psi}(x,y) \geq 0\}} - p$.

2.3.2 PDE characterization in the domain

In view of Proposition 2.3.2, the PDE characterization of Theorem 2.2.5 can be extended to the problem (2.3.1). Let us first introduce notations associated to the augmented system. For $\hat{u} = (u, \alpha, \pi) \in \hat{U}$ and $\hat{x} = (x, p) \in \hat{\mathbf{X}}$, set

$$\hat{\mu}(\hat{x}, \hat{u}) := \begin{pmatrix} \mu_X(x, u) \\ -\int_E \pi(e)\lambda(de) \end{pmatrix}, \quad \hat{\sigma}(\hat{x}, \hat{u}) := \begin{pmatrix} \sigma_X(x, u) \\ \alpha^T \end{pmatrix},$$
$$\hat{\beta}(\hat{x}, \hat{u}(e), e) := \begin{pmatrix} \beta_X(x, u(e), e) \\ \pi(e) \end{pmatrix}.$$

We also introduce the following operators

$$\hat{\mathcal{L}}^{\hat{u}}\varphi(t, \hat{x}) := \partial_t\varphi(t, \hat{x}) + \hat{\mu}(\hat{x}, \hat{u}) \cdot D\varphi(t, \hat{x}) + \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}\hat{\sigma}^T(\hat{x}, \hat{u}) D^2\varphi(t, \hat{x})\right]$$
$$\hat{\mathcal{G}}^{\hat{u},e}\varphi(t, \hat{x}) := \beta_Y(x, \varphi(t, \hat{x}), u(e), e) - \varphi\left(t, \hat{x} + \hat{\beta}(\hat{x}, \hat{u}(e), e)\right) + \varphi(t, \hat{x}).$$

Recalling point 3 of Remark 2.3.3, we also introduce, for $(x, k, q, A) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$, $\hat{u} = (u, \alpha, \pi) \in \hat{U}$, $\varepsilon > 0$ and $\eta \in [-1, 1]$,

$$\hat{N}^{\hat{u}}(\hat{x}, y, q) := \sigma_Y(x, y, u) - \hat{\sigma}(\hat{x}, \hat{u})^T q = N^u(x, y, q_x) - q_p\alpha, \text{ for } q = (q_x, q_p) \in \mathbb{R}^d \times \mathbb{R},$$
$$\widehat{\Delta}^{\widehat{u},e}(t, \widehat{x}, y, \psi) := \beta_Y(x, y, u, e) - \psi\left(t, \widehat{x} + \widehat{\beta}(\widehat{x}, \widehat{u}(e), e)\right) + y$$
$$\hat{\mathcal{N}}_{\varepsilon,\eta}(t, \hat{x}, y, q, \psi) := \left\{ \begin{array}{l} \hat{u} \in \hat{U} \text{ s.t. } \left|\hat{N}^{\hat{u}}(\hat{x}, y, q)\right| \leq \varepsilon, \quad p + \pi(e) \in [m, M] \\ \quad \text{and } \widehat{\Delta}^{\widehat{u},e}(t, \widehat{x}, y, \psi) \geq \eta \quad \text{for } \lambda\text{-a.e. } e \in E \end{array} \right\} \tag{2.3.5}$$
$$\hat{H}_{\varepsilon,\eta}(\hat{\Theta}, \varphi) := \sup_{\hat{u} \in \hat{\mathcal{N}}_{\varepsilon,\eta}(t, \hat{x}, y, q, \varphi)} \hat{\mathbf{A}}^{\hat{u}}(\hat{\Theta}) \tag{2.3.6}$$

where

$$\hat{\Theta} := (t, \hat{x}, y, k, q, A)$$
$$\hat{\mathbf{A}}^{\hat{u}}\left(\hat{\Theta}\right) := -k + \mu_Y(x, y, u) - \hat{\mu}(\hat{x}, \hat{u}) \cdot q - \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}\hat{\sigma}^T(\hat{x}, \hat{u}) A\right]$$

and

$$\hat{\mathbf{N}}(t, \widehat{x}, y, q, \psi) := \left\{ \begin{array}{l} (r, s) \in \mathbb{R}^d \times \mathbb{R} : \exists\, \widehat{u} \in \widehat{U} \text{ s.t. } r = \widehat{N}^{\widehat{u}}(\widehat{x}, y, q) \\ \quad \text{and } s \leq \widehat{\Delta}^{\widehat{u},e}(t, \widehat{x}, y, \psi) \quad \text{for } \lambda\text{-a.e. } e \in E \end{array} \right\},$$
$$\hat{\delta} := \mathrm{dist}\left(0, \hat{\mathbf{N}}^c\right) - \mathrm{dist}\left(0, \hat{\mathbf{N}}\right).$$

The operators $\hat{H}^*, \hat{H}_*\widehat{\delta}^*$ and $\widehat{\delta}_*$ are constructed from $\hat{H}_{\varepsilon,\eta}$ and $\widehat{\delta}$ exactly as H^*, H_*, δ^* and δ_* are defined from $H_{\varepsilon,\eta}$ and δ. Finally, we define the function

$$\hat{g}(\hat{x}) := \inf\left\{y \geq -\kappa : \hat{V}(\hat{x}, y) \geq 0\right\}, \quad \hat{x} = (x, p) \in \mathbf{X} \times [m, M].$$

As an almost direct consequence of Theorems 2.2.5 and 2.3.2, we obtain the viscosity property of $\hat{v}$ under the following assumptions, which are the analogs of Assumptions

2.2.3 and 2.2.4 for the augmented control system $\hat{\mathbf{X}}$. Define then as previously for any $\varphi \in C^{1,2,2}([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R})$, $\hat{u}\in\hat{U}$ and $(t,\hat{x},z_1,z_2)\in[0,T]\times\mathbb{R}^{2d+3}$

$$\hat{\mathcal{L}}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z) := \mathcal{L}^u\varphi(t,\hat{x}) - \hat{\mu}(\hat{x},u)\cdot z_1 - \mu_Y(\hat{x},y,u)z_2, \tag{2.3.7}$$

where $z =: (z_1,z_2)\in\mathbb{R}^{d+1}\times\mathbb{R}$ and $\bar{\varphi}(t,\hat{x},z) := \varphi(t,\hat{x}) - |z|^2$.

Assumption 2.3.4. *For all $\varepsilon>0,\eta\in[-1,1],(t_o,x_o,p_o)\in[0,T]\times\mathbb{R}^d\times[m,M],\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ and finite C_1 satisfying*

$$\sup_{\hat{u}\in\mathcal{N}_{\varepsilon,\eta}(t,\hat{x},y,D\varphi,\varphi)}\left\{\mu_Y(x,y,u)-\hat{\mathcal{L}}^{\hat{u}}\varphi(t,\hat{x})\right\}\le 2C_1$$
$$\text{for all } (t,\hat{x})\in B_\varepsilon(t_o,\hat{x}_o) \text{ and } y\in\mathbb{R} \text{ s.t. } |y-\varphi(t,\hat{x})|\le\varepsilon,$$

there exists $\varepsilon'>0,\eta'\in[-1,1]$ and a finite C_2 such that

$$\begin{aligned}&\sup_{\hat{u}\in\mathcal{N}_{\varepsilon',\eta'}(t,\hat{x},y,D\varphi,\varphi)}\left\{\mu_Y(x,y,u)-\mathcal{L}^u_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z)\right\}\le 2C_1+|C_1|\\ &\text{for all } (t,\hat{x},y,z)\in[0,T]\times\mathbb{R}^{2d+4} \text{ s.t. } \begin{cases}(t,\hat{x},z)\in B_{\varepsilon'}(t_o,\hat{x}_o,0)\\ y\in\mathbb{R} \text{ s.t. } |y-\bar{\varphi}(t,\hat{x})|\le\varepsilon',\end{cases}\end{aligned}\tag{2.3.8}$$

and

$$\begin{aligned}&\frac{\left[\mu_Y(x,y,u)-\mathcal{L}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}(t,\hat{x},z)\right]^+}{1+|N^u(x,y,D\varphi)|}\le C_2\left(1+|\sigma_Y(x,y,u)|+\sum_{i=1}^d\left|\hat{\sigma}^{i,\cdot}(\hat{x},u)\right|\right)\\ &\text{for all } (t,\hat{x},y,z)\in[0,T]\times\mathbb{R}^{2d+4} \text{ s.t. } \begin{cases}(t,\hat{x},z_1,z_2)\in B_{\varepsilon'}(t_o,\hat{x}_o,0)\\ y\in\mathbb{R} \text{ s.t. } |y-\bar{\varphi}(t,\hat{x},z)|\le\varepsilon',\end{cases}\\ &\text{and } u\in U \text{ such that } \Delta^{u,\cdot}(t,\hat{x},y,\varphi)\ge\eta\ \lambda\text{-a.e.}\end{aligned}\tag{2.3.9}$$

Assumption 2.3.5. *(Continuity of $\hat{\mathcal{N}}_{0,\eta}(t,x,p,y,q,f)$) Let B be a subset of $[0,T]\times\boldsymbol{X}\times[m,M]\times\mathbb{R}\times\mathbb{R}^{d+1}$, $f\in\mathcal{C}^0([0,T]\times\boldsymbol{X}\times[m,M])$ and $\eta>0$ such that $\hat{\mathcal{N}}_{0,2\eta}(\cdot,f)\neq\emptyset$ on B. Then, for every $\varepsilon>0$, $(t_0,x_0,p_0,y_0,q_0)\in Int(B)$ and $\hat{u}_0\in\hat{\mathcal{N}}_{0,2\eta}(t_0,x_0,p_0,y_0,q_0,f)$, there exists an open neighborhood B' of (t_0,x_0,p_0,y_0,q_0) and a locally Lipschitz map $\hat{\nu}$ defined on B' such that $|\hat{\nu}(t_0,x_0,p_0,y_0,q_0)-\hat{u}_0|\le\varepsilon$, and $\hat{\nu}(t,x,p,y,q)\in\hat{\mathcal{N}}_{0,\eta}(t,x,y,p,q,f)$ on B'.*

As in Section 2.2.2, we shall need to define the definition of viscosity solution we shall use in this framework.

Definition 2.3.6. *We say that a l.s.c. (resp. u.s.c.) function U (resp. V) is a viscosity supersolution of $\hat{H}^*U\ge0$ (resp. subsolution of $\hat{H}_*V\le0$) on $[0,T)\times\mathbb{R}^d\times\mathbb{R}$ if for every smooth function $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$ of linear growth and $(t_o,x_o,p_o)\in[0,T)\times\mathbb{R}^d\times\mathbb{R}$ such that $\min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}}(U-\varphi)=(U-\varphi)(t_o,x_o,p_o)=0$ (resp. $\max_{[0,T]\times\mathbb{R}^d\times\mathbb{R}}(V-\varphi)=(V-\varphi)(t_o,x_o,p_o)=0$), we have*

$$H^*\varphi(t_o,x_o,p_o)\ge0 \qquad (\textit{resp. } H_*\varphi(t_o,x_o,p_o)\le0).$$

Corollary 2.3.7. *Under Assumption 2.3.4, the function $\hat{v}_*$ is a viscosity supersolution of*

$$\hat{H}^* \hat{v}_* \geq \ \text{on } [0,T) \times \hat{\boldsymbol{X}}. \tag{2.3.10}$$

Under the additional 2.3.5, $\hat{v}^*$ is a viscosity subsolution of

$$\min \left\{ \hat{v}^* + \kappa, \hat{H}_* \hat{v}^* \right\} \leq 0 \ \text{on } [0,T) \times \hat{\boldsymbol{X}}. \tag{2.3.11}$$

The supersolution property is a direct consequence of Theorem 2.2.5, the representation (2.3.2) and point 3 of Remark 2.3.3. The subsolution property is obtained similarly.

Example 2.3.3. In the context of both Examples 2.3.1 and 2.3.2, with the dynamics of Example 2.2.1, if the conditions of Corollary 2.3.7 are satisfied. By direct computations, we then have that both $\tilde{v}_*$ and $\tilde{v}_*$ are viscosity supersolution on $[0,T) \times \mathbf{X}$ of

$$\begin{aligned} 0 \leq & -\partial_t \varphi - \frac{1}{2}\sigma^2 D_{xx}\varphi \\ & - \inf_{\substack{\pi \in \Pi(p) \\ \alpha \in \mathbb{R}^d}} \left\{ \frac{1}{2} |\alpha|^2 D_{pp}\varphi + \operatorname{Tr}\left[\sigma \alpha D_{xp}\varphi\right] - \alpha \left(D_p \varphi\right)^T \sigma^{-1}\mu - D_p\varphi \int_E \pi(e)\lambda(de) \right\}, \end{aligned} \tag{2.3.12}$$

whenever $D_{pp}\varphi > 0$, and with

$$\Pi(p) := \left\{ \begin{array}{c} \pi \in \mathbb{L}^2_\lambda \text{ s.t., for } \lambda\text{-a.e. } e \in E, \ p + \pi \in [0,1] \\ \text{and } \left(D_x\varphi + \sigma^{-1} D_p \varphi \alpha\right) \beta(\cdot, e) - \varphi\left(\cdot, \cdot + \beta(\cdot, e), \cdot + \pi(e)\right) + \varphi \geq 0 \end{array} \right\}.$$

Notice in this particular context that the process X is not influenced by the control ν. Hence, as in Example 2.2.2, the integrability issues due to the possible unboundedness of ν are handled by controlling $\frac{\mu_Y(u)}{\sigma_Y(u)}$. We refer to Section 2.2.3.1 and Chapters 3 and 4 for the arguments used for the controls χ and α.

Remark 2.3.8. Let us comment in this remark on the crucial role played by Assumption 2.3.4. Recalling the arguments of step 4 in Section 2.2.3.1, one need the counterpart in the present context of the process L^n to be an uniformly integrable martingale. By [Kazamaki 94, Theorem 2.3], a sufficient condition is that the process $\int \hat{\alpha}^n_s dW_s$ is a BMO-martingale. However, without Assumption 2.3.4, the process $\hat{\alpha}^n$ may fail to have enough integrability for the stochastic integral to exist.

In order to fix the ideas, consider the case where $d = 1$, the more general case being handled with some linear algebra. For a given control $(\nu, \alpha, \chi) \in \hat{\mathcal{U}}$, we are then interested in the integrability of

$$\frac{\left(\mu_Y - \varphi_t - \mu_X \varphi_x + \varphi_p \int_E \chi^n_s(e)\lambda(de) - \sigma_X \alpha^n \varphi_{xp} - \frac{1}{2}\sigma^2_X \varphi_{x^2} - \frac{1}{2}(\alpha^n_s)^2 \varphi_{p^2}\right)^+}{|\sigma_Y - \sigma_X \varphi_x - \alpha^n_s \varphi_p|},$$

s pas s

where $\varphi_t, \varphi_x, \varphi_p$ denotes the partial derivatives of φ in its t, x and p variables, and $\varphi_{x^2}, \varphi_{xp}$ and φ_{p^2} the second derivatives with respect to x^2, x and p and p^2. We also omitted the parameters in the diffusion coefficients for sake of clarity, but have to keep in mind that μ_Y, σ_Y, μ_X and σ_X are controlled by ν^n which is possibly unbounded.

Roughly speaking, in order to deal with the possible unboundedness of ν and α, one would expect for (2.3.9) to hold to have "some compensation" between μ_Y (resp. σ_X^2, α^2) and σ_Y (resp. $\sigma_X \varphi_x, \alpha\varphi_p$).

Consider the case where X is not influenced by the control ν, and that μ_Y and σ_Y are linear in ν. The compensation between μ_Y and σ_Y is then obvious, and it remains only to deal with the unboundedness of χ^n and α^n. Equation (2.3.8) will then play an important role, since it will imply that $\varphi_{pp} > 0$ on some neighborhood of (t_o, x_o, p_o), which will give us some coercivity and continuity, and will allow us to control by $|\sigma_Y|$ and $|\sigma_X|$ with $\int_{t_n}^{\cdot} \mu_X$ and $\int_{t_n}^{\cdot} \mu_Y$ bounded, which is of important use in step 5 of Section 2.2.3.1.

2.3.3 Boundary conditions and state constraint

In our general context, the natural domain of P is $[m, M]$. In the case where m or M are finite, we need to specify the boundary conditions at the end points m and M. By definition of the stochastic target problem with controlled expected loss, we have

$$\hat{v}(\cdot, M) = v \text{ and } \hat{v}(\cdot, m) = -\kappa, \tag{2.3.13}$$

where

$$v(t,x) := \inf\left\{y \geq -\kappa : \Phi\left(X_{t,x}^{\nu}(T), Y_{t,x,y}^{\nu}(T)\right) \geq 0 \text{ for some } \nu \in \mathcal{U}\right\},$$

with

$$\Phi(x,y) := \Psi(x,y) - M. \tag{2.3.14}$$

Also, since Ψ is non-decreasing in y, we know that $\hat{v}$ is non-decreasing in p. Hence,

$$\begin{aligned} -\kappa \leq \hat{v}_*(\cdot, m) \leq \hat{v}^*(\cdot, p) \leq \hat{v}^*(\cdot, M) \leq v^* \quad \text{for} \quad p \in [m, M], \\ \hat{v}^*(\cdot, p) = -\kappa \quad \text{for} \quad p < m \quad \text{and} \quad \hat{v}^*(\cdot, p) = \infty \quad \text{for} \quad p > M, \end{aligned} \tag{2.3.15}$$

and one can naturally expect that $\hat{v}_*(\cdot, m) = -\kappa$ and $\hat{v}^*(\cdot, M) = v^*$. However, the function $\hat{v}$ may have discontinuities at $p = m$ or $p = M$ and, in general, the boundary conditions have to be stated in a weak form, see (2.3.20) and (2.3.54) below. This corresponds to the classical state-space constraint problems, see [Barles 94, Fleming 06, Soner 86a, Soner 86b] and the references therein.

To obtain a characterization of $\hat{v}$ on these boundaries, we shall appeal to the following additional assumptions. Assumptions 2.3.11 and 2.3.12 already appeared in [Bouchard 09]. Assumptions 2.3.9, 2.3.10 and 2.3.13 will be used to handled the non-local operator. Also notice that Assumption 2.3.12 linked with Assumption 2.3.4.

Assumption 2.3.9. *The following hold.*

(H1) *For some integer $\gamma \geq 1, \hat{v}^*(\cdot, m)^+$ satisfies the growth condition*

$$\sup_{[0,T]\times\mathbb{R}^d} \frac{|w(t,x)|}{1+|x|^\gamma} < \infty. \tag{2.3.16}$$

(H2) *There is a function Λ on $\mathbb{R}^d$ satisfying*

(H2-i) *For all $x \in \boldsymbol{X}$ and $y > \Lambda(x)$, there exists $\bar{u} \in U$ such that*

$$\beta_Y\left(x, y, \bar{u}(e), e\right) - \Lambda\left(x + \beta_X\left(x, \bar{u}(e), e\right)\right) + \Lambda(x) > 0 \quad \textit{for } \lambda\textit{-a.e. } e \in E.$$

(H2-ii) $\Lambda\left(x\right) / \left|x\right|^\gamma \to +\infty$ *as* $|x| \to \infty$.

(H2-iii) $\Lambda \leq -\kappa$ *on* $\boldsymbol{X}$.

Assumption 2.3.10. *The set E is finite and $\lambda(e) > 0$ for all $e \in E$.*

Assumption 2.3.11. *For all $(x, y, q) \in \boldsymbol{X} \times (-\kappa, \infty) \times \mathbb{R}^d$, we have*

$$\{u \in U : N^u(x, y, q) = 0\} \subsetneq U.$$

We need for the next assumption to introduce the following set, for $(x, y, q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$:

$$\tilde{\mathcal{N}}_\varepsilon(x, y, q) := \{u \in U : |N^u(x, y, q)| \leq \varepsilon\}\,. \tag{2.3.17}$$

Assumption 2.3.12. *For all compact subset D of $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, there exists $C > 0$ such that*

$$\sup_{u \in \tilde{\mathcal{N}}_\varepsilon(x,y,q)} \left\{\mu_Y(x, y, u) - k - \mu_X(x, u)\cdot q - \frac{1}{2} Tr\left[\sigma_X \sigma_X^T(x, u) A\right]\right\} \leq C\left(1 + \varepsilon^2\right)$$

for all $\varepsilon > 0$ and $(x, y, k, q, A) \in D$.

Assumption 2.3.13. *The maps β_X, β_Y are continuous on $\boldsymbol{X} \times E$ and $\boldsymbol{X} \times \mathbb{R} \times E$ uniformly in $u \in U$. Moreover, β_X, β_Y and σ_X satisfy the following condition*

$$\operatorname*{ess\,sup}_{u\in U, e\in E} \left\{|\beta_X(\cdot, u(e), e)| + |\beta_Y(\cdot, u(e), e)| + |\sigma_X(\cdot, u)|\right\} \textit{ is locally bounded}$$

Since the main concern of this paper is the analysis of the stochastic target problem under controlled loss with jumps, we do not establish a comparison result of viscosity supersolutions of (2.2.10)-(2.2.13) and subsolutions of (2.2.11)-(2.2.14). Nonetheless, as in [Bouchard 09], we need such a comparison result in order to establish the boundary conditions of this section.

Assumption 2.3.14. *There is a class of functions $\mathcal{C}$ containing all $[-\kappa, +\infty)$ valued functions dominated by v^* such that, for every*

- $v_1 \in \mathcal{C}$, *lower semi-continuous viscosity supersolution of* (2.2.10)-(2.2.13) *on* $[0,T] \times \boldsymbol{X}$
- $v_2 \in \mathcal{C}$, *upper semi-continuous viscosity subsolution of* (2.2.11)-(2.2.14) *on* $[0,T] \times \boldsymbol{X}$

we have $v_1 \geq v_2$.

The main results of this section shows that the natural boundary conditions (2.3.13) indeed holds true, whenever the comparison principle of Assumption 2.3.14 holds and under the above additional conditions.

Theorem 2.3.15. *Assume that Assumptions 2.3.5, 2.3.10 and 2.3.13 hold true.*

(i) *Assume that* $m > -\infty$. *Under Assumptions 2.3.9, and 2.3.11, we have* $\hat{v}^*(\cdot, m) = -\kappa$ *on* $[0,T) \times \boldsymbol{X}$ *and* $\hat{v}_*(\cdot, m) = -\kappa$ *on* $[0,T] \times \boldsymbol{X}$.

(ii) *Assume that* $M < \infty$. *Under Assumptions 2.3.12 and 2.3.4,* $\hat{v}^*(\cdot, M)$ *is a viscosity supersolution of* (2.2.10)-(2.2.13) *on* $[0,T] \times \boldsymbol{X}$. *In particular, if in addition the comparison principle of Assumption 2.3.14 is satisfied, then* $\hat{v}^*(\cdot, M) = \hat{v}_*(\cdot, M) = v_* = v^*$ *on* $[0,T] \times \boldsymbol{X}$.

The proof is reported in Section 2.3.5.

Remark 2.3.16. 1. This subsection is similar to the one in [Bouchard 09], where the authors studied the boundary conditions at $p = 0$ and $p = 1$ in the case of target reachability under controlled probability, i.e. Ψ is of the form $\Psi(x, y) = \mathbf{1}_{\{y \geq g(x)\}}$. In this paper, the natural domain of P is $[0, 1]$, and the authors studied the behavior of the value function $\hat{v}$ when $p \to 0$ and $p \to 1$.

2. Observe that under Assumption 2.3.12, one might omit Assumption 2.3.4 and follow the same reasoning as in Chapters 3 or 4. We however report the proof that Assumption 2.3.4 holds on this particular context in the quoted Chapters.

2.3.4 On the Terminal Condition

The boundary condition at T for $\hat{v}_*$ and $\hat{v}^*$ can be easily derived from the characterization of Theorem 2.2.9.

Corollary 2.3.17. *Under Assumption 2.3.4, the function* $\hat{x} \in \hat{\boldsymbol{X}} \mapsto \hat{v}_*(T, \hat{x})$ *is a viscosity supersolution of*

$$\min\left\{(\hat{v}_*(T,\cdot) - \hat{g}_*)\,\mathbf{1}_{\{\hat{H}^*\hat{v}_*(T,\cdot)<\infty\}}, \hat{\delta}^*\hat{v}_*(T,\cdot)\right\} \geq 0 \text{ on } \hat{\boldsymbol{X}}.$$

If in addition, Assumption 2.3.5 holds, then $\hat{x} \in \hat{\boldsymbol{X}} \mapsto \hat{v}^*(T, \hat{x})$ *is a viscosity subsolution of*

$$\min\left\{\hat{v}^*(T,\cdot) - \hat{g}^*, \hat{\delta}_*\hat{v}^*(T,\cdot)\right\} \leq 0 \text{ on } \hat{\boldsymbol{X}}.$$

The condition $\hat{H}^*\hat{v}_*(T,\cdot) < \infty$ may not be satisfied because the control (α,χ) appearing in the definition of $\hat{H}$ may not be bounded. It follows that the above boundary condition may be useless in most examples.

The rest of this section is devoted to the discussion of conditions under which a precise boundary condition can be specified.

Proposition 2.3.18. (i) *Assume that for all sequence $(t_n, x_n, y_n, p_n, \nu_n)_{n\geq 1}$ of $[0,T)\times \boldsymbol{X}\times\mathbb{R}_+\times[m,M]\times\mathcal{U}$ such that $(t_n,x_n,y_n,p_n)\to(T,x,y,p)\in\{T\}\times\boldsymbol{X}\times\mathbb{R}_+\times[m,M]$, there exists a sequence of $\mathbb{P}$-absolutely continuous probability measure $(\mathbb{Q}^n)_{n\geq1}$, defined by $\frac{d\mathbb{Q}^n}{d\mathbb{P}} =: H^n$ for some sequence of non-negative random variable $(H^n)_{n\geq 1}$, such that*

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}^n}\left[Y^{\nu_n}_{t_n,x_n,y_n}\right] &\leq y,\\ \limsup_{n\to\infty}\mathbb{E}\left[\left|H^n D_p^+ \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right) - D_p^+\circledcirc\hat{g}(x_n,p_n)\right|\right] &= 0 \qquad (2.3.18)\\ \text{and}\quad \liminf_{n\to\infty}\mathbb{E}\left[H^n\circledcirc\hat{g}\left(X^{\nu_n}_{t_n,x_n}(T),p_n\right)\right] &\geq \circledcirc\hat{g}(x,p), \end{aligned}$$

where D_p^+ stands for the right derivative in p. Then, $\hat{v}_*(T,x,p)\geq\circledcirc\hat{g}(x,p)$ for all $(x,p)\in\boldsymbol{X}\times[0,1]$.

(ii) *Let the conditions (ii) of Theorem 2.3.15 hold true and assume that $\hat{v}^*$ is convex in its p-variable and that $v^*(T,x)\leq g(x)$. Then $\hat{v}^*(T,x,p)\leq\circledcirc\hat{g}(x,p)$ for all $(x,p)\in\boldsymbol{X}\times[m,M]$.*

Proof. (i) Given a sequence $(t_n,x_n,p_n)_{n\geq1}$ in $[0,T)\times\mathbf{X}\times(m,M)$ such that $(t_n,x_n,p_n)\to(T,x,p)$ and $\hat{v}(t_n,x_n,p_n)\to\hat{v}_*(T,x,p)$ as $n\to\infty$, we can find $\hat{\nu}_n=(\nu_n,\alpha_n,\chi_n)\in\hat{\mathcal{U}}$ such that

$$\hat{V}\left(\hat{X}^{\hat{\nu}_n}_{t_n,x_n,p_n}(T),Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\right)\geq 0,$$

where $y_n := \hat{v}(t_n,x_n,p_n)+n^{-1}\to\hat{v}_*(T,x,p)$, recall (2.3.2). This implies that

$$Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\geq\hat{g}\left(\hat{X}^{\hat{\nu}_n}_{t_n,x_n,p_n}(T)\right),$$

and, by the definition of the convex hull of $\hat{g}$,

$$H^nY^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\geq H^n\circledcirc\hat{g}\left(\hat{X}^{\hat{\nu}_n}_{t_n,x_n,p_n}(T)\right).$$

Using the convexity of $\circledcirc \hat{g}$ then leads to

$$
\begin{aligned}
& H^n Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T) \\
& \geq H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) + H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right)\left(P^{\alpha_n,\chi_n}_{t_n,p_n}(T) - p_n\right) \\
& = H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) + D^+_p \circledcirc \hat{g}\left(x_n, p_n\right) P^{\alpha_n,\chi_n}_{t_n,p_n}(T) \\
& \quad - H^n p_n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) \\
& \quad + P^{\alpha_n,\chi_n}_{t_n,p_n}(T)\left[H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) - D^+_p \circledcirc \hat{g}\left(x_n, p_n\right)\right] \\
& \geq H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) + D^+_p \circledcirc \hat{g}\left(x_n, p_n\right) P^{\alpha_n,\chi_n}_{t_n,p_n}(T) \\
& \quad - H^n p_n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) \\
& \quad - M\left|H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) - D^+_p \circledcirc \hat{g}\left(x_n, p_n\right)\right|,
\end{aligned}
$$

where the last inequality follows from the fact that we can always assume that $P^{\alpha_n,\chi_n}_{t_n,p_n}$ takes values in $[m, M]$, see (2.3.3). Taking the expectation under $\mathbb{P}$ and using the fact that $P^{\alpha_n,\chi_n}_{t_n,p_n}$ is a $\mathbb{P}$-martingale, we obtain

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}^n}\left[Y^{\hat{\nu}_n}_{t_n,x_n,y_n}(T)\right] \\
& \geq \mathbb{E}\left[H^n \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) + p_n\left(D^+_p \circledcirc \hat{g}\left(x_n, p_n\right) - H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right)\right)\right. \\
& \quad \left. - M\left|H^n D^+_p \circledcirc \hat{g}\left(X^{\nu_n}_{t_n,x_n}(T), p_n\right) - D^+_p \circledcirc \hat{g}\left(x_n, p_n\right)\right|\right].
\end{aligned}
$$

Passing to the limit, and using (2.3.18) leads to $\hat{v}_*(T, x, p) \geq \circledcirc \hat{g}(x, p)$.

(ii) Using (2.3.15) and the convexity of $\hat{v}^*$ together with the definition of the convex hull of a function lead to the required result. □

Example 2.3.4. In the context of Example 2.3.1, we may easily notice that the generalized inverse of $\check{V}$ at 0,

$$\check{g}(x, p) := \inf\left\{y \geq -\kappa : \check{V}(x, p, y) \geq 0\right\},$$

satisfies

$$\check{g}(x, p) = pg(x)$$

and is convex in p. Moreover, for the dynamics of Example 2.2.1, the convexity of $\check{v}$ in its p-variable is quite obvious, since $Y^{\nu}_{t,x,\mu y}(T) = \mu Y^{\nu}_{t,x,y}(T)$ for any $\mu \in [0, 1]$, and the expectation operator is linear.

We have already shown in Section 2.3.2 that $\check{v}_*$ is a supersolution of (2.3.12). If the condition of Corollary 2.3.7 ((see Chapter 3),) and (i) of Proposition 2.3.18 are satisfied we deduce that $\check{v}_*$ satisfies the boundary conditions

$$
\begin{aligned}
\check{v}_*(\cdot, 1) = v \text{ and } \check{v}_*(\cdot, 0) = 0 \text{ on } [0, T) \times \mathbf{X} \\
\text{and } \check{v}_*(T, x, p) \geq pg(x) \text{ on } \mathbf{X} \times [0, 1].
\end{aligned}
\tag{2.3.19}
$$

Example 2.3.5. In the context of Example 2.3.2, we define the function

$$\tilde{g}(x, p) := \inf\left\{y \geq -\kappa : \tilde{V}(x, p, y) \geq 0\right\}$$

and let $\tilde{\psi}$ be the generalized inverse of $\tilde{\Psi}$ at 0, i.e.

$$\tilde{\psi}(x) := \inf\left\{y \geq -\kappa : \tilde{\Psi}(x,y) \geq 0\right\}.$$

Then, $\tilde{g}(x,p) = \tilde{\psi}(x)\mathbf{1}_{\{p>0\}}$ for $x \in \mathbf{X}$ and $p \in [0,1]$. The convexity of $\tilde{v}$ is far from being obvious. However, one may notice that the convex hull of $\tilde{g}$ in p is $\circledcirc(\hat{g})(x,p) = pg(x)$, with $g = \tilde{\psi}^{-1}$, and that the condition of Corollary 2.3.7 and (i) of Proposition 2.3.18 are satisfied. It follows that, as for the expected success ratio problem of Example 2.3.4 above, $\tilde{v}_*$ is a viscosity supersolution on $[0,T] \times \mathbf{X} \times [0,1]$ of (2.3.12) - (2.3.19).

Remark 2.3.19. In [Bouchard 09], the authors considered the case $\hat{g}(x,p) = g(x)\mathbf{1}_{\{p>0\}}$, so that $\circledcirc\hat{g}(x,p) = pg(x)$, and therefore $D_p^+ \circledcirc \hat{g}(x,p) = g(x)$. Then, Assumption 2.3.18, in the case of [Bouchard 09], should take the form:

$$\limsup_{n\to\infty} \mathbb{E}\left[\left|H^n g\left(X^{\nu_n}_{t_n,x_n}(T)\right) - g(x)\right|\right] = 0.$$

The Assumption 2.3.18 is then almost the counterpart of the one made in their proposition 3.2. The difference comes from a slight error in their proof [2] where they use the fact that $P^{\alpha_n,\chi_n}_{t_n,p_n}$ is a $\mathbb{Q}$-martingale while it is only a $\mathbb{P}$-martingale, *a priori*.

2.3.5 Derivation of the boundary conditions for the stochastic target with controlled expected loss

We now prove Theorem 2.3.15. These boundary conditions need only to be specified in the case where m and/or M are finite.

2.3.5.1 The endpoint $p = M$, finite

In order to show that $\hat{v}_*(\cdot, M)$ is a viscosity supersolution of (2.2.10)-(2.2.13), it suffices to show that $\hat{v}_*(\cdot, M)$ is a viscosity supersolution on $[0,T) \times \mathbf{X}$ of

$$\max\left\{\hat{v}_*(\cdot, M) - v_*, H^*\hat{v}_*(\cdot, M)\right\} \geq 0, \tag{2.3.20}$$

and that $\hat{v}_*(T, \cdot, M)$ is a viscosity supersolution on $\mathbf{X}$ of

$$\max\left\{\begin{array}{c}\hat{v}_*(T,\cdot,M) - v_*, \\ \min\left\{\left(\hat{v}_*(T,\cdot,M) - j_*\right)\mathbf{1}_{\{H^*\hat{v}_*(T,\cdot,M)<\infty\}}, \delta^*\hat{v}_*(T,\cdot,M)\right\}\end{array}\right\} \geq 0, \tag{2.3.21}$$

where j is the generalized inverse of Φ at 0:

$$j(x) := \inf\left\{y \geq -\kappa : \Phi(x,y) \geq 0\right\},$$

recall (2.3.14).

To convince ourself, let us show for instance that (2.3.20) implies (2.2.10). Let (t_0, x_0) be a local minimizer of $\hat{v}_*(\cdot, M) - \varphi$ for some smooth function φ of linear growth. Then

[2] The author would like to thank Bruno Bouchard, Romuald Elie and Nizar Touzi for pointing out this issue and for their explanations on how to fix it in their particular context.

- either $\hat{v}_*(t_0, x_0, M) < v_*(t_0, x_0)$ and then (2.2.10) holds for φ at (t_0, x_0)
- or $\hat{v}_*(t_0, x_0, M) = v_*(t_0, x_0)$ so that (t_0, x_0) is a local minimizer of $v_* - \varphi$, and (2.2.10) holds for φ at (t_0, x_0) by the viscosity property of v_*, see Theorem 2.2.5.

step 1: We first show that for any smooth function φ of linear growth on $[0,T] \times \mathbf{X} \times [m, M]$ and $(t_0, x_0) \in [0, T) \times \mathbf{X}$ such that

$$\text{(strict)} \min_{[0,T)\times\mathbf{X}\times[m,M]} (\hat{v}_* - \varphi) = (\hat{v}_* - \varphi)(t_0, x_0, M) = 0, \tag{2.3.22}$$

we have

$$\max\left\{\varphi(t_0, x_0, M) - v_*(t_0, x_0), \hat{H}^*\varphi(t_0, x_0, M)\right\} \geq 0.$$

If not, we can find $\eta, \varepsilon, \iota > 0$ such that

$$\begin{aligned} &\max\left\{\varphi_\iota - v_*(t,x), \mu_Y(x,y,u) - \hat{\mathcal{L}}^{\hat{u}}\varphi_\iota(t,x,p), \right\} \leq -2\eta \\ &\quad \text{for all} \quad \hat{u} := (u, \alpha, \pi) \in \hat{\mathcal{N}}_{\varepsilon,-\eta}(t, x, y, D\varphi_\iota(t,x,p), \varphi_\iota) \\ &\quad \text{and} \quad (t, x, p, y) \in [0, T) \times \mathbf{X} \times (m, M] \times \mathbb{R} \\ &\text{s.t.} \quad (t, x, p) \in B_\varepsilon(t_0, x_0) \times [M - \varepsilon, M] \quad \text{and} \quad |y - \varphi_\iota(t, x, p)| \leq \frac{\eta}{2}, \end{aligned} \tag{2.3.23}$$

with $\varphi_\iota(t, x, p) := \varphi(t, x, p) - f_\iota(x) - g_\iota(p)$, f_ι defined as in (2.2.18), and

$$g_\iota : p \in [m, M] \longmapsto \frac{2\iota}{\pi} \int_0^{\pi|p-M|} \sin^2 u du \mathbb{1}_{\{|p-M|\leq 1\}} + \iota \mathbb{1}_{\{|p-M|>1\}},$$

recall (2.2.19), and observe that the same results hold for g_ι. We now define as previously for all $z \in \mathbf{X} \times [m, M] \times \mathbb{R}$

$$\bar{\varphi}_\iota(t, \hat{x}, z) := \varphi_\iota(t, \hat{x}) - |z|^2.$$

By Assumption 2.3.4, there exists a finite constant $C > 0$ such that, after possibly changing $\varepsilon, \eta > 0$, we have

$$\begin{aligned} &\mu_Y(x, y, u) - \hat{\mathcal{L}}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}_\iota(t, \hat{x}, z) \leq -\eta \\ &\text{for all } (t, \hat{x}, z, y) \in [0, T) \times (\mathbf{X} \times [m, M])^2 \times \mathbb{R}^2 \text{ s.t. } \begin{cases} (t, \hat{x}, z) \in B_\varepsilon(t_0, \hat{x}_0, 0) \\ |y - \bar{\varphi}_\iota(t, \hat{x}, z)| \leq \frac{\eta}{4}, \end{cases} \\ &\text{for all } u \in \mathcal{N}_{\varepsilon,-\eta}(t, \hat{x}, y, D\varphi_\iota(t, x), \varphi_\iota) \end{aligned}$$

and

$$\begin{aligned} &\frac{\left[\mu_Y(x, y, u) - \hat{\mathcal{L}}^{\hat{u}}_{\hat{X},Z}\bar{\varphi}_\iota(t, \hat{x}, z)\right]^+}{1 + |N^u(x, y, D\varphi_\iota|} \leq C\left(1 + |\sigma_Y(x, y, u)| + \sum_{i=1}^d \left|\hat{\sigma}^{i,\cdot}(\hat{x}, u)\right|\right) \\ &\text{for all } (t, \hat{x}, z) \in B_\varepsilon(t_o, \hat{x}_o, 0) \text{ and } y \in \mathbb{R} \text{ s.t. } |y - \bar{\varphi}_\iota(t, \hat{x}, z)| \leq \frac{\eta}{4} \\ &\text{and for all } u \in U \text{ s.t. } \Delta^{u,\cdot}(t, x, y, \varphi_\iota) \geq -\eta \ \lambda\text{-a.e.} \end{aligned}$$

Let $(t_n, x_n, p_n, z_n)_n$ be a sequence in $[0,T) \times (\mathbf{X} \times (m, M))^2$ which converges to $(t_0, x_0, M, 0)$ and such that $\hat{v}(t_n, x_n, p_n) \to \hat{v}_*(t_0, x_0, M)$. Set $y_n := \hat{v}(t_n, x_n, p_n) + n^{-1}$ and observe that

$$\gamma_n := y_n - \varphi_\iota(t_n, x_n, p_n) \to 0.$$

For each $n \geq 1$, we have $y_n > \hat{v}(t_n, x_n, p_n)$. Then, by **(GDPj1)**, there exists some $\hat{\nu}^n := (\nu^n, \alpha^n, \chi^n) \in \hat{\mathcal{U}}$ such that

$$Y^n(\theta_n) \geq \hat{v}_*\left(\theta_n, X^n(\theta_n), P^n(\theta_n)\right) \geq \bar{\varphi}_\iota\left(\theta_n, X^n(\theta_n), P^n(\theta_n), Z^n(\theta_n)\right)$$

where

$$\theta_n^o := \{s \geq t_n : (s, X^n(s), P^n(s), Z^n(s)) \in D\}$$
$$\theta_n := \left\{s \geq t_n : |Y^n(s) - \varphi_\iota\left(s, X^n(s), P^n(s)\right)| \geq \frac{\eta}{4}\right\} \wedge \theta_n^o$$

together with

z_n ...

$$(X^n, P^n, Y^n, Z^n) := \left(X^{\nu^n}_{t_n,x_n}, P^{\alpha^n,\chi^n}_{t_n,p_n}, Y^{\nu^n}_{t_n,x_n,y_n}, Z^{\hat{\nu}^n}_{t_n,\hat{x}_n,z_n}\right),$$

$$Z^{\hat{\nu}^n}_{t_n,\hat{x}_n,z_n}(s) = z_n + \int_{t_n}^{s} \begin{pmatrix} \hat{\mu}\left(\hat{X}^n(u), \hat{\nu}^n_u\right) \\ \mu_Y\left(\hat{X}^n(u), Y^n(u), \nu^n(u)\right) \end{pmatrix} du$$

and

$$\mathcal{V}_\varepsilon(t_0, x_0, 0) := (\{t_0 + \varepsilon\} \times B_\varepsilon(x_0, 0)) \cup ([t_0, t_0 + \varepsilon) \times \partial B_\varepsilon(x_0, 0))$$
$$D := (\mathcal{V}_\varepsilon(t_0, x_0, 0) \times [M - \varepsilon, M]) \cup (B_\varepsilon(t_0, x_0) \times [M - \varepsilon, M])^c \times B_\varepsilon(0).$$

It follows from (2.3.23) and (2.3.22), recall (2.2.20), that

$$\zeta := \inf_D \left(\hat{v} - \bar{\varphi}_\iota\right) > 0.$$

Using the definition of θ_n and $\zeta > 0$, this implies that

$$Y^n(\theta_n) - \bar{\varphi}_\iota\left(\theta_n, X^n(\theta_n), P^n(\theta_n), Z^n(\theta_n)\right) \geq \zeta \wedge \frac{\eta}{4}.$$

By arguing as in Section 2.2.3.1, this leads to a contradiction.

step 2: We now show (2.3.20), i.e. for any smooth function φ on $[0,T] \times \mathbf{X}$ and $(t_0, x_0) \in [0,T) \times \mathbf{X}$ such that

$$\text{(strict)} \min_{[0,T)\times\mathbf{X}} \left(\hat{v}_*(\cdot, M) - \varphi\right) = \left(\hat{v}_*(\cdot, M) - \varphi\right)(t_0, x_0) = 0,$$

we have

$$\max\left\{\varphi(t_0, x_0) - v_*(t_0, x_0), H^*\varphi(t_0, x_0)\right\} \geq 0. \tag{2.3.24}$$

a. The first step is similar as in [Bouchard 09], up to modifications due the need for linear growth test function in x. For every k, we introduce the smooth function

$$\varphi_k(t, x, p) := \varphi(t, x) - \left(f(x) + (t - t_0)^2 + \psi_k(p)\right),$$

where f is defined as in (2.2.18) with $\iota = 1$, and for some $\rho > 0$,

$$\psi_k(p) := -\rho k \int_p^M \frac{e^{2kM}}{e^{k(r+M)} - e^{2kM+1}} dr, \qquad k > 0. \tag{2.3.25}$$

Observe that

$$\begin{aligned} \psi_k(p) \geq 0 \quad \text{for all} \quad k > 0,\ p \in [m, M], \\ -2\rho k \leq \psi_k'(p) = \rho k \frac{e^{2kM}}{e^{k(p+M)} - e^{2kM+1}} \leq -\frac{\rho k}{2(e-1)} \\ \text{for } k \text{ large enough,} \end{aligned} \tag{2.3.26}$$

$$\psi_k''(p) = -\rho k^2 \frac{e^{k(p+3M)}}{\left(e^{k(p+M)} - e^{2kM+1}\right)^2} < 0 \quad \text{for all } k > 0, \tag{2.3.27}$$

$$\begin{aligned} \lim_{k\to\infty} \frac{(\psi_k'(p_k))^2}{\left|\psi_k''(p_k)\right|} = \rho \quad \text{if } (p_k)_k \text{ is a sequence in } [m, M] \\ \text{s.t.} \quad \lim_{k\to\infty} k(M - p_k) = 0. \end{aligned} \tag{2.3.28}$$

Let (t_k, x_k, p_k) be a minimizer of $\hat{v}_* - \varphi_k$ on $[0, T] \times \overline{B_1^{\mathbf{X}}(x_0)} \times [m, M]$, where $B_1^{\mathbf{X}}(x_0) := B_1(x_0) \cap \mathbf{X}$ and $B_1(x_0)$ is the open unit ball centered at x_0. Observe that, by definition of (t_k, x_k, p_k) and (t_0, x_0),

$$\begin{aligned} &(\hat{v}_*(\cdot, M) - \varphi)(t_0, x_0) \\ &= (\hat{v}_* - \varphi_k)(t_0, x_0, M) \\ &\geq (\hat{v}_* - \varphi_k)(t_k, x_k, p_k) \\ &= (\hat{v}_*(\cdot, p_k) - \varphi)(t_k, x_k) + \left(f(x_k) + (t_k - t_0)^2 + \psi_k(p_k)\right) \\ &\geq (\hat{v}_*(\cdot, p_k) - \varphi)(t_k, x_k) + \left(f(x_k) + (t_k - t_0)^2 + \frac{\rho k}{2(e-1)}(M - p_k)\right), \end{aligned}$$

where the last inequality follows from (2.3.26), for k large enough, and the fact that $\psi_k(M) = 0$. Since $\hat{v}_* \geq -\kappa$ by construction and φ is bounded, this implies that the sequence $(t_k, x_k, p_k)_{k\geq 1}$ is bounded, and therefore converges to some (t_*, x_*, p_*) up to a subsequence. Clearly, $p_* = M$, since otherwise we would have $k(M - p_k) \to \infty$. By definition of (t_0, x_0), this implies that

$$\begin{aligned} &(\hat{v}_*(\cdot, M) - \varphi)(t_0, x_0) \\ &\geq \liminf_{k\to\infty} (\hat{v}_* - \varphi_k)(t_k, x_k, p_k) \\ &\geq (\hat{v}_*(\cdot, M) - \varphi)(t_*, x_*) + \left(f(x_*) + (t_* - t_0)^2 + \liminf_{k\to\infty} \frac{\rho k}{2(e-1)}(M - p_k)\right) \\ &\geq (\hat{v}_*(\cdot, M) - \varphi)(t_0, x_0) + \left(f(x_*) + (t_* - t_0)^2 + \liminf_{k\to\infty} \frac{\rho k}{2(e-1)}(M - p_k)\right). \end{aligned}$$

This shows that, after possibly passing to a subsequence,

$$\begin{aligned} (t_k, x_k, p_k) \to (t_0, x_0, M), \quad k(M - p_k) \to 0, \\ \text{and} \quad \hat{v}_*(t_k, x_k, p_k) \to \hat{v}_*(t_0, x_0, M). \end{aligned} \tag{2.3.29}$$

b. We now go on with the arguments of [Bouchard 09], up to a non trivial adaptation required by the non-local parts of the operator. In order to prove (2.3.20), we assume

$$\hat{v}_* (t_0, x_0, M) - v_* (t_0, x_0) < 0 \tag{2.3.30}$$

and we intend to prove that

$$H^* \varphi (t_0, x_0) \geq 0. \tag{2.3.31}$$

By (2.3.29) and the lower semicontinuity of $\hat{v}_*$, it follows from (2.3.30) that the sequence $(t_k, x_k, p_k)_{k\geq 1}$ of minimizers of the difference $\hat{v}_* - \varphi_k$ satisfies $\varphi_k (t_k, x_k, p_k) - v_* (t_k, x_k) < 0$, after possibly passing to a subsequence. By Corollary 2.3.7 together with the result of step 1, Remark 2.2.10, Assumptions 2.3.13 and 2.3.4, and the fact φ_k is of linear growth in x and p, we deduce that

$$\hat{H}^* \left(t_k, x_k, p_k, \varphi_k, \partial_t \varphi_k, D\varphi_k, D^2\varphi_k, \hat{v}_*\right) \geq 0 \quad \text{for every} \quad k > 1.$$

Now observe that, by (2.3.29), and the definition of φ_k:

$$\begin{aligned} \left(\partial_t \varphi_k, D_x \varphi_k, D^2_{xx}\varphi_k\right)(t_k, x_k, p_k) &\underset{k\to\infty}{\longrightarrow} \left(\partial_t \varphi, D_x \varphi, D^2_{xx}\varphi\right)(t_0, x_0) \\ \left(D_p \varphi_k, D^2_{xp}\varphi_k, D^2_{pp}\varphi_k\right)(t_k, x_k, p_k) &= \left(-\psi'_k (p_k), 0, -\psi''_k (p_k)\right) \quad \forall\ k > 1. \end{aligned} \tag{2.3.32}$$

By definition of $\hat{H}^*$, we can find sequences $(\varepsilon_k)_{k\geq 1}$, $\left(\hat{x}^0_k\right)_{k\geq 1}$, $(y_k)_{k\geq 1}$, $(q_k)_{k\geq 1}$, $(A_k)_{k\geq 1}$ such that $\varepsilon_k > 0, \hat{x}^0_k = \left(x^0_k, p^0_k\right) \in \mathbf{X} \times [m, M], y_k \geq -\kappa, q_k = \left(q^x_k, q^p_k\right) \in \mathbb{R}^d \times \mathbb{R}$, A_k is a symmetric matrix of $\mathbb{S}^{d+1}$, with rows $\left(A^{xx}_k, A^{xp}_k\right) \in \mathbb{S}^d \times \mathbb{R}^d$ and $\left(A^{xpT}_k, A^{pp}_k\right) \in \mathbb{R}^d \times \mathbb{R}$,

$$\begin{aligned} &\varepsilon_k \to 0, \quad \hat{x}^0_k \to (x_0, M) \\ &\text{and} \left|(y_k, q_k, A_k) - \left(\varphi_k, D\varphi_k, D^2\varphi_k\right)(t_k, x_k, p_k)\right| \leq k^{-1}, \end{aligned} \tag{2.3.33}$$

where $\left(t_k, \hat{x}^0_k\right)$ belongs to a compact neighborhood of (t_0, x_0, M), and

$$\hat{H}_{\varepsilon_k, -k^{-1}} \left(t_k, \hat{x}^0_k, y_k, \partial_t \varphi(t_0, x_0), q_k, A_k, \hat{v}_*\right) \geq -k^{-1}. \tag{2.3.34}$$

By the definition of $\hat{H}_{\varepsilon_k, -k^{-1}}$, we may find a sequence

$$(u_k, \alpha_k, \pi_k) \in \hat{\mathcal{N}}_{\varepsilon_k, -2k^{-1}} \left(t_k, \hat{x}^0_k, y_k, q_k, \widehat{v}_*\right)$$

such that

$$\begin{aligned} -\partial_t \varphi (t_0, x_0) + \mu_Y \left(x^0_k, y_k, u_k\right) - \mu_X \left(x^0_k, u_k\right) \cdot q^x_k - \frac{1}{2} \mathrm{Tr}\left[\sigma_X \sigma^T_X \left(x^0_k, u_k\right) A^{xx}_k\right] \\ \geq -2k^{-1} + \frac{1}{2} |\alpha_k|^2 A^{pp}_k + \sigma^T_X \left(x^0_k, u_k\right) A^{xp}_k \cdot \alpha_k - \int_E \pi_k(e) \lambda(de) q^p_k \end{aligned} \tag{2.3.35}$$

and

$$\beta_Y \left(x^0_k, y_k, u_k(e), e\right) - \hat{v}_* \left(t_k, x^0_k + \beta_X \left(x^0_k, u_k(e), e\right), p^0_k + \pi_k(e)\right) + y_k \geq -2k^{-1}$$
$$\text{for } \lambda\text{-a.e. } e \in E. \tag{2.3.36}$$

Recalling (2.3.17), we observe that $(u_k, \alpha_k, \pi_k) \in \hat{\mathcal{N}}_{\varepsilon_k, -2k^{-1}}(t_k, \hat{x}_k, y_k, q_k, \widehat{v}_*)$ implies that $u_k \in \tilde{\mathcal{N}}_{\varepsilon_k + |q_k^p \alpha_k|}(x_k^0, y_k, q_k^x)$. We deduce then from Assumption 2.3.12 and (2.3.35) that, for some constant $C > 0$, (which may change from line to line but does not depend on k or ρ),

$$C\left(1 + |q_k^p \alpha_k|^2\right) \geq \frac{1}{2}|\alpha_k|^2 A_k^{pp} + \sigma_X^T(x_k^0, u_k) A_k^{xp} \cdot \alpha_k - \int_E \pi_k(e)\lambda(de) q_k^p$$
$$\geq \frac{1}{2}|\alpha_k|^2 A_k^{pp} - C|A_k^{xp}||\alpha_k| - \int_E \pi_k(e)\lambda(de) q_k^p \tag{2.3.37}$$

where we have used the condition that $\sup_{u \in U} |\sigma_X(\cdot, u)|$ is locally bounded. From (2.3.26), (2.3.27), (2.3.28), (2.3.29), (2.3.32) and (2.3.33), it follows that

$$A_k^{pp} \to +\infty, \quad A_k^{xp} \to 0, \quad q_k^p \to +\infty \quad \text{and} \quad \frac{(q_k^p)^2}{|A_k^{pp}|} \to \rho \text{ as } k \to \infty. \tag{2.3.38}$$

Recall from (2.3.5) that

$$\pi_k \leq M - p_k \quad \lambda\text{-a.e.}, \tag{2.3.39}$$

where $p_k \in [m, M]$. We may hence consider that $(\pi_k)_{k \geq 1}$ is bounded from above, so that, by (2.3.37) and the fact that $q_k^p, A_k^{pp} > 0$

$$C\left(\frac{1}{A_k^{pp}} + \frac{|q_k^p|^2}{A_k^{pp}}|\alpha_k|^2\right) \geq \frac{1}{2}|\alpha_k|^2 - C\frac{|A_k^{xp}|^2}{A_k^{pp}}|\alpha_k| - C\frac{q_k^p}{A_k^{pp}}.$$

Hence, (2.3.38) leads to

$$0 \geq \limsup_{k \to \infty}\left(\left(\frac{1}{2} - C\rho\right)|\alpha_k|^2 - C\frac{|A_k^{xp}|^2}{A_k^{pp}}|\alpha_k|\right).$$

Taking ρ small enough implies that

$$|\alpha_k| \underset{k \to \infty}{\to} 0. \tag{2.3.40}$$

Moreover, since $k(M - p_k) \to 0$, see (2.3.29), there exists $\epsilon_k \downarrow 0$ such that $k(M - p_k) \leq \epsilon_k$. Recalling (2.3.39), this implies that $\pi_k \leq \frac{\epsilon_k}{k}$, so that, by (2.3.26),

$$q_k^p(\pi_k(e))^+ \to 0 \quad \text{as} \quad k \to \infty \quad \text{for all } e \in E. \tag{2.3.41}$$

Recalling the fact that $\lambda(E) < \infty$ and that $q_k^p > 0$, the above inequalities lead to

$$\left(\int_E \pi_k(e)\lambda(de) q_k^p\right)^+ \to 0. \tag{2.3.42}$$

Also recall that $\frac{|q_k^p|^2}{A_k^{pp}} \to \rho$, see (2.3.38), which combined with (2.3.37), (2.3.38), (2.3.40) and (2.3.42), implies that

$$C\left(1 + \rho A_k^{pp}|\alpha_k|^2\right) \geq \frac{1}{2}|\alpha_k|^2 A_k^{pp} + \left(\int_E \pi_k(e)\lambda(de) q_k^p\right)^-$$

or equivalently

$$C\left(1+\left|q_k^p\right|^2\left|\alpha_k\right|^2\right) \geq \frac{1}{2}\left|\alpha_k\right|^2 \frac{\left|q_k^p\right|^2}{\rho}+\left(\int_E \pi_k(e)\lambda(de)q_k^p\right)^-$$

for some $\rho>0$. Taking ρ small enough leads to

$$\begin{aligned} \left|A_k^{pp}\right|\left|\alpha_k\right|^2 \leq C, \quad \left|q_k^p\right|^2\left|\alpha_k\right|^2 \leq C\rho \\ \text{and} \quad C+C\rho \geq \left(\int_E \pi_k(e)\lambda(de)q_k^p\right)^- . \end{aligned} \tag{2.3.43}$$

We then deduce from the right hand side bound of (2.3.26) and (2.3.33) that

$$0 \geq \limsup_{k\to+\infty}\left(\int_E \pi_k(e)\lambda(de)\right)^- .$$

Combined with (2.3.41), this shows that

$$\int_E \pi_k(e)\lambda(de) \to 0 \quad \text{and} \quad \pi_k(e)\to 0 \quad \text{for } \lambda\text{-a.e. } e\in E. \tag{2.3.44}$$

c. We now return to (2.3.35) and the middle inequality in (2.3.43) to deduce that

$$\begin{aligned} -\partial_t\varphi\left(t_0,x_0\right)+\mu_Y\left(x_k^0,y_k,u_k\right)-\mu_X\left(x_k^0,u_k\right)\cdot q_k^x-\frac{1}{2}\mathrm{Tr}\left[\sigma_X\sigma_X^T\left(x_k^0,u_k\right)A_k^{xx}\right] \\ \geq -2k^{-1}+\sigma_X^T\left(x_k^0,u_k\right)A_k^{xp}\cdot\alpha_k-\left(\int_E \pi_k(e)\lambda(de)q_k^p\right)^+, \end{aligned} \tag{2.3.45}$$

and

$$u_k \in \tilde{\mathcal{N}}_{\varepsilon_k+\sqrt{C\rho}}\left(x_k^0,y_k,q_k^x\right). \tag{2.3.46}$$

since $A_k^{pp}>0$.

Consider now (2.3.36), i.e.

$$\begin{aligned} \beta_Y\left(x_k^0,y_k,u_k(e),e\right)-\hat{v}_*\left(t_k,x_k^0+\beta_X\left(x_k^0,u_k(e),e\right),p_k^0+\pi_k(e)\right)+y_k \\ \geq -2k^{-1} \quad \text{for } \lambda\text{-a.e. } e\in E, \end{aligned} \tag{2.3.47}$$

Using the upper semi-continuity of $-\hat{v}_*$, the fact that β_Y is continuous, (2.3.44), together with $p_k^0\to M$ as $k\to\infty$, we obtain

$$\begin{aligned} \beta_Y\left(x_k^0,y_k,u_k(e),e\right)-\hat{v}_*\left(t_k,x_k^0+\beta_X\left(x_k^0,u_k(e),e\right),M\right)+y_k \geq -2k^{-1}-\vartheta_k^e \\ \text{for } k \text{ large enough and for } \lambda\text{-a.e. } e\in E, \end{aligned}$$

with $\vartheta_k^e\geq 0$ such that $\vartheta_k^e\to 0$ as $k\to\infty$ for all $e\in E$. We now use Assumption 2.3.10 to deduce that there exists $\vartheta_k>0$ with $\vartheta_k\to 0$ as $k\to\infty$ such that, for all $e\in E$ and k large enough,

$$\beta_Y\left(x_k^0,y_k,u_k(e),e\right)-\hat{v}_*\left(t_k,x_k^0+\beta_X\left(x_k^0,u_k(e),e\right),M\right)+y_k \geq -2k^{-1}-\vartheta_k. \tag{2.3.48}$$

By combining (2.3.45) (2.3.46) and (2.3.48), we finally obtain

$$\begin{aligned} & H_{\varepsilon_k+\sqrt{C\rho},-2k^{-1}-\vartheta_k}\left(t_k, x_k^0, y_k, \partial_t\varphi(t_0,x_0), q_k^x, A_k^{xx}, \hat{v}_*(\cdot, M)\right) \\ & \geq -2k^{-1} - \left(\sigma_X^T\left(x_k^0, u_k\right) A_k^{xp}\cdot\alpha_k\right)^- - \left(\int_E \pi_k(e)\lambda(de)q_k^p\right)^+, \end{aligned}$$

and we deduce the required result (2.3.31) by sending $k\to\infty$ and then $\rho\to 0$, and recalling that $\left(|\alpha_k|, A_k^{xp}, \left(\int_E \pi_k(e)\lambda(de)q_k^p\right)^+\right)\to 0$, that σ_X is locally bounded uniformly in the control u, and that $\hat{v}_*\geq\varphi$.

step 3: It remains to prove (2.3.21). The fact that $\hat{v}_*(T,\cdot,M)$ is a viscosity supersolution

$$\max\left\{\hat{v}_*(T,\cdot,M) - v_*(T,\cdot), \delta^*\hat{v}_*(T,\cdot,M)\right\}\geq 0$$

is deduced from (2.3.24) of the previous step by using the same arguments as in the proof of (2.2.1) in Section 2.2.3.2. It remains to show that $\hat{v}_*(T,\cdot,M)$ is a viscosity supersolution of

$$\max\left\{\hat{v}_*(T,\cdot,M) - v_*(T,\cdot), \left(\hat{v}_*(T,\cdot,M) - j_*\right)\mathbf{1}_{\{H^*\hat{v}_*(T,\cdot,M)<\infty\}}\right\}\geq 0.$$

By combining the arguments of step 1 with those of Section 2.2.3.2, we first show that for any smooth function $\hat{\varphi}$ on $\mathbf{X}\times[m,M]$ and $x_0\in\mathbf{X}$ such that

$$\text{(strict)}\min_{\mathbf{X}\times[m,M]}\left(\hat{v}_*(T,\cdot)-\hat{\varphi}\right) = \left(\hat{v}_*(T,\cdot)-\hat{\varphi}\right)(x_0,M) = 0,$$

we have

$$\max\left\{\hat{\varphi}(x_0,M) - v_*(T,x_0), \left(\hat{\varphi}(x_0,M)-\hat{g}_*(x_0)\right)\mathbf{1}_{\{\hat{H}^*\hat{\varphi}(x_0,M)<\infty\}}\right\}\geq 0. \quad (2.3.49)$$

We then consider a smooth function φ on $\mathbf{X}$ and $x_0\in\mathbf{X}$ such that

$$\text{(strict)}\min_{\mathbf{X}}\left(\hat{v}_*(T,\cdot,M)-\varphi\right) = \left(\hat{v}_*(T,\cdot,M)-\varphi\right)(x_0) = 0 \quad (2.3.50)$$

and

$$\varphi(x_0) < \hat{v}(T,x_0), \quad (2.3.51)$$

and we assume that

$$H^*\varphi(T,x_0)<\infty.$$

We next follow the construction of step 2 of the modified test functions

$$\varphi_k := \varphi(x) - \left(f(x)+\psi_k(p)\right), \quad (2.3.52)$$

where ψ_k is defined in (2.3.25). As in the above step 2, one can prove that the difference $\hat{v}_*(T,\cdot)-\varphi_k$ has a local minimizer $\hat{x}_k=(x_k,p_k)$ satisfying all estimates derived in the above step 2 (forgetting about the t variable). In particular, since $H^*\varphi_k(x_k)\leq C$ for some constant $C>0$ independent of k, recall (2.3.51), we deduce from the same estimates than in step 2 that $\hat{H}^*\varphi_k(\hat{x}_k)\leq 2C$ for all large k. It then follows from Corollary 2.3.17, (2.3.49) and (2.3.51) that $\hat{v}_*(T,\hat{x}_k)\geq\hat{g}_*(\hat{x}_k)$. Sending $k\to\infty$, this provides $\hat{v}_*(T,x_0,M)\geq\hat{g}_*(x_0,M)$, and the proof is completed by observing that $\hat{g}_*(x_0,M)=j_*(x_0)$, by definition of j. □

2.3.5.2 The endpoint $p = m$, finite

We organize the proof in four steps. As in the previous section, steps 1, 2 and 3 focus on $t < T$ while step 4 concentrates on $t = T$. Steps 1 and 4 are similar to arguments used in [Bouchard 09]. The main difference comes from steps 2 and 3.

step 1: We first show that for any smooth function $\hat{\varphi}$ on $[0,T) \times \mathbf{X} \times [m,M]$ and $(t_1,x_1) \in [0,T) \times \mathbf{X}$ such that

$$\text{(strict)} \max_{[0,T)\times\mathbf{X}\times[m,M]} (\hat{v}^* - \hat{\varphi}) = (\hat{v}^* - \hat{\varphi})(t_1,x_1,m) = 0, \tag{2.3.53}$$

we have

$$\min\left\{\hat{v}^* + \kappa, \hat{H}_*\hat{\varphi}\right\}(t_1,x_1,m) \le 0. \tag{2.3.54}$$

The proof is very similar to that of Sections (2.2.3.3) up to the modification explained in the proof of Corollary 2.3.17, and the fact that we have to handle the state constraint $p = m$. For completeness, we report here the entire argument. Assume to the contrary that

$$4\eta := \min\left\{\hat{v}^* + \kappa, \hat{H}_*\hat{\varphi}\right\}(t_1,x_1,m) > 0$$

i.e., for some $\varepsilon > 0$, and after possibly changing $\eta > 0$,

$$\begin{aligned}
&\min\left\{\hat{\varphi}_\iota(t,\hat{x}) + \kappa, \mu_Y(x,y,\widehat{u}) - \hat{\mathcal{L}}^{\hat{u}}\hat{\varphi}_\iota(t,\hat{x})\right\} \ge 2\eta \\
&\text{for some} \quad \hat{u} = (u,\alpha,\pi) \in \hat{\mathcal{N}}_{0,\eta}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}),\widehat{\varphi}_\iota) \\
&\text{for all} \quad (t,\hat{x},y) \in [0,T) \times \hat{\mathbf{X}} \times \mathbb{R} \\
&\text{s.t.} \quad (t,\hat{x}) \in B_\varepsilon(t_1,x_1) \times [m,m+\varepsilon], |y - \hat{\varphi}_\iota(t,\hat{x})| \le \varepsilon,
\end{aligned} \tag{2.3.55}$$

where $\hat{\varphi}_\iota(t,\hat{x}) := \hat{\varphi}(t,\hat{x}) + f_\iota(x) + g_\iota(p)$ with ι small enough, for f_ι and g_ι defined as in (2.2.18) with x_1 and m respectively in place of x_o. Then, Assumptions 2.3.5 and 2.3.10 imply that

$$\begin{aligned}
&\min\left\{\begin{array}{c}
\hat{\varphi}_\iota(t,\hat{x}) + \kappa, \\
\mu_Y(x,y,\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}))) - \hat{\mathcal{L}}^{\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}))}\hat{\varphi}_\iota(t,\hat{x}), \\
\min_{e\in E}\hat{\mathcal{G}}^{\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x})),e}\hat{\varphi}_\iota(t,\hat{x})
\end{array}\right\} \ge \eta \\
&\text{for} \ (t,\hat{x},y) \in [0,T] \times \hat{\mathbf{X}} \times \mathbb{R} \text{ s.t.} \\
&(t,\hat{x}) \in B_\varepsilon(t_1,x_1) \times [m,m+\varepsilon] \quad \text{and} \quad |y - \hat{\varphi}_\iota(t,\hat{x})| \le \frac{\eta}{4},
\end{aligned} \tag{2.3.56}$$

where $\hat{\nu}$ is a locally Lipschitz map satisfying

$$\begin{aligned}
&\hat{\nu}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x})) \in \hat{\mathcal{N}}_{0,\eta}(t,\hat{x},y,D\hat{\varphi}_\iota(t,\hat{x}),\widehat{\varphi}_\iota) \\
&\text{on} \quad B_\varepsilon(t_1,x_1) \times [m,m+\varepsilon].
\end{aligned} \tag{2.3.57}$$

Observe that, since (t_1, x_1, m) is a strict maximizer in (2.3.53) and by (2.2.20), we have

$$-\xi := -(\zeta \wedge \gamma_{\varepsilon,\iota}) := \max_D (\hat{v}_* - \hat{\varphi}_\iota) < 0, \tag{2.3.58}$$

where

$$D := \left(\{t_1+\varepsilon\} \times \overline{B}_\varepsilon(x_1) \times [m, m+\varepsilon]\right) \cup \left([t_1, t_1+\varepsilon) \times (B_\varepsilon(x_1) \times [m, m+\varepsilon))^c\right).$$

Also, we deduce from (2.3.55) and the fact that $\hat{v}(\cdot, m) = -\kappa$ by definition, that

$$0 > -\eta \geq \max_{B_\varepsilon(t_1,x_1)} (\hat{v} - \hat{\varphi})(\cdot, m). \tag{2.3.59}$$

By following the arguments in step 2 of Section 2.2.3.3, we see that (2.3.56), (2.3.57), (2.3.58) and (2.3.59) lead to a contradiction of **(GDPj2)**.

<u>step 2:</u> Let φ be a smooth function on $[0,T] \times \mathbf{X}$ and $(t_0, x_0) \in [0,T) \times \mathbf{X}$ such that

$$(\text{strict}) \max_{[0,T)\times\mathbf{X}} (\hat{v}^*(\cdot, m) - \varphi) = (\hat{v}^*(\cdot, m) - \varphi)(t_0, x_0) = 0.$$

By definition, we have $\hat{v}^*(t_0, x_0, m) \geq -\kappa$. Let us assume that

$$\hat{v}^*(t_0, x_0, m) + \kappa =: 4\eta > 0, \tag{2.3.60}$$

and work towards a contradiction. Define the function ψ_k as in (2.3.25) with m in place M:

$$\psi_k(p) := \rho k \int_m^p \frac{e^{2km}}{e^{k(r+m)} - e^{2km+1}} dr, \qquad k > 0,$$

and for f defined as in (2.2.18) for $\iota = 1$,

$$\varphi_k(t,x,p) := \varphi(t,x) + \left(f(x) + (t-t_0)^2 + \psi_k(p)\right).$$

Arguing as in step 2 of the preceding section, we see that the difference $\hat{v}^* - \varphi_k$ has a local maximizer (t_k, x_k, p_k) on $\overline{([0,T] \times \mathbf{X} \times [m,M])}$ satisfying

$$(t_k, x_k, p_k) \to (t_0, x_0, m), \quad k(p_k - m) \to 0 \quad \text{and} \quad \hat{v}^*(t_k, x_k, p_k) \to \hat{v}^*(t_0, x_0, m),$$

so that

$$\left(\partial_t\varphi_k, D_x\varphi_k, D^2_{xx}\varphi_k\right)(t_k, x_k, p_k) \to \left(\partial_t\varphi, D_x\varphi, D^2_{xx}\varphi\right)(t_0, x_0) \quad \text{as} \quad k \to \infty$$
$$\left(D_p\varphi_k, D^2_{xp}\varphi_k, D^2_{pp}\varphi_k\right)(t_k, x_k, p_k) = \left(\psi_k'(p_k), 0, \psi_k''(p_k)\right).$$

Since $\hat{v}^*(t_0, x_0, m) > -\kappa$, we have $\hat{v}^*(t_k, x_k, p_k) > -\kappa$ for all k, after possibly passing to a subsequence. Then, it follows from Corollary 2.3.7, step 1 and the arguments of Remark 2.2.10 under Assumption 2.3.13, that

$$\hat{H}_*(\cdot, \varphi_k, \partial_t\varphi_k, D\varphi_k, D^2\varphi_k, \hat{v}^*)(t_k, x_k, p_k) \leq 0 \text{ for } k > 1.$$

By the definition of $\hat{H}_*$, we deduce that there exist sequences $(\varepsilon_k)_{k\geq 1}$, $(\hat{x}_k)_{k\geq 1}$, $(y_k)_{k\geq 1}$, $(q_k)_{k\geq 1}$ and $(A_k)_{k\geq 1}$ such that $\varepsilon_k > 0$, $\hat{x}_k^0 = (x_k^0, p_k^0) \in \mathbf{X} \times [m, M]$, $y_k \geq -\kappa$, $q_k = (q_k^x, q_k^p) \in \mathbb{R}^d \times \mathbb{R}$, and $A_k \in \mathbb{S}^{d+1}$ with rows $(A_k^{xx}, A_k^{xp}) \in \mathbb{S}^d \times \mathbb{R}^d$ and $\left(A_k^{xpT}, A_k^{pp}\right) \in \mathbb{R}^d \times \mathbb{R}$ satisfying

$$\begin{aligned} &\varepsilon_k \to 0, \quad \hat{x}_k^0 \to (x_0, m), \\ \text{and} \quad &\left|(y_k, q_k, A_k) - \left(\varphi_k, D\varphi_k, D^2\varphi_k\right)(t_k, x_k, p_k)\right| \leq k^{-1} \end{aligned} \tag{2.3.61}$$

for which

$$\hat{H}_{\varepsilon_k, k^{-1}}(t_k, \hat{x}_k, y_k, \partial_t\varphi(t_0, x_0), q_k, A_k, \hat{v}^*) \leq k^{-1}. \tag{2.3.62}$$

Fix $u \in U$, $\pi = 0$ and set $\alpha_k := N^u(x_k^0, y_k, q_k^x)/q_k^p$. Since $\pi = 0$, it follows from (2.3.62) together with (2.3.5), (2.3.6) and Assumption 2.3.10 that either $(u, \alpha_k, \pi) \in \widehat{\mathcal{N}}_{\varepsilon_k, k^{-1}}(t, \widehat{x}_k, y_k, q_k, \widehat{v}^*)$ and then

$$\begin{aligned} &\mu_Y\left(x_k^0, y_k, u\right) - \partial_t\varphi(t_0, x_0) - \mu_X\left(x_k^0, u\right) \cdot q_k^x \\ &\quad - \frac{1}{2}\left(\mathrm{Tr}\left[\sigma_X\sigma_X^T(x_k^0, u)A_k^{xx}\right] + |\alpha|^2 A_k^{pp} + 2\sigma_X^T(x_k^0, u)A_k^{xp} \cdot \alpha\right) \leq k^{-1} \end{aligned} \tag{2.3.63}$$

or

$$\beta_Y(x_k^0, y_k, u(e_k), e_k) - \hat{v}^*\left(t_k, x_k^0 + \beta_X\left(x_k^0, u(e_k), e_k\right), p_k^0\right) + y_k \leq k^{-1}, \tag{2.3.64}$$

for some sequence $(e_k)_{k\geq 1} \subseteq E$. Using the same kind of arguments as in step 2 of the previous section leads to

$$A_k^{pp} < 0, \quad q_k^p < 0 \text{ for large } k, \quad \lim_{k\to\infty} A_k^{xp} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\left(q_k^p\right)^2}{\left|A_k^{pp}\right|} = \rho. \tag{2.3.65}$$

Consider first the case where (2.3.63) holds along a subsequence. Using (2.3.63) and (2.3.65), we then deduce that

$$\left|A_k^{pp}\right| |\alpha_k|^2 = \frac{\left|A_k^{pp}\right|}{\left(q_k^p\right)^2} \left|N^u\left(x_k^0, y_k, q_k^x\right)\right|^2 \leq C,$$

for some $C > 0$ independent of k and ρ. Sending $k \to \infty$ in the above inequality, we then deduce from (2.3.61) and (2.3.65) that

$$\rho^{-1}\left|N^u\left(x_0, \varphi(t_0, x_0), D\varphi(t_0, x_0)\right)\right|^2 \leq C.$$

Since $\rho > 0$ can be chosen arbitrarily close to 0, this shows that $N^u\left(x_0, \varphi(t_0, x_0), D\varphi(t_0, x_0)\right) = 0$, and the arbitrariness of $u \in U$ is in contradiction with Assumption 2.3.11. This contradicts (2.3.60). Hence, if (2.3.60) holds, then (2.3.64) holds along a subsequence, i.e.

$$\beta_Y(x_k^0, y_k, u(e_k), e_k) - \hat{v}^*\left(t_k, x_k^0 + \beta_X\left(x_k^0, u(e_k), e_k\right), p_k^0\right) + y_k \leq k^{-1}.$$

Sending $k \to \infty$, using the arbitrariness of $u \in U$ and Assumption 2.3.10 then leads to

$$\check{G}\hat{v}^*(t_0, x_0, m) \leq 0,$$

where

$$\check{G}\varphi = \sup_{u\in U} \min_{e\in E} \{\beta_Y(\cdot, u(e), e) - \varphi(\cdot + \beta_X(\cdot, u(e), e)) + \varphi\}.$$

Hence

$$\min\left\{\hat{v}^* + \kappa, \check{G}\hat{v}^*\right\}(t_0, x_0, m) \leq 0 \tag{2.3.66}$$

on $[0, T) \times \mathbf{X}$.

step 3: Now observe that, by standard arguments, for every $(t, x) \in [0, T) \times \mathbf{X}$, we may find a sequence of smooth functions $(\varphi^n)_{n\geq 1}$ such that $\varphi^n \downarrow \hat{v}^*$, $(t_n, x_n, p_n)_{n\geq 1}$ converging towards (t, x, m) and such that $(\varphi^n - \hat{v}^*)$ achieves a maximum at (t_n, x_n, p_n). We refer to [Bouchard 02, Lemma 6.1] for the approximation argument by continuous functions. The extension to an approximation by smooth functions is straightforward.

It thus follows from step 2, that $\hat{v}^*(\cdot, m)$ is a classical subsolution of (2.3.66) on $[0, T) \times \mathbf{X}$. In order to conclude the proof, we now appeal to the following easy lemma.

Lemma 2.3.1. *Assume that **H2** holds. Let w be a upper semi-continuous subsolution of*

$$\min\left\{w + \kappa, \check{G}w\right\} \leq 0 \text{ on } \boldsymbol{X} \tag{2.3.67}$$

such that w^+ satisfies the growth condition (2.3.16). Then, $w \leq -\kappa$ on $\boldsymbol{X}$.

Applying Lemma 2.3.1 to $\hat{v}^*(t_0, \cdot, m)$ for an arbitrary $t_0 \in [0, T)$ then leads to $\hat{v}^*(\cdot, m) = -\kappa$, since $\hat{v}^*(\cdot, m) \geq -\kappa$ and $\hat{v}^{*-}$ satisfies (2.3.16) by assumption.

step 4: We finally show that $\hat{v}_*(T, \cdot, m) = -\kappa$ on $\mathbf{X}$. Since $\hat{v}^*(t, x, m) = -\kappa$ for $t < T$ and $x \in \mathbf{X}$, we can find a sequence $(t_n, x_n, p_n)_{n\geq 1}$ in $[0, T) \times \mathbf{X} \times (m, M)$ such that $(t_n, x_n, p_n) \to (T, x, m)$ and $-\kappa \leq \hat{v}(t_n, x_n, p_n) \leq -\kappa + \frac{1}{n}$ for all $n \geq 0$. Passing to the limit leads to the required result.

□

Proof of Lemma 2.3.1.

We assume that $\sup_{\mathbf{X}}(w + \kappa) > 0$ and work towards a contradiction. It follows from the growth condition (2.3.16) on w, **(H2-ii)** and **(H2-iii)** that there is some $x_0 \in \mathbf{X}$ such that

$$\max_{\mathbf{X}}(w - \Lambda) = (w - \Lambda)(x_0) =: \xi > 0. \tag{2.3.68}$$

By **(H2-i)**, Assumption 2.3.10 and (2.3.68), there exists some $\bar{u} \in U$ such that

$$\min_{e\in E} \beta_Y(x_0, w(x_0), \bar{u}(e), e) - \Lambda(x_0 + \beta_X(x_0, \bar{u}(e), e)) + \Lambda(x_0) > 0. \tag{2.3.69}$$

Since w is a subsolution on $\mathbf{X}$ of (2.3.67), we have $\check{G}w(x_0) \leq 0$. Recalling Assumption 2.3.10, we may then find $\hat{e} \in E$ such that

$$\beta_Y(x_0, w(x_0), \bar{u}(\hat{e}), \hat{e}) - w(x_0 + \beta_X(x_0, \bar{u}(\hat{e}), \hat{e})) + w(x_0) \leq 0.$$

Combining the last inequality with (2.3.69) leads to

$$w(x_0) - \Lambda(x_0) < w\left(x_0 + \beta_X\left(x_0, \bar{u}(\hat{e}), \hat{e}\right)\right) - \Lambda\left(x_0 + \beta_X\left(x_0, \bar{u}(\hat{e})\right)\right),$$

which contradicts the definition of x_0 in (2.3.68).

$\square$

Chapter 3

Stochastic Target With Controlled Loss in Jump Diffusion Models - Example

Contents

In this chapter, we specialized the discussion of the controlled expected loss (utility) introduced in Section 1.1.3. We will consider here the settings of Chapter 2, in a particular case.

3.1 Introduction

3.1.1 Market Model

For sake of simplicity, we shall assume that the state space of the process X is $\mathbf{X} := (0, \infty)$. The case of $\mathbf{X} = (0, \infty)^d$ can be easily obtained from this example with some linear algebra. We assume moreover that the marked point process J is a Poisson measure of constant intensity $\lambda \times dt$, such that $E = \{e\}$ and then $\beta(\cdot, e) \equiv \beta$ does not depend on $e \in E$. Let Γ be a set of controls such that

$$\mathbb{E}\left[\int_0^T |\gamma_s|^2 ds\right] < \infty.$$

qui est U ?

Given $\nu \in \mathcal{U}$, the dynamics of (X, Y^ν) are given by

$$X_{t,x}(s) = x + \int_t^s \mu\left(X_{t,x}(r)\right) dr + \int_t^s \sigma\left(X_{t,x}(r)\right) dW_r + \int_t^s \beta\left(X_{t,x}(r)\right) N(dr)$$
$$Y_{t,x,y}(s) = y + \int_t^s \nu_r \mu\left(X_{t,x}(r)\right) dr + \int_t^s \nu_r \sigma\left(X_{t,x}(r)\right) dW_r + \int_t^s \nu_r \beta\left(X_{t,x}(r)\right) N(dr),$$

where μ, σ and β are Lipschitz-continuous functions The process X models the evolution of a risky asset subject to jumps, and Y^ν stands for the wealth process associated to an investment policy ν. Notice that in this chapter, the size of the jump is uniquely determined by the position of X in the state space. The risk free interest is set to 0 for sake of simplicity.

3.1.2 Risk Averse Agent

We consider that the preference of the Agent is characterized with a function $\rho : \mathbb{R} \to \mathbb{R}$, which is assumed to be strictly increasing, concave, continuously differentiable, of linear growth and such that $\rho(\mathbb{R}) = \mathbb{R}$. We assume furthermore that ρ^{-1} is continuous with linear growth on $\mathbb{R}$. We also introduce a continuous function g of linear growth. The function ρ may represent in this context the utility function of the Agent (or, up to the sign, its loss function), while $g(X_{t,x}(T)$ denotes the random payoff of a European option written on the risky asset sold by the agent. The aim of the agent is to find the minimal amount of money $w(t,x,p)$ (above $-\kappa$) he has to invest in a dynamic strategy ν in order to reach its target in expectation above its given threshold p, where he has to deliver the payoff $g(X_{t,x}(T))$ at terminal time T. Set

his

$$\Psi(x,y) := \rho(y - g(x)),$$

so that

ajouter contrainte Y\ge -\kappa et dire qu'est ok aussi par rapport au cadre general

$$w(t,x,p) := \inf\left\{y \geq -\kappa : \exists\, \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}\left[\rho\left(Y^\nu_{t,x,y}(T) - g\left(X_{t,x}(T)\right)\right)\right] \geq p\right\}.$$

In the remainder of this chapter, we intend to give an explicit characterization of the value function w in terms of $\tilde{g}$ defined as

$$\tilde{g} : (x,p) \in (0,\infty) \times \mathbb{R} \longmapsto \inf\left\{y \in \mathbb{R} \text{ s.t. } \Psi(x,y) \geq p\right\} \vee (-\kappa).$$

je t'ai deja dit qu'il y a un pb avec le GDP si tu fais ca.

Observe that

$$\tilde{g}(x,p) = \left(g(x) + \rho^{-1}(p)\right) \vee (-\kappa). \tag{3.1.1}$$

Remark 3.1.1. Under minor modifications of the calculations of this chapter, we could also build the function ℓ in order to represent the success ratio of [Föllmer 99]. We however don't provide a characterization of the quantile hegding price of the claim $g(X_{t,x}(T)$ since, as it has already been discussed in the Example (2.3.5) of Chapter 2, we failed in establishing the convexity of the value function.

3.2 Explicit Resolution in the case of Poisson Process

In the present context, we may state the following corollary. We introduce in this sense, for any smooth function φ defined on $[0,T]\times(0,\infty)\times\mathbb{R}$ and every $(\alpha,\pi)\in\mathbb{R}^2$, the operators

$$\mathcal{H}_1^{\alpha,\pi}\varphi := -\frac{\partial\varphi}{\partial t} - \frac{1}{2}\sigma^2\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial p}\pi\lambda_t + \alpha\mu\sigma^{-1}\frac{\partial\varphi}{\partial p} - \sigma\alpha\frac{\partial^2\varphi}{\partial xp} - \frac{1}{2}\alpha^2\frac{\partial^2\varphi}{\partial p^2}$$
$$\mathcal{H}_2^{\alpha,\pi}\varphi := \left(\frac{\partial\varphi}{\partial x} + \sigma^{-1}\frac{\partial\varphi}{\partial p}\alpha\right)\beta(\cdot) - \varphi\left(\cdot, \cdot+\beta(\cdot), \cdot+\pi\right) + \varphi.$$

Corollary 3.2.1. *The function w is a viscosity super solution of*

$$\begin{cases} \sup_{(\alpha,\pi)\in\mathbb{R}^2} \min\{\mathcal{H}_1^{\alpha,\pi}\varphi\,,\ \mathcal{H}_2^{\alpha,\pi}\varphi\} \geq 0 & \text{on } [0,T)\times(0,\infty)\times\mathbb{R} \\ w_*(T,\cdot) \geq \tilde{g} & \text{on } (0,\infty)\times\mathbb{R}. \end{cases} \tag{3.2.1}$$

The function w is a viscosity subsolution of

$$\begin{cases} \sup_{(\alpha,\pi)\in\mathbb{R}^2} \min\{\mathcal{H}_1^{\alpha,\pi}\varphi\,,\ \mathcal{H}_2^{\alpha,\pi}\varphi\} \leq 0 & \text{on } [0,T)\times(0,\infty)\times\mathbb{R} \\ w^*(T,\cdot) \leq \tilde{g} & \text{on } (0,\infty)\times\mathbb{R}. \end{cases} \tag{3.2.2}$$

Pour la sous solution, je devrais être en mesure d'écrire une edp plus sympa, à voir selon ce que j'arrive à avoir sur le problème dual...

Proof. A sufficient condition for the results to hold is that conditions of Corollary 2.3.7 and Proposition 2.3.18 hold. We divide the proof into several steps.

First recall from Section 2.3.2 the definition of the Kernel and of the HJB operator in this context. For every $\varepsilon>0, \eta\in[-1,1]$, $\varphi\in C^0([0,T]\times(0,\infty)^2)$ and

$$\Theta := (t,x,p,y,q_t,q_x,q_p,q_{xx},q_{xp},q_{pp},\varphi)\in[0,T]\times(0,\infty)\times\mathbb{R}^8$$

$$\mathcal{N}_{\varepsilon,\eta}(t,x,p,y,q_x,q_p,\varphi) = \left\{\begin{matrix} (u,a,\pi)\in\mathbb{R}^3 \text{ s.t. } |u\sigma(x)-aq_p-q_x\sigma(x)|\leq\varepsilon \\ \text{and } u\beta(x)-\varphi(t,x+\beta(x),p+\pi)+\varphi(t,x,p)\geq\eta \end{matrix}\right\}$$

and

$$H_{\varepsilon,\eta}(\Theta,\varphi) = \sup_{\hat{u}\in\mathcal{N}_{\varepsilon,\eta}(\Theta)}\left\{\mu(x)(u-q_x)-q_t+\lambda\pi q_p-\sigma a q_{xp}-\frac{1}{2}\sigma(x)^2q_{xx}-\frac{1}{2}q_{pp}a^2\right\},$$

where $\hat{u}$ denotes (u,a,π).

Step 1: *Assumption 2.3.5 holds.*

Let $B\subset[0,T]\times(0,\infty)\times\mathbb{R}^8$, $\eta>0$ and $\varphi\in C^0([0,T]\times(0,\infty)^2)$ such that $\mathcal{N}_{0,2\eta}(\cdot,\varphi)\neq\emptyset$ on B. Fix $\Theta^o\in B$ with

$$\Theta^o := (t^o,x^o,p^o,y^o,q_t^o,q_x^o,q_p^o,q_{xx}^o,q_{xp}^o,q_{pp}^o),$$

and $(u_o, a_o, \pi_o) \in \mathcal{N}_{0,2\eta}(\Theta^o, \varphi)$. We denote without the superscript any $\Theta \in [0,T] \times (0,\infty) \times \mathbb{R}^8$ and define the map

$$(\hat{u}, \hat{a}, \hat{\pi}) : \Theta \in [0,T] \times (0,\infty) \times \mathbb{R}^8 \longmapsto \left(q_x + a_o \frac{q_p}{\sigma(x)} \ , a_o \ , \ \pi_o \right),$$

which is trivially Lipschitz-continuous since $\sigma > 0$ and satisfies

$$(\hat{u}, \hat{a}, \hat{\pi})(\Theta^o) = (u_o, a_o, \pi_o),$$

recall that $(u_o, a_o, \pi_o) \in \mathcal{N}_{0,2\eta}(\Theta, \varphi)$ implies in particular that

$$u_o = q_x^o + a_o \frac{q_p^o}{\sigma(x^o)}.$$

Moreover, for every $\Theta \in [0,T] \times (0,\infty) \times \mathbb{R}^8$, we have

$$|\hat{u}\sigma(x) - \hat{a}q_p - \sigma(x)q_x| = 0.$$

By continuity of φ and β, there is a neighborhood B_ε of (u_o, a_o, π_o) such that

$$\hat{u}(\Theta)\beta(x) - \varphi(t, x + \beta(x), p + \hat{\pi}(\theta)) + \varphi(t,x,p) \geq \eta,$$

and so follows the result.

Step 2: *Assumption 2.3.4 holds.*

$$\mathcal{N}_{\varepsilon,\eta} := \left\{ \begin{array}{c} (u,a,\pi) \in \mathbb{R}^2 \times \mathbb{L}^2_\lambda \text{ s.t. } |u\sigma(x) - q_x\sigma(x) - q_p a| \leq \varepsilon \\ u\beta(x) - \varphi(t, x + beta(x), p+\pi) + \varphi(t,x) \geq \eta \end{array} \right\}.$$

Let (t_o, x_o, p_o) and φ such that

$$H^*\varphi(t_o, x_o, p_o) < \infty.$$

By definition of H^*, there is $\varepsilon > 0$ and a finite $C > 0$ such that

$$u\mu(x) - \varphi_t - \varphi_x\mu(x) + \varphi_p \int_E \pi(e)\lambda(de) - \tfrac{1}{2}\varphi_{xx}\sigma(x)^2 - \tfrac{1}{2}\varphi_{pp}a^2_{x,u} - \sigma(x)a_{x,u}\varphi_{px} \leq C$$

$$\text{for all } (t,x,p) \in B_\varepsilon(t_o,x_o,p_o),\ \zeta_{t,x,p}, q_p \in \mathbb{R} \text{ s.t. } \left\{ \begin{array}{c} |\zeta_{t,x,p}\sigma(x)| \leq \varepsilon \\ q_p \in B_\varepsilon(\varphi_p(t_o,x_o,p_o)) \\ |q_p| \geq \varepsilon/2 \end{array} \right.$$

$$\text{and } (u,\pi) \in \mathbb{R} \times L^2_\lambda \text{ s.t. } u\beta(x) - \varphi(t, x+\beta(x), p+\pi) + \varphi(t,x) \geq \eta,$$

indexation super lourde et qui ne sert a rien ?

where

$$a_{x,u} := \frac{\sigma(x)}{q_p}\left(u - \varphi_x - \zeta_{t,x,p}\right).$$

Hence we have

$$\varphi_p \int_E \pi(e)\lambda(de) + A_{t,x,p} + B_{t,x,p}u - \tfrac{1}{2}\varphi_{pp}\frac{\sigma(x)^2}{q_p^2}u^2 \leq C$$

$$\text{for all } (t,x,p) \in B_\varepsilon(t_o,x_o,p_o),\ \zeta_{t,x,p}, q_p \in \mathbb{R} \text{ s.t. } \left\{ \begin{array}{c} |\zeta_{t,x,p}\sigma(x)| \leq \varepsilon \\ q_p \in B_\varepsilon(\varphi_p(t_o,x_o,p_o)) \\ |q_p| \geq \varepsilon/2 \end{array} \right.$$

$$\text{and } (u,\pi) \in \mathbb{R} \times L^2_\lambda \text{ s.t. } u\beta(x) - \varphi(t, x+\beta(x), p+\pi) + \varphi(t,x) \geq \eta,$$

where

$$A_{t,x,p} := -\varphi_t - \varphi_x\mu(x) - \tfrac{1}{2}\varphi_{xx}\sigma(x)^2 + \frac{\sigma(x)^2}{q_p}\varphi_{px}(\varphi_x + \zeta_{t,x,p}) - \frac{\sigma(x)^2}{q_p^2}\varphi_{pp}(\varphi_x + \zeta_{t,x,p})$$

$$B_{t,x,p} := \mu(x) - \frac{\sigma(x)^2}{q_p}\varphi_{px} + \varphi_{pp}\frac{\sigma(x)^2}{q_p^2}(\varphi_x + \zeta_{t,x,p}).$$

Recalling that $\sigma > 0$, there is finite $C' > 0$ such that

$$\left(\varphi_p \int_E \pi(e)\lambda(de)\right)^+ \leq C'\left(1 + u^2\right) \text{ for all } \ldots$$

u \pi such that

j'ai essaye de lire la suite... impossible... essaie de l'ecrire plus proprement...

deja fait

Fix now $(t_o,x_o,p_o)\in[0,T)\times(0,\infty)^2$ and $\varphi\in C^{1,2,2}([0,T]\times(0,\infty)^2)$ such that

$$H^*\varphi(t_o,x_o,p_o)<\infty. \tag{3.2.3}$$

a. Assume first that

$$\frac{\partial\varphi}{\partial p}(t_o,x_o,p_o)=0. \tag{3.2.4}$$

Using the definition of $\mathcal{N}$ and H^*, there is $\varepsilon>0$ and $\eta\in[-1,1]$ such that

$$\left\{-\frac{\partial\varphi}{\partial t}-\mu(x)\frac{\partial\varphi}{\partial x}-\frac{1}{2}\sigma(x)^2\frac{\partial^2\varphi}{\partial x^2}-\sigma(x)a\frac{\partial^2\varphi}{\partial xp}-\frac{1}{2}a^2\left(\frac{\partial^2\varphi}{\partial p^2}-\varepsilon\right)\right\}(t_o,x_o,p_o)\le 2C_1$$

for all $(a,\pi)\in\mathbb{R}\times\Delta_\eta(a;t_o,x_o,p_o)$, with

$$\Delta_\eta(a;t,x,p):=\left\{\pi\in\mathbb{R}\text{ s.t. }\left\{\begin{array}{c}\left(\frac{\partial\varphi}{\partial x}(\cdot)+\frac{a\frac{\partial\varphi}{\partial p}(\cdot)}{\sigma(x)}\right)\beta(x)\\ -\varphi\left(t,x+\beta(x),p+\pi\right)+\varphi(\cdot)\end{array}\right\}(t,x,p)\ge\eta\right\}.$$

Recall from (3.2.4) that $\Delta_\eta(a;t_o,x_o,p_o)$ does not depend on $a\in\mathbb{R}$ so that, for $\pi\in\Delta_\eta(a;t_o,x_o,p_o)$ fixed, we have for all $a\in\mathbb{R}$

$$\left\{-\frac{\partial\varphi}{\partial t}-\mu(x)\frac{\partial\varphi}{\partial x}-\frac{1}{2}\sigma(x)^2\frac{\partial^2\varphi}{\partial x^2}-\sigma(x)a\frac{\partial^2\varphi}{\partial xp}-\frac{1}{2}a^2\left(\frac{\partial^2\varphi}{\partial p^2}-\varepsilon\right)\right\}(t_o,x_o,p_o)\le 2C_1,$$

and there is then a finite positive constant C such that

$$-\frac{1}{2}a^2\left(\frac{\partial^2\varphi}{\partial p^2}-\varepsilon\right)\le C\left(1+|a|\right).$$

Taking a large enough gives then $\frac{\partial^2\varphi}{\partial p^2}\ge\varepsilon$, and hence, by smoothness of φ, we have $\frac{\partial^2\varphi}{\partial p^2}>0$ on some neighborhood B of (t_o,x_o,p_o).

We now intend to characterize the set of controls (u,a,π) such that

$$u\beta(x)-\varphi(t,x+\beta(x),p+\pi)+\varphi(t,x,p)\ge\eta. \tag{3.2.5}$$

Recall from the definition of $\mathcal{N}$ and (3.2.3) again that, we may find a neighborhood B' of (t_o,x_o,p_o) such that $B'\subset B$, for q_p small enough, ζ_x such that $|\sigma(x)\zeta_x|\le\varepsilon$ and all $(u,\pi)\in\mathbb{R}$ satisfying (3.2.5) on B', we have

pb...

$$u\mu(x)-\frac{\partial\varphi}{\partial t}(t,x,p)-\frac{\partial\varphi}{\partial x}(t,x,p)+\frac{\partial\varphi}{\partial p}(t,x,p)\int_E\pi(e)\lambda(de)-\frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}(t,x,p)\sigma(x)^2-\frac{\partial^2\varphi}{\partial xp}(t,x,p)\sigma(x)a_{x,u}-\frac{1}{2}\frac{\partial^2\varphi}{\partial p^2}$$

for some finite $C>0$ where

$$a_{x,u}:=\frac{\sigma(x)}{q_p}\left(u-\frac{\partial\varphi}{\partial x}(t,x,p)-\zeta_x\right).$$

Recalling the particular form of $\mathcal{N}$ and H^* in that case allows then to reduce to the case where a takes values in a compact set of $\mathbb{R}$. We than have

def qqpart ?

$$\frac{\left[u\mu(x)-\mathcal{L}^{a,\pi}\varphi(t,x,p)\right]^+}{\left|u\sigma(x)-\sigma(x)\frac{\partial\varphi}{\partial x}-a\frac{\partial\varphi}{\partial p}\right|(t,x,p)}\le\frac{1}{\varepsilon}\left[u\mu(x)-\mathcal{L}^{a,\pi}\varphi(t,x,p)\right]^+$$

for all $(u, a, \pi) \in \mathbb{R}$ such that

$$\left| u\sigma(x) - \sigma(x)\frac{\partial \varphi}{\partial x} - a\frac{\partial \varphi}{\partial p} \right| (t, x, p) > \varepsilon,$$

and by smoothness of φ, μ, σ, there exists thus a finite constant $C \geq 0$ such that

$$\frac{[u\mu(x) - \mathcal{L}^{a,\pi}\varphi(t, x, p)]^+}{\left| u\sigma(x) - \sigma(x)\frac{\partial \varphi}{\partial x} - a\frac{\partial \varphi}{\partial p} \right| (t, x, p)} \leq C(1 + |\pi|),$$

and so hold (**??**).

Consider now the case where $\frac{\partial \varphi}{\partial p}(t_o, x_o, p_o) \neq 0$. Then there is a neighborhood B of (t_o, x_o, p_o) such that $\frac{\partial \varphi}{\partial p} \neq 0$ on B. Hence, by a continuity argument, there is a finite constant $C \geq 0$ such that

$$\frac{[u\mu(x) - \mathcal{L}^{a,\pi}\varphi(t, x, p)]^+}{\left| u\sigma(x) - \sigma(x)\frac{\partial \varphi}{\partial x} - a\frac{\partial \varphi}{\partial p} \right| (t, x, p)} \leq C(1 + |\pi| + |a|),$$

so that (**??**) stands.

Equation (**??**) trivially stands by definition of H^*.

Depending on the case, combining the above results gives that Assumption 2.3.4 holds.

Step 3: *We have $w^*(T, \cdot) \leq \tilde{g}$ on $(0, \infty) \times \mathbb{R}$..*

Fix $(x, p) \in (0, \infty) \times \mathbb{R}$, and et $y > \tilde{g}(x, y)$, so that $\Psi(x, y) > p$, recall that Ψ is strictly increasing. Let $(t_n, x_n, p_n) \to (T, x, p)$ as $n \to \infty$ such that

$$w(t_n, x_n, p_n) \underset{n\to\infty}{\longrightarrow} w^*(T, x, p).$$

We claim that

$$\mathbb{E}\left[\Psi\left(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T)\right)\right] \underset{n\to\infty}{\longrightarrow} \Psi(x, y).$$

Recalling that $y > \Psi(x, y) > p$ implies that $\mathbb{E}[\Psi(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T))] > p$ for n large enough, and so $y \geq w(t_n, x_n, p_n)$ for n large enough, and thus $y \geq w^*(T, x, p)$. We conclude by arbitrariness of $y > \tilde{g}(x, p)$.

We finally prove the claim. Classical estimates give

$$\mathbb{E}\left[\left|\left(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T)\right) - (x, y)\right|^p\right] \leq C\left(|T - t_n|^{\frac{p}{2}} + |x_n - x|^p\right) \tag{3.2.6}$$

for all $p \geq 1$ and some finite constant C which does not depend on n. Recall now from assumption on ρ and g that Ψ is continuous and of linear growth, and therefore uniformly continuous on compact sets. Hence, for every $\varepsilon > 0$ and $k \geq 1$, there is $\eta^\varepsilon_k > 0$ such that

$$\sup_{(x,y,x',y')\in\Gamma^\varepsilon_k} \left|\Psi(x, y) - \Psi(x', y')\right| \leq \varepsilon,$$

where

$$\Gamma_k^\varepsilon := \left\{ (x,x',y,y') \in (0,\infty)^2 \times \mathbb{R}^2 \text{ s.t. } \begin{cases} |(x,y)| \vee |(x',y')| \leq k \\ |(x,y)-(x',y')| \leq \eta_k^\varepsilon \end{cases} \right\}$$

is compact. Note that standard estimates imply that

$$\mathbb{P}\left[\left|\left(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T)\right)\right| > k\right] \leq \frac{1}{k}\mathbb{E}\left[\left|\left(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T)\right)\right|\right] \leq \frac{C}{k},$$

for some finite constant C. Moreover, it follows from (3.2.6) that

$$\mathbb{P}\left[\left|\left(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T)\right) - (x,y)\right| > \eta_k^\varepsilon\right] \leq r_n^{\varepsilon,k}$$

where $r_n^{\varepsilon,k} \to 0$ as $n \to \infty$ for all fixed ε, k. Recalling that Ψ is of linear growth, we have for all $k \geq |(x,y)|$

$$\mathbb{E}\left[\left|\Psi\left(X_{t_n,x_n}(T), Y^0_{t_n,x_n,y}(T)\right) - \Psi(x,y)\right|\right] \leq \varepsilon + \frac{C}{k} + r_n^{\varepsilon,k}.$$

We conclude by sending $n \to \infty$, and then $k \to \infty$ and $\varepsilon \to 0$.

Step 4: *Condition (i) of Proposition 2.3.18 holds.*

Classical estimates and the same reasoning as in the previous step leads the result, recall that $\rho \in C^1(\mathbb{R})$.

□

We also define, for every $(t,x,q) \in [0,T] \times (0,\infty) \times (0,\infty)$, the process $(Q^\gamma_{t,x,q}, \bar{X}_{t,x})$ by the dynamic

$$\begin{aligned} \frac{dQ^\gamma_{t,x,q}}{Q^\gamma_{t,x,q}}(s) &= \left(\frac{\mu(\cdot)}{\sigma(\cdot)}\left(\bar{X}_{t,x}(s)\right) + \gamma_s \frac{\beta(\cdot)}{\sigma(\cdot)}\left(\bar{X}_{t,x}(s)\right)\right) dW(s) + \gamma_s \lambda \tilde{J}(ds), \\ d\bar{X}_{t,x}(s) &= \sigma\left(\bar{X}_{t,x}(s)\right) dW(s) + \beta\left(\bar{X}_{t,x}(s)\right) \tilde{J}(ds), \\ \left(\bar{X}_{t,x}(t), Q^\gamma_{t,x,q}(t)\right) &= (x,q) \in (0,\infty)^2. \end{aligned} \tag{3.2.7}$$

We finally define the function $\mathrm{y} : (t,x,p) \in [0,T] \times (0,\infty) \times \mathbb{R} \mapsto \mathrm{y}(t,x,p)$ as:

$$\mathrm{y}(t,x,p) = \mathbb{E}\left[g\left(\bar{X}_{t,x}(T)\right)\right] + \sup_{\gamma\in\Gamma} \mathbb{E}\left[\rho^{-1}\left(I\left(Q^\gamma_{t,x,\hat{q}_\gamma(t,x,p)}(T)\right)\right)\right].$$

3.2.1 Characterization of a lower bound

We define now, as in [Bouchard 09] the Fenchel-Legendre dual with respect to the p-variable of the lower semi-continuous envelop w_* of w:

$$\tilde{w} : (t,x,q) \in [0,T] \times (0,\infty)^2 \mapsto \sup_{p\in\mathbb{R}} \left\{pq - w_*(t,x,p)\right\}. \tag{3.2.8}$$

We shall need for the characterization of a lower bound for w to define the function

$$\underline{w} : (t,x,q) \mapsto \inf_{\gamma\in\Gamma} \mathbb{E}\left[Q^\gamma_{t,x,q}(T) I\left(Q^\gamma_{t,x,q}(T)\right) - \rho^{-1}\left(I\left(Q^\gamma_{t,x,q}(T)\right)\right) - g\left(\bar{X}_{t,x}(T)\right)\right],$$

where $I := \left((\rho^{-1})'\right)^{-1}$. We define the following operator, for every smooth function $\psi \in C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$:

$$F\psi := \sup_{\gamma \in (0,\infty)} \left\{ \begin{array}{c} -\psi_t + \gamma\beta\psi_x + (q\lambda - \gamma q)\,\psi_q - \frac{1}{2}\sigma^2\psi_{xx} \\ -(q\mu + \gamma q\beta)\,\psi_{qx} - \frac{1}{2}\left(q\frac{\mu}{\sigma} + \gamma q\frac{\beta}{\sigma}\right)^2 \psi_{qq} \\ -\gamma\left[\psi\left(t, \cdot + \beta, \cdot + \left(\frac{1}{\gamma}q\lambda - q\right)\right) - \psi\right] \end{array} \right\}.$$

We introduce now the PDE system for the dual problem:

$$\begin{cases} F\psi(t,x,q) \geq 0 & \text{on } [0,T) \times (0,\infty) \times (0,\infty) \\ \psi(T,x,q) \geq qI(q) - \rho^{-1}(I(q)) - g(x) & \text{on } (0,\infty) \times (0,\infty); \end{cases} \tag{3.2.9}$$

$$\begin{cases} F\psi(t,x,q) \leq 0 & \text{on } [0,T) \times (0,\infty) \times (0,\infty) \\ \psi(T,x,q) \leq qI(q) - \rho^{-1}(I(q)) - g(x) & \text{on } (0,\infty) \times (0,\infty); \end{cases} \tag{3.2.10}$$

Assumption 3.2.2. *There is a class of functions $\mathcal{C}$ containing all $[-\kappa, +\infty)$ valued functions dominated by $\underline{w}$ such that, for every*

- *$v_1 \in \mathcal{C}$, lower semi-continuous viscosity supersolution of (3.2.9) on $[0,T] \times \boldsymbol{X}$;*
- *$v_2 \in \mathcal{C}$, upper semi-continuous viscosity subsolution of (3.2.10) on $[0,T] \times \boldsymbol{X}$;*

we have $v_1 \geq v_2$.

Lemma 3.2.1. *The function $\tilde{w}$ is an upper semi-continuous viscosity subsolution on $[0,T] \times (0,\infty) \times (0,\infty)$ of (3.2.10).*

We refer the proof to the end of this section.

Proposition 3.2.3. *Under Assumption 3.2.2, we have*

$$w \geq \mathrm{y}.$$

Proof.

By Lemma 3.2.1 and (Bonne réf), under Assumption 3.2.2, and since $\underline{w}$ is a viscosity supersolution of (3.2.9) (vérifier la semicontinuity de $\underline{w}$?), we have $\tilde{w} \leq \underline{w}$. We can now provide a lower bound to the primal function w by using (3.2.8). Define for every $\gamma \in \Gamma$ the function $\tilde{w}_\gamma : (t,x,q) \in [0,T] \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$\tilde{w}_\gamma(t,x,q) := \mathbb{E}\left[Q^\gamma_{t,x,q}(T)I\left(Q^\gamma_{t,x,q}(T)\right) - \rho^{-1}\left(I\left(Q^\gamma_{t,x,q}(T)\right)\right) - g\left(\bar{X}_{t,x}(T)\right)\right].$$

Clearly the function $\tilde{w}_\gamma$ is convex in q, so that there is an unique solution $\hat{q}_\gamma(t,x,p)$ to the equation Vérifier la convexité et les calculs, mais tout doit être ok...

$$\frac{\partial \tilde{w}_\gamma}{\partial q}(t,x,\hat{q}_\gamma) = \mathbb{E}\left[Q^\gamma_{t,x,1}(T)I\left(Q^\gamma_{t,x,\hat{q}_\gamma(t,x,p)}(T)\right)\right] = p. \tag{3.2.11}$$

Hence, we have

$$\begin{aligned} w(t,x,p) &\geq w_*(t,x,p) \geq \sup_{q>0}\left(pq-\tilde{w}(t,x,q)\right) \\ &\geq \sup_{q>0}\left(pq-\underline{w}(t,x,q)\right) \\ &\geq \sup_{\gamma\in\Gamma}\left\{\sup_{q>0}\left(pq-\tilde{w}_\gamma(t,x,q)\right)\right\} \\ &\geq \sup_{\gamma\in\Gamma}\left\{p\hat{q}_\gamma(t,x,p)-\tilde{w}_\gamma\left(t,x,\hat{q}_\gamma(t,x,p)\right)\right\} \\ &\geq \sup_{\gamma\in\Gamma}\left\{\hat{q}_\gamma(t,x,p)\left[p-\mathbb{E}\left[Q^\gamma_{t,x,1}(T)I\left(Q^\gamma_{t,x,\hat{q}_\gamma(t,x,p)}(T)\right)\right]\right]\right. \\ &\qquad\qquad \left.+\mathbb{E}\left[\rho^{-1}\left(I\left(Q^\gamma_{t,x,\hat{q}_\gamma(t,x,p)}(T)\right)\right)+g\left(\bar{X}_{t,x}(T)\right)\right]\right\} \\ &\geq \mathbb{E}\left[g\left(\bar{X}_{t,x}(T)\right)\right]+\sup_{\gamma\in\Gamma}\mathbb{E}\left[\rho^{-1}\left(I\left(Q^\gamma_{t,x,\hat{q}_\gamma(t,x,p)}(T)\right)\right)\right] \\ &= \mathrm{y}(t,x,p), \end{aligned} \tag{3.2.12}$$

where the last inequality follows from (3.2.11).

□

We conclude this section by the proof of Lemma 3.2.1.

Proof of Lemma 3.2.1.

The function u is clearly upper semi-continuous on $[0,T]\times(0,\infty)\times(0,\infty)$ recalling that $\widehat{v}_*$ is lower semi-continuous and (**??**). The boundary condition (**??**) is an immediate consequence of the right-hand side in (2.3.19) and (**??**).

Now let φ be a smooth function with bounded derivatives and $(t_0,x_0,q_0)\in[0,T)\times(0,\infty)\times(0,\infty)$ be a local maximizer of $(u-\varphi)$ such that $(u-\varphi)(t_0,x_0,q_0)=0$.

a. As in [Bouchard 09], we first show that one can reduce to the case where the test function φ is strictly convex. Indeed, since u is convex and (t_0,x_0,q_0) is a local maximizer of $(u-\varphi)$, we have $D_{qq}\varphi(t_0,x_0,q_0)\geq 0$. Define $\varphi_{\varepsilon,\eta}$ by $\varphi_{\varepsilon,\eta}(t,x,q):=\varphi(t,x,q)+\varepsilon\left|q-q_0\right|^2+\varepsilon\eta\left|q-q_0\right|^2\left(\left|q-q_0\right|^2+\left|t-t_0\right|^2+\left|x-x_0\right|^2\right)$ for some $\varepsilon,\eta>0$. Then (t_0,x_0,q_0) is still a local minimizer of $(u-\varphi_{\varepsilon,\eta})$, and, since $D_{qq}\varphi(t_0,x_0,q_0)\geq 0$ and φ has bounded derivatives, we have $D_{qq}\varphi_{\varepsilon,\eta}\geq D_{qq}\varphi+2\eta\varepsilon\left(\left|t-t_0\right|^2+\left|x-x_0\right|^2\right)$. We may thus choose η large enough so that $D_{qq}\varphi_{\varepsilon,\eta}>0$ on a neighborhood of (t_0,x_0,q_0). Next we observe that, if $\varphi_{\varepsilon,\eta}$ satisfies the first line in (3.2.10) at (t_0,x_0,q_0) for all $\varepsilon>0$, then (3.2.10) holds for φ at this point too. Indeed, the derivatives up to order 2 of $\varphi_{\varepsilon,\eta}$ at (t_0,x_0,q_0) converge to the corresponding derivatives of φ as $\varepsilon\to 0$, and we have

$$
\begin{aligned}
&-\gamma\left(\varphi_{\varepsilon,\eta}\left(t_0,x_0+\beta,q_0+\left(\frac{1}{\gamma}q_0\lambda-q_0\right)\right)-\varphi_{\varepsilon,\eta}(t_0,x_0,q_0)\right)\\
&\quad=-\gamma\left(\varphi\left(t_0,x_0+\beta,q_0+\left(\frac{1}{\gamma}q_0\lambda-q_0\right)\right)-\varphi(t_0,x_0,q_0)+\varepsilon\left|\pi^q\right|^2+\varepsilon\eta\left|\pi^q\right|^2\left(\left|\pi^q\right|^2+|\beta|^2\right)\right)\\
&\quad=-\gamma\left(\varphi\left(t_0,x_0+\beta,q_0+\left(\frac{1}{\gamma}q_0\lambda-q_0\right)\right)-\varphi(t_0,x_0,q_0)\right)-\gamma\varepsilon\left|\pi^q\right|^2-\gamma\varepsilon\eta\left|\pi^q\right|^2\left(\left|\pi^q\right|^2+|\beta|^2\right),
\end{aligned}
$$

with $\pi^q := \frac{1}{\gamma}q_0\lambda - q_0$. Recalling (**??**), we know that we can restrict the choice of the control α, χ to the controls leading to $P^{\alpha,\chi} \in [0,1]$, implying that π^q is bounded. Hence,

$$
-\gamma\left(\varphi_{\varepsilon,\eta}\left(t_0,x_0+\beta,q_0+\left(\frac{1}{\gamma}q_0\lambda-q_0\right)\right)-\varphi_{\varepsilon,\eta}(t_0,x_0,q_0)\right)
$$

converges toward $-\gamma\left(\varphi\left(t_0,x_0+\beta,q_0+\left(\frac{1}{\gamma}q_0\lambda-q_0\right)\right)-\varphi(t_0,x_0,q_0)\right)$ as $\varepsilon \to 0$.

b. We may now assume that φ is strictly convex in its q-variable. Let $\widetilde{\varphi}$ be the Fenchel transform of φ with respect to the q variable

$$
\widetilde{\varphi}(t,x,p) := \sup_{p\in\mathbb{R}}\left\{pq-\varphi(t,x,q)\right\}.
$$

since φ is strictly convex in q and smooth on its domain, $\widetilde{\varphi}$ is strictly convex in p and smooth on its domain, see [Rockafellar 97]. Moreover, we have

$$
\begin{aligned}
\varphi(t,x,q) &= \sup_{p\in\mathbb{R}}\left\{pq-\widetilde{\varphi}(t,x,p)\right\}\\
&= p^*(t,x,q)q-\widetilde{\varphi}\left(t,x,p^*(t,x,q)\right) \text{ on } (0,T)\times(0,\infty)\times(0,\infty)\subset \operatorname{int}\left(\operatorname{dom}(\varphi)\right)
\end{aligned}
\tag{3.2.13}
$$

where $p^*(t,x,q) = J(t,x,q)$ and $q \mapsto J(\cdot,q)$ denotes the inverse of $p \mapsto D_p\widetilde{\varphi}(\cdot,p)$. From the fact that $q_0 > 0$ and by (**??**), there exists $p_0 \in [0,1]$ such that $u(t_0,x_0,q_0) = p_0q_0 - \widehat{v}_*(t_0,x_0,p_0)$, which, using (3.2.8) and the definition of (t_0,x_0,p_0,q_0) leads to

$$
\begin{aligned}
u(t_0,x_0,q_0) &= p_0q_0-\widehat{v}_*(t_0,x_0,p_0)\\
\varphi(t_0,x_0,q_0) &= \sup_{p\in\mathbb{R}}\left\{pq_0-\widetilde{\varphi}(t_0,x_0,p)\right\} \geq p_0q_0-\widetilde{\varphi}(t_0,x_0,p_0)\\
0 &= u(t_0,x_0,q_0)-\varphi(t_0,x_0,q_0) \geq u(t,x,q)-\varphi(t,x,q).
\end{aligned}
$$

Then we have

$$
p_0q_0-\widehat{v}_*(t_0,x_0,p_0)-p_0q_0+\widetilde{\varphi}(t_0,x_0,p_0) \geq 0 \geq u(t,x,q)-\varphi(t,x,q),
$$

which leads to

$$\widehat{v}_*(t_0,x_0,p_0) - \widetilde{\varphi}(t_0,x_0,p_0) \le \widehat{v}_*(t,x,p^*) - \widetilde{\varphi}(t,x,p^*),$$

by using (3.2.13) and the definition of u. We finally have

$$(t_0,x_0,p_0) \text{ is a local minimizer of } \widehat{v}_* - \widetilde{\varphi} \text{ such that } \left(\widehat{v}_* - \widetilde{\varphi}\right)(t_0,x_0,p_0) \le 0. \tag{3.2.14}$$

We conclude the proof by discussing three alternative cases depending on the value of p_0.

1. If $p_0 \in (0,1)$, then (3.2.14) implies that $\widetilde{\varphi}$ satisfies (**??**) at (t_0,x_0,p_0), i.e.

$$\begin{aligned} 0 &\le \sup_{\substack{\pi\in\mathbb{L}^2_\lambda \\ \alpha\in\mathbb{R}}} \left\{\min\left\{\mathcal{H}_1^{\widetilde{\varphi}}(\alpha,\pi), \mathcal{H}_2^{\widetilde{\varphi}}(\alpha,\pi)\right\}\right\} \\ &= \sup_{\substack{\pi\in\mathbb{L}^2_\lambda \\ \alpha\in\mathbb{R}}} \left\{\inf_{\varepsilon\in[0,1]} \left\{\varepsilon\mathcal{H}_1^{\widetilde{\varphi}}(\alpha,\pi) + (1-\varepsilon)\mathcal{H}_2^{\widetilde{\varphi}}(\alpha,\pi)\right\}\right\} \\ &= \inf_{\varepsilon\in[0,1]} \left\{\sup_{\substack{\pi\in\mathbb{L}^2_\lambda \\ \alpha\in\mathbb{R}}} \left\{\varepsilon\mathcal{H}_1^{\widetilde{\varphi}}(\alpha,\pi) + (1-\varepsilon)\mathcal{H}_2^{\widetilde{\varphi}}(\alpha,\pi)\right\}\right\}, \end{aligned} \tag{3.2.15}$$

where the last inequality is obtained by the minimax theorem. Indeed, denoting $f^{\widetilde{\varphi}} : (\varepsilon,(\alpha,\pi)) \in [0,1]\times(\mathbb{R}\times\mathbb{L}^2_\lambda) \mapsto \varepsilon\mathcal{H}_1^{\widetilde{\varphi}}(\alpha,\pi) + (1-\varepsilon)\mathcal{H}_2^{\widetilde{\varphi}}(\alpha,\pi) \in \mathbb{R}$, we have that $f^{\widetilde{\varphi}}$ is convex and lower semi-continuous in ε, and, using the convexity in p and the smoothness of $\widetilde{\varphi}$, that $f^{\widetilde{\varphi}}$ is concave and upper semi-continuous in (α,π). Moreover, using the concavity and the coercivity of $f^{\widetilde{\varphi}}$, we may find a compact subset $\mathcal{K}$ of $\mathbb{R}\times\mathbb{L}^2_\lambda$ such that

$$\sup_{(\alpha,\pi)\in\mathbb{R}\times\mathbb{L}^2_\lambda} f^{\widetilde{\varphi}}(\varepsilon,\alpha,\pi) = \max_{(\alpha,\pi)\in\mathcal{K}} f^{\widetilde{\varphi}}(\varepsilon,\alpha,\pi).$$

Now observe that the supremum in α is obtained in α^* satisfying

$$\alpha^* = \frac{\varepsilon\left(\widetilde{\varphi}_p\frac{\mu}{\sigma} - \sigma\widetilde{\varphi}_{xp}\right) + (1-\varepsilon)\frac{\widetilde{\varphi}_p}{\sigma}\beta}{\varepsilon\widetilde{\varphi}_{pp}}, \tag{3.2.16}$$

recall that $\widetilde{\varphi}$ is strictly convex. Using the fact that $a\widehat{x}^2 + b\widehat{x} = -a\widehat{x}^2$ where $\widehat{x}$ is such that $\widehat{x} := \arg\max\left\{ax^2+bx+c\right\}$, (3.2.15) becomes

$$\begin{aligned} 0 \le \inf_{\varepsilon\in[0,1]} \Bigg\{ &\sup_{\pi\in\mathbb{L}^2_\lambda} \left\{\varepsilon\widetilde{\varphi}_p\lambda\pi - (1-\varepsilon)\widetilde{\varphi}(\cdot+\beta,p+\pi)\right\} - \varepsilon\left(\widetilde{\varphi}_t + \frac{1}{2}\sigma^2\widetilde{\varphi}_{xx}\right) \\ &+(1-\varepsilon)\left(\widetilde{\varphi}_x\beta + \widetilde{\varphi}\right) + \varepsilon\frac{\widetilde{\varphi}_{pp}}{2}\alpha^{*2}\Bigg\}. \end{aligned} \tag{3.2.17}$$

We consider now the supremum in $\pi \in \mathbb{L}^2_\lambda$

$$\begin{aligned}\sup_{\pi\in\mathbb{L}^2_\lambda}\{\varepsilon\widetilde{\varphi}_p\lambda_t\pi-&(1-\varepsilon)\widetilde{\varphi}(\cdot+\beta,p+\pi)\}\\ &=(1-\varepsilon)\sup_{\pi\in\mathbb{L}^2_\lambda}\left\{\frac{\varepsilon\widetilde{\varphi}_p\lambda_t}{1-\varepsilon}(p+\pi)-\widetilde{\varphi}(\cdot+\beta,p+\pi)\right\}-\varepsilon\widetilde{\varphi}_p\lambda_t p\\ &=(1-\varepsilon)\varphi\left(t,x+\beta,q+\left(\frac{\varepsilon\widetilde{\varphi}_p\lambda_t}{1-\varepsilon}-q\right)\right).\end{aligned} \tag{3.2.18}$$

In view of (3.2.16), (3.2.18) and the usual link between the derivatives of a smooth function and its Fenchel transform, namely

$$\begin{aligned}\widetilde{\varphi}_t&=\varphi_t\\ \widetilde{\varphi}_x&=-\varphi_x\\ \widetilde{\varphi}_{xx}&=-\varphi_{xx}-\varphi_{xq}q^*_x\\ \widetilde{\varphi}_p&=q^*\\ \widetilde{\varphi}_{pp}&=\frac{1}{\varphi_{qq}}\\ \widetilde{\varphi}_{xp}&=q^*_x\\ q^*_x&=-\frac{\varphi_{xq}}{\varphi_{qq}},\end{aligned} \tag{3.2.19}$$

(3.2.17) can be written, for $\gamma:=\frac{1-\varepsilon}{\varepsilon}$,

$$\begin{aligned}0\le\inf_{\gamma\in\Gamma}\Bigg\{&\gamma\left(\varphi\left(\cdot+\beta,\cdot+q\left(\frac{\lambda}{\gamma}-1\right)\right)-\varphi\right)+(\gamma q-q\lambda)\,\varphi_q\\ &+\varphi_t+\frac{1}{2}\sigma^2\varphi_{xx}-\gamma\beta\varphi_x+\frac{1}{2}\left(q\frac{\mu}{\sigma}+\gamma q\frac{\beta}{\sigma}\right)^2\varphi_{qq}+(q\mu+\gamma q\beta)\varphi_{qx}\Bigg\},\end{aligned} \tag{3.2.20}$$

which is (**??**).

2. If $p_0=1$, then the first boundary condition in (2.3.19) and (3.2.14) imply that (t_0,x_0) is a local minimizer of $(\widehat{v}_*-\widetilde{\varphi})(\cdot,1)=(v-\widetilde{\varphi}(\cdot,1))$ such that $(v-\widetilde{\varphi}(\cdot,1))(t_0,x_0)\le 0$. This implies that $\widetilde{\varphi}(\cdot,1)$ satisfies (**??**) at (t_0,x_0), and therefore that $\widetilde{\varphi}$ satisfies (**??**) for $\alpha=0$ and $\pi=0$ at $(t_0,x_0,1)$. We then conclude as in **1.** above.

3. If $p_0=0$, then the second boundary condition in (2.3.19) and (3.2.14) imply that (t_0,x_0) is a local minimizer of $(\widehat{v}_*(\cdot,0)-\widetilde{\varphi}(\cdot,0))=0-\widetilde{\varphi}(\cdot,0)$ such that $(0-\widetilde{\varphi}(t_0,x_0,0))\le 0$. In particular, (t_0,x_0) is a local maximum point for $\widetilde{\varphi}(\cdot,0)$ so that we have $\partial_t\widetilde{\varphi}(t_0,x_0)\le 0, D_x\widetilde{\varphi}(t_0,x_0)=0$ and $D_{xx}\widetilde{\varphi}(t_0,x_0,0)\le 0$. This implies that $\widetilde{\varphi}(\cdot,0)$ satisfies (**??**), and thus (**??**) at $(t_0,x_0,0)$ for $\alpha=0$ and $\pi=0$. We then argue as in **1.**.

□

3.2.2 Characterization of an upper bound

Chapter 4

Stochastic Target Games

Chapter 5

Lipschitz selection

Chapter 6

Utility

Part II

Example

Chapter 7

4 author papers

Chapter 8

Application in the hybrid case

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