Super-linear spreading in local and non-local cane toads equations

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Abstract

In this paper, we show super-linear propagation in a nonlocal reaction-diffusion-mutation equation modeling the invasion of cane toads in Australia that has attracted attention recently from the mathematical point of view. The population of toads is structured by a phenotypical trait that governs the spatial diffusion. In this paper, we are concerned with the case when the diffusivity can take unbounded values, and we prove that the population spreads as $t^{3/2}$. We also get the sharp rate of spreading in a related local model.

1 Introduction

The invasion of cane toads in Australia has interesting features different from the standard spreading observed in most other species. The experimental data [42, 45] show that the invasion speed has steadily increased during the eighty years since the toads were introduced in Australia. In addition, the younger individuals at the edge of the invasion front have a significantly different morphology compared to other populations – their legs tend to be on average much longer than away from the front. This is just one example of a non-uniform space-trait distribution – see, for instance, a study on the expansion of bush crickets in Britain [46]. Several works have addressed the front invasions in ecology, where the trait is related to the dispersal ability [3, 18]. It has been observed that selection of more mobile individuals can occur, even if they have no advantage in their reproductive rate, due to the spatial sorting [1, 35, 42, 43].

In this paper, we focus on the super-linear in time propagation in a model of the cane toads invasion proposed in [5], based on the classical Fisher-KPP equation [24, 36]. The population density is structured by a spatial variable, $x \in \mathbb{R}$, and a motility variable $\theta \in \Theta \overset{\text{def}}{=} [\underline{\theta}, \infty)$, with a fixed $\underline{\theta} > 0$. This population undergoes diffusion in the trait variable θ, with a constant diffusion coefficient $\alpha > 0$, representing mutation, and in the spatial variable, with the diffusion coefficient θ, representing the effect of the trait on the spreading rates of the species. Thus, neglecting the competition and reproduction, the population model for the population density $u(t, x, \theta)$ would be

$$u_t = \theta u_{xx} + \alpha u_{\theta\theta}. \tag{1.1}$$

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In addition, each toad competes locally in space with all other individuals for resources. If the competition is local in the trait variable, has a saturation level S, and a growth rate r, then a Fisher-KPP type generalization of (1.1) is

$$u_t = \theta u_{xx} + \alpha u_{\theta\theta} + ru(S - u). \tag{1.2}$$

It is also natural to consider a non-local in trait competition (but still local in space), which leads to

$$n_t = \theta n_{xx} + \alpha n_{\theta\theta} + rn(S - \rho), \tag{1.3}$$

where

$$\rho(t, x) = \int_{\underline{\theta}}^{\infty} n(t, x, \theta)d\theta \tag{1.4}$$

is the total population at the position x. Both (1.2) and (1.3) are supplemented by the Neumann boundary condition at $\theta = \underline{\theta}$:

$$u_\theta(t, x, \underline{\theta}) = 0, \quad t > 0, \ x \in \mathbb{R}. \tag{1.5}$$

The non-dimensional versions of (1.2) and (1.3) are, respectively,

$$u_t = \theta u_{xx} + u_{\theta\theta} + u(1 - u), \quad t > 0, \ x \in \mathbb{R}, \ \theta \in \Theta, \tag{1.6}$$

and

$$n_t = \theta n_{xx} + n_{\theta\theta} + n(1 - \rho), \quad \rho(t, x) = \int_{\underline{\theta}}^{\infty} n(t, x, \theta)d\theta, \quad t > 0, \ x \in \mathbb{R}, \ \theta \in \Theta. \tag{1.7}$$

In general, the speed of propagation in the Fisher-KPP type equations is determined by the linearization around zero, that is, with the terms $u(1 - u)$ in (1.6) and $n(1 - \rho)$ in (1.7) replaced by u and n, respectively. Since the linearizations of (1.6) and (1.7) are identical, we expect both models to have the same propagation speed.

Models involving non-local reaction terms have been the subject of intense study in recent years due to the complexity of the dynamics – see, for example, [2, 14, 23, 33, 38, 39] and references therein. The cane toads equation has similarly attracted recent interest, mostly when the motility set Θ is a finite interval. An Hamilton-Jacobi framework has been formally applied to the non-local model in [11], and rigorously justified in [47]. In these works, the authors obtain the speed of propagation and the expected repartition of traits at the edge of the front by solving a spectral problem in the trait variable. The existence of traveling waves has been proved in [9]. The precise asymptotics of the front location for the Cauchy problem, up to a constant shift has been obtained in [13] by using a Harnack-type inequality that allows one to compare the solutions of the non-local equation to those of a local equation, whose dynamics are well-understood [15].

As far as unbounded traits are concerned, a formal argument in [11] using a Hamilton-Jacobi framework predicted front acceleration and spreading rate of $O(t^{3/2})$. In this paper, we give a rigorous proof of this spreading rate both in the local and non-local models. This is an addition to the growing list of "accelerating fronts" that have attracted some interest in recent years [12, 16, 17, 19, 26, 32, 34, 37].

The local case

Our first result concerns the local equation (1.6).

Theorem 1.1 (**Acceleration in the local cane toads model**)**.** *Let $u(t, x)$ be the solution of the local equation (1.6), with the boundary condition (1.5), and the initial condition $u(0, x) = u_0(x) \geq 0$. Assume that $u_0(x)$ is compactly supported, and fix any constant $m \in (0, 1)$, then*

$$\lim_{t \to \infty} \frac{\max\{x \in \mathbb{R} : \exists \theta \in \Theta, u(t, x, \theta) = m\}}{t^{3/2}} = \frac{4}{3}, \tag{1.8}$$

The limit is uniform in $m \in [\varepsilon, 1 - \varepsilon]$, for any $\varepsilon > 0$ fixed.

The assumption that u_0 is compactly supported is made purely for convenience, one could allow more general rapidly decaying or front-like initial conditions. The proof of Theorem 1.1 is in two steps. First, we show that the Hamilton-Jacobi framework provides a super-solution to (1.6), and gives the upper bound of the limit in (1.8) – see Proposition 3.1. Second, we prove the lower bound to the limit in (1.8) by constructing a sub-solution to u. This is done in Proposition 4.2. The argument involves building a sub-solution to (1.8) on a traveling ball with the Dirichlet boundary conditions, whose path comes from the Hamilton-Jacobi framework discussed above. These sub-solutions are arbitrarily small initially but become bounded uniformly away from zero on any compact subset of the traveling ball for large time. The analysis is complicated by the fact that the diffusivity is unbounded in the θ direction. It is crucial to match the upper bound that we use the optimal paths coming from the Hamilton-Jacobi framework.

The non-local case

Our second main result shows that the full non-local model (1.7) exhibits a similar front acceleration.

Theorem 1.2 (**Acceleration in the non-local cane toads model**)**.** *Let $n(t, x)$ be the solution of the non-local cane toads equation (1.7), with the Neumann boundary condition (1.5), and the initial condition $n(0, x) = n_0(x) \geq 0$. Assume that $n_0(x)$ is compactly supported, and fix any $\varepsilon > 0$. There exists a positive constant c_ε, depending only on ε, such that*

$$\frac{8}{3\sqrt{3\sqrt{3}}}(1 - \varepsilon) \leq \limsup_{t \to \infty} \frac{\max\{x \in \mathbb{R} : \rho(t, x) \geq c_\varepsilon\}}{t^{3/2}}. \tag{1.9}$$

In addition, if m is any constant in $(0, 1)$, then we have that

$$\limsup_{t \to \infty} \frac{\max\{x \in \mathbb{R} : \rho(t, x) \geq m\}}{t^{3/2}} \leq \frac{4}{3}. \tag{1.10}$$

In contrast to the sharp bound provided by Theorem 1.1, Theorem 1.2 does not have matching lower and upper bounds. This is due to the lack of a comparison principle for (1.7). The proof of the lower bound involves arguing by contradiction in order to compare (1.7) to a solution of the local, linear equation defined on a moving ball with the Dirichlet boundary conditions. This causes the lim sup to appear in (1.9), as opposed to the more desirable lim inf. In addition, because the optimal Hamilton-Jacobi trajectories trajectories are initially almost purely in the θ direction, we can not use them to move the "Dirichlet ball", and are forced to use non-optimal trajectories,

leading to the sub-optimal constant $8/(3\sqrt{3\sqrt{3}})$ in (1.9). We comment on this in Section 4.2. We believe that the sharp result would have the lower bound matching the upper bound (1.10). The proof of the upper bound is given by Proposition 3.1 below, as in the local case, while the proof of the lower bound and an explicit bound on c_ε are given by Proposition 4.3. We also note that in general non-local Fisher-KPP type equations the stability of the steady state $u \equiv 1$ may fail [6, 27, 29]. Thus, it is not surprising that we are restricted to working with the level sets of certain heights c_ε in (1.9) – this mirrors the propagation results in [33].

We should mention the concurrent work by Berestycki, Mouhot, and Raoul [7], who use a mix of probabilistic and analytic methods to prove the same sharp result in the local model (1.6). In addition, they prove the sharp asymptotics in a non-local model where ρ is replaced by a windowed non-local. Ecologically, the windowed term models a situation where individuals compete for resources only with individuals of a similar trait. The techniques of [7] do not seem to apply to the full non-local model that we address here.

The rest of the paper is organized as follows. In Section 2 we recall some facts from [11] on the Hamilton-Jacobi framework. Then we derive in Section 3 the upper bound common to Theorems 1.1 and 1.2 using an explicit super-solution that arises from the work in Section 2. The lower bound is then proved in Section 4. There, we first derive a general propagation result on the linearized equation by using the optimal paths from Section 2, re-scaling the equation appropriately, and using precise spectral estimates. We then use this result to obtain the lower bounds for the local and non-local models. Section 5 contains the proofs of some auxiliary results.

2 The Hamilton-Jacobi solutions

In this section, we recall how a suitable scaling limit of the cane toads equation can formally give the acceleration rate [11]. The analysis of this section will be used in the rest of the paper to construct "good" sub- and super-solutions to the local and local equations (1.6) and (1.7). We will focus on the linearized equation

$$u_t = \theta u_{xx} + u_{\theta\theta} + u. \tag{2.1}$$

Writing

$$u(t, x) = e^t v(t, x),$$

we reduce it to

$$v_t = \theta v_{xx} + v_{\theta\theta}. \tag{2.2}$$

One obvious solution to this equation is

$$v(t, \theta) = \frac{1}{\sqrt{4\pi t}} e^{-\theta^2/(4t)},$$

which also provides a spatially uniform super-solution to (2.1):

$$\bar{v}(t,\theta) = e^{t-\theta^2/(4t)}. \tag{2.3}$$

However, the function $\bar{v}(t,\theta)$ has no decay in x, and is not useful in the spatial regions where we expect the solution of the full cane toads equation to be small.

In order to construct another super-solution to (2.2), with decay in space, we rescale (2.2) setting

$$v^{\varepsilon}(t,x,\theta) = v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{3/2}}, \frac{\theta}{\varepsilon}\right).$$

The function $u^{\varepsilon}(t,x,\theta)$ satisfies

$$\varepsilon\partial_t u^{\varepsilon} = \varepsilon^2 \theta \partial_{xx}^2 u^{\varepsilon} + \varepsilon^2 \partial_{\theta\theta}^2 u^{\varepsilon}.$$

We perform the *Hopf-Cole transformation*

$$u^{\varepsilon} = \exp\{-\varphi^{\varepsilon}/\varepsilon\},$$

so that

$$\partial_t \varphi^{\varepsilon} + \theta|\varphi_x^{\varepsilon}|^2 + |\varphi_\theta^{\varepsilon}|^2 = \varepsilon\theta\varphi_{xx}^{\varepsilon} + \varepsilon\varphi_{\theta\theta}^{\varepsilon}, \tag{2.4}$$

and obtain, in the formal limit as $\varepsilon \to 0$, the Hamilton-Jacobi equation

$$\partial_t \psi + \theta|\psi_x|^2 + |\psi_\theta|^2 = 0. \tag{2.5}$$

We will use the solutions of this Hamilton-Jacobi equation to construct sub- and super-solutions to the original problem. The Hopf-Cole transformation is an effective tool in the analysis of front propagation for reaction-diffusion equations – see, for instance, [4, 22, 25], including parabolic integro-differential equations modeling populations structured by a phenotypical trait (see [14, 21, 41]).

A "heat kernel" solution of the Hamilton-Jacobi equation

Let us consider the Hamilton-Jacobi equation (2.5) with the initial condition

$$\psi(t=0,x,\theta) = \begin{cases} 0 & \text{if } (x,\theta) = (0,0) \\ +\infty & \text{if } (x,\theta) \neq (0,0). \end{cases} \tag{2.6}$$

The reason for this choice of the initial data is clear: for the standard heat equation, the Hamilton-Jacobi equation would be

$$\psi_t + |\psi_x|^2 = 0, \tag{2.7}$$

and the solution is simply

$$\psi(t,x) = \frac{x^2}{4t}, \tag{2.8}$$

leading to the standard heat kernel. The Hamiltonian for (2.5) is

$$H((x,\theta),(p_x,p_\theta)) = \theta|p_x|^2 + |p_\theta|^2, \tag{2.9}$$

and the corresponding Lagrangian is

$$L((x,\theta),(v_x,v_\theta)) = \frac{v_x^2}{4\theta} + \frac{v_\theta^2}{4}. \tag{2.10}$$

Using the Lax-Oleinik formula to solve (2.5), we get

$$\psi(t,x,\theta) = \inf_{w\in C^1} \left\{ \int_0^t L(w(s),\dot{w}(s))ds \mid w(0) = (x,\theta), w(t) = (0,0) \right\}. \tag{2.11}$$

The optimal trajectory given by the Hamiltonian flow is the solution of

$$\frac{dX(s)}{ds} = 2P_x(s)\theta(s), \quad \frac{d\theta(s)}{ds} = 2P_\theta(s), \quad \frac{dP_x(s)}{ds} = 0, \quad \frac{dP_\theta(s)}{ds} = -P_x(s)^2.$$

hence

$$\frac{d}{ds}\left(\frac{1}{\theta(s)}\frac{dX(s)}{ds}\right) = 0, \quad \frac{d^2\theta(s)}{ds^2} = -\frac{1}{2}\left(\frac{1}{\theta(s)}\frac{dX(s)}{ds}\right)^2.$$

Thus, there is a constant C such that

$$\dot{X}(s) = C\theta(s), \qquad \ddot{\theta}(s) = -\frac{C^2}{2}, \quad X(0) = x, \ \theta(0) = \theta, \ X(t) = 0, \ \theta(t) = 0. \tag{2.12}$$

Plugging this into the expression for ψ gives

$$\psi(t,x,\theta) = \int_0^t L(w(s),\dot{w}(s))ds = \int_0^t \frac{1}{4}\left(\frac{\dot{X}(s)^2}{\theta(s)} + \dot{\theta}(s)^2\right)ds$$

$$= \frac{C}{4}\int_0^t \dot{X}(s)ds + \frac{t\dot{\theta}(t)^2}{4} - \frac{1}{2}\int_0^t s\dot{\theta}(s)\ddot{\theta}(s)ds = -\frac{Cx}{4} + \frac{t\dot{\theta}(t)^2}{4} + \frac{C^2}{4}\int_0^t s\dot{\theta}(s)ds \tag{2.13}$$

$$= -\frac{Cx}{4} + \frac{t\dot{\theta}(t)^2}{4} - \frac{C^2}{4}\int_0^t \theta(s)ds = -\frac{Cx}{4} + \frac{t\dot{\theta}(t)^2}{4} + \frac{Cx}{4} = \frac{t\dot{\theta}(t)^2}{4}.$$

We now compute $\dot{\theta}(t)$. From (2.12), we find that

$$\theta(s) = -\frac{C^2}{4}s^2 + \left(\frac{C^2 t}{4} - \frac{\theta}{t}\right)s + \theta, \tag{2.14}$$

which implies that

$$\dot{\theta}(t) = -\frac{\theta}{t} - \frac{C^2}{4}t = -\frac{1}{t}\left(\theta + \frac{C^2 t^2}{4}\right). \tag{2.15}$$

To obtain a closed form for ψ, we need to find C. We use (2.12) to obtain

$$-x = \int_0^t \dot{X}(s)ds = C\int_0^t \theta(s)ds = C\left(-\frac{C^2 t^3}{12} + \left(\frac{C^2 t}{4} - \frac{\theta}{t}\right)\frac{t^2}{2} + \theta t\right) = \frac{(Ct)^3}{24} + \frac{\theta(Ct)}{2}.$$

It follows that $Z = Ct/2$ is the unique real solution of the cubic equation

$$Z^3 + 3\theta Z + 3x = 0. \tag{2.16}$$

Combining (2.13), (2.15), and (2.16), we obtain an explicit formula for $\psi(t,x)$:

$$\psi(t,x,\theta) = \frac{1}{4t}\left(\theta + Z(x,\theta)^2\right)^2, \tag{2.17}$$

the analog of (2.8) for our problem.

Super-solutions with the diffusion

The function $\psi(t, x)$ was obtained neglecting the right side in (2.4). It turns out, that, when the diffusion is taken into account, it still leads to a super-solution to the linearized cane toads equation

$$u_t - \theta u_{xx} - u_{\theta\theta} - u \geq 0, \tag{2.18}$$

of the form

$$\bar{u}(t, x, \theta) = \exp\{t - \psi(t, x, \theta)\}, \tag{2.19}$$

or, more explicitly showing the difference with (2.3):

$$\bar{u}(t, x, \theta) = \exp\left\{t - \frac{1}{4t}(\theta + Z^2(x, \theta))^2\right\}. \tag{2.20}$$

This function satisfies

$$\bar{u}_t - \theta\bar{u}_{xx} - \bar{u}_{\theta\theta} - \bar{u} = \bar{u}\big(-\psi_t + \theta\psi_{xx} - \theta|\psi_x|^2 + \psi_{\theta\theta} - |\psi_\theta|^2\big) = \bar{u}\big(\theta\psi_{xx} + \psi_{\theta\theta}\big). \tag{2.21}$$

Observe that

$$4t\big(\theta\psi_{xx} + \psi_{\theta\theta}\big) = 4\theta((\theta + Z^2)ZZ_x)_x + 2((\theta + Z^2)(1 + 2ZZ_\theta))_\theta$$
$$= 4(\theta + Z^2)(\theta(ZZ_x)_x + (ZZ_\theta)_\theta) + 8\theta(ZZ_x)^2 + 2(1 + 2ZZ_\theta)^2 \geq 4(\theta + Z^2)Z\,(\theta Z_{xx} + Z_{\theta\theta}). \tag{2.22}$$

We claim that, somewhat miraculously,

$$\theta Z_{xx} + Z_{\theta\theta} = 0. \tag{2.23}$$

Indeed, we have

$$Z_\theta = -\frac{Z}{Z^2 + \theta}, \quad Z_x = -\frac{1}{Z^2 + \theta}, \tag{2.24}$$

so that

$$Z_{\theta\theta} = -\frac{Z_\theta(Z^2 + \theta) - Z(2ZZ_\theta + 1)}{(Z^2 + \theta)^2} = \frac{Z(Z^2 + \theta) + \left(\theta - Z^2\right)Z}{(Z^2 + \theta)^3} = \frac{2Z\theta}{(Z^2 + \theta)^3},$$

and

$$Z_{xx} = -\frac{2Z}{(Z^2 + \theta)^3}.$$

Thus, (2.23) holds and, as a consequence, (2.18) follows.

A level set of the super-solution and the optimal trajectories

The level set $\{\bar{u}(t, x, \theta) = 1\}$ is given by

$$\theta + Z(x, \theta)^2 = 2t. \tag{2.25}$$

Multiplying this equation by Z, and using (2.16), we get

$$3\theta Z + 3x = -(2t - \theta)Z, \tag{2.26}$$

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or

$$Z(x, \theta) = -\frac{3x}{2(\theta + t)}. \tag{2.27}$$

Inserting this back into (2.25), gives the equation for the level set $\{\bar{u}(t, x, \theta) = 1\}$:

$$x^2 = \frac{4}{9}(2t - \theta)(\theta + t)^2. \tag{2.28}$$

In order to compute the rightmost point of this level set at a given time $t > 0$, we differentiate (2.28) in θ. The critical points are determined by

$$0 = 2x\frac{\partial x}{\partial \theta} = \frac{4}{9}\left[-(\theta + t)^2 + 2(2t - \theta)(\theta + t)\right] = \frac{4}{3}(t + \theta)(t - \theta),$$

and the maximum

$$x_{\text{edge}}(t) = \frac{4}{3}t^{3/2} \tag{2.29}$$

is attained at $\theta_{\text{edge}}(t) = t$.

With the maximal end-points in hand, we now compute the Lagrangian trajectories $X(s)$, $\theta(s)$ which travel to the far edge $(x_{\text{edge}}(t), \theta_{\text{edge}}(t))$. Using the general form (2.14) with the choice $C = 2Z/t$ for $\theta(s)$, we obtain

$$\theta(s) = -Z(x_{\text{edge}}(t), \theta_{\text{edge}}(t))^2\left(\frac{s}{t}\right)^2 + (Z(x_{\text{edge}}(t), \theta_{\text{edge}}(t))^2 - \theta_{\text{edge}})\frac{s}{t} + \theta_{\text{edge}}(t).$$

Then using the expressions $x_{\text{edge}}(t) = 4t^{3/2}/3$ and $\theta_{\text{edge}}(t) = t$ along with the expression (2.27) for Z, we obtain

$$\theta(s) = -\frac{9x_{\text{edge}}(t)^2}{16t^2}\left(\frac{s}{t}\right)^2 + \left(\frac{9x_{\text{edge}}(t)^2}{16t^2} - t\right)\frac{s}{t} + t = t\left(1 - \left(\frac{s}{t}\right)^2\right). \tag{2.30}$$

We similarly deduce the trajectory for $X(s)$. Indeed, we have from (2.12) and the definition of Z that

$$X(s) = x_{\text{edge}}(t) + \frac{2Z}{t}\int_0^s \theta(r)dr.$$

Using expression (2.29) for $x_{\text{edge}}(t)$ and (2.30), we get

$$X(s) = \frac{4}{3}t^{3/2} - \frac{1}{t}\left(\frac{3x_{\text{edge}}}{\theta_{\text{edge}} + t}\right)\int_0^s t\left(1 - \left(\frac{r}{t}\right)^2\right)dr = \frac{4}{3}t^{3/2} - 2t^{1/2}\left(s - \frac{s^3}{3t^2}\right)$$

$$= \frac{4}{3}t^{3/2}\left(1 - \frac{3s}{2t}\left(1 - \frac{s^2}{3t^2}\right)\right).$$

To obtain the forward trajectories, we reverse time, that is, change variables from $s \mapsto t - s$, and with a slight abuse of notation, write

$$X(s) = \left(\frac{3}{2} - \frac{s}{2t}\right)\left(\frac{s}{t}\right)^2\frac{4}{3}t^{3/2}, \quad \text{and} \quad \theta(s) = s\left(2 - \frac{s}{t}\right). \tag{2.31}$$

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The minimum of $\psi(t, x, \theta)$

In the sequel, we also need the minimum $\theta_*(t, x)$ of $\psi(t, x, \theta)$ in $\theta \in \Theta$, for t and x fixed. We differentiate expression (2.17) for $\psi(t, x, \theta)$ with respect to θ:

$$\psi_\theta = \frac{1}{2t} \left(\theta + Z^2 \right) \left(1 + 2ZZ_\theta \right). \tag{2.32}$$

Hence, the critical points satisfy

$$ZZ_\theta = -\frac{1}{2}. \tag{2.33}$$

Using expression (2.24) for Z_θ leads to (note that Z has the opposite sign of x)

$$Z = -\sqrt{\theta_*(x)} \operatorname{sgn}(x).$$

We insert this value into (2.16), and find

$$\theta_*(x) = \left(\frac{3}{4} |x| \right)^{2/3}, \tag{2.34}$$

and

$$\psi(t, x, \theta_*(x)) = \frac{1}{t} \left(\frac{3}{4} |x| \right)^{4/3}. \tag{2.35}$$

Note that the internal minimum exists only if $|x|$ is sufficiently large:

$$|x| > r_c \overset{\text{def}}{=} \frac{4}{3} \underline{\theta}^{3/2}, \tag{2.36}$$

so that $\theta_*(x) \geq \underline{\theta}$, otherwise the minimum of $\psi(t, x, \theta)$ is attained at $\theta = \underline{\theta}$.

3 An upper bound

In this section, we construct an explicit super-solution for the local and nonlocal versions of the cane toads equation. This provides an upper bound on the spreading rate.

A super-solution

Ideally, we would take as a super-solution the function

$$u(t, x, \theta) = \bar{u}(t + a, x, \theta),$$

with $\bar{u}$ as in (2.19) with some suitably chosen $a > 0$. There is an obstacle: a super-solution, in addition to (2.18), should satisfy the boundary condition

$$u_\theta(t, x, \underline{\theta}) \leq 0. \tag{3.1}$$

For a function of the form (2.19), this condition is equivalent to

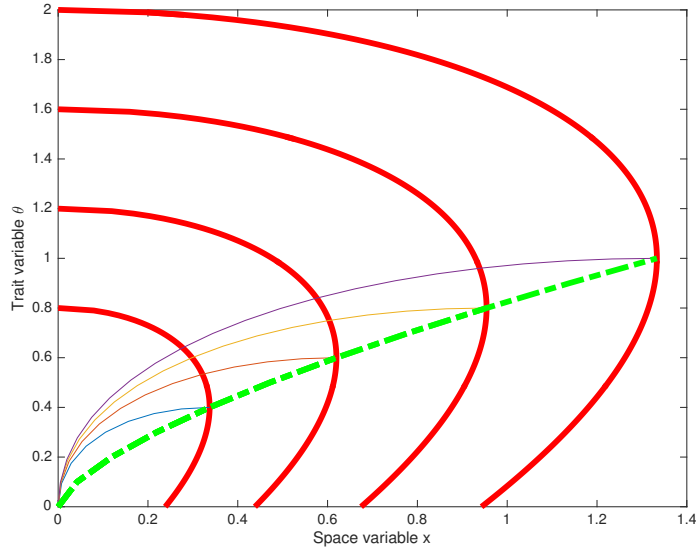$$\psi_\theta(t, x, \underline{\theta}) \geq 0. \tag{3.2}$$

Figure 2.1: The level set $\{\bar{u}(t, x, \theta) = 1\}$ in the phase space (x, θ) for different values of time and the optimal trajectory, see also [11]. We also plot the trajectories that lead to the edge of the front for various values of time. These latter trajectories are crucial for the paper.

The function $\psi(t, x, \theta)$ has a single minimum, either at $\theta = \underline{\theta}$, if $|x| < r_c$, or at $\theta = \theta_*(x)$ if $|x| > r_c$, where we recall r_c from (2.36). Hence, (3.2) can not hold for $|x| > r_c$, and we need to modify $u(t, x, \theta)$ to turn it into a true super-solution. To this end, we recall the other family of super-solutions, $\bar{v}(t, \theta)$ given by (2.3):

$$\bar{v}(t, \theta) = e^{t - \theta^2/(4t)}, \tag{3.3}$$

that do satisfy the boundary condition (3.1). As, on the other hand, $\bar{v}(t, \theta)$ has no decay in x, we will only use it for $x < 0$. Let us define the set

$$\Omega = \{(x, \theta) : \ x \geq r_c, \text{ and } \underline{\theta} \leq \theta \leq \theta_*(x)\}. \tag{3.4}$$

We define our super-solution first on $\{x \leq 0\}$ and on $\{x \geq 0\} \cap \Omega^c$, but we extend it to all of $\mathbb{R} \times \Omega$ below. First, we define

$$\tilde{u}(t, x, \theta) = \begin{cases} C(a)\bar{v}(t + a, \theta), & x \leq 0, \\ C(a)\bar{u}(t + a, x, \theta), & x \geq 0, \ (x, \theta) \in \Omega^c, \end{cases} \tag{3.5}$$

with the constant $C(a)$ to be chosen later. Note that we have

$$\bar{v}(t + a, \theta) \geq \bar{u}(t + a, x, \theta), \tag{3.6}$$

with the equality holding only at $x = 0$, where we recall that $Z(t, x = 0, \theta) = 0$. Hence, the function $\tilde{u}(t, x, \theta)$ is C^1 in Ω^c. It is a super-solution in the sense that

$$\tilde{u}_t - \theta\tilde{u}_{xx} - \tilde{u}_{\theta\theta} - \tilde{u} \geq 0 \text{ in } \Omega^c, \tag{3.7}$$

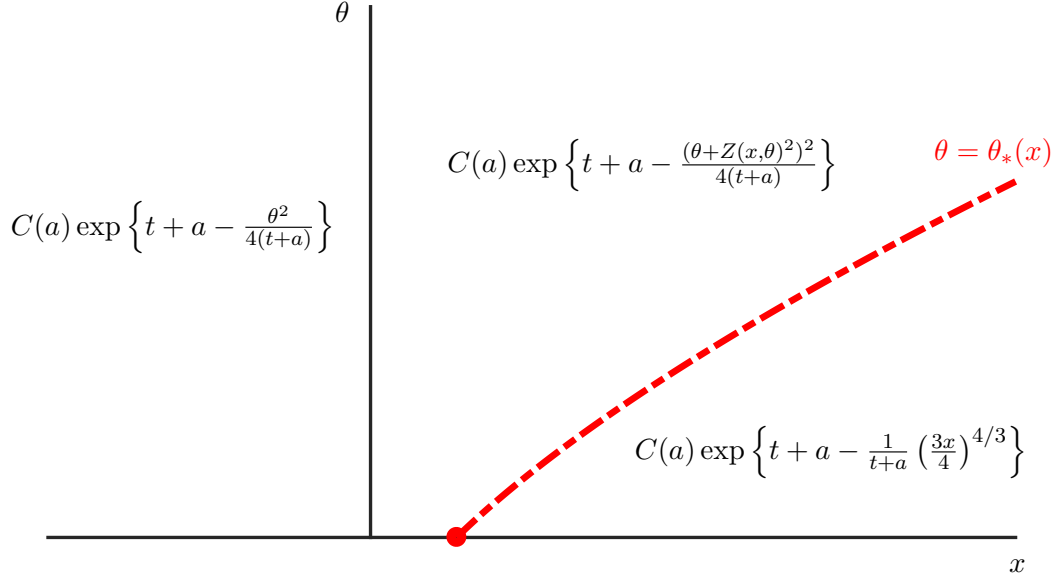$$\bar{u}_\theta(t, x, \underline{\theta}) \leq 0 \text{ for all } x \leq r_c.$$

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Figure 3.1: Definition of the super-solution in the three domains $\{x < 0\}$, Ω and $\Omega^c \cap \{x \geq 0\}$. The red line gives the curve $(x, \theta_*(x))$.

In order to extend $\tilde{u}$ into Ω as a C^1-supersolution we simply set

$$\tilde{u}(t, x, \theta) = C(a)\bar{u}(t, x, \theta_*(x)), \quad \text{for } (x, \theta) \in \Omega, \tag{3.8}$$

or, more explicitly

$$\tilde{u}(t, x, \theta) = C(a) \exp\left\{ t + a - \frac{1}{t+a}\left(\frac{3x}{4}\right)^{4/3} \right\}, \quad \text{for } (x, \theta) \in \Omega. \tag{3.9}$$

We need to verify that the extended function is C^1 and that it satisfies (3.7). Before we verify this, we point out that in order to apply the maximum principle, we only require that $\tilde{u}$ is C^1 and is C^2 except on a smooth 1D set. As such, we need not worry that $\tilde{u}$ is likely not C^2 on $\partial\Omega$. The boundary condition in (3.7) is automatic since $\tilde{u}$ does not depend on the variable θ in Ω. As $\theta_*(x)$ is the minimum of $\psi(t, x, \theta)$ in θ, we have $\tilde{u}_\theta = 0$ on both sides of the curve

$$\Gamma = \partial\Omega = \{(x, \theta_*(x)), \ x \geq r_c\}.$$

It is easy to see that the x-derivatives of $\tilde{u}$ match across Γ for the same reason, and the extended function is C^1. We now compute in Ω:

$$\tilde{u}_t - \theta\tilde{u}_{xx} - \tilde{u}_{\theta\theta} - \tilde{u} = \tilde{u}\left[\frac{1}{(t+a)^2}\left(\frac{3x}{4}\right)^{4/3} + 1 - \frac{\theta}{(t+a)^2}\left(\frac{3x}{4}\right)^{2/3} + \frac{\theta}{t+a}\left(\frac{3}{2x}\right)^{2/3} - 1 \right]$$

$$\geq \frac{\tilde{u}}{(t+a)^2}\left(\frac{3x}{4}\right)^{2/3}\left[\left(\frac{3x}{4}\right)^{2/3} - \theta\right] = \frac{\tilde{u}}{(t+a)^2}\left(\frac{3x}{4}\right)^{2/3}(\theta_*(x) - \theta) \geq 0,$$

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as $\theta \leq \theta_*(x)$ in Ω. Thus, the function $\tilde{u}(t,x,\theta)$ is a super-solution in the sense of (3.7) in the whole domain $(x,\theta) \in \mathbb{R} \times \Theta$:

$$\tilde{u}_t - \theta\tilde{u}_{xx} - \tilde{u}_{\theta\theta} - \tilde{u} \geq 0 \text{ for all } x \in \mathbb{R} \text{ and } \theta \in \Theta, \tag{3.10}$$
$$\tilde{u}_\theta(t,x,\underline{\theta}) \geq 0 \text{ for all } x \in \mathbb{R}.$$

An upper bound

We now use the above super-solution to give an upper bound for the speed of propagation for the local and non-local cane toads equations.

Proposition 3.1. *Let $u_0 \in L^\infty(\mathbb{R} \times \Theta)$ be a non-zero, non-negative function such that*

$$u_0 \leq \mathbb{1}_{[-\infty,x_r] \times [\underline{\theta},\theta_u]}$$

for some $x_r \in \mathbb{R}$, $\theta_u > \underline{\theta}$. Fix any $m > 0$. Let u be a solution of either the local cane toads equation (1.6) or the non-local cane toads equation (1.7). Then the following upper bounds hold

$$\lim_{t \to \infty} \frac{\max\{x \in \mathbb{R} : \exists \theta \in \Theta, u(t,x,\theta) \geq m\}}{t^{3/2}} \leq \frac{4}{3}, \quad and$$
$$\lim_{t \to \infty} \frac{\max\{x \in \mathbb{R} : \rho(t,x,\theta) \geq m\}}{t^{3/2}} \leq \frac{4}{3}.$$

Proof. Whether u solves (1.6) or (1.7), it satisfies

$$u_t - \theta u_{xx} - u_{\theta\theta} - u \leq 0, \qquad (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta,$$
$$u_\theta(t,x,0) = 0,$$
$$u(t=0,x,\theta) = u_0(x,\theta).$$

As a consequence, u is a sub-solution of the linearized cane toads equation. On the other hand, the function $\tilde{u}(t,x,\theta)$ defined by (3.5) and (3.9) is a super-solution, in the sense that (3.10) holds. Let us choose a constant $C(a)$ large enough so that, for all $(x,\theta) \in \mathbb{R} \times \Theta$,

$$\tilde{u}(0,x,\theta) \geq u_0(x,\theta).$$

We deduce from the parabolic comparison principle that

$$u(t,x,\theta) \leq \tilde{u}(t,x,\theta), \tag{3.11}$$

for all $t \geq 0$, $x \in \mathbb{R}$, and $\theta \in \Theta$.

We now use the explicit expression for $\tilde{u}$ to obtain an upper bound on the location of the level sets of u. To this end, fix any $m \in (0,1)$. As $\theta_*(x)$ is the minimum of the function $\psi(t,x,\theta)$, the rightmost point $\tilde{x}_m(t)$ of the level set $\{\tilde{u}(t,x,\theta) = m\}$ is where it intersects the curve $\theta = \theta_*(x)$:

$$\tilde{x}_m(t) = \frac{4}{3}(t+a)^{3/2}\left(1 - \frac{1}{t+a}\log\left(\frac{m}{C(a)}\right)\right)^{3/4}. \tag{3.12}$$

Using (3.11) and passing to the limit $t \to +\infty$, we obtain from (3.12):

$$\lim_{t \to \infty} \frac{\max\{x \in \mathbb{R} : \exists \theta \in \Theta, u(t,x,\theta) \geq m\}}{t^{\frac{3}{2}}} \leq \frac{4}{3}.$$

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This gives us an upper bound on the level sets of u, but we also need an upper bound on ρ. As follows from (3.11), it suffices to bound

$$\tilde{\rho}(t,x) = \int_{\underline{\theta}}^{\infty} \tilde{u}(t,x,\theta)d\theta.$$

Let us use $\tilde{m} = m/10(t+a)$ in (3.12), so that

$$\tilde{u}(t,x_{m,t},\theta) \leq \frac{m}{10(t+a)}, \quad \text{for all } \theta \geq \underline{\theta},$$

where we define $x_{m,t} = \tilde{x}_{m/10(t+a)}(t)$. We recall that

$$\theta_*(x_{m,t}) = \left(\frac{3x_{m,t}}{4}\right)^{2/3}. \tag{3.13}$$

Note that, for t large enough we have

$$\frac{5}{6}(t+a) \leq \theta_*(x_{m,t}) \leq \frac{7}{6}(t+a). \tag{3.14}$$

Then we have

$$\int_0^{5\theta_*(x)} \tilde{u}(t,x_{m,t},\theta)d\theta \leq 5\theta_*(x)\frac{m}{10(t+a)} \leq \frac{5m}{6}.$$

We now consider the integral from $5\theta_*(x_{m,t})$ to ∞. We write, using (3.6) and (3.14):

$$\int_{5\theta_*(x_{m,t})}^{\infty} \tilde{u}(t,x_{m,t},\theta)d\theta \leq C(a)\int_{5\theta_*(x_{m,t})}^{\infty} \exp\left(t+a-\frac{\theta^2}{4(t+a)}\right)d\theta$$

$$\leq C'(a)\frac{t+a}{\theta_*(x_{m,t})}\exp\left\{t+a-\frac{25\theta_*^2(x_{m,t})}{4(t+a)}\right\} \leq C'(a)e^{-(t+a)/10}.$$

If t is sufficiently large, depending only on a and m, the last two estimates give us

$$\limsup_{t\to\infty}\int_{\underline{\theta}}^{\infty} \overline{u}(t,x_{m,t},\theta)d\theta \leq m.$$

Since $\overline{u}$ is monotonic in the spatial variable x, we have, for any $x \geq 0$

$$\limsup_{t\to\infty}\rho(t,x+x_{m,t}) \leq \limsup_{t\to\infty}\int_{\underline{\theta}}^{\infty} \overline{u}(t,x+x_{m,t},\theta)d\theta \leq \limsup_{t\to\infty}\int_{\underline{\theta}}^{\infty} \overline{u}(t,x_{m,t},\theta)d\theta \leq m.$$

Noticing that $x_{m,t}/t^{3/2}$ tends to $4/3$ as $t \to +\infty$ finishes the proof. $\qquad\square$

4   The lower bound

In this section, we obtain a lower bound on the propagation in the local and non-local cane toads equations. As we have mentioned, the idea is to construct sub-solutions of the linearized problem with the Dirichlet boundary condition on a moving boundary of a domain $E(t)$, and then use them to deduce a lower bound on the solution of the nonlinear problem. The goal is to have $E(t)$ move

as fast as possible while ensuring that the solution of the linearized problem is $O(1)$ – it neither grows too much, nor decays. This strategy is inspired by the proof of the Freidlin-Gärtner formula for the standard KPP equation by J.-M. Roquejoffre [44]; however, in contrast, the coefficients that arise in our formulation are non-periodic and so new estimates are required.

To this end, given some constants x_c, θ_c, and r, we define the ellipse

$$E_{x_c,\theta_c,R} := \left\{ (x,\theta) \in \mathbb{R} \times [\underline{\theta}, \infty] : \frac{(x-x_c)^2}{\theta_c} + (\theta - \theta_c)^2 \leq R^2 \right\}. \tag{4.1}$$

Given a large time T, we will move such an ellipse along some trajectories $X_T(t)$ and $\Theta_T(t)$ on the time interval $[0,T]$, starting at a point $(X_T(0), \Theta_T(0))$. We will denote $\Theta_T(0) = H \geq \underline{\theta}$. Note that the equation is translationally invariant in x, so the starting point $X_T(0)$ is not important. The trajectories will satisfy certain conditions: first, they move "up and to the right":

$$\dot{X}_T(t) \geq 0, \quad \dot{\theta}_T(t) \geq 0, \quad \text{for all } 0 \leq t \leq T. \tag{4.2}$$

Next, for all $0 \leq t \leq T$, with some fixed $\varepsilon > 0$ we assume that either

$$L(X_T(t), \theta_T(t), \dot{X}_T(t), \dot{\theta}_T(t)) \leq 1 - 2\varepsilon \quad \text{or} \quad \dot{X}(t) = 0 = \dot{\Theta}(t). \tag{4.3}$$

Here, $L(x, \theta, v_x, v_\theta)$ is the Lagrangian given by (2.10):

$$L(x, \theta, v_x, v_\theta) = \frac{v_x^2}{4\theta} + \frac{v_\theta^2}{4}. \tag{4.4}$$

Finally, we assume that

$$\lim_{T,H \to \infty} \left[\left| \ddot{\Theta}_T(t) \right| + \frac{|\ddot{X}_T(t)|}{\sqrt{\Theta_T(t)}} \right] = 0, \quad \text{uniformly in } t \in [0,T]. \tag{4.5}$$

We now state our main lemma, which we use in both the local and non-local settings.

Lemma 4.1. *Consider any trajectories $X_T(t)$ and $\Theta_T(t)$ on $[0,T]$ which satisfy the above assumptions, and fix $\varepsilon > 0$, $\delta > 0$, and $\gamma > 0$ sufficiently small. There exist constants R_ε, $T_{\varepsilon,\delta,\gamma}$, and $H_{\varepsilon,\gamma}$ such that for all $R \geq R_\varepsilon$, $T \geq T_{\varepsilon,\delta,\gamma}$, and $H \geq H_{\varepsilon,\gamma}$, there is a function v which solves*

$$\begin{cases} v_t - \theta v_{xx} - v_{\theta\theta} \leq (1-\varepsilon)v, & (t,x,\theta) \in \mathbb{R}^+ \times E_{X_T(t),\Theta_T(t),R}, \\ v(t,x,\theta) \leq 1, & (t,x,\theta) \in [0,T] \times E_{X_T(t),\Theta_T(t),R}, \\ v(t,x,\theta) = 0, & (t,x,\theta) \in [0,T] \times \partial E_{X_T(t),\Theta_T(t),R}, \\ v(0,x,\theta) < \delta, & (x,\theta) \in E_{X_T(0),\Theta_T(0),R}, \end{cases} \tag{4.6}$$

and such that $\|v(T,\cdot,\cdot)\|_{L^\infty} \geq 1 - \gamma$, and $v(T,x,\theta) \geq C_R$ for all $(x,\theta) \in E_{X_T(T),\Theta_T(T),R/2}$, with a constant $C_R > 0$ that depends only on R. The constant R_ε depends only on ϵ, the constant $H_{\varepsilon,\gamma}$ depends only on ε and γ, and $T_{\varepsilon,\delta,\gamma}$ depends only on ε, δ, γ, and the rate of the limit in (4.5).

We apply Lemma 4.1 as follows. First, we use it to build a sub-solution along a sufficiently large ellipse moving along a suitably chosen trajectory $(X_T(t), \Theta_T(t))$. In this step, we choose δ such that

we may fit εv underneath the solution u so that u always stays above εv. Hence, u is at least of the size ϵC_R near the point $(X_T(T), \Theta_T(T))$, after time T. Then we re-apply the lemma, with the trivial trajectory that remains fixed at the point $(X_T(T), \Theta_T(T))$ for all time, to build a sub-solution to u on the ellipse $E_{X_T(T),\Theta_T(T),R/2}$ that grows from εC_R to any prescribed height $m \in (0,1)$ in $O(1)$ time, depending on ϵ and m. It follows that u is at least of height m near $(X_T(T), \Theta(T))$ after time $T + O(1)$.

The proof of Lemma 4.1 involves estimates of the solution to a spectral problem posed on the moving domain $E_{X_T(t),\Theta_T(t),R}$ after a suitable change of variables and a rescaling. We prove this lemma in Section 5 below.

4.1 The lower bound in the local equation

Here, we show that Lemma 4.1 allows us to propagate a constant amount of mass along trajectories that we choose carefully. Our main result in this section is the following.

Proposition 4.2. *Suppose that u satisfies the local cane toads equation* (1.5) *with any initial data $u_0 > 0$. Then, for any $m \in (0,1)$, we have*

$$\frac{4}{3} \leq \liminf_{T \to \infty} \frac{\max \{x : \exists \theta \in \Theta, u(T,x,\theta) \geq m\}}{T^{3/2}}.$$

The assumption that u_0 is positive is not restrictive since any solution with a non-zero, non-negative initial condition becomes positive for all $t > 0$ as a consequence of the maximum principle. In particular, as the initial condition u_0 in Theorem 1.1 is compactly supported, non-negative, and non-zero, we may apply Proposition 4.2 to $\tilde{u}$, the solution to the cane toads equation with the initial condition $\tilde{u}_0(x,\theta) = u(t=1,x,\theta)$.

Take any function v given by Lemma 4.1. Since $v \leq 1$, then $\epsilon v \leq \epsilon$ and, hence, satisfies

$$(\epsilon v)_t - \theta(\epsilon v)_{xx} - (\epsilon v)_{\theta\theta} - (\epsilon v)(1 - (\epsilon v)) \leq -\epsilon(\epsilon v) + (\epsilon v)^2 \leq 0.$$

In particular, if we choose δ and our trajectories well, we will have that

$$u(t,x,\theta) \geq \varepsilon v(t,x,\theta) \text{ for all } 0 \leq t \leq T \text{ and } (x,\theta) \in E_{X_T(t),\Theta_T(t),R}. \tag{4.7}$$

Here, $u(t,x)$ is the solution to (1.5). In particular, we have

$$\forall (x,\theta) \in E_{X_T(T),\Theta_T(T),R/2}, \qquad u(T,x,\theta) \geq \varepsilon C_R, \tag{4.8}$$

after a sufficiently long time T. However, we do not have control on the constant C_R. To remedy this, we again apply Lemma 4.1, getting a sub-solution v' for u, in order to show that we can quickly grow the solution from this small constant to $O(1)$. As such, we make the choices $R' = R/2$, and $\delta' = \epsilon C_R$, $\epsilon' = m$.

Proof of Proposition 4.2

Let us now present the details of the argument. We fix $\varepsilon > 0$ and any $m \in (0,1)$, and let u be the solution to (1.5) with the initial condition u_0. Given T, R, and H to be determined later, we

will use the trajectories which are a slightly slowed down version of the optimal Hamilton-Jacobi trajectories (2.31):

$$X_T(t) = (1 - 2\varepsilon)^{3/4} \left(\frac{2t^2}{T^{1/2}} - \frac{2t^3}{3T^{3/2}} \right), \tag{4.9}$$

$$\Theta_T(t) = (1 - 2\varepsilon)^{1/2} \left(2t - \frac{t^2}{T} \right) + H.$$

It is straightforward to verify to that X_T and Θ_T satisfy the assumptions above Lemma 4.1: for all $0 \le t \le T$, we have $\dot{X}_T(t) \ge 0$, $\dot{\Theta}_T \ge 0$ and

$$L(X_T(t), \Theta_T(t), \dot{X}_T(t), \dot{\Theta}_T(t)) \le 1 - 2\varepsilon$$

while

$$\ddot{X}_T(t) = \frac{4(1 - 2\varepsilon)^{3/4}}{T^{1/2}} \left(1 - \frac{t}{T} \right), \quad \ddot{\Theta}_T(t) = -\frac{2(1 - 2\varepsilon)^{1/2}}{T},$$

so that (4.3) and (4.5) hold as well.

We set $\gamma = \epsilon$ and

$$\delta \overset{\text{def}}{=} \inf_{E_{X_T(0), \Theta_T(0), R}} u_0(x, \theta).$$

Note that δ depends on R and H but not on T. Applying Lemma 4.1, we may find R_ε, $T_{\varepsilon, \delta}$ and H_ε such that if $R \ge R_\varepsilon$, $T \ge T_{\varepsilon, \delta}$ and $H \ge H_\varepsilon$, then there exists a function v which satisfies (4.6). Hence, as we have discussed, the function εv is a sub-solution to u on $E_{X_T(t), \Theta_T(t), R}$ for all $t \in [0, T]$, and (4.7)-(4.8) hold. In particular, we have that

$$u(T, x, \theta) \ge \epsilon v(T, x, \theta) \ge \epsilon C_R$$

for all $(x, \theta) \in E_{X_T(T), \Theta_T(T), R/2}$.

Next, we use Lemma 4.1 a second time, with the new choices

$$R' = R/2, \quad \delta' = \frac{\epsilon C_R}{m}, \quad \epsilon' = \frac{1 + m}{2} \quad \text{and} \quad \gamma' = \frac{1 - m}{1 + m},$$

to find H_m, R_m and $T_{m, \delta'}$ such that if $R/2 = R' \ge R_m$ and $H \ge H_m$ then we may find a solution w to (4.6) on $[0, T_{m, \delta'}] \times E_{X_T(T), \Theta_T(T), R'}$. We shift in time and scale to define

$$\tilde{w}(t, x, \theta) \overset{\text{def}}{=} \epsilon' w(t - T, x, \theta).$$

By our previous work and our choice of δ', it follows that $\tilde{w}(T, x, \theta) \le u(T, x, \theta)$. In addition, the partial differential equation for w, (4.6), with our choice of m, guarantees that $\tilde{w}$ is a sub-solution to u on $[T, T + T_{m, \delta'}] \times E_{X_T(T), \Theta_T(T), R'}$. This implies that

$$\|u(T + T_{m, \delta'}, \cdot, \cdot)\|_{L^\infty(E_{X_T(T), \Theta_T(T), R'})} \ge \|\tilde{w}(T + T_{m, \delta'}, \cdot, \cdot)\|_{L^\infty(E_{X_T(T), \Theta_T(T), R'})}$$

$$= \epsilon' \|w(T_{m, \delta'}, \cdot, \cdot)\|_{L^\infty(E_{X_T(T), \Theta_T(T), R'})} \ge \frac{1 + m}{2} (1 - \gamma) = \frac{1 + m}{2} \left(1 - \frac{1 - m}{1 + m} \right) \ge m. \tag{4.10}$$

The first inequality is a consequence of the fact that $\tilde{w}$ is a sub-solution of u, the first equality is a consequence of the definition of $\tilde{w}$, and the final equalities is a consequence of Lemma 4.1 and the choices of ϵ' and γ.

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The above, (4.10), implies that u achieves a value at least as large as m for some

$$x \geq X_T(T) - \sqrt{\Theta_T(T)}R'$$

at time $T + T_{m,\delta'}$. As a consequence, we have

$$\frac{4}{3}(1-2\varepsilon)^{3/4}\left(\frac{T}{T+T_{m,\delta'}}\right)^{3/2} - \frac{\sqrt{T+H}}{(T+T_{m,\delta'})^{3/2}}R' \leq \frac{\max\{x : \exists\theta \in \Theta, u(T+T_{m,\delta}, x, \theta) = m\}}{(T+T_{m,\delta'})^{3/2}}. \quad (4.11)$$

Here, we used the definition of $X_T(T)$ and $\Theta_T(T)$, and the upper bound

$$\Theta_T(T) \leq T + H + R.$$

Since (4.11) holds for all T sufficiently large, and since H, R, and $T_{m,\delta}$ are fixed, we may take the limit as T tends to infinity to obtain

$$\frac{4}{3}(1-2\varepsilon)^{3/4} \leq \liminf_{T\to\infty} \frac{\max\{x : \exists\theta \in \Theta, u(T, x, \theta) \geq m\}}{T^{3/2}}.$$

Since ε is arbitrary, this finishes the proof of Proposition 4.2. $\square$

4.2 The lower bound in the non-local equation

In this section we prove the following proposition.

Proposition 4.3. *Suppose that n is a solution of the non-local cane toads equation (1.7) with a positive initial condition $n_0 \in L^\infty(\mathbb{R} \times \Theta)$. Define*

$$\rho_0(x) \overset{\text{def}}{=} \int_{\underline{\theta}}^\infty n_0(x, \theta)d\theta,$$

and assume that $\rho_0 \in L^\infty(\mathbb{R})$. Then, for any $\varepsilon > 0$ and $\gamma \in (0,1)$ we have

$$c_1 \overset{\text{def}}{=} \frac{8}{3\sqrt{3\sqrt{3}}}(1-2\varepsilon)^{3/4} \leq \limsup_{T\to\infty} \frac{\max\{x : \rho(T, x) \geq \gamma\varepsilon\}}{T^{3/2}}. \quad (4.12)$$

Our strategy here is similar to the local case, though this time we argue by contradiction. Suppose that the result does not hold. Then there exists $\varepsilon > 0$, a time $t_\varepsilon > 0$ and $\gamma \in (0,1)$ such that, for all $t \geq t_\varepsilon$,

$$\sup_{x \geq c_1 t^{3/2}} \rho(t, x) < \gamma\varepsilon. \quad (4.13)$$

Our goal is to construct a sub-solution v to n which will satisfy

$$\int_{\underline{\theta}}^\infty v(t, x, \theta)d\theta \geq \varepsilon > \gamma\varepsilon,$$

for some $x \geq c_1 t^{3/2}$. This will push $\rho(t, x)$ to be greater than ε as well, yielding a contradiction to (4.13). Note that if (4.13) holds, then any solution to

$$v_t - \theta v_{xx} - v_{\theta\theta} \leq v(1 - \gamma\varepsilon), \quad (4.14)$$

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defined for $t \geq t_\varepsilon$ and which is supported on $x \geq c_1 t^{3/2}$ satisfies

$$n(t, x, \theta) \geq v(t, x, \theta) \text{ for all } t \geq t_\varepsilon, \ (t, x, \theta) \in \text{supp } v, \tag{4.15}$$

provided that this inequality holds at $t = t_\varepsilon$. This is because (4.13) implies

$$n_t - \theta n_{xx} - n_{\theta\theta} = n(1 - \rho) \geq n(1 - \gamma\varepsilon), \tag{4.16}$$

for all $t \geq t_\varepsilon$ and $x \geq c_1 t^{3/2}$. Note that the nature of the argument by contradiction requires us now to have the "Dirichlet ball" completely to the right of $x = c_1 t^{3/2}$. On the other hand, the "nearly optimal" Hamilton-Jacobi trajectories (4.9) that we have used in the local case, initially move mostly in the θ-direction when T is large, and violate this condition. This forces us to choose sub-optimal trajectories for the center of the "Dirichlet ball", leading to the non-sharp constant c_1 in (4.12). We assume now (4.13) and define the trajectories

$$X_T(t) = c_\gamma (t + t_\varepsilon)^{3/2}, \quad \Theta_T(t) = (1 - 2\gamma\varepsilon)^{1/2} \frac{2}{\sqrt{3}} (t + t_\varepsilon) + H, \tag{4.17}$$

with

$$c_\gamma = \frac{8}{3\sqrt{3\sqrt{3}}} (1 - 2\gamma\varepsilon)^{3/4}.$$

As $c_\gamma > c_1$, we have

$$X_T(t) > c_1 (t + t_\varepsilon)^{3/2} \text{ for all } t > 0.$$

The constant H will be determined below. We note that X_T and Θ_T satisfy the assumptions preceding Lemma 4.1. Indeed, both the non-negativity of $\dot{X}_T$ and $\dot{\Theta}_T$ in (4.2) and the limit in (4.5) are obvious from (4.17). Hence, we only need to check the condition on the Lagrangian in (4.3). To this end, we compute

$$\frac{\dot{X}_T^2}{4\Theta_T} + \frac{\dot{\Theta}_T^2}{4} = \frac{4(1 - 2\gamma\varepsilon)(t + t_\varepsilon)}{6(t + t_\varepsilon) + \frac{3\sqrt{3}}{\sqrt{1-2\gamma\epsilon}} H} + \frac{(1 - 2\gamma\varepsilon)}{3} < (1 - 2\gamma\varepsilon).$$

We now build a sub-solution on $E_{X_T(t),\Theta_T(t),R}$ for suitably chosen H, R, and T which grows and forces ρ to be larger than $\gamma\varepsilon$, giving us a contradiction. The aforementioned condition on the support, i.e. that $(x, \theta) \in \text{supp}(v)$ only if $x \geq c_1 t^{3/2}$, is equivalent to

$$X_T(t - t_\varepsilon) - R\sqrt{\Theta_T(t - t_\varepsilon)} \geq c_1 t^{3/2}, \tag{4.18}$$

for all $t \in [t_\varepsilon, t_\varepsilon + T]$. In other words, the left-most point of a ball centered at $X_T(t - t_\varepsilon)$ of the radius $R\sqrt{\Theta_T(t - t_\varepsilon)}$ must be to the right of $c_1 t^{3/2}$. Since $c_\gamma > c_1$, we can clearly arrange for this to be satisfied by increasing, if necessary, t_ε in a way depending only on R and H.

Fix $M > 0$ to be determined later. Let us define

$$\delta \overset{\text{def}}{=} \frac{1}{M} \inf_{E_{X_T(0),\Theta_T(0),R}} n(t_\varepsilon, x, \theta).$$

Note that δ depends on R and M but not on T. Applying Lemma 4.1 with ϵ and δ defined above and $\gamma = 1/2$, we may find H_ε and R_ε, depending only on ε, and $T_{\varepsilon,\delta}$, that depends only on ε, δ, such that, if $H \geq H_\varepsilon$, $R \geq R_\varepsilon$, and $T \geq T_{\varepsilon,\delta}$, we may find v which satisfies (4.6). Define

$$\tilde{v}(t, x, \theta) = Mv(t - t_\epsilon, x, \theta).$$

By virtue of the discussion above and by (4.6), we see that $\tilde{v}$ is a sub-solution of u on $[t_\epsilon, t_\epsilon + T_{\epsilon,\delta}] \times E_{X_T(t), \Theta_T(t), R}$. In particular, we have that

$$
\inf_{E_{X_T(T), \Theta_T(T), R/2}} u(t_\epsilon + T, \cdot, \cdot) \geq \inf_{E_{X_T(T), \Theta_T(T), R/2}} \tilde{v}(t_\epsilon + T, \cdot, \cdot)
$$
$$
= \frac{M}{2} \inf_{E_{X_T(T), \Theta_T(T), R/2}} v(t_\epsilon + T, \cdot, \cdot) \geq \frac{M}{2} C_R. \tag{4.19}
$$

We emphasize here that C_R depends only on R and not on M.

At the expense of possibly increasing $T_{\epsilon,\delta}$, we can now specify $M = \frac{2\varepsilon}{RC_R}$. Using now (4.19), we obtain

$$
\int_{\underline{\theta}}^{\infty} u(t_\varepsilon + T, X_T(T), \theta) d\theta > \int_{\Theta_T(T) - R/2}^{\Theta_T(T) + R/2} u(t_\varepsilon + T, X_T(T), \theta) d\theta
$$
$$
\geq \int_{\Theta_T(T) - R/2}^{\Theta_T(T) + R/2} M C_R d\theta = M C_R R = 2\varepsilon.
$$

Hence we have $\rho(t_\varepsilon + T, X_T(T)) > \varepsilon$. As we have

$$
X_T(T) > c_1 (t_\varepsilon + T)^{3/2},
$$

this contradicts (4.13), finishing the proof. $\square$

5 Proofs of the auxiliary lemmas

In this section we prove the auxiliary results needed in the construction of the sub-solutions. Some of them are quite standard, we present the proofs for the convenience of the reader.

5.1 Existence of a sub-solution along trajectories – the proof of Lemma 4.1

In this subsection we prove Lemma 4.1 by suitably re-scaling the equation and then using careful spectral estimates. Recall that our goal is to show that there exist constants R_ε, $T_{\varepsilon,\delta,\gamma}$, and $H_{\varepsilon,\gamma}$ such that for all $R \geq R_\varepsilon$, $T \geq T_{\varepsilon,\delta,\gamma}$, and $H \geq H_{\varepsilon,\gamma}$, there is a function v which satisfies

$$
\begin{cases}
v_t - \theta v_{xx} - v_{\theta\theta} \leq (1 - \varepsilon)v, & \text{for all } t > 0, \text{ and } (x, \theta) \in E_{X_T(t), \Theta_T(t), R}, \\
v(t, x, \theta) = 0, & \text{for all } 0 \leq t \leq T \text{ and } (x, \theta) \in \partial E_{X_T(t), \Theta_T(t), R}, \\
v(0, x, \theta) < \delta, & \text{for all } (x, \theta) \in E_{X_T(0), \Theta_T(0), R}, \\
v(t, x, \theta) \leq 1, & \text{for all } 0 \leq t \leq T \text{ and } (x, \theta) \in E_{X_T(t), \Theta_T(t), R},
\end{cases} \tag{5.1}
$$

and such that $\|v(T, \cdot, \cdot)\|_{L^\infty} \geq 1 - \gamma$, and $v(T, x, \theta) \geq C_R$ for all $(x, \theta) \in E_{X_T(T), \Theta_T(T), R/2}$, with a constant $C_R > 0$ that depends only on R and $\delta > 0$. To construct the desired sub-solution, we first go into the moving frame, and rescale the spatial variable:

$$
v(t, x, \theta) = \tilde{v}\left(t, \frac{x - X_T}{\sqrt{\Theta_T}}, \theta - \Theta_T\right), \quad y = \frac{x - X_T}{\sqrt{\Theta_T}}, \quad \text{and} \quad \eta = \theta - \Theta_T. \tag{5.2}
$$

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Then (5.1) yields

$$\tilde{v}_t - \left(\frac{y}{2}\frac{\dot{\Theta}_T}{\Theta_T} + \frac{\dot{X}_T}{\sqrt{\Theta_T}}\right)\tilde{v}_y - \dot{\Theta}_T\tilde{v}_\eta \le \left(1 + \frac{\eta}{\Theta_T}\right)\tilde{v}_{yy} + \tilde{v}_{\eta\eta} + (1-\varepsilon)\tilde{v}, \tag{5.3}$$

with the boundary conditions

$$\tilde{v}(t, y, \eta) = 0, \qquad \text{for all } (y, \eta) \in \partial B_R.$$

Here, $B_R \overset{\text{def}}{=} B_R(0,0)$ is a ball of radius R centered at $(y, \eta) = (0, 0)$. As in [30, 31], the next step is to remove an exponential, setting,

$$w(t, y, \eta) = e^{\frac{\dot{X}}{2\sqrt{\Theta_T}}y + \frac{\dot{\Theta}_T}{2}\eta}\tilde{v}(t, y, \eta).$$

Note that if T and H are sufficiently large, there is a constant M_R, depending only on R, such that

$$\frac{1}{M_R}w(t, y, \eta) \le \tilde{v}(t, y, \eta) \le M_R w(t, y, \eta) \tag{5.4}$$

holds for all t, y, and η, because of the uniform bound (4.3) on the Lagrangian. Changing variables in (5.3), we see that w must satisfy the inequality

$$w_t - \underbrace{\left(\frac{y\dot{\Theta}_T}{2\Theta_T} - \frac{\dot{X}_T}{\sqrt{\Theta_T}}\frac{\eta}{\Theta_T}\right)}_{\overset{\text{def}}{=}A}w_y \le \underbrace{\left(1 + \frac{\eta}{\Theta_T}\right)}_{\overset{\text{def}}{=}1+D}w_{yy} + w_{\eta\eta}$$

$$+ w\underbrace{\left(1 - \varepsilon - \frac{\dot{X}_T^2}{4\Theta_T} - \frac{\dot{\Theta}_T^2}{4} + \left(\frac{\ddot{X}_T}{2\sqrt{\Theta_T}} - \frac{\dot{X}_T\dot{\Theta}_T}{4\Theta_T^{\frac{3}{2}}}\right)y + \left(\frac{\dot{X}_T^2}{4\Theta_T^2} + \frac{\ddot{\Theta}_T}{2}\right)\eta\right)}_{\overset{\text{def}}{=}G}.$$

Note that by choosing T and H large enough and using (4.3) and (4.5), we may ensure that

$$\left|\left(\frac{\ddot{X}_T}{2\sqrt{\Theta_T}} - \frac{\dot{X}_T\dot{\Theta}_T}{4\Theta_T^{\frac{3}{2}}}\right)y + \left(\frac{\dot{X}_T^2}{4\Theta_T^2} + \frac{\ddot{\Theta}_T}{2}\right)\eta\right| \le \min\left\{\frac{\epsilon}{4}, \frac{1-\varepsilon}{2}\right\}.$$

Hence, using (4.3), we have that

$$G \ge 1 - \varepsilon - \frac{\dot{X}_T^2}{4\Theta_T} - \frac{\dot{\Theta}_T^2}{4} - \min\left\{\frac{\epsilon}{4}, \frac{1-\varepsilon}{2}\right\} \ge \min\left\{\frac{1-\epsilon}{2}, \frac{3\varepsilon}{4}\right\}.$$

The two cases in the minimum correspond to the two cases in (4.3). Hence, if we construct w which satisfies

$$w_t - Aw_y \le (1+D)w_{yy} + w_{\eta\eta} + \min\left\{\frac{1-\epsilon}{2}, \frac{3\varepsilon}{4}\right\}w. \tag{5.5}$$

then w also satisfies

$$w_t - Aw_y \le (1+D)w_{yy} + w_{\eta\eta} + Gw.$$

Returning to the original variables, v would satisfy the desired differential inequality. With this in mind, we seek to construct w satisfying (5.5) which has the desired bounds.

We define w using the principal Dirichlet eigenfunction of the operator

$$L_{T,H}(t) \stackrel{\text{def}}{=} -A\partial_y - (1+D)\partial_{yy} - \partial_{\eta\eta}. \tag{5.6}$$

To this end, for each $t \in [0,T]$, define $\varphi_{T,H}(t,x,\theta)$ and $\lambda_{T,H}(t)$ to be the principal Dirichlet eigenfunction and eigenvalue of $L_{T,H}(t)$ (depending on t as a parameter) in the ball B_R, with the normalization

$$\|\varphi_{T,H}\|_{L^2(B_R)} = 1.$$

We state two lemmas regarding these quantities which will allow us to finish the proof. First, we need to understand the behavior of $\lambda_{T,H}$ for T and H large. We recall the following result.

Lemma 5.1. *Fix $s > 1$. Consider the operator*

$$L_{a,b} = -\nabla \cdot a\nabla - b \cdot \nabla,$$

defined on a smooth, bounded domain $\Omega \subset \mathbb{R}^d$ with $a, b \in C^s(\Omega)$ and where a is a uniformly positive definite matrix. Let $\lambda_{a,b,\Omega}$ be the principal Dirichlet eigenvalue of $L_{a,b}$ with eigenfunction $\varphi_{a,b,\Omega}$ having L^∞-norm one. Then $\lambda_{a,b,\Omega}$ and $\varphi_{a,b,\Omega}$ are continuous in a and b, when considered as maps from $(L^\infty)^{d^2} \times (L^\infty)^d$ to $\mathbb{R}$ and to H^s, respectively.

The proof of Lemma 5.1 will be presented in section 5.2 below. Lemma 5.1 implies that, as T and H tend to infinity, $\lambda_{T,H}(t)$ becomes bounded above and below by a constant multiple of R^{-2} since the principal eigenvalue of $-\Delta$ on B_R equals c_1/R^2 for some positive constant c_1. This convergence is uniform in t. Hence, choosing first R sufficiently large, we may choose H and T, depending only on ε, R, the convergence rate of the limit in (4.5), such that

$$\lambda_{T,H}(t) < \frac{1}{4} \min\left\{\frac{1-\epsilon}{2}, \frac{\varepsilon}{4}\right\}, \qquad \text{for all } t. \tag{5.7}$$

We will also need the behavior of the time derivative of $\varphi_{T,H}$.

Lemma 5.2. *Using the notation above, $\partial_t\varphi_{T,H}$ is a smooth function in y and η, and*

$$\lim_{T,H\to\infty} \left\|\frac{\partial_t\varphi_{T,H}}{\varphi_{T,H}}\right\|_{L^\infty} = 0.$$

Lemma 5.2 implies that for fixed R, we may choose T and H, depending only on ε and the convergence rate of the limit in (4.5), such that

$$|\partial_t\varphi_{T,H}| \leq \frac{1}{4} \min\left\{\frac{1-\epsilon}{2}, \frac{\varepsilon}{4}\right\} \varphi_{T,H}, \qquad \text{for all } (t,y,\eta) \in [0,T] \times B_R. \tag{5.8}$$

With this set-up, we can now conclude the proof of Lemma 4.1. We define

$$w_{T,H} \stackrel{\text{def}}{=} \delta e^{rt} \frac{\varphi_{T,H}}{N},$$

with

$$r \overset{\text{def}}{=} -\frac{1}{T} \log(\delta) \quad \text{and} \quad N \overset{\text{def}}{=} \sup_{t \in [0,T]} \|\varphi_{T,H}(t)\|_\infty.$$

Fix T large enough, depending on ε and δ, such that

$$r < \frac{1}{4} \min \left\{ \frac{1-\epsilon}{2}, \frac{\varepsilon}{4} \right\}.$$

Then, (5.7) and (5.8) and our choice of r imply that

$$\partial_t w_{T,H} - L_{T,H} w_{T,H} = \partial_t w_{T,H} + \lambda_{T,H} w_{T,H}$$

$$\leq \frac{\partial_t \varphi_{T,H}}{\varphi_{T,H}} w_{T,H} + r w_{T,H} + \lambda_{T,H} w_{T,H} \leq \frac{3}{4} \min \left\{ \frac{1-\epsilon}{2}, \frac{\varepsilon}{4} \right\} w_{T,H},$$

and $w_{T,H}$ is a sub-solution to (4.6). In addition, by construction, we know that $w_{T,H}(0, y, \eta) \leq \delta$ for all $(y, \eta) \in B_R$, and $w_{T,H}(t, y, \eta) \leq 1$ for all $t \in [0, T]$ and $(y, \eta) \in B_R$.

Finally, due to the uniform convergence of $\varphi_{T,H}$ to the principal Dirichlet eigenfunction of B_R, if T and H are sufficiently large, there exists c_R such that

$$c_R \leq \min_{(y,\eta) \in B_{R/2}} \frac{\varphi_{T,H}(y, \eta)}{N} = \min_{(y,\eta) \in B_{R/2}} w_{T,H}(T, y, \eta).$$

In view of (5.4) and by undoing the change of variables, this implies

$$c_R \leq \min_{(y,\eta) \in B_{R/2}} w_{T,H}(T, y, \eta) \leq M_R \min_{(x,\theta) \in E_{X_T(T), \Theta_T(T), R/2}} v(T, x, \theta),$$

which finishes the proof of the lower bound on $\tilde{v}$.

The fact that $\|\tilde{v}\|_\infty \geq 1 - \gamma$ follows by noting the following: $\varphi_{T,H}/N$ converges to the Dirichlet eigenfunction with L^∞ norm 1, which takes a maximum of 1 at $(y, \eta) = (0, 0)$. Choosing T and H large enough, we have that $\varphi_{T,H}(T, 0, 0)/N \geq 1 - \gamma$. Using the definition of $w_{T,H}$ finishes the proof. $\square$

5.2 The spectral problem – the proofs of Lemma 5.1 and Lemma 5.2

The proof of Lemma 5.1.

To simplify the notation, we show continuity at $\lambda_{I,0,\Omega}$ and $\varphi_{I,0,\Omega}$, but the same argument works for all matrices a and vector fields b. Fix any sequence a_n and b_n which tend to I and 0, respectively. Let φ_n and λ_n be the principal eigenfunction and eigenvalue of the operator $L_n := L_{a_n, b_n, \Omega}$, with the normalization $\|\varphi_n\|_{L^2(\Omega)} = 1$.

First, we show that λ_n is bounded. Suppose by contradiction that (up to a subsequence that we omit) λ_n tends to infinity, and write

$$\frac{1}{\lambda_n} L_n \varphi_n = \varphi_n.$$

Multiplying by a test function $\eta \in C_0^\infty(\Omega)$, integrating over Ω and passing to the limit $n \to +\infty$ we see that φ_n converges weakly to zero as $n \to +\infty$. and the right side is bounded in L^2 uniformly

in n, then elliptic regularity implies that $\tilde{\varphi}_n$ is bounded uniformly in H^2. Let $\tilde{\varphi}_\infty$ be the strong H^1 limit of $\tilde{\varphi}_n$, taking a subsequence if necessary. Multiplying the equation for $\tilde{\varphi}_n$ by $\tilde{\varphi}_n$ and integrating, we see that

$$\int_\Omega \langle \nabla \tilde{\varphi}_n | a_n \nabla \tilde{\varphi}_n \rangle \, dx + \int_\Omega \tilde{\varphi} b_n \cdot \nabla \tilde{\varphi}_n \, dx = 1.$$

Taking the limit as n tends to infinity yields

$$\|\nabla \varphi_n\|_{L^2(\Omega)} = 1.$$

On the other hand, since $\|\varphi_n\|_{L^2(\Omega)} = 1$ and λ_n tends to infinity, it is clear that $\tilde{\varphi}_\infty = 0$. This is a contradiction.

Since λ_n is bounded uniformly, then φ_n is bounded uniformly in H^2. Hence, we may, taking a subsequence if necessary, find a strong H^{1+s} limit of φ_n, φ_∞, and a limit of λ_n, λ_∞. Since a_n and b_n converge in L^∞, then we see that φ_∞ must satisfy

$$-\Delta \varphi_\infty = \lambda_\infty \varphi_\infty.$$

The maximum principle and the fact that $\varphi_n > 0$ imply that $\varphi_\infty > 0$. Hence φ_∞ must be the principal eigenfunction of $-\Delta$ and λ_∞ must be the principal eigenvalue.

We have shown that every sequence φ_n and λ_n has a subsequence which must converge to the $\varphi_{I,0,\Omega}$ and $\lambda_{I,0,\Omega}$, respectively. This implies that $\varphi_{a,b,\Omega}$ and $\lambda_{a,b,\Omega}$ are continuous at $a = I$ and $b = 0$, finishing the proof. $\square$

The proof of Lemma 5.2.

First we show that $\partial_t \varphi_{T,H}$ is well-defined. To do this, we need only show that $\lambda_{T,H}(t)$ is Lipschitz continuous in t. Indeed, with this shown, if $\lambda_{T,H}(t)$ is in $W^{1,\infty}$ as a function of t, we may take a derivative in t of the equation for $\varphi_{T,H}$, allowing us to write down an equation for $\partial_t \varphi_{T,H}$, showing that $\partial_t \varphi_{T,H}$ is well-defined.

Before we begin, we recall that we may characterize $\lambda_{T,H}$ as

$$\lambda_{T,H}(t) = \sup_{0 < \psi \in C_0^2(B_R)} \inf_{(y,\eta) \in B_R} \frac{L_{T,H}(t)\psi(y,\eta)}{\psi(y,\eta)} = \inf_{0 < \psi \in C_0^2(B_R)} \sup_{(y,\eta) \in B_R} \frac{L_{T,H}^*(t)\psi(y,\eta)}{\psi(y,\eta)}.$$

Using this, we now estimate $\lambda_{T,H}(t+h) - \lambda_{T,H}(t)$ from below – the upper bound may be found similarly. For any h small enough and p_h to be determined, we have that, as long as $\varphi_{T,H} + h p_h > 0$,

$$
\begin{aligned}
\lambda_{T,H}(t+h) &\geq \inf_{(y,\eta)} \left(\frac{L_{T,H}(t+h)\varphi_{T,H}(t) + h L_{T,H}(t+h) p_h}{\varphi_{T,H}(t) + h p_h} \right) \\
&= \inf_{(y,\eta)} \left(\frac{\lambda_{T,H}(t)\varphi_{T,H}(t)}{\varphi_{T,H}(t) + h p_h} + \frac{(L_{T,H}(t+h) - L_{T,H}(t))\varphi_{T,H}(t) + h L_{T,H}(t+h) p_h}{\varphi_{T,H}(t) + h p_h} \right).
\end{aligned}
\tag{5.9}
$$

This suggests that we define p_h as the unique solution of

$$
\begin{aligned}
L_{T,H}(t+h) p_h &= -\frac{L_{T,H}(t+h) - L_{T,H}(t)}{h} \varphi_{T,H}(t), \quad \text{in } B_R \\
p_h|_{\partial B_R} &= 0.
\end{aligned}
\tag{5.10}
$$

With the following lemma, we may choose h small enough that $\psi + h p_h > 0$, and $\psi + h p_h$ is admissible in the formula above (5.9).

Lemma 5.3. *For h sufficiently small, $\|p_h/\varphi_{T,H}(t)\|_\infty \leq C$. Further, $\|p_h/\varphi_{T,H}\|_\infty \to 0$ as $T, H \to \infty$ uniformly in t and h.*

Returning to (5.9) and using our choice of p_h, we obtain the inequality:

$$\lambda_{T,H}(t+h) \geq \lambda_{T,H}(t) \inf_x \left(\frac{1}{1 + h\frac{p_h}{\varphi_{T,H}(t)}} \right) \geq \lambda_{T,H}(t) \left(1 - C \left\| \frac{p_h}{\varphi_{T,H}(t)} \right\|_{L^\infty} h \right),$$

with a universal constant C. Since $\lambda_{T,H}(t)$ is bounded independently of t, T, and H by a constant which we may denote by M, we arrive at

$$\lambda_{T,H}(t+h) - \lambda_{T,H}(t) \geq -\lambda_{T,H}(t) C \left\| \frac{p_h}{\varphi_{T,H}(t)} \right\|_{L^\infty} h \geq -CM \left\| \frac{p_h}{\varphi_{T,H}(t)} \right\|_{L^\infty} h.$$

Hence, $\lambda_{T,H}(t)$ is Lipschitz and its Lipschitz bound is linear in $\|p_h/\varphi_{T,H}(t)\|_{L^\infty}$. Since we may easily obtain a similar upper bound, we obtain

$$|\partial_t \lambda_{T,H}(t)| \leq CM \left\| \frac{p_h}{\varphi_{T,H}(t)} \right\|_{L^\infty}.$$

Using Lemma 5.3 and the above bound implies that $\partial_t \lambda_{T,H}(t)$ tends to zero in $L^\infty([0,T])$ as T and H tend to infinity.

We now show that $\partial_t \varphi_{T,H}$ tends uniformly to zero in L^2. We argue by contradiction; assume that $\|\partial_t \varphi_{T,H}\|_2$ is bounded from below, passing to a subsequence if necessary. Let us define

$$\psi_{T,H} \overset{\text{def}}{=} \frac{\partial_t \varphi_{T,H}}{\|\partial_t \varphi_{T,H}\|_{L^2}}.$$

Taking the t derivative of the equation for $\varphi_{T,H}$ yields the equation

$$L_{T,H} \psi_{T,H} - \lambda_{T,H} \psi_{T,H} = \frac{(\partial_t \lambda_{T,H})\varphi_{T,H} + (\partial_t A)\partial_y \varphi_{T,H} + (\partial_t D)\partial_{yy}\varphi_{T,H}}{\|\partial_t \varphi_{T,H}\|_{L^2}}. \tag{5.11}$$

The explicit forms of A and D and the fact that $\partial_t \lambda_{T,H}$ shows that, if $\|\partial_t \varphi_{T,H}\|_{L^2}$ is bounded uniformly below, then the right hand side of this equality tends uniformly to zero in L^2 as T and H tend to infinity. In addition, calling λ_{B_R} the principal Dirichlet eigenvalue of $-\Delta$ on B_R, we have that $\lambda_{T,H}$ converges to λ_{B_R} by Lemma 5.1.

By elliptic regularity, see e.g. [28, Theorem 8.13], $\psi_{T,H}$ is uniformly bounded in H^2 since it has L^2 norm 1. Up to taking a subsequence if necessary, we see that $\psi_{T,H}$ converges to some function ψ with $\|\psi\|_{L^2} = 1$ that, owing to (5.11), solves

$$-\Delta\psi - \lambda_{B_R}\psi = 0. \tag{5.12}$$

Let φ_{B_R} be the L^2-normalized Dirichlet eigenfunction corresponding to λ_{B_R}. By Lemma 5.1, it follows that $\varphi_{T,H}$ converges to φ_{B_R}. Since λ_{B_R} is a principal eigenvalue, it is simple, see e.g. [20, Chapter VIII]. Hence, $\psi = a\varphi_{B_R}$. Note that $|a| = 1$ since $\|\psi\|_{L^2} = 1$. On the other hand, we have that

$$a = \int_{B_R} \psi \varphi_{B_R} dy d\eta = \lim_{T,H\to\infty} \int_{B_R} \left(\frac{\partial_t \varphi_{T,H}}{\|\partial_t \varphi_{T,H}\|_{L^2}} \right) \varphi_{T,H} dy d\eta = \lim_{T,H\to\infty} \frac{1}{2\|\partial_t \varphi_{T,H}\|_{L^2}} \partial_t \|\varphi_{T,H}\|^2 = 0.$$

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The last equality holds since $\|\varphi_{T,H}\|_{L^2} = 1$ for all t. This contradicts the fact that $|a| = 1$. Hence, it must be that $\|\partial_t \varphi_{T,H}\|_{L^2}$ tends to zero uniformly as T and H tend to infinity.

Knowing that $\partial_t \varphi_{T,H}$ and $\partial_t \lambda_{T,H}$ tend to zero, we may now conclude. First, we note that

$$L_{T,H}(\partial_t \varphi_{T,H}) - \lambda_{T,H}(\partial_t \varphi_{T,H}) = (\partial_t \lambda_{T,H})\varphi_{T,H} + (\partial_t A)\partial_y \varphi_{T,H} + (\partial_t D)\partial_{yy}\varphi_{T,H}.$$

Since the right side tends to zero in H^2 and since $\partial_t \varphi_{T,H}$ tends to zero in L^2, it follows that $\partial_t \varphi_{T,H}$ tends to zero in H^4, see e.g. [28, Theorem 8.13]. Using Morrey's inequality [28, Theorem 7.22], we may strengthen this to show that $\partial_t \varphi_{T,H}$ converges to zero in $C^1(B_R)$ uniformly in t. On the other hand, by Lemma 5.1 $\varphi_{T,H}$ converges in H^4 to φ_{B_R}. Again, using Morrey's inequality, we have that $\varphi_{T,H}$ converge to φ_{B_R} in $C^1(B_R)$ uniformly in t.

It follows from this C^1 convergence that when T and H are sufficiently large, $\varphi_{T,H}$ is uniformly positive for any compact subset of B_R and $\partial_n \varphi_{T,H}$ is uniformly negative, where n is the outward unit normal of ∂B_R since φ_{B_R} has these properties. On the other hand $\partial_t \varphi_{T,H}$ converges uniformly to zero on B_R and $\partial_n(\partial_t \varphi_{T,H})$ converges uniformly to zero on ∂B_R. This yields

$$\lim_{T,H \to \infty} \left\| \frac{\partial_t \varphi_{T,H}}{\varphi_{T,H}} \right\|_{L^\infty} = 0,$$

which finishes the proof. $\square$

In order to conclude, we must now prove Lemma 5.3. We do that here.

Proof of Lemma 5.3. Since $\|\varphi_{T,H}(t)\|_{L^2(B_R)} = 1$ and since the coefficients of $L_{T,H}$ are bounded in C^2, then estimates as in, e.g. [28, Theorem 8.10 and Theorem 8.13], imply that $\|\varphi_{T,H}(t)\|_{H^4(B_R)} \leq C_R$, for a constant independent of T and H. As the coefficients in $L_{T,H}(t)$ are have two bounded derivatives in y and η, we may apply these same estimates to obtain that

$$\|p_h\|_{H^4(B_R)} \leq C \left(\|p_h\|_{L^2(B_R)} + \left\| \frac{L_{T,H}(t+h) - L_{T,H}(t)}{h} \varphi_{T,H}(t) \right\|_{H^2(B_R)} \right)$$

$$\leq C \left(\|p_h\|_{L^2} + \epsilon_{T,H} C_R \right),$$

where, due to the fact that $\partial_t D$ and $\partial_t A$ tend to zero as T and H tend to infinity, $\epsilon_{T,H}$ is a constant which tends to zero as T and H tend to infinity. Of course, using the definitions of p_h and $L_{T,H}$ and integrating $L_{T,H}(t+h)p_h$ by parts yields

$$-\int p_h A \nabla p_h \, dy d\eta + (1 - \|D\|_\infty) \int |\nabla p_h|^2 dy d\eta \leq \int p_h L_{T,H}(t) p_h \, dy d\eta$$

$$\leq \left\| \frac{L_{T,H}(t+h) - L_{T,H}(t)}{h} \varphi_{T,H}(t) \right\|_{L^2} \|p_h\|_{L^2}.$$

Using that A tends to zero as T and H tend to infinity and that $\|p_h\|_{L^2} \leq C_R \|\nabla p_h\|_{L^2}$, we obtain

$$\|\nabla p_h\|_{L^2} \leq C_R \epsilon_{T,H}.$$

Here $\epsilon_{T,H}$ is as above. By the Poincaré inequality and our H^4 bound of p_h, we obtain

$$\|p_h\|_{H^4} \leq C_R \epsilon_{T,H}.$$

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Hence, p_h is bounded in H^4 uniformly in h. By Morrey's inequality, see e.g. [28, Theorem 7.22], p_h and $\varphi_{T,H}(t)$ are bounded in $C^{2,\alpha}$ for some $\alpha > 0$. Moreover, we see that the p_h tends to zero in $C^{2,\alpha}$ as T and H tend to infinity.

We now suppose that $p_h(y_n, \eta_n)/\varphi_{T,H}(t_n, y_n, \eta_n)$ as n tends to infinity for some sequences $h_n \to 0$, T_n, H_n, and (t_n, y_n, η_n). Since p_h and $\varphi_{T,H}(t)$ are bounded in $C^{2,\alpha}$, we may define

$$p_0(y, \eta) = \lim_{n \to \infty} p_{h_n}(y, \eta), \quad \text{and} \quad \varphi(t, y, \eta) = \lim_{n \to \infty} \varphi_{T_n, H_n}(t + t_n, y, \eta).$$

In addition, using that B_R is compact, we let $y_n \to y_\infty$ and $\eta_n \to \eta_\infty$. Since $\|\varphi_{T,H}(t)\|_{L^2(B_R)} = 1$, then by compactness, $\|\varphi\|_{L^\infty(B_R)} = 1$.

Since p_h is bounded in $C^{2,\alpha}$, we must have that either $\varphi(y_\infty, \eta_\infty) = 0$ or that $(y_\infty, \eta_\infty) \in \partial B_R$ and the normal derivative of φ is zero. The first violates the maximum principle and the second violates the Hopf maximum principle. This is a contradiction and thus $p_h/\varphi_{T,H}$ is bounded uniformly above.

To see that $p_h/\varphi_{T,H}$ tends uniformly to zero. We may argue exactly as we did to show that $\partial_t \varphi_{T,H}/\varphi_{T,H}$ tends to zero, using the fact that p_h tends to zero in C^1. This concludes the proof. $\square$

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