**Université Paris-Dauphine - PSL**

# Quelques contributions à l'étude qualitative et quantitative de modèles de la physique et de la biologie

**Mémoire d'habilitation à diriger les recherches**

**Discipline : Mathématiques**

présenté et soutenu publiquement le 22 février 2022

par

Emeric Bouin

devant le jury composé de

| | |
|---|---|
| *Coordinateur :* | Jean Dolbeault (CEREMADE - Université Paris-Dauphine) |
| *Rapporteurs :* | Laurent Desvillettes (IMJ - Université Paris Diderot)<br>François Hamel (I2M - Université Aix-Marseille)<br>Antoine Mellet (Department of Mathematics - University of Maryland) |
| *Examinateurs :* | Vincent Calvez (ICJ - Université Lyon 1)<br>Frédéric Hérau (LMJL - Université de Nantes)<br>Clément Mouhot (DPMMS - University of Cambridge)<br>Benoît Perthame (LJLL - Sorbonne Université) |

**Centre de Recherche en Mathématiques de la Décision — UMR 7534**

Mis en page avec la classe thesul.

# Remerciements

Ce mémoire n'existerait pas sans le grand nombre de personnes qui m'ont incité, autour d'un café, au gré d'un couloir, à la fin d'un Zoom ou d'un Skype, à assumer de l'écrire. Je ne suis pas certain[1] que le résultat scientifique sera à la hauteur de leurs attentes, mais que ces lignes soient alors à leur hauteur : en premier. Bien évidemment, tout cela doit beaucoup à tous les collègues et amis avec qui j'ai eu la chance de partager un verre, un café, un tableau, des maths, un ou plusieurs articles, une organisation de conférence, un enseignement, des discussions administratives à n'en plus finir. Je remercie d'ores et déjà[2] l'ensemble de mes collaborateurs (et mentors pour certains, je ne l'oublie pas, rassurez-vous !) depuis le début de ma thèse, avec qui il a fait bon vivre sans pression de résultat : Nils Caillerie, Vincent Calvez, Jérôme Coville, Jean Dolbeault, Jimmy Garnier, Emmanuel Grenier, Christopher Kling Henderson, Franca Hoffmann, Laurent Lafleche, Guillaume Legendre, Yuan Lou, Sepideh Mirrahimi, Stéphane Mischler, Grégoire Nadin, Florian Patout, Benoît Perthame, Lenya Ryzhik, Christian Schmeiser. J'ai aussi en tête à cet instant Amic Frouvelle, Pierre Gabriel et Adrian Lam, pour toutes les discussions que nous avons eues et qui je suis sûr ne manqueront pas d'aboutir un jour.

Mes mots vont maintenant tout naturellement à Jean Dolbeault, qui a accepté d'être le coordinateur de cette habilitation à diriger des recherches. Cela est pour moi très symbolique et tout sauf juste un rôle administratif. Dès mon arrivée au CEREMADE, Jean m'a réservé un accueil exceptionnel, a même accepté de passer un temps infini à m'expliquer beaucoup de mathématiques, s'est beaucoup soucié de mon évolution. J'apprécie beaucoup nos échanges et collaborations scientifiques et j'espère qu'elles continueront longtemps, et même qui sait en haut d'une montagne.

Ma plus sincère gratitude va à Laurent Desvillettes, François Hamel et Antoine Mellet pour avoir accepté et subi le travail quelque peu cauchemardesque qu'est de faire un rapport sur ce manuscrit. Leurs travaux mathématiques m'ont énormément inspiré depuis le début de ma thèse et sont à mon avis d'une grande influence sur la communauté mathématique d'aujourd'hui. Je suis très honoré de leur présence à la soutenance et suis très reconnaissant de l'intérêt qu'ils ont porté à mes travaux.

L'impulsion de la recherche et l'envie de faire des mathématiques appliquées m'a été donnée par Vincent Calvez et Benoît Perthame. Je suis très heureux qu'ils aient accepté de perdre de leur temps pour prendre part au jury. Je me souviens encore de mon stage de fin de licence à Chevaleret, ou je m'émerveillais devant des simulations MATLAB[3] qui montraient des arbres phylogénétiques. Treize ans plus tard, les phénotypes sont encore dans ce manuscrit. Merci, Benoît, pour tout ce que tu m'as apporté pendant toutes ces années. Les trois ans de thèse sous la direction de Vincent ont été extrêmement formateurs et ont manifestement eu un impact sur la coloration scientifique des mes travaux à l'heure actuelle. Merci, Vincent, de m'avoir montré comment il aurait fallu faire, de m'avoir appris à chercher et à toujours vouloir trouver.

Je dois un remerciement tout particulier à Clément Mouhot, certes d'abord pour avoir accepté de faire partie du jury, mais surtout pour m'avoir accueilli à Cambridge quasiment

[1] Je suis certain que non, en fait !

[2] Certains se verront apparaître dans ces remerciements pour plusieurs raisons... Hé oui, on peut faire des maths en buvant des bières !

[3] Oui, Amic, c'est promis, je vais me mettre à Julia ☺

tous les ans depuis le milieu de ma thèse et intégré au Kinetic Group. Les IPA, burgers, Fish & Chips et Sticky Toffee Pudding du Eagle's sont toujours un plaisir. Les dimanches à faire des maths sur Zoom resteront dans ma mémoire (et dans mes rotules !) bien longtemps. Merci pour tous les conseils que tu m'as donnés, pour tous les bons moments passés et pour ton enthousiasme à transmettre des mathématiques.

Ses travaux mathématiques ont guidé mes reflexions ces dernières années, et sa gentillesse lors de nos échanges m'a beaucoup marqué. Merci à Frédéric Hérau d'avoir accepté de participer à ce jury d'habilitation.

Il est temps de rendre hommage à l'excellent environnement de travail que j'ai trouvé à Paris-Dauphine et qui a largement contribué à la préparation des travaux présentés dans ce manuscrit. César, Gilles, Isabelle, Marie et Thomas s'occupent très bien de nous et les directeurs du laboratoire successifs Olivier et Vincent ont mon respect éternel pour supporter mes lubies constantes. Merci à tous les collègues du CEREMADE pour l'ambiance bon enfant qui règne là-haut. J'ai une pensée toute particulière pour ... Daniela qui s'est tristement échappée à Padoue, ce qui changera la bièrADE à jamais; Robin et toute la Ryder family; Amic, Sandrine et les filles; Alessandra et le bon accent; Julien qui ne m'a toujours pas appris à danser, avec ou sans chapeau; X'ian et son bon thé divin; Marc pour sa confiance, son humour mathématique, ses coups de boost et sa passion pour mon HDR; BZ depuis Lyon pour ses câlins; François H. pour m'avoir supporté comme co-organisateur du séminaire d'Analyse et Probabilités, et pour la magistrale tour de boites de capsules dans son bureau; François S. pour ses piqures anti-déprime; (Big) David pour ses jouets imprimés en 3D; Jacques, le plus grand mathématicien que je connaisse, pour son soutien et les bons moments ensemble, avec et sans MIDO; Yating pour son sourire et son chinois si chantonnant; Pierre L. pour m'avoir fait connaitre des bons bars à Paris; Antonin pour la dolce mathematica italiana; Paul P. parce c'est tellement classe de faire de la théorie de la mesure et de s'appeler p.p.; Olga pour avoir eu un intérêt pour l'élaboration de ce manuscrit; Paul G. pour son flegme inimitable; Pierre C. pour sa confiance, sa gentillesse, son souci des étudiants et son intérêt pour mon évolution; Julien et Lucile pour les soirées miam et les parties de switch endiablées, et pour le soutien dans les moments moyens; Mon héros José pour tous les bons moments du quotidien, en maths, en chocolat ou en espagnol; Alexandre parce que Nancy c'est beaucoup trop loin; Juliette enfin moins loin; Denis Capello pour sa classe éternelle; Cyril pour les discussions géniales; Stéphane pour toutes ses idées géniales mais parfois si difficiles à suivre; Béatrice mon exemple de sérieux et de gentillesse.

Amusément, Guillaume V., Guillaume L., Benjamin et moi partageons le même numéro de bureau, et seul au plus un mur en carton nous sépare. Pour votre présence quotidienne, parce c'est vous qui subissez mes jeux de mots constants et les heures d'étudiants dans mon bureau, vous méritez bien quelques mots à part : merci pour tout, vraiment.

Many thanks to the Cambridge Kinetic Group! Angeliki, Helge, Iván, Jessica, Jo, Kleber, Ludovic, Marc, Sara, Thomas, among others, have made my tentative learning of kinetic theory very enjoyable. I am always very happy of the time we spend altogether. El mismo placer con el grupo cubano que crece con años, ¡seguimos los mojitos!

Merci à tous les collègues de la communauté mathématique en général avec qui il fait toujours bon rire en conférence ou lors de visites. J'ai une pensée spéciale pour Didier Bresch, Jérôme Coville (encore lui, qui me fait faire chauffer mon Skype, mais ce n'est que plaisir !), Anne-Laure Dalibard, Francis (Caliméro) Filbet[4], Pierre Gabriel, Jimmy Garnier, Christopher Kling Henderson, Florence Hubert, Otared Kavian et Idriss Mazari (dont l'enthousiasme est maintenant arrivé au CEREMADE !). Un remerciement spécial à l'ami François Bolley avec qui la randonnée en montagne est juste un paradis, parmi tant d'autres bons moments. La vie de la communauté c'est aussi Opération Postes, initiative fantastique à laquelle j'ai la chance de pouvoir contribuer : coucou à tous les posteurs ! Je remercie aussi les thésards avec qui j'ai (eu) le plaisir de travailler pendant leur thèse : Alain, Florian, Nils et Laurent. Comment pourrais-je oublier Sylvie Benzoni, ma maman de Lyon, qui m'a tant rassuré quand il a fallu commencer.

Special thanks to Yuan Lou and Adrian Lam for all mathematical discussions but also for the wonderful time we had in the US, in France or in China. My stay at Stanford right after my PhD defence was a very significant boost, made possible thanks to Lenya Ryzhik. Thank you very much, Lenya, for the opportunity and all mathematical discussions. Vielen Dank Christian für das Vertrauen und den Austausch, den wir in den letzten Jahren hatten.

Parce que cela n'est pas monnaie courante, j'aimerais dédier ces quelques lignes aux collègues pour qui l'enseignement n'est pas la corvée et qui se soucient des étudiants. Parmi ceux là, j'ai une amitié toute particulière pour Alexandre, Benjamin, Denis, Emmanuel, Maude, François S., Guillaume, Guillaume, Jacques, José, Julien S., Julien P., Juliette, Katia, Marc, Moulka, Pierre C., Pierre L., parmi d'autres, avec qui échanger au sujet des enseignements ou d'organisation administrative et pédagogique de MIDO est toujours agréable, intéressant et constructif. Il est grand temps de saluer Anne-Marie, dont la présence aura marqué plus d'un étudiant ... et d'un collègue ! Merci de venir encore nous voir. La responsabilité pédagogique de la L3 maths n'est pas une mince affaire, mais lorsque vous travaillez avec un cadeau sur terre en la personne de Samira, tout est tellement plus facile... Merci, Samira ! Merci aussi à Ariane, Siham et Dorothée qui rendent mon quotidien tellement plus agréable ! Les soirées Dauphinoises sont au top grâce à Patricia, Florent et Jean-Paul, il est hors de question de les oublier !

Mes pensées vont maintenant à tous ceux que j'ai rencontrés sur les bancs de l'école alors que c'est moi qui faisait les blagues au tableau. Que ce soit pendant une heure, un semestre, ou des années, dans une salle de cours, un couloir, par échange de mails ou des heures dans mon bureau, toutes mes interactions avec eux m'ont fait apprendre beaucoup, travailler sur moi-même et parfois (re-)réaliser ce qu'est la vraie vie. Je ne pourrai pas tous les lister, mais tout de même ... Bravo à Florian pour être passé de taupin à chargé de recherche (en étant toujours bougon !). Une pensée pour Alexandre avec qui je rêve de kayak à la Réunion. Merci à Mathieu, Tom et Samuel pour tous les bons moments passés ensemble depuis Lyon, à Paris, à la montagne, ou ailleurs ! Je crois que c'est la pure nostalgie des séances de TP d'option B qui m'a fait écrire ma pensée pour vous à cet endroit des remerciements ! Et maintenant, tous ces Dauphinois qui poussent et qui me poussent[5] : Antoine qui a bien grossi, comme dirait José. Ariel le footeux vietnamien a fui à Polytechnique; qui sait notre prochain voyage sera au fin fond du Mexique ! Bon vent à Etienne qui a les cheveux de plus en plus courts lui aussi. Grégoire m'impressionne un peu plus chaque jour, tout en candeur et gentillesse. Du début de la licence à la thèse, Ariane est le rêve de tout enseignant-chercheur. Merci pour ton sourire et ton amitié. Raphaël fait lui aussi le même très joli chemin et enchante mon

[4]né un 8 juillet, n'oubliez pas !

[5]à bout, parfois ☺ !

quotidien avec ses estimateurs, mais pas que ! Merci pour ta proche présence, si revigorante ! Je rêve secrètement d'aller à Walt Disney Studios avec MJ, parce qu'il paraît qu'elle est encore plus en extase là-bas qu'en visitant R-Studio ! Mention spéciale à mon handballeur préféré Hugo pour être passé du hoodie jogging au costume d'actuaire en un temps très bref[6] et merci à Jarod de veiller à assurer une part suffisante d'OM dans tout ce petit groupe infernal ! Merci à Maximilien aussi, même si voir les choses si différemment est souvent vraiment très dur, mais c'est la vie comme on dit. Evidemment, vouloir tirer des plus jeunes vers le haut avec affection expose à des horreurs et réserve parfois des surprises durablement fracassantes. Je pense à cet instant très fort à mon petit frère Bastien ... évaporé ... merci gros loulou !

Lyon n'a jamais été aussi près de Paris. Merci à Magalie (et toute la petite famille) d'être toujours là pour moi et de ne pas m'avoir oublié. J'attends la prochaine fois à chaque fois. Tant qu'on est à Lyon, j'ai une pensée pour tous mes copains de promo de l'ENS, et adhérents, maintenant partis vaquer à leurs occupations mathématiques ça et là du monde, et qui m'ont tant appris. Jessica, je crois que tu vas devoir me faire une petite place pour des week-ends à Montpellier ! Merci pour ta présence toutes ces années. Bérénice et les jumeaux sont aussi là bas, d'ailleurs ! Gros bisous ! Coucou à mes médecins préférées, Lauriane et Sophie, toujours là, même à distance. Les soirées pyjama à l'internat de Saint-Louis sont maintenant bien loin, mais j'ai toujours la chance et le plaisir de voir Soafara sourire de temps en temps. Merci, So ! Merci à mon double Thomas pour tous les bons moments passés ensemble, l'entente indéfectible et tous nos supers voyages. J'espère secrètement que tu ne te lasseras pas trop trop vite du vieux ...

Rendre visite à ma marraine sur le bassin est toujours un grand bonheur, et donne mêmes des idées mathématiques en bonus ! Mon parrain a suivi toutes mes études et je l'en remercie de nouveau grandement. Merci à Zhe pour tous les bons moments, en France ou en Chine. Il est temps que tu reviennes en France ! Les derniers mots iront à mes parents, parce que c'est difficile d'avoir un fils enseignant-chercheur que l'on trouve toujours totalement dans son monde; à mes grands parents pour qui je fais toujours mes devoirs; à ma soeur Morgane, Gaël et au petit Marin, pour rendre la famille si joviale et vivante.

[6]Même plus d'histoire de $\varepsilon > 0$...

*À mes grands-parents et à mon oncle,*

# Contents

**Introduction** **1**

**List of publications** **5**

**I Spreading in nonlocal models from biology** **7**

**1** Bramson delay in nonlocal models 9

1.1 The way analysts do it: the Hamel, Nolen, Roquejoffre and Ryzhik method. 9

1.2 Bramson correction in the cane toads equation with bounded traits . . . . . 12

1.3 The nonlocal Fisher-KPP equation: when the competition kernel influences the Bramson correction. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14

1.4 Closing a gap in the non-linearities: a Fisher-KPP equation with a logarithmic type nonlinearity . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18

1.5 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

**2** Dispersal evolution and front acceleration in cane toads equations 23

2.1 A quantitative proof of acceleration in the cane toads equation with unbounded traits . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24

2.2 Super-linear spreading in local bistable cane toads equations . . . . . . . . . 28

2.3 A mortality trade-off and its quantitative influence . . . . . . . . . . . . . . . 33

2.4 Perspectives and related topics . . . . . . . . . . . . . . . . . . . . . . . . . 38

**3** Sharp spreading in integro-differential equations 41

3.1 More precise rates of expansion when $f$ is of KPP type . . . . . . . . . . . . 42

3.2 The case of a weak Allee effect . . . . . . . . . . . . . . . . . . . . . . . . . 45

3.3 The lower bound with ignition nonlinearities . . . . . . . . . . . . . . . . . 52

3.4 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54

**II Long time behaviour and scaling limits in kinetic theory** **55**

**4** Spreading in kinetic reaction transport equations: local and nonlocal Hamilton-Jacobi equations 57
4.1 Propagation in higher dimensions : influence of the velocity set . . . . . . . 58
4.2 Large deviations and nonlocal Hamilton-Jacobi equations . . . . . . . . . . . 61
4.3 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 68

**5** A unified approach to fluid approximations of linear kinetic models 71
5.1 The question at hand . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 71
5.2 The history of the mathematical treatment of the problem . . . . . . . . . . . 72
5.3 The spectral approach and the result . . . . . . . . . . . . . . . . . . . . . 74
5.4 Application to concrete models . . . . . . . . . . . . . . . . . . . . . . . . 80
5.5 A word about the proofs . . . . . . . . . . . . . . . . . . . . . . . . . . . . 82
5.6 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 83

**6** Contributions to hypocoercivity 85
6.1 Hypocoercivity without confinement . . . . . . . . . . . . . . . . . . . . . . 87
6.2 The influence of a weak confinement . . . . . . . . . . . . . . . . . . . . . 95
6.3 Hypocoercivity when the steady state is not known . . . . . . . . . . . . . . 98
6.4 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 100

**References** **103**

# Introduction

This memoir is the synthesis of the research work I have done after the defence of my PhD thesis under the supervision of Prof. Vincent Calvez. Since then, I have been working on applied analysis, more particularly studying partial differential equations, with a strong interest in developing and using tools to get qualitative and quantitative results on models which arise from physics and biology. Modelling and analysis are crucial in such disciplines and providing robust tools to tackle challenging problems from there is definitely of great importance.

My work is focused on reaction-diffusion, integro-differential, Hamilton-Jacobi and kinetic equations. Even though each type of equations has his own mathematical toolbox, various bridges and transfers of methodologies between *a priori* unrelated models exist and are often at the core of very interesting mathematical issues. I have tried to take advantage of these links to put an emphasis on nonlocal models, for which getting sharp quantitative results is difficult and open in most cases. I have also tried to get universal results, to try to put forward common structures.

This document is divided into two parts. The first one is devoted to the study of invasions in biology. Mathematical analysis has given a better understanding of many aspects of population dynamics and often helps for the interpretation of experimental data. The second one is about long time behaviour and trend to equilibrium in kinetic theory. Kinetic theory has its origin in particle physics, with some recent applications in biology. I shall now review the contents of both parts.

In the first part, we are seeking for precise rates of spreading in nonlocal models that arise very naturally from the modelling of biological invasions. The issue of survival of species under climate change, the intermingling of population groups and its genetic consequences, the propagation of viruses in epidemiology, the transmission of diseases (dengue fever, zika, chikungunya for instance) through mobile hosts (mosquitoes, among others) illustrate the need of studying quantitatively invasions to understand their outcomes. The analysis of nonlocal models in biology is very active for now a couple of decades (at least). Population dynamics have revealed their necessity at least to explain large distance migrations or demogenetics, for instance. However, getting efficient tools to attack them and get precise long time behaviour as a result is still relatively recent and difficult. The main phenomena I am interested in are front propagation or accelerations. The first contribution of my work was to extend and modify analytical techniques to obtain rigorously Bramson corrections in nonlocal models. Then, I have put a lot of efforts on the accelerations of fronts: the mathematical community has started a strong focus on it around 2010 (see for example [113]), and I found it relatively exciting. I review my contributions in three chapters.

- In Chapter 1, we are interested in logarithmic corrections of Bramson type that appear in reaction diffusion models of KPP type. From the analysis point of view, important contributions are due to Hamel, Nolen, Roquejoffre and Ryzhik. After describing briefly their approach, I introduce several of my results which generalise it to nonlocal models of interest and to related local models with rather specific nonlinearities. More precisely, I present:

  – a result coming from [B20] on the cane toads equation with bounded traits,
  – a study of the non-local Fisher-KPP equation with fail tailed competition kernel [B22] showing a drastic influence of the tails of the kernel on the Bramson correction,
  – a more precise study of a local KPP type equation with logarithmic nonlinearity [B19], for which a connexion with large deviations and Hamilton-Jacobi equations appear.

- Chapter 2 is devoted to a quantitative study of the evolution of dispersal. Dispersal is a basic characteristic of many living organisms, and its adaptive significance has been widely investigated. One obvious benefit of dispersal is the potential to find a better habitat. Population geneticists are interested in genetic structures and focus on how dispersal changes allele frequencies among populations. One iconic example of the interaction between ecology and evolution is the spread of cane toads in Australia. I explain in this chapter how I proved with Henderson and Ryzhik that the fully nonlocal cane toads equation exhibits an acceleration phenomenon [B21]. As an application of this technology, I also present a result about the bistable cane toads equation, showing that the bistable nonlinearity does not prevent acceleration [B18] as it does for a fractional Laplace reaction diffusion equation, for instance. Finally, to get more robust tools, I have worked on a cane toads equation with a mortality trade-off, to show the quantitative influence of the latter on rates of propagation [B9].

- Chapter 3 gathers some results I obtained about acceleration phenomena in nonlocal equations of integro-differential or fractional types. This is more or less a discussion on how to get sharp rates of acceleration (which is not easy for nonlocal models) depending on the form of the nonlinearities and jump rates involved. First, with Garnier, Henderson and Patout in [B17], we sharpen a result by Garnier [98] on integro-differential equations with KPP-type nonlinearities. We show a rate of acceleration for a wide range of jump kernels. This chapter also contains two results obtained with Coville and Legendre on exponents of acceleration for a very general range of integro-differential equations with weak Allee effect [B11] or ignition temperature [B10].

The second part of the memoir is about kinetic theory. At the end of the $19^{th}$ century, in 1872, Boltzmann has proved his $H$-theorem, that describes the tendency of the entropy to decrease despite reversible microscopic mechanics: a statement about fundamentally irreversible processes. The question of convergence to equilibrium states then arises, and while physics would broadly predict exponential decay, it appears that this is in fact not clear at all. Cercignani's conjecture for instance, questioning the comparison between Boltzmann's entropy production functional and the relative $H$ entropy functional, is sometimes true and always almost true [180]. All of this is part of the largest so-called Hilbert sixth's problem, which is to axiomatise those branches of physics in which mathematics are prevalent. Regularity of solutions and quantitative rates of convergence are thus widely open central issues in kinetic theory. After my PhD thesis, I have been working on long time behaviour and trend to equilibrium for kinetic equations, which involves various scaling limits and tools from the theory of hypocoercivity. This latter theory exists now for a couple of decades, and has been strongly promoted by Villani [182]. It takes roots and intuitions in diffusive limits, that consequently helps the understanding of long time behaviour in kinetic theory. One aim was to obtain a systematic and structured approach to rates of convergence (in a broad sense) and fluid approximations for linear kinetic models in local or nonlocal settings, standard or anomalous macroscopic limit models, exponential or polynomial decay frameworks. Added to the fact that it provides new methods and contributions, I also view this as an effort to exhibit and highlight mathematical structures. This is very important from the point of view of applications, to be able to handle a growing diversity of models. I also believe that analytical approaches also originate and complement probabilistic and numerical counterparts. My contributions are again collected in chapters.

- I have been interested in studying propagation phenomena in kinetic models. These models have actually demonstrated to be very accurate to describe some biological phenomena, such as bacterial pulses for instance, see [47, 169, 43]. In Chapter 4, I develop two main ideas. First, I show how to extend the study of finite speed propagation to multidimensional velocities, showing the influence of the geometry on speeds of propagation. With Calvez, Grenier and Nadin [B5], we have developed an Hamilton-Jacobi type framework to handle large deviations regimes in kinetic equations for a class of Piecewise Deterministic Markov Processes (PDMP) in the full multidimensional space. This setup is a new type of nonlocal Hamilton-Jacobi equations, which allows to get sharp rates of acceleration in kinetic reaction transport equations.

- A unified framework to study fluid approximations of linear kinetic equations in the full space follows in Chapter 5. I present a paper written with Mouhot [B26] in which we develop a spectral approach of Ellis and Pinsky [82] type in order to derive the ad hoc fluid limit for a quite large class of linear kinetic models. This allows to deal simultaneously with standard and fractional type macroscopic equations and obtain quantitative coefficients and rates of convergence. Moreover, we take into account a large class of linear collision operators, including Fokker-Planck, Levy-Fokker-Planck and scattering operators.

- Chapter 6 gathers my work about hypocoercivity of linear kinetic equations. I have focused on extending the so-called Dolbeault-Mouhot-Schmeiser method [80] that requires a rather restrictive mathematical framework compared to the potential field of applications. More precisely, I report on papers [B12, B13, B14] where we find rates of decay to equilibrium for some linear kinetic equations in the full space with microscopic equilibrium having rather large types of decay, ranging from very fast (typically gaussian) to very slow (*e.g.* algebraic). We use pseudo-differential operators to extend the approach in [80] that needs strong confinement in space and velocity. Another extension comes from [B15], where we take into account a very weak confinement in space, that prevents from using Fourier techniques that were central (at first glance) for [B13, B12, B14]. I conclude this chapter with an example of application/extension of the theory for which hypocoercivity functionals can help showing existence of stationary states in a model coming from fiber lay-down processes [B23].

Both parts can be read independently, but a lot of connexions between them can be made: hypocoercive techniques can be useful to study precise spreading (such as Bramson corrections) in kinetic reaction transport models, for instance. Each chapter includes some research perpectives is presented. I have also tried to make the chapters accessible independently readers interested in specific results.

This memoir does not review in details my papers [B16] and [B24]. Even though they are very close in terms of themes and connected to what is exposed here, I view them as independent contributions.

# List of publications


[B1] ⋆ E. Bouin. A Hamilton-Jacobi approach for front propagation in kinetic equations. *Kinet. Relat. Models*, 8(2):255–280, 2015.

[B2] E. Bouin and N. Caillerie. Spreading in kinetic reaction-transport equations in higher velocity dimensions. *European J. Appl. Math.*, 30(2):219–247, 2019.

[B3] ⋆ E. Bouin and V. Calvez. A kinetic eikonal equation. *C. R. Math. Acad. Sci. Paris*, 350(5-6):243–248, 2012.

[B4] ⋆ E. Bouin and V. Calvez. Travelling waves for the cane toads equation with bounded traits. *Nonlinearity*, 27(9):2233–2253, 2014.

[B5] E. Bouin, V. Calvez, E. Grenier, and G. Nadin. Large deviations for velocity-jump processes and non-local Hamilton-Jacobi equations. *hal-01344939*, Sept. 2019.

[B6] ⋆ E. Bouin, V. Calvez, N. Meunier, S. Mirrahimi, B. Perthame, G. Raoul, and R. Voituriez. Invasion fronts with variable motility: phenotype selection, spatial sorting and wave acceleration. *C. R. Math. Acad. Sci. Paris*, 350(15-16):761–766, 2012.

[B7] ⋆ E. Bouin, V. Calvez, and G. Nadin. Hyperbolic traveling waves driven by growth. *Math. Models Methods Appl. Sci.*, 24(6):1165–1195, 2014.

[B8] ⋆ E. Bouin, V. Calvez, and G. Nadin. Propagation in a kinetic reaction-transport equation: travelling waves and accelerating fronts. *Arch. Ration. Mech. Anal.*, 217(2):571–617, 2015.

[B9] E. Bouin, M. H. Chan, C. Henderson, and P. S. Kim. Influence of a mortality trade-off on the spreading rate of cane toads fronts. *Comm. Partial Differential Equations*, 43(11):1627–1671, 2018.

[B10] E. Bouin, J. Coville, and G. Legendre. Acceleration in integro-differential combustion equations, *arXiv:2105.09946*, May 2021.

[B11] E. Bouin, J. Coville, and G. Legendre. Sharp exponent of acceleration in integro-differential equations with weak allee effect, *arXiv:2105.09911*, May 2021.

[B12] E. Bouin, J. Dolbeault, and L. Lafleche. Fractional hypocoercivity. *Comm. Math. Phys.*, to appear, 2022.

[B13] E. Bouin, J. Dolbeault, L. Lafleche, and C. Schmeiser. Hypocoercivity and sub-exponential local equilibria. *Monatsh. Math.*, 194(1):41–65, 2021.

[B14] E. Bouin, J. Dolbeault, S. Mischler, C. Mouhot, and C. Schmeiser. Hypocoercivity without confinement. *Pure Appl. Anal.*, 2(2):203–232, 2020.

[B15] E. Bouin, J. Dolbeault, and C. Schmeiser. Diffusion and kinetic transport with very weak confinement. *Kinet. Relat. Models*, 13(2):345–371, 2020.

[B16] E. Bouin, J. Dolbeault, and C. Schmeiser. A variational proof of Nash's inequality. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 31(1):211–223, 2020.

[B17] E. Bouin, J. Garnier, C. Henderson, and F. Patout. Thin front limit of an integro-differential Fisher-KPP equation with fat-tailed kernels. *SIAM J. Math. Anal.*, 50(3):3365–3394, 2018.

[B18] E. Bouin and C. Henderson. Super-linear spreading in local bistable cane toads equations. *Nonlinearity*, 30(4):1356–1375, 2017.

[B19] E. Bouin and C. Henderson. The Bramson delay in a Fisher-KPP equation with log-singular nonlinearity. *Nonlinear Anal.*, 213:Paper No. 112508, 30, 2021.

[B20] E. Bouin, C. Henderson, and L. Ryzhik. The Bramson logarithmic delay in the cane toads equations. *Quart. Appl. Math.*, 75(4):599–634, 2017.

[B21] E. Bouin, C. Henderson, and L. Ryzhik. Super-linear spreading in local and non-local cane toads equations. *J. Math. Pures Appl. (9)*, 108(5):724–750, 2017.

[B22] E. Bouin, C. Henderson, and L. Ryzhik. The Bramson delay in the non-local Fisher-KPP equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(1):51–77, 2020.

[B23] E. Bouin, F. Hoffmann, and C. Mouhot. Exponential decay to equilibrium for a fiber lay-down process on a moving conveyor belt. *SIAM J. Math. Anal.*, 49(4):3233–3251, 2017.

[B24] E. Bouin, G. Legendre, Y. Lou, and N. Slover. Evolution of anisotropic diffusion in two-dimensional heterogeneous environments. *J. Math. Biol.*, 82(5):Paper No. 36, 34, 2021.

[B25] $\star$ E. Bouin and S. Mirrahimi. A Hamilton-Jacobi approach for a model of population structured by space and trait. *Commun. Math. Sci.*, 13(6):1431–1452, 2015.

[B26] E. Bouin and C. Mouhot. Quantitative fluid approximation in transport theory: a unified approach, *arXiv:2011.07836*, 2020.

[B27] $\star$ S. Mirrahimi, B. Perthame, E. Bouin, and P. Millien. Population formulation of adaptative meso-evolution: theory and numerics. In *The mathematics of Darwin's legacy*, Math. Biosci. Interact., pages 159–174. Birkhäuser/Springer Basel AG, Basel, 2011.

**Important note:** Articles written during my PhD are not presented in this memoir (but cited when it is relevant). They are identified by a symbol $\star$.

# Part I

# Spreading in nonlocal models from biology

# Chapter 1

# Bramson delay in nonlocal models

**Contents**


**1.1 The way analysts do it: the Hamel, Nolen, Roquejoffre and Ryzhik method. . . . 9**

**1.2 Bramson correction in the cane toads equation with bounded traits . . . 12**

**1.3 The nonlocal Fisher-KPP equation: when the competition kernel influences the Bramson correction. . . . 14**

**1.4 Closing a gap in the non-linearities: a Fisher-KPP equation with a logarithmic type nonlinearity . . . 18**

**1.5 Perspectives . . . 21**


The Bramson correction in reaction-diffusion equations has attracted a lot of attention in the last decade in the applied analysis community. One reason[7] for this could be that propagation phenomena is a very important topic for analysts and that even though the first probabilistic evidence dates back to 1978, no analytic proof with the same precision of its occurence for the standard Fisher-KPP equation was known until the 2010's.

This chapter is devoted to reporting on my contributions around the Bramson delay in the context of non-local equations, after a quick presentation of what it is on the standard case of the Fisher-KPP equation and how Hamel, Nolen, Ryzhik and Roquejoffre have proved it analytically in [111, 112]. What follows is based on the papers [B19, B20, B22].

## 1.1 The way analysts do it: the Hamel, Nolen, Roquejoffre and Ryzhik method.

The Fisher-KPP equation

$$u_t = u_{xx} + u(1-u) \tag{1.1}$$

is one of the simplest models for population spreading, accounting for a competition for resources. After the pioneering works by Fisher [91], Kolmogorov, Petrovskii and Piskunov [131], it is well known that this latter model has travelling wave solutions. The minimal speed of existence is $c^* = 2$. As a consequence, if one starts with an initial data that is exactly one

[7] This does not say that it is a good reason!

of these waves, up to a shift, then the solution translates at finite speed. Note that such a waves cannot realistically be taken as initial data since they are not compactly supported (to the right). Our interest is in the spreading of the solutions of when the initial density $u_0$ is localised. The classical result of [91, 131] says a solution to the scalar KPP equation with a non-negative compactly supported initial condition propagate with the speed $c^* = 2$ in the sense that

$$\lim_{t \to +\infty} u(t, ct) = 0,$$

for all $c \geq c^*$, and

$$\lim_{t \to +\infty} u(t, ct) = 1,$$

for all $c \in [0, c^*)$. This indicates that diffusion and reaction together can create propagation even though diffusion alone cannot. The solution of the Cauchy problem thus spreads asymptotically with the minimal speed of the travelling waves, but eventually with lower order terms in the rate of expansion of the level sets.

The Fisher-KPP result for the solutions of (1.1) has been refined by Bramson in [35, 36]. He has shown the following: for any $m \in (0,1)$, let

$$X_m(t) = \sup\{x : \; u(t,x) = m\},$$

This level set has the asymptotics

$$X_m(t) = 2t - \frac{3}{2}\log t + x_m + o(1), \quad \text{as } t \to +\infty.$$

Here, $x_m$ is a constant that depends on $m$ and the initial condition $u_0$. Bramson's original proof was probabilistic, using Branching Brownian Motions. A shorter probabilistic proof can be found in a recent paper by Roberts [166]. One may think of $\bar{X}(t) = 2t$ as the position of a traveling wave, and $d(t) = \frac{3}{2}\log t$ as the delay due to the fact that the initial condition $u_0$ is compactly supported, so that the solution lags behind the traveling wave because it needs time to mollify.

Actually, an earlier PDE type proof due to Uchiyama [134, 178] exists but with less precise asymptotics. In the last decade, Hamel, Nolen, Ryzhik and Roquejoffre carried out a program around getting, with analytic arguments, very refined estimates on the dynamics of the level sets for the Fisher-KPP equation. In [111], a short proof of the Bramson result for the standard Fisher-KPP is proposed. A more involved approach, necessary to reach more general contexts, is developed in a periodic framework in [112]. The interested reader may go read [156] to learn how to get very high order expansions. In this paragraph, we will stick to the $O(1)$ precision, since this is what we will be seeking for on nonlocal models later.

The intuitions and methods behind the proofs of Hamel, Nolen, Ryzhik and Roquejoffre in [111, 112] are of crucial influence for the next sections. Let us highlight the most important ingredients of these works that we will adapt later on. We stick to the most basic framework for this description. Since the Fisher-KPP enjoys a maximum principle, one may sandwich a solution to the Cauchy problem between well-behaved sub- and super-solutions. Both constructions are actually interesting and are related to solutions to a Dirichlet problem for the linearised Fisher-KPP equation in a well chosen shifted frame, that is

$$\begin{cases} w_t - w_{xx} = 0, & \text{for } x > x(t), \\ w(t, x(t)) = 0, & \text{for } t > 0, \end{cases}$$

where $x(t)$ is suitably chosen.

- *The supersolution.* Constructing a super-solution from a Dirichlet problem needs an extra argument, since the value zero at the boundary prevents from being above the solution of the Cauchy problem, that is positive. Actually, the shift $2t - \frac{3}{2}\ln(t)$ is the only one that makes an order one solution $w$ at $2t - \frac{3}{2}\log t + O(1)$. One can then truncate properly around there to get a super-solution.

- *The subsolution.* Put the Dirichlet boundary condition at $2t$. Show that very far ahead, at $2t + \sqrt{t}$, the function $w$ is of order $\exp(-\sqrt{t})/t$. Thanks to the asymptotics at infinity of a travelling wave solution to the Fisher-KPP equation, one can fit a travelling wave underneath the solution $u$ at $2t + \sqrt{t}$. Tracing back the travelling wave around $2t - \frac{3}{2}\log t + O(1)$ yields an order bound on $u$.

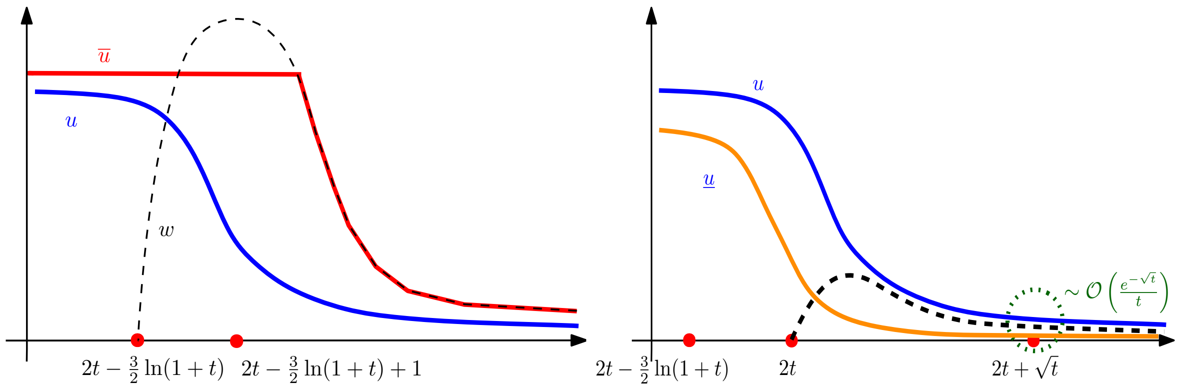



Figure 1.1: Left: a sketch of the construction of the super-solution $\overline{u}$ and the solution $u$. Right: a sketch of the sub-solution $\underline{u}$ (a shifted travelling wave), the solution $u$ of the Fisher-KPP problem, and of the solution of the linearised problem with the Dirichlet boundary condition at $x = 2t$.

A large part of the work is then to show the necessary quantitative estimates on $w$ with the necessary various choices of $x(t)$. The way to obtain such estimates may differ when the standard Fisher-KPP equation is set in a periodic media, see the important technical differences in [112] compared to [111].

Nonlocal models have proved to be necessary to model biological invasions. However, by now, not much quantitative work is done in the case of nonlocal equations. The previous strategy taken *stricto sensu* would immediately fail with nonlocal terms due to the absence of maximum principle, and probabilistic techniques are not as much developed in this context. In this chapter, I shall review three contributions related to the study of the Bramson delay in nonlocal reaction-diffusion models.

## 1.2 Bramson correction in the cane toads equation with bounded traits

My first contribution, carried out with Henderson and Ryzhik in [B20], was to study the existence of a Bramson correction in the case of the (nonlocal) cane toads equation with bounded traits, for which I had proved existence of travelling waves solutions with Calvez in [B4] during my PhD thesis. Again, the latter typically means finite speed propagation for the Cauchy problem when starting with a reasonable initial data, and thus the issue of adding a correction or not to the linear term appears. Of course, due to the similarity and the resemblance with the standard KPP-equation, the positive answer is somewhat expected. Nevertheless, getting a rigorous proof requires to adapt the proof by Hamel, Nolen, Roquejoffre and Ryzhik in the periodic setting given in [112] to the case of this structured model, and more importantly to develop an Harnack inequality to tackle the nonlocality.

Cane toads were introduced in Queensland, Australia in 1935, to control the native cane beetles in sugar-cane fields. Initially, about one hundred cane toads were released, and by now, their population is estimated to be about two hundred million, leading to disastrous ecological effects. Their invasion has interesting features different from the standard spreading observed in most other species [162]. Toads with longer legs move faster and are the first to arrive to new areas, followed later by those with shorter legs. In addition, those at the front have longer legs than toads in the long-established populations – the typical leg length of the advancing population at the front grows in time. The leg length is greatest in the new arrivals and then declines over a sixty year period. The cane toads are just one example of a non-uniform space-trait distribution – one other is the expansion of the bush crickets in Britain [175]. There, the difference is between the long-winged and short-winged crickets, with similar conclusions. In all such phenomena, modelling of the spreading rates has to include the trait structure of the population.

Figure 1.2: Photo taken after shaking hands with him in 2014 (Rick Shine's lab - University of Sydney).

A now standard model of the cane toads invasion is based on the classical Fisher-KPP equation. The population of toads is $u(t, x, \theta)$ is structured by a spatial variable $x$, and a motility variable $\theta$. This population undergoes diffusion in the trait variable $\theta$, with a constant diffusion coefficient, representing mutation, and in the spatial variable, with the diffusion coefficient $\theta$, representing the effect of the trait on the spreading rates of the species. In addition, each toad competes locally in space with all other individuals for resources (that is, independently of its trait). This leads to the nonlocal model proposed in [38] (see also [12], importantly):

$$u_t = \theta u_{xx} + u_{\theta\theta} + ru(1-\rho), \tag{1.2}$$

where

$$\rho(t,x) = \int_\Theta u(t,x,\theta)d\theta$$

is the total population at the position $x$. Here, $\Theta = [\underline{\theta}, \overline{\theta}]$ is the set of all possible traits, with traits bounded away from zero, $\underline{\theta} > 0$. The model is supplemented by Neumann boundary conditions at $\theta = \underline{\theta}$ and $\theta = \overline{\theta}$:

$$u_\theta(t,x,\underline{\theta}) = u_\theta(t,x,\overline{\theta}) = 0, \quad t > 0, \ x \in \mathbb{R}.$$

The study of the spreading of solutions to the cane toads equations started with a Hamilton-Jacobi framework that was formally developed in [B6], and rigorously justified in [177]. However, at that stage, no results on the propagation for the Cauchy problem was known. We have proved the following.

**Theorem 1.1** (B., Henderson, Ryzhik [B20])**.** *Let $u(t,x,\theta)$ satisfy the cane toads equation* (1.2)*, with the initial condition $u_0 \geq 0$ being compactly supported to the right: there exists $x_0$ such that $u_0(x) = 0$ for all $x \geq x_0$. There exists $m_0$ such that for all $\varepsilon \in (0, m_0)$, there is a positive constant $C_\varepsilon$ such that*

$$\liminf_{t\to\infty} \inf_{x \leq c^* t - \frac{3}{2\lambda^*}\log(t) - C_\varepsilon} u(t,x) \geq m_0 - \varepsilon,$$

$$\limsup_{t\to\infty} \sup_{x \geq c^* t - \frac{3}{2\lambda^*}\log(t) + C_\varepsilon} u(t,x) \leq \varepsilon.$$

A precise characterisation of the minimal speed $c^*$ and the decay rate $\lambda^*$ are given in my paper with Calvez [B4].

Actually, the first and most crucial step of the proof consists in relating the cane toads equation to a local model of cane toads type, for which one may be able to carry out the strategy initiated by Hamel, Nolen, Roquejoffre and Ryzhik. The main tool for this is a *local in time* parabolic Harnack inequality that we have derived and is of independent interest. Consider an operator

$$Lv = \sum_{ij} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}.$$

Here, $a(x) := \left(a_{ij}(x)\right)_{i,j}$ is a Hölder continuous, uniformly elliptic matrix.

**Proposition 1.2.** *Suppose that u is a positive solution of*

$$\begin{cases} v_t - Lv = 0, \ t > 0, & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ v(0,\cdot) = v_0. \end{cases}$$

*For any $t_0 > 0$, $R > 0$, and $p > 1$, there exists a constant $C$ such that if $t \geq t_0$ and $|x-y| \leq R$, then*

$$v(t,x) \leq C \|v_0\|_\infty^{1-1/p} v(t,y)^{1/p}.$$

*Moreover, $C$ depends only on $v$, $t_0$, $R$, $p$ and uniform ellipticity bounds on $a$.*

Thanks to the uniform bound on solutions to the cane toads equation (1.2) proved by Turanova in [177], it is possible to show that the Harnack inequality holds for $u$ solving the cane toads equation as well. This actually implies that for $t \geq 1$ the function $u$ is a super-solution to the equation

$$\underline{u}_t - \theta \underline{u}_{xx} - \underline{u}_{\theta\theta} = \underline{u}(1 - C\underline{u}^{1/p}),$$

and a sub-solution to the equation

$$\overline{u}_t - \theta \overline{u}_{xx} - \overline{u}_{\theta\theta} = \overline{u}\left(1 - C^{-p}\overline{u}^p\right).$$

Here, $\underline{u}$ and $\overline{u}$ satisfy the same Neumann boundary conditions as $u$.

The result of the main theorem is thus proved once the similar result is proved for both equations there above. Indeed, being local, they both obey the comparison principle, and sharing with the cane toads equation the same linearised problem, we expect a spreading with similar level sets.

The next step is to adapt the Hamel, Nolen, Ryzhik and Roquejoffre strategy to these structured local equations. This requires dedicated technical estimates that can be tedious. To stay concise, we will not recall the arguments in this document but better refer to the original paper [B20].

## 1.3 The nonlocal Fisher-KPP equation: when the competition kernel influences the Bramson correction.

As a very basic model, the Fisher-KPP only accounts for a local competition between individuals. When this competition is nonlocal, one is led to the so-called nonlocal Fisher-KPP equation

$$\begin{aligned} &u_t - u_{xx} = u(1 - \varphi \star u), \qquad t > 0,\ x \in \mathbb{R}, \\ &u(0, \cdot) = u_0. \end{aligned} \tag{1.3}$$

Here, $\varphi$ is a probability density that represents the strength of the competition between individuals a given distance apart. Equation (1.3) has garnered much interest recently, mostly for two reasons. First, it does not admit a comparison principle, leading to inherent technical difficulties – even proving a uniform upper bound on $u$ is non-trivial [114]. Second, unusual behaviour may occur, such as the existence of oscillating wave trains behind the front [88, 99, 102, 151]. The model (1.3) considered here was first introduced by Britton [37] and has a quite involved history, see the introduction of [28] for a brief overview.

My second contribution concerns the study of propagation in the nonlocal Fisher-KPP equation. As far as the behaviour at the front is concerned, the main results before this work were that travelling waves of speed $c = 2$ exist [87, 102] and that solutions to the Cauchy problem with an initial data compactly supported to the right propagate with speed $c(t) = 2 + o(1)$ as $t \to +\infty$ [114]. With Henderson and Ryzhik, we refine the propagation result obtained in 2014 by Hamel and Ryzhik [114]. We show that tracking more precisely the front position (that is, getting a Bramson type correction) involves the rate of decay of the kernel $\varphi$ at infinity. When $\varphi$ decays fast enough, solutions to (1.3) spread as those of the local equation: the front is at a position as in the standard Fisher-KPP equation, up to a constant order error. However, when $\varphi$ decays slowly, and the competition at large distances is relatively strong, the delay behind the "traveling wave" position $2t$ is not logarithmic but algebraic, of the order $O(t^\beta)$, with $\beta$ depending only on the rate of decay of $\varphi$.

Let us make our assumptions more precise. First, we assume that $\varphi$ is an even, continuous, and bounded probability density:

$$\int_{\mathbb{R}} \varphi(x)dx = 1, \quad \text{and} \quad \varphi(x) = \varphi(-x) \text{ for all } x \in \mathbb{R}. \tag{1.4}$$

In addition, $\varphi$ has some “mass” near the origin, that is, there exists $\sigma_\varphi > 0$ such that

$$\varphi(x) \geq \sigma_\varphi \mathbb{1}_{[-\sigma_\varphi, \sigma_\varphi]}. \tag{1.5}$$

The behaviour of $u$ depends strongly on the tail behaviour of $\varphi$. Here we make two different assumptions, that are helpful for the upper and lower bounds, respectively. The first assumption, an upper bound on the tail of $\varphi$, is that there exists $A_\varphi > 0$ and $r > 1$ such that, for all $R \geq 1$,

$$\int_R^\infty \varphi(x)dx \leq A_\varphi R^{-r+1}. \tag{1.6}$$

Sometimes we will need to complement this with a lower bound on the tail: for all $R \geq 1$, we have

$$\int_R^\infty \varphi(x)dx \geq A_\varphi^{-1} R^{-r+1}. \tag{1.7}$$

Roughly, (1.6) and (1.7) mean that $\varphi \sim x^{-r}$ for $x \gg 1$.

For the initial condition, we assume that $u_0$ is localized to the left of some point $x_0$:

$$0 \leq u_0 \leq 1, \qquad \exists x_0 \text{ such that } u_0(x) = 0 \text{ for all } x \geq x_0, \quad \text{and} \quad \liminf_{x \to -\infty} u_0(x) > 0. \tag{1.8}$$

The main result is the following.

**Theorem 1.3** (B., Henderson, Ryzhik [B22]). *Suppose that $u$ satisfies* (1.3), *with $u_0$ as in* (1.8) *and $\varphi$ satisfying* (1.4), (1.5), *and* (1.6).

*If $r > 3$, then the solution $u$ propagates with a logarithmic delay:*

$$\liminf_{t\to\infty} \inf_{x \leq 0} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) > 0,$$

*and*

$$\lim_{L\to\infty} \limsup_{t\to\infty} \sup_{x \geq L} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) = 0.$$

*If $r = 3$, then the solution $u$ propagates with a larger logarithmic delay: there exists $S_\varphi > s_\varphi > 3/2$ such that*

$$\liminf_{t\to\infty} \inf_{x \leq 0} u\Big(t, 2t - S_\varphi \log t + x\Big) > 0,$$

*and*

$$\lim_{t\to\infty} \sup_{x \geq 0} u\Big(t, 2t - s_\varphi \log t + x\Big) = 0.$$

*If $r \in (1, 3)$, then the delay is algebraic: there exists $C_\varphi > 0$, depending only on $r$, $\sigma_\varphi$, and $A_\varphi$, such that*

$$\liminf_{t\to\infty} \inf_{x \leq 0} u\Big(t, 2t - C_\varphi t^{\frac{3-r}{1+r}} + x\Big) > 0,$$

*and, if additionally* (1.7) *holds, then there exists* $c_\varphi \in (0, C_\varphi)$*, depending only on* $\sigma_\varphi$*,* $r$*, and* $A_\varphi$*, such that*

$$\lim_{t\to\infty} \sup_{x\geq 0} u\Big(t, 2t - c_\varphi t^{\frac{3-r}{1+r}} + x\Big) = 0.$$

This result is interesting for at least two reasons. First, getting algebraic types of delay with explicit powers is not so common in reaction-diffusion frameworks. Second, it puts forward that even though linearised problems are of constant importance in the proofs, the nonlinearity cannot be totally forgotten in general and plays a crucial quantitative role on the dynamics. We note that a parallel work by Penington [159] with probabilistic techniques and roughly similar but eventually weaker results appeared at the same period as ours.

Let us now comment on the proof of the main result of this section, Theorem 1.3. First of all, the main tool that allows us to get "reasonably sharp" asymptotics for the front position is again a local-in-time Harnack inequality that is of independent interest.

**Proposition 1.4.** *Suppose that* $v \in L^\infty([0,T]\times\mathbb{R})$ *is a non-negative function that solves*

$$v_t = v_{xx} + c(t,x)v,$$

*on* $[0,T]\times\mathbb{R}$ *with* $c \in L^\infty([0,T]\times\mathbb{R})$ *and* $T > 0$*. Then, for any* $p \in (1,\infty)$*, there exist positive constants* $\alpha$*,* $\beta$*, and* $C$*, that depend only on* $\|c\|_{L^\infty([0,T]\times\mathbb{R})}$ *and* $p$*, such that, for all* $x, y \in \mathbb{R}$ *and* $t \in (0,T]$*, we have*

$$v(T, x+y) \leq C\|v\|_{L^\infty([t,T])\times\mathbb{R}}^{1-\frac{1}{p}} v(T,x)^{\frac{1}{p}} e^{\alpha t + \frac{\beta y^2}{t}}. \tag{1.9}$$

We have used a less precise form of it in the previous section (see Proposition 1.2) to obtain the logarithmic delay for solutions of the cane toads equation in [B20]. With this inequality at hand, it is possible to relate the nonlocal Fisher-KPP equation to some local Fisher-KPP equation with a time dependent logarithmic type nonlinearity. Of course, the form of this latter nonlinearity induces technical difficulties that are more involved than for the cane toads equation.

Since the scales at hand are unusual, we pursue our discussion with explaining heuristically how to get the various forms of delays and how relevant scales appear. The upper bound for $r > 3$ is obtained by a rather direct adaptation of the arguments in [111]. Let us outline a heuristic argument leading to the upper bound for $r \in (1,3)$. It also explains how the exponent $\frac{3-r}{1+r}$ comes about.

Let the front have a delay $d(t)$ behind $2t$, so that

$$\inf_{x\leq 2t-d(t)} u(t,x) \geq \delta_0, \tag{1.10}$$

with some $\delta_0 > 0$. We expect that the solution looks like an exponential to the right of $x = 2t - d(t)$ and until the "front edge" at $x = 2t + e(t)$:

$$u(t,x) \sim \exp\{-(x - 2t + d(t))\}, \quad \text{for } x \in (2t - d(t), 2t + e(t)). \tag{1.11}$$

The diffusive Gaussian decay dominates the exponential "travelling wave" decay for $x > 2t + e(t)$. Using (1.10) and then (1.7), one may estimate $\varphi \star u(t,x)$ when $x \in (2t - d(t), 2t + e(t))$ as

$$\varphi \star u(t,x) \geq \delta_0 \int_{-\infty}^{2t-d(t)} \varphi(x-y)dy \gtrsim (x - (2t - d(t)))^{1-r} \gtrsim (e(t) + d(t))^{1-r}.$$

Thus, in order for the exponential in (1.11) to be a super-solution to (1.3) inside $(2t - d(t), 2t + e(t))$, we need

$$(e(t) + d(t))^{1-r} \gtrsim d'(t). \tag{1.12}$$

We also need the exponential to be above $u(t,x)$ at the front edge. To control $u$ there, we use that, letting $h = e^{-t}u$, $h$ is a sub-solution to the heat equation. In other words,

$$h_t \leq h_{xx}$$

and, hence, for all $x \gg 1$,

$$e^{-t}u(t,x) = h(t,x) \leq \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \lesssim \frac{\sqrt{t}}{x} e^{-\frac{x^2}{4t}}. \tag{1.13}$$

Thus, for $u$ to sit below the exponential super-solution at $x = 2t + e(t)$, we require

$$\exp\left\{t - \frac{(2t + e(t))^2}{4t}\right\} \lesssim \exp\{-(e(t) + d(t))\},$$

that is,

$$e(t)^2 \geq 4td(t). \tag{1.14}$$

Since $e(t)$ should be $o(t)$, we get

$$\lim_{t\to+\infty} \frac{d(t)}{e(t)} = 0. \tag{1.15}$$

Combining (1.12), (1.14) and (1.15) gives, for $t$ large,

$$d'(t) \lesssim e(t)^{1-r} \lesssim t^{\frac{1-r}{2}} d(t)^{\frac{1-r}{2}},$$

and thus necessarily

$$d(t) \lesssim t^{(3-r)/(1+r)}.$$

We deduce also $e(t) \gtrsim t^{2/(1+r)}$.

A way to estimate the solution from below, to get the lower bounds, is to study the linearised Fisher-KPP equation with a Dirichlet boundary condition at $2t + e(t)$, as in [111]. The problem that comes up after removing the exponential factor is

$$\begin{aligned} &z_t = z_{xx} + e'(t)(z_x - z), \quad t > 0, \, x > 0, \\ &z(t,0) = 0. \end{aligned}$$

Once again, the case $r > 3$ is treated similarly to [111]. In particular, while the term $e'(t)z$ is important and is responsible for the $\frac{3}{2}$ pre-factor in the logarithmic correction, the drift $e'(t)z_x$ is negligible. Roughly, we estimate $z(t,x)$ at $x \sim \sqrt{t}$, and use a "tracing back to a shifted travelling wave" argument, to construct a sub-solution for $u$. In the case when $r = 3$, the diffusive scale and the induced drift have the same order. Here, the balance of these two scales causes a somewhat larger delay, but the strategy is more or less the same. When $r < 3$, we choose $e(t) = t^\gamma$. Since now $\gamma > \frac{1}{2}$, the drift $e'(t)z_x$ can no longer be neglected, and the choice of the exact exponent $\gamma$ is necessary to get matching asymptotics. We explicitly construct a sub-solution to $u$ to estimate the solution at the far edge, and then perform a "tracing back" argument with a travelling wave.

For the detailed proofs, we refer to the paper [B22].

## 1.4 Closing a gap in the non-linearities: a Fisher-KPP equation with a logarithmic type nonlinearity

As briefly explained in the previous paragraph, the study of the nonlocal Fisher-KPP equation had led to investigating local Fisher-KPP type equations with rough logarithmic nonlinearities. It was thus very natural to focus more precisely on the following model

$$u_t = u_{xx} + u\left(1 - A\left(\log\left(\frac{\nu}{u}\right)\right)^{1-r}\right), \tag{1.16}$$

for $r > 1$, $A > 0$, and $\nu = e^{A^{-1/(r-1)}}$. We note that $\nu$ is a normalisation constant that may be removed by scaling; however, it ensures that 1 is a steady state of (1.16). Our goal with Henderson in [B19] was to understand the effect of the parameter $r$, which quantifies the singularity of the reaction term, on the behaviour of solutions. In particular, we study the shape of the minimal speed traveling wave solutions of (1.16) and the spreading of the solutions of (1.16) when the initial density $u_0$ is localised.

Our first result is about the behaviour at infinity of critical (minimal speed) traveling wave solutions of (1.16); that is, solutions to

$$-2U' = U'' + U\left(1 - A\left(\log\left(\frac{\nu}{U}\right)\right)^{1-r}\right) \tag{1.17}$$

with the far-field conditions $U(-\infty) = 1$ and $U(+\infty) = 0$. In the study of invasion in the Fisher-KPP equation, critical traveling waves attract any reasonable initial data and the shape of their profile at $+\infty$ is crucial to quantify the position of level sets [110]. Interestingly, the nonlinearity influences the decay of the waves. Recently, a paper by Giletti [100] reveals similar features for some Fisher-KPP equations with degenerate monostable nonlinearities.

**Theorem 1.5** (B., Henderson [B19])**.** *Let $U$ be a traveling wave solution of* (1.16)*, that is solving* (1.17)*. Then $U$ is monotonically decreasing, less than* 1*, and has the following behaviour at infinity:*

*(i) If $r > 3$, then $\lim_{\xi\to\infty} U(\xi)/(\xi e^{-\xi}) = \kappa$ for some $\kappa > 0$.*

*(ii) If $r = 3$, then $\lim_{\xi\to\infty} U(\xi)/(\xi^{\alpha} e^{-\xi}) = \kappa$ for some $\kappa > 0$ and where $\alpha$ is the unique solution of $\alpha(\alpha-1) = A$ such that $\alpha > 1$.*

*(iii) If $r \in (1,3)$, then*
$$\lim_{\xi\to\infty} \frac{\log(e^{\xi}U(\xi))}{\xi^{\frac{3-r}{2}}} = \frac{2\sqrt{A}}{3-r}.$$

Importantly, this theorem shows that the transition between the standard Fisher-KPP regime and the regime for which the nonlinearity plays a crucial role in the decay of the waves happens at $r = 3$. This transition is new in the literature up to my knowledge.

Having this in mind, we are interested in deriving as explicitly as possible the Bramson correction. The earlier result with Ryzhik in [B22], presented in Theorem 1.3, gives its existence and order of magnitude (that is, either logarithmic of algebraic) but explicit constants in front were not found in that work. The main result here is the following.

**Theorem 1.6** (B., Henderson [B19])**.** *Suppose that $u$ satisfies* (1.16)*.*

*(i) If $r > 3$, then the solution $u$ propagates with a logarithmic delay:*

$$\lim_{L\to\infty} \liminf_{t\to\infty} \inf_{x\leq -L} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) = 1$$

$$\text{and} \lim_{L\to\infty} \limsup_{t\to\infty} \sup_{x\geq L} u\Big(t, 2t - \frac{3}{2}\log t + x\Big) = 0.$$

*(ii) If $r = 3$, then the solution $u$ propagates with larger logarithmic delay: defining $\alpha > 1$ to be the solution of $\alpha(\alpha - 1) = A$,*

$$\lim_{L\to\infty} \liminf_{t\to\infty} \inf_{x\leq -L} u\Big(t, 2t - \frac{2\alpha+1}{2}\log t + x\Big) = 1,$$

$$\text{and} \quad \lim_{x\to\infty} \limsup_{t\to\infty} u\Big(t, 2t - \frac{2\alpha+1}{2}\log(t) + x\Big) = 0.$$

*(iii) If $r \in (1,3)$, then the delay is algebraic: there exists $\Theta_r > 0$, such that*

$$\lim_{t\to\infty} \inf_{x\leq 2t - \Theta_r A^\gamma t^\beta + o(t^\beta)} u(t,x) = 1 \quad \text{and} \quad \lim_{t\to\infty} \sup_{x\geq 2t - \Theta_r A^\gamma t^\beta - o(t^\beta)} u(t,x) = 0,$$

*with $\gamma := \frac{2}{1+r}$ and $\beta := \frac{3-r}{1+r}$. Further, $\Theta_r = \psi(0)$ where $\psi$ solves*

$$\begin{cases} \psi' = \frac{\gamma}{2} y - \sqrt{\frac{\gamma^2 y^2}{4} + A y^{1-r} - \beta\psi} \\ \\ \psi(\bar{y}_r) = \frac{\gamma^2 \bar{y}_r^2}{4\beta} + \frac{A\bar{y}_r^{1-r}}{\beta} \qquad \text{where } \bar{y}_r = (1+r)^\gamma. \end{cases} \tag{1.18}$$

Each delay coefficient has a heuristic meaning behind it that we shall now comment on. As described in the introduction of [B22] and in the previous section of this manuscript, the main length scale on which the nonlinearity acts is $x \sim t^{\frac{2}{1+r}}$.

When $r > 3$, then $\gamma < 1/2$ and the diffusive length scale $x \sim t^{1/2}$ dominates, allowing us to ignore the nonlinearity. In this case, the heuristics are as in the standard case of a Fisher-KPP nonlinearity: roughly, following [111, Section 1], the $3/2$ coefficient arises because the time decay of the (Dirichlet) heat kernel on the half-line $[0,\infty)$ is $t^{-\frac{3}{2}}$.

When $r = 3$, $\gamma = 1/2$ and both scales balance and the nonlinearity is relevant. In this case, the coefficient $\alpha + 1/2$ comes from the fact that solutions of $h_t = h_{xx} - Ax^{-2}h$ have time decay $t^{-\alpha-1/2}$ (note that, from Theorem 1.5, we expect $\log(\nu/h)^{1-r} \sim x^{-2}$ when $r = 3$). The analysis is done in self-similar diffusive variables. The proof uses the same techniques as in [111], by writing an expansion of the rescaled solution on a suitable Hilbertian basis. Due to the added induced drift, operators are a bit modified compared to [111] but the spirit is the same. Moreover, the nonlinearity requires additional estimates in the rescaled variables.

When $r \in (1,3)$, the scale of the nonlinearity is larger than the scale of the diffusion. Thus, to understand the dynamics, it is required to understand the large deviations rate function appearing on that scale. Since the phenomenon is a bit newer in that case, we expand a bit more. We begin by changing variables to the moving frame and removing an exponential factor; that is, we let $u(t, x + 2t - s(t)) = \nu e^{-x} v(t,x)$. Then

$$v_t + \dot{s}(t)(v_x - v) = v_{xx} - A(x + \log(1/v))^{-(r-1)} v.$$

From [B22], we know that the correct length scale to look on is $x \sim t^{\gamma}$. Hence, let

$$\varphi(\tau, y) = -\frac{1}{\tau} \log v(\tau^{1/\beta}, y\tau^{\gamma/\beta}).$$

Note very importantly the large deviations flavour of this ansatz: this is where the mixing of scales appears and where $\gamma > \frac{1}{2}$ plays a role. The correct length scale to look is larger than the diffusive one.

Recall from [B22] and from Theorem 1.3 that the delay should be $O(t^{\beta})$. Set (again, heuristically) $s(t) = \theta t^{\beta}$, where the goal is to determine $\theta$ so that $w = O(1)$ near the origin (since we expect $u$ to be $O(1)$ near the front). This requires $\varphi(\tau, 0) = O(1/\tau)$ and gives the following

$$\beta\tau\varphi_\tau - \gamma y \varphi_y + \beta\varphi + \theta\beta \left(\tau^{-\frac{r-1}{3-r}} \varphi_y + 1\right) = \tau^{-1}\varphi_{yy} - |\varphi_y|^2 + A\left(y + \tau^{-\frac{r-1}{3-r}}\varphi\right)^{-(r-1)}.$$

Formally taking $\tau \to \infty$ (and assuming that $\tau\varphi_\tau \to 0$ since we expect equilibrium dynamics), we obtain the limit equation

$$\begin{aligned}
&|\varphi_y^\infty|^2 - \gamma y \varphi_y^\infty - (Ay^{1-r} - \beta(\theta + \varphi^\infty)) = 0 \qquad \text{in } (0, \infty),\\
&\varphi^\infty(0) = 0.
\end{aligned}$$

We now explain how to guess the correct value of $\theta$. This comes by comparing the asymptotics of the solutions of such an ODE to expected asymptotics of $v$.

One solution of this quadratic polynomial is given by the solution of the following ODE.

$$\begin{aligned}
&\varphi_y^\infty(y) = \frac{\gamma y}{2} - \sqrt{\frac{\gamma^2 y^2}{4} + Ay^{1-r} - \beta\varphi^\infty}, \qquad y > 0,\\
&\varphi^\infty(0) = 0.
\end{aligned}$$

If we expect convergence of $u$ to a travelling wave in the moving frame, it is then natural in view of Theorem 1.5 to expect that $v(t, x) \sim \exp\left\{\frac{2A^{\frac{1}{2}}}{3-r} x^{\frac{3-r}{2}}\right\}$ close to zero. This fits exactly with the asymptotics of $\varphi^\infty$ near $y = 0$ for any compatible value of $\theta$. We make two observations from this. First, since this works for all $\theta$, $\theta$ cannot be defined at this stage. Second, we arrived at this $\varphi^\infty$ via a quadratic formula, which involves choosing a root. Had we chosen the *other* root, the traveling wave asymptotics would *not* hold, allowing us to conclude that we have chosen the correct root for small $y$.

However, we expect $v(t, x) \sim e^{-\frac{x^2}{4t}}$ very far ahead of the front. This corresponds to $\varphi^\infty \sim y^2/4$ when $y \gg 1$. Unfortunately, if $\theta \gg 1$ then $\varphi^\infty$ cannot be extended as a solution to large $y$, and, even for those $\theta$ for which it can, $\varphi$ does not grow quadratically. Thus this $\varphi^\infty$ cannot the expected $\varphi^\infty$ when $\tau$ goes to infinity. To solve this issue, define

$$\Theta = \sup\left\{\theta \in \mathbb{R} : \frac{\gamma^2 y^2}{4} + Ay^{1-r} - \beta\varphi^\infty > 0 \text{ for all } y \geq 0\right\}.$$

By a continuity argument, if $\theta = \Theta$, $\varphi^\infty$ touches the curve $\Gamma$ "tangentially". There is only one touching point $\bar{y}_r$. Thus, when $\theta = \Theta$, we construct the $\mathcal{C}^1$ globally defined solution $\Phi$ which is equal to $\varphi$ to the left of $\bar{y}_r$ and solves

$$\Phi_y(y) = \frac{\gamma}{2} y + \sqrt{\beta(\Gamma - \Phi)} \qquad \text{in } (\bar{y}_r, \infty).$$

With this definition, $\Phi$ solves (1.18), $\Phi \sim \Theta - \frac{2\sqrt{A}}{3-r} y^{\frac{3-r}{2}}$ when $y \sim 0$, and $\Phi \sim y^2/4$ when $y \gg 1$. Hence $\Phi$ is the $\varphi^\infty$ that should arise from $\varphi$ when taking $\tau \to \infty$, above. Making this precise choice $\theta = \Theta$ is the only way to make this happen.

## 1.5 Perspectives

1. *Convergence to a travelling wave.*

   It has been proved by Bramson [35, 36] that in the shifted frame including the logarithmic delay, a solution to Fisher-KPP equation converges to a translate of the travelling wave corresponding to the minimal speed $c^* = 2$. His proof is quite elaborate, and based on probabilistic arguments. Later, and actually rather recently, Nolen, Roquejoffre and Ryzhik have given a proof using purely analytic techniques [155]. This opens the possibility of working on the similar question for more involved local and nonlocal models. Since the value of delays is known explicitly for the cane toads equation (1.1) and the Fisher-KPP type equation with logarithmic nonlinearity (1.16), one could think of the behaviour of the solutions of these equations in the relevant shifted frames. Note that the case of nonlocal equations might be more involved since the behaviour at the back of a front is sometimes very subtle.

2. *Harnack inequalities with kernel operators.*

   An important tool in our proofs above when the models were nonlocal was a local in time Harnack inequality to recover a local model. We could achieve it for a diffusion operator of parabolic type. Actually, for a modelling point of view, having mutations represented by a kernel operator would be very relevant. Getting Harnack inequalities in this framework is likely to be more involved since heat kernels are less precisely known and such operators are less regularising. Elliptic type Harnack inequalities exist for kernel operators [62], but investigations seem to have to be made to have general inequalities in the time dependent case.

3. *Numerical schemes to catch the Bramson correction.*

   Up to my knowledge, getting a good numerical approximation of the Bramson correction is not that well documented. Since several different types of order of magnitude of Bramson type corrections appear now in the literature, I believe that this lacks to the toolbox. Especially in the perspective of the next chapter, concerning accelerated profiles, for which having precise schemes is very open.

4. *A word about the Bramson correction for integro-differential operators.*

   The strategy developed by Hamel *et al.* can be applied to nonlocal dispersion cases where individuals can make large jumps. In such a situation, the Laplace operator is often replaced by a convolution operator, see Chapter 3. Graham [103] has proved, under reasonable restriction on the thin tailed jump kernel, that the Bramson correction happens in the integro-differential KPP equation. However, even though the base of the proof is the Hamel *et al.* strategy, part of the techniques for the estimates use probabilistic arguments that are not necessarily extendable to more involved situations, like heterogeneous media for instance, or with various nonlinearities. Cleaning this would be interesting for further development.

# Chapter 2

# Dispersal evolution and front acceleration in cane toads type equations

**Contents**


**2.1 A quantitative proof of acceleration in the cane toads equation with unbounded traits** . . . . . . . . . . 24

**2.2 Super-linear spreading in local bistable cane toads equations** . . . . . . 28

**2.3 A mortality trade-off and its quantitative influence** . . . . . . . . . . . . 33

**2.4 Perspectives and related topics** . . . . . . . . . . . . . . . . . . . . . . . . . 38


This chapter mainly investigates super-linear spreading in structured equations of cane toads type. The invasion of cane toads has interesting features different from the standard spreading observed in most other species [162]. Rather than invading at a constant speed, the annual rate of progress of the toad invasion front has increased by a factor of about five since the toads were first introduced: the toads expanded their range by about 10 km a year during the 1940s to the 1960s, but were invading new areas at a rate of over 50 km a year by 2006. This is exactly an accelerated invasion.

The cane toads equation presented in the previous chapter can model an acceleration when considered with unbounded phenotypical traits. A formal argument in [B6] using a Hamilton-Jacobi framework predicted front acceleration and spreading rate of $O(t^{3/2})$. In this chapter, we will present the paper [B21], that gives a rigorous proof of this spreading rate with quantitative estimates. Later on, we will show how the technology used in [B21] can be also used to get an interesting feature for a bistable cane toads equation: the acceleration still occurs despite the bistability [B18]. Lastly, we study the influence of a mortality trade-off on the spreading rate of cane toads fronts [B9].

## 2.1 A quantitative proof of acceleration in the cane toads equation with unbounded traits

Let us consider jointly the following two models. The first one is the (nonlocal) cane toads equation as exposed in Chapter 1,

$$u_t = \theta u_{xx} + u_{\theta\theta} + u(1-\rho), \quad \rho(t,x) = \int_{\underline{\theta}}^{\infty} n(t,x,\theta)d\theta, \qquad t>0,\ x\in\mathbb{R},\ \theta\in\Theta. \tag{2.1}$$

The second one is the natural local version of the former,

$$u_t = \theta u_{xx} + u_{\theta\theta} + u(1-u), \quad t>0,\ x\in\mathbb{R},\ \theta\in\Theta. \tag{2.2}$$

Both are supplemented by the Neumann boundary condition at $\theta = \underline{\theta}$:

$$u_\theta(t,x,\underline{\theta}) = 0, \ \ t>0,\ x\in\mathbb{R}.$$

Our first result concerns the local equation (2.2).

**Theorem 2.1** (B., Henderson, Ryzhik [B21])**.** *Let $u$ be the solution of the local equation* (2.2)*, with a compactly supported initial data. Fix any constant $m\in(0,1)$, then*

$$\lim_{t\to\infty} \frac{\max\{x\in\mathbb{R} : \exists\theta\in\Theta, u(t,x,\theta)=m\}}{t^{3/2}} = \frac{4}{3},$$

*The limit is uniform in $m\in[\varepsilon,1-\varepsilon]$, for any $\varepsilon>0$ fixed.*

The assumption that $u_0$ is compactly supported is made purely for convenience, one could allow more general rapidly decaying or front-like initial conditions. The theorem gives the exact propagation rate $\frac{4}{3}t^{\frac{3}{2}}$ as was imagined in [B6].

Our second main result is for the full non-local model (2.1). We obtained the first front acceleration result for this equation, with non-matching lower and upper bounds at that time.

**Theorem 2.2** (B., Henderson, Ryzhik [B21])**.** *Let $u$ be the solution of the non-local cane toads equation* (2.1)*, with a compactly supported initial data. Fix any $\varepsilon>0$, there exists a positive constant $c_\varepsilon$, depending only on $\varepsilon$, such that*

$$\frac{8}{3\sqrt{3\sqrt{3}}}(1-\varepsilon) \le \limsup_{t\to\infty} \frac{\max\{x\in\mathbb{R} : \rho(t,x)\ge c_\varepsilon\}}{t^{3/2}}. \tag{2.3}$$

*In addition, if $m$ is any constant in $(0,1)$, then we have that*

$$\limsup_{t\to\infty} \frac{\max\{x\in\mathbb{R} : \rho(t,x)\ge m\}}{t^{3/2}} \le \frac{4}{3}. \tag{2.4}$$

Let us note that taking inspiration from the strategy of [B21], Calvez and co-authors found the exact constant involved. Their study is based on finding more refined trajectories related to the problem to replace the ones we will briefly describe below. It is important to notice that both lower and upper bounds are in fact not sharp, revealing that the lack of comparison principle due to the nonlocal nature of the equation is not the only reason for having unsharp results. Of course, this was not understood before the contribution [B21].

We now describe the strategy of the proof of Theorem 2.1 (as done in [B21]). We construct sub- and super- solutions, and the difficulty is to design nicely both of them. Nevertheless, a common point to both is to take advantage of an Hamilton-Jacobi framework, formally presented in [B6], that we shall briefly present below.

Formally rescale the linearised cane toads equation (common to both local and nonlocal models!) the following way,

$$(t, x, \theta) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{3/2}}, \frac{\theta}{\varepsilon}\right).$$

One is led to solve

$$\varepsilon w_t^\varepsilon = \varepsilon^2 \theta w_{xx}^\varepsilon + \varepsilon^2 w_{\theta\theta}^\varepsilon + w^\varepsilon.$$

The Hopf-Cole transformation is an effective tool in the analysis of front propagation for reaction-diffusion equations – see, for instance, [85, 92, 21], including parabolic integro-differential equations modelling populations structured by a phenotypical trait (see [76, 160, B25]). It is not a surprise to perform

$$w^\varepsilon = \exp\left(-\frac{\varphi^\varepsilon}{\varepsilon}\right),$$

so that

$$\varphi_t^\varepsilon + \theta|\varphi_x^\varepsilon|^2 + |\varphi_\theta^\varepsilon|^2 = \varepsilon\theta\varphi_{xx}^\varepsilon + \varepsilon\varphi_{\theta\theta}^\varepsilon + 1,$$

and obtain, in the formal limit as $\varepsilon \to 0$, the Hamilton-Jacobi equation

$$\varphi_t + \theta|\varphi_x|^2 + |\varphi_\theta|^2 = 1.$$

Miraculously, we obtain an explicit formula for $\varphi(t, x, \theta)$, $\varphi(t, x, \theta) = \frac{1}{4t}\left(\theta + Z(x, \theta)^2\right)^2$, where $Z^3 + 3\theta Z + 3x = 0$.

This offers a serious candidate for a super-solution to (2.1) and (2.2)

$$\bar{u}(t, x, \theta) = \exp\left\{t - \frac{1}{4t}(\theta + Z^2(x, \theta))^2\right\}.$$

For some technical reasons developed with greater detail in [B21], this has to be modified a bit to get a proper super-solution, as shown in Figure 2.1.

To get lower bounds, the idea is to construct sub-solutions of the linearised problem with the Dirichlet boundary condition on a moving boundary of a domain $\mathcal{E}(t)$, and then use them to deduce a lower bound on the solution of the linear and nonlinear problems. The goal is to have $\mathcal{E}(t)$ move as fast as possible while ensuring that the solution of the linearised problem is $O(1)$ – it neither grows too much, nor decays.

The domain $\mathcal{E}(t)$ moves on suitable trajectories. Roughly, we choose the optimal trajectories associated to the Lagrangian of the cane toads diffusion operator,

$$L(x, \theta, v_x, v_\theta) = \frac{v_x^2}{4\theta} + \frac{v_\theta^2}{4}.$$

They solve Euler-Lagrange type equations.

$$\frac{dX(s)}{ds} = 2P_x(s)\theta(s), \quad \frac{d\theta(s)}{ds} = 2P_\theta(s), \quad \frac{dP_x(s)}{ds} = 0, \quad \frac{dP_\theta(s)}{ds} = -P_x(s)^2.$$

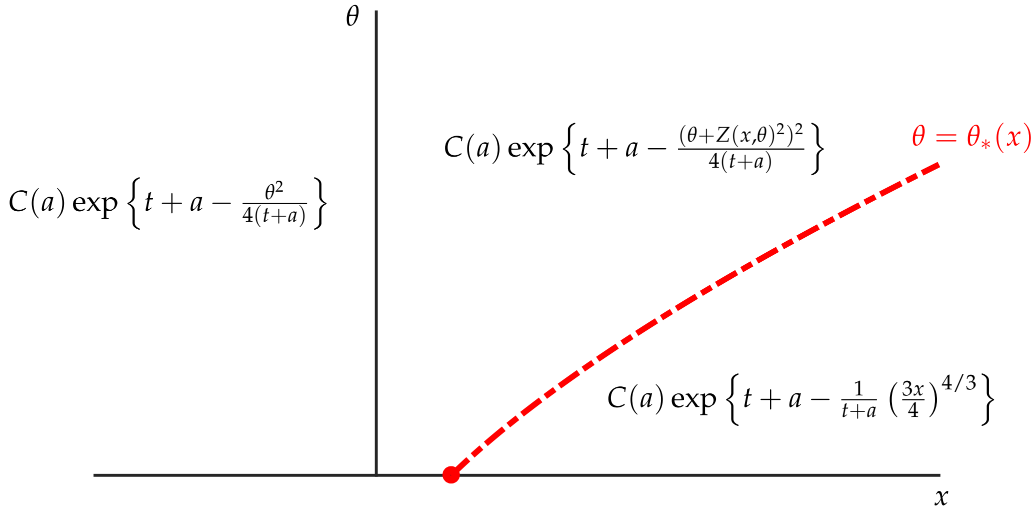



Figure 2.1: Sketch of the super-solution that achieves the upper bound in Theorem 2.1 and Theorem 2.2.

Luckily, here they are,

$$X(s) = \left(\frac{3}{2} - \frac{s}{2t}\right)\left(\frac{s}{t}\right)^2 \frac{4}{3} t^{3/2}, \quad \text{and} \quad \theta(s) = s\left(2 - \frac{s}{t}\right),$$

taking into account the fact that we want the trajectories that travel from the origin to the far edge $(x_{\text{edge}}(t), \theta_{\text{edge}}(t)) = (\frac{4}{3}t^{3/2}, t)$ of the level set $\{\bar{u}(t,x,\theta) = 1\}$, given by

$$\theta + Z(x,\theta)^2 = 2t.$$

To satisfy some technical requirements, the exact trajectories we choose in the paper [B21] are slight modifications of these, that we omit for the sake of conciseness.

The domain we slide along a trajectory is an ellipse of the form

$$\mathcal{E}_{R,T}(s) := \left\{ (x,\theta) \in \mathbb{R} \times [\underline{\theta}, \infty] : \frac{(x - X_T(s))^2}{\Theta_T(s)} + (\theta - \Theta_t(s))^2 \le R^2 \right\}.$$

Given a large time $T$, it is clear from its definition that such an ellipse moves along the trajectory $(X_T(s), \Theta_T(s))$ on the time interval $[0,T]$, starting at a point $(X_T(s), \Theta_T(0))$. The bump we define on $\mathcal{E}(s)$ is a function $v$ which solves

$$\begin{cases} v_t - \theta v_{xx} - v_{\theta\theta} \le (1-\varepsilon)v, & (s,x,\theta) \in \mathbb{R}^+ \times \mathcal{E}_{R,T}(s)\,, \\ v(s,x,\theta) \le 1, & (s,x,\theta) \in [0,t] \times \mathcal{E}_{R,T}(s)\,, \\ v(s,x,\theta) = 0, & (s,x,\theta) \in [0,t] \times \partial\mathcal{E}_{R,T}(s)\,, \\ v(0,x,\theta) < \delta, & (x,\theta) \in \mathcal{E}_{R,T}(0)\,, \end{cases}$$

and such that $\|v(T,\cdot,\cdot)\|_{L^\infty} \ge 1 - \gamma$, and $v(T,x,\theta) \ge C_R$ for all $(x,\theta) \in \mathcal{E}_{\frac{R}{2},T}(T)$, with a constant $C_R > 0$ that depends only on $R$. Of course, all parameters appearing here have to be fitted suitably to get a complete argument, we refer to the original paper [B21] for the full proof, especially for the construction of the function $v$ [B21, Lemma 4.1]. We only want to present ideas here. We summarise this discussion in Figure 2.2.

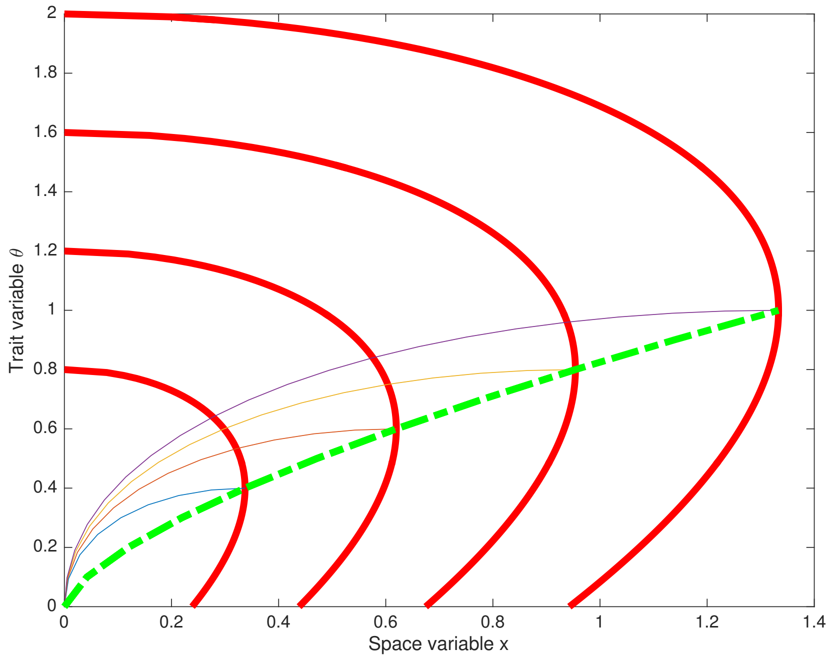



Figure 2.2: The level set $\{\bar{u}(t,x,\theta) = 1\}$ in the phase space $(x,\theta)$ for different values of time and the optimal trajectory, see also [B6]. We also plot the trajectories that lead to the edge of the front for various values of time.

The previous strategy gives the sharp rate for the local model. For the nonlocal model, since the argument goes by contradiction due to the lack of maximum principle, it is necessary to use sub-optimal trajectories, leading to the sub-optimal constant $\frac{8}{3\sqrt{3\sqrt{3}}}$ in (2.3). Actually, as revealed by a later work by Calvez *et al.* [46], the good trajectories to take are more involved and getting the sharp rate needs to take an Hamilton-Jacobi equation that takes into account the nonlinearity in a crucial way.

We should mention the concurrent work by Berestycki, Mouhot, and Raoul [29], who use a mix of probabilistic and analytic methods to prove the same sharp result in the local model (2.2). In addition, they prove the sharp asymptotics in a non-local model where $\rho$ is replaced by a windowed non-local term. Ecologically, the windowed term models a situation where individuals compete for resources only with individuals of a similar trait. The techniques of [29] do not seem to apply to the full non-local model that we address here.

## 2.2 Super-linear spreading in local bistable cane toads equations

We study the influence of an Allee effect on the spreading rate in a local reaction-diffusion-mutation equation modelling the invasion of cane toads in Australia. We are, in particular, concerned with the case when the diffusivity can take unbounded values. We show that the acceleration feature presented in Theorem 2.1 and Theorem 2.2 that arises in this model with a Fisher-KPP, or monostable, non-linearity still occurs when this non-linearity is instead bistable, despite the fact that this kills the small populations. This is in stark contrast to the work of Alfaro [4], Gui and Huan [108], and Mellet and Roquejoffre and Sire [147] in related models, where the change to a bistable non-linearity prevents acceleration. This shows that, for the cane toads equation, acceleration is a bulk phenomenon, i.e. that it is not driven by small populations far from the origin.

Fix a non-linearity $f : [0,1] \to [0,\infty]$ that is a Lipschitz continuous function such that there exists $\alpha \in (0, \frac{1}{2})$ with

$$f(u) \geq u(u-\alpha)(1-u).$$

We assume also that $f(0) = f(1) = 0$. Note that this assumption allows to cover all three of the bistable, ignition and monostable cases in one argument. However, our interest here is not in the monostable case, already extensively discussed in the previous section. As a matter of fact, the result we will obtain here is not quantitatively sharp: only the occurence of the acceleration phenomenon is discussed.

We are interested in the long-time asymptotics of solutions to the Cauchy problem

$$\begin{cases} u_t = \theta u_{xx} + u_{\theta\theta} + f(u), & (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta, \\ u_\theta(t,x,\underline{\theta}) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x,\theta) = u_0(x,\theta) & (x,\theta) \in \mathbb{R} \times \Theta. \end{cases} \tag{2.5}$$

The initial data function $u_0$ is assumed to satisfy

$$u_0 \geq \mathbb{1}_{\mathbb{R}^- \times [\underline{\theta},(1+\lambda)\underline{\theta}]},$$

where $\lambda > 0$.

Using the same equation with a monostable non-linearity as a super-solution to (2.5), the results from [29, B21] show that no level set can move faster than $O(t^{3/2})$. Our main result is to show that a lower bound of the same order holds as well.

**Theorem 2.3** (B., Henderson [B18]). *Fix any $m \in (0,1)$. There exists $\lambda_0 \in \mathbb{R}_+^*$, depending only on $\alpha$ and $\underline{\theta}$, and $\gamma \in \mathbb{R}_+^*$, depending only on $\alpha$, such that if $\lambda \geq \lambda_0$ then*

$$\liminf_{t\to\infty} \frac{\max\{x : \exists \theta \in \Theta,\ u(t,x,\theta) = m\}}{t^{3/2}} \geq \gamma.$$

To prove this theorem, we follow a similar strategy as exposed in the previous section: in rough words, we slide a suitable "bump" along suitable trajectories in the phase plane $\mathbb{R} \times \Theta$, making sure that the ball remains below the solution of the original Cauchy problem (2.5). Once the trajectories are well chosen, this will imply the acceleration phenomena claimed in theorem 2.7. However, the "bump" sub-solution is significantly more complicated to create in this setting than in the monostable one. Indeed, in [B21], the sub-solution is created almost entirely with the linearised (around zero) problem. In our setting, of course, the linearised problem decays to zero on any traveling ball. We describe how to overcome this difficulty below.

To begin, we fix a large $T > 0$ and any level set height $m \in (0,1)$. We then define a trajectory in the space $\mathbb{R} \times \Theta$ by

$$s \mapsto (X_T(s), \Theta_T(s)),$$

for any $s \in [0, T]$. The coordinate functions $(X_T, \Theta_T)$ are depicted in Figure 2.3.

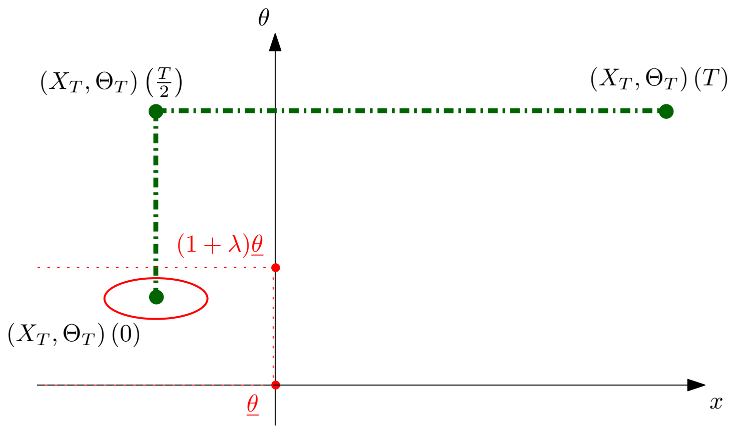



Figure 2.3: Plot of the trajectory in the phase plane $(x, \theta)$. We emphasize that the first part of the trajectory corresponds to a movement towards large traits only in the $\theta$-direction, whereas the second part of the trajectory corresponds to a movement towards large space positions only in the $x$-direction, *at an accelerated spreading rate*. The red dotted line is the support of the initial condition. The red bold line is the initial support of the sub-solution $w$ : it sits inside the support of the initial condition.

In order to slide a bump along this trajectory, we define the moving (growing) ellipse

$$\mathcal{E}_{R,T}(s) = \left\{ (x, \theta) \in \mathbb{R} \times \Theta : \frac{(x - X_T(s))^2}{\Theta_T(s)} + (\theta - \Theta_T(s))^2 \leq R^2 \right\}$$

(observe that it is the same as in the previous section) and the moving (growing) annulus

$$\mathcal{A}_{R,T}(s) = \left\{ (x, \theta) \in \mathbb{R} \times \Theta : \Lambda^2 \leq \frac{(x - X_T(s))^2}{\Theta_T(s)} + (\theta - \Theta_T(s))^2 \leq 4R^2 \right\},$$

where $R$ is a positive constant to be chosen later that encodes the sizes of this two objects.

We now build a sub-solution to (2.5) on the bigger ellipse $\mathcal{A}_{R,T} \cup \mathcal{E}_{R,T}$, what we called a "bump" above. We shall patch together a solution $w^+ \geq \alpha$ on $\mathcal{E}_{R,T}$ of

$$\begin{cases} w_t^+ = \theta w_{xx}^+ + w_{\theta\theta}^+ + f_r(w^+), & (x, \theta) \in \mathcal{E}_{R,T}(s), \\ w^+ \equiv \alpha, & (x, \theta) \in \partial\mathcal{E}_{R,T}(s), \end{cases}$$

and a positive solution $w^- \le \alpha$ on $\mathcal{A}_{R,T}$ of

$$\begin{cases} w^-_t = \theta w^-_{xx} + w^-_{\theta\theta} + f_r(w^-), & (x,\theta) \in \mathcal{A}_{R,T}, \\ w^- \equiv \alpha, & (x,\theta) \in \partial\mathcal{E}_{R,T}, \\ w^- \equiv 0, & (x,\theta) \in \partial\mathcal{A}_{R,T} \setminus \partial\mathcal{E}_{R,T}. \end{cases}$$

The new non-linearity $f_r$ is defined as follows for any $r \in (2\alpha, 1]$:

$$f_r(u) := \begin{cases} u(u-\alpha)(1-u), & \text{for } u \le \alpha, \\ c_r u(u-\alpha)(r-u), & \text{for } u \ge \alpha. \end{cases}$$

where we define $c_r = (1-\alpha)(r-\alpha)^{-1}$. It is easily verified that $f_r(u) \le f(u)$ for all $u$, and it is clear that $f_r$ is Lipschitz continuous. For technical reasons, we are required to take $r$ to be slightly less than 1 in the sequel.

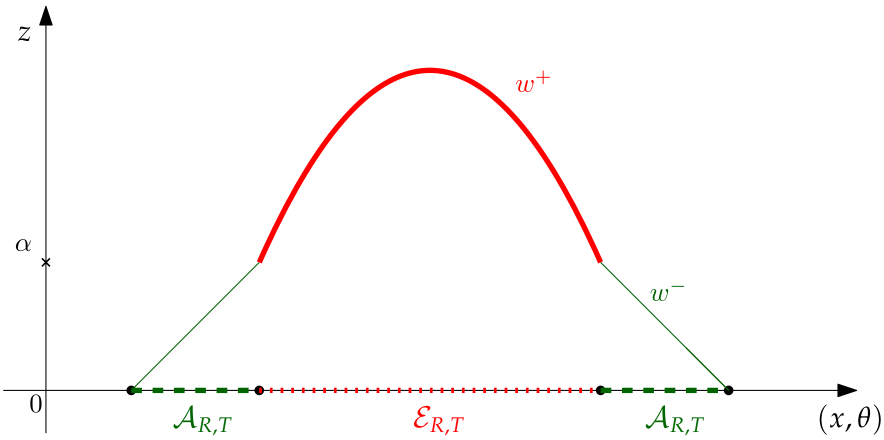



Figure 2.4: Schematic plot of the flying saucer like sub-solution $w$ that is slid along the trajectory. The bolded green line denotes the set $\mathcal{A}_{R,T}$, the red dotted one the set $\mathcal{E}_{R,T}$.

Now that we have defined $w^\pm$, we obtain a sub-solution $w$ of (2.5) on $\mathbb{R}^+ \times \mathbb{R} \times \Theta$ defined by

$$w = w^+ \mathbb{1}_{\mathcal{E}_{R,T}} + w^- \mathbb{1}_{\mathcal{A}_{R,T}}.$$

By construction, $w$ is Lipschitz continuous. However, to make sure that it is indeed a sub-solution at the boundary $\partial\mathcal{E}_{R,T}$ we need to check properly that the normal derivatives along this boundary are well ordered:

$$|\partial_n w^+| \ge |\partial_n w^-| \qquad \text{on } \partial\mathcal{E}_{R,T},$$

where $\partial_n$ is the (outward) normal derivative to the boundary of $\mathcal{E}_{R,T}$. This is the main technical issue at hand in the proof.

We conclude this discussion with numerical simulations in Figure 2.5 showing this acceleration.

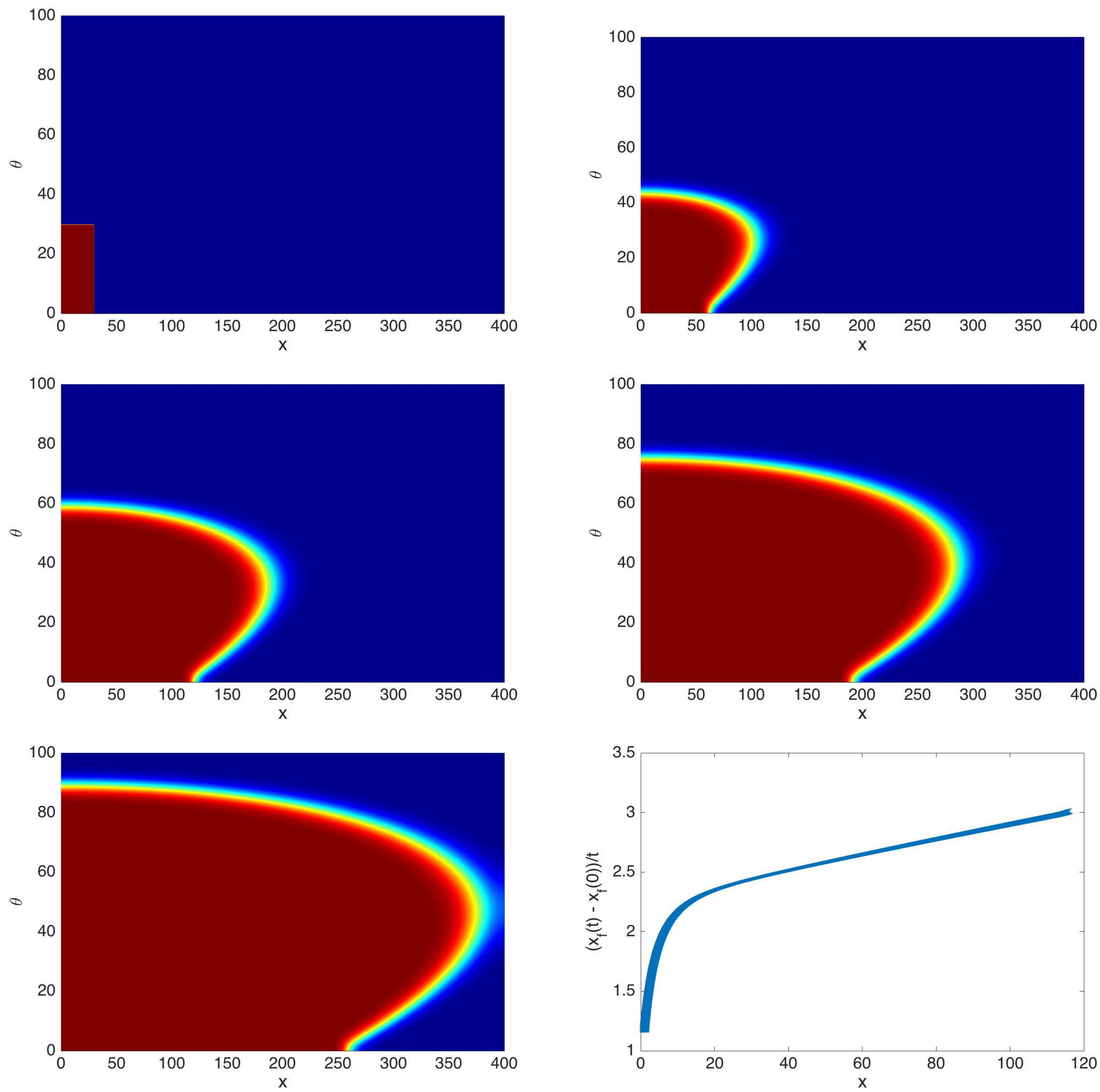



Figure 2.5: Numerical simulations of the Cauchy problem of equation (2.5) at a fixed time, in the phase space $\mathbb{R} \times \Theta$. From top left to bottom right, the first five plots are: *the initial data*, $t = 30$, $t = 60$, $t = 90$, $t = 116$ (when the simulation terminated). One can track the accelerated behavior, for example on the space axis. The invasion at the back in the $\theta$-direction is expected to be linear in time. This pattern is very similar as for the monostable cane toads equation, see also [B6, B21, 29]. The final plot is the distance the front has travelled divided by time – here, we define the location of the front $x_f(t) := \max\{x \in \mathbb{R} : \exists \theta, u(t, x, \theta) = 1/2\}$. The positive slope shows that the front accelerates; were the front moving linearly, the curve would be asymptotically constant.

Since our interest in this part of the memoir is on nonlocal models, we conclude by discussing a nonlocal version of (2.5):

$$\begin{cases} u_t = \theta u_{xx} + u_{\theta\theta} + u(\rho - \alpha)(1-\rho), & (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta, \\ \rho(t,x) := \int_{\underline{\theta}}^{\infty} u(t,x,\theta)\, d\theta, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u_\theta(t,x,\underline{\theta}) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x,\theta) = u_0(x,\theta) & (x,\theta) \in \mathbb{R} \times \Theta. \end{cases} \tag{2.6}$$

Of course, the propagation can be no faster than $O(t^{3/2})$. At this time, we conjecture that the model (2.6) exhibits acceleration and we provide some numerics in Figure 2.6 to support this conjecture. We see the distinctive "up-and-over" behaviour that leads to acceleration for the non-local, monostable model, see [29].

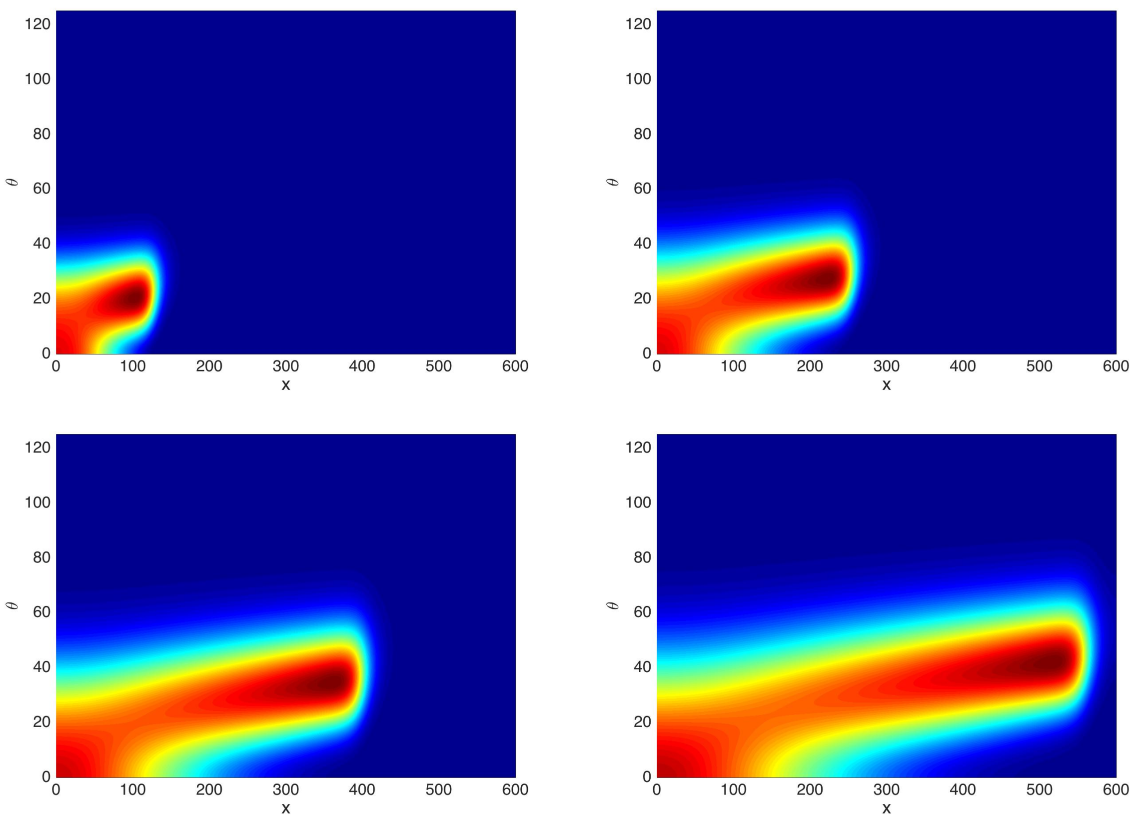



Figure 2.6: Numerical simulations of the Cauchy problem of equation (2.6) at a fixed time, in the phase space $\mathbb{R} \times \Theta$. From top left to bottom right, the first five plots are: $t = 45$, $t = 90$, $t = 135$, $t = 180$. One can track the accelerated behaviour, for example on the space axis.

## 2.3 A mortality trade-off and its quantitative influence

It has been demonstrated by biologists that increased dispersal is often associated with reduced investment in reproduction, for example in populations of the peckled wood butterfly, *Pararge aegeria* [119]. Some physiological description of this trade-off between dispersal and fecundity has been reported in [148]. There, two morphs of the cricket *Gryllus rubens* (Orthoptera, Gryllidae) are studied: a fully-winged (flight-capable morph) and a short-winged morph (that cannot fly). It turns out that the short-winged morph is substantially more fecund than the fully-winged one. This widespread occurrence of dispersal polymorphisms among insects is consistent with the fact that fitness costs are associated with flight capability. It is now well documented that for both males and females (for example for the planthopper *Prokelisia dolus*) there is a strong trade-off between flight capability and reproduction [132]. See also [106] where the physiological differences between the male and female of two morphs of crickets that may lead to such a trade-off are discussed. We refer to [32] for an extensive review on the different cost types that occur during dispersal in a wide array of organisms, ranging from micro-organisms to plants and invertebrates to vertebrates.

In view of these biological issues, we are interested in the influence of a mortality trade-off on the rate of spreading of a structured cane toads population. Namely, our goal is to estimate of the effect of this penalisation; that is, depending on the strength of the trade-off, does the population go extinct or still propagate, and in the latter case, what is the effect on the acceleration seen in [29, B21]?

To answer these questions, we focus on a cane toads equation with a mortality trade-off.

$$\begin{cases} u_t = \theta u_{xx} + u_{\theta\theta} + u\left(1 - m(\theta) - \rho\right), & (t,x,\theta) \in \mathbb{R}^+ \times \mathbb{R} \times \Theta, \\ u_\theta(t,x,\underline{\theta}) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \end{cases} \tag{2.7}$$

We assume that $m$ depends only on $\theta$, that $m(\underline{\theta}) = 0$ and that $m \in \mathcal{C}^2(\overline{\Theta})$ increases to $+\infty$ as $\theta$ tends to $\infty$. Moreover, we suppose that $\lim_{\theta\to\infty} m(\theta)/\theta$ exists and is an element of $\mathbb{R}^+ \cup \{+\infty\}$ and that if $m(\theta)/\theta$ tends to zero as $\theta$ tends to $+\infty$, then $m'' \in L^\infty\left(\overline{\Theta}\right)$ and there exists $\theta_d > \underline{\theta}$ such that $m(\theta)/\theta$ is decreasing for all $\theta \geq \theta_d$. An important class of examples of such $m$'s are $m(\theta) \propto \theta^p - \underline{\theta}^p$ for $p > 0$.

### Presentation of the results

In order to expect propagation at any speed, the average growth rate $\gamma_\infty$ in time should be positive. This is necessary to expect any positive steady state at the back of a traveling front. In other words, letting $Q$ and $\gamma_\infty$ be the principal eigenfunction and eigenvalue of the linearised equation

$$\begin{cases} Q'' + (1-m)\,Q = \gamma_\infty Q, & \text{on } \Theta, \\ Q'\left(\underline{\theta}\right) = 0, \quad Q > 0, \end{cases} \tag{2.8}$$

we expect propagation *only* in the case that $\gamma_\infty > 0$. Notice that, since $m \geq 0$ and $m \not\equiv 0$, $\gamma_\infty < 1$. The sign of $\gamma_\infty$ does not depend strongly on the growth of $m$ at $\infty$, but on having sufficiently many traits with sufficiently large growth rates[8]. Our expectation above is confirmed by the

[8] Indeed, consider the following simple example. When $m(\theta) = \sigma^2(\theta - \underline{\theta})^2$ for any $\sigma > 0$, one can find the principal eigenfunction and eigenvalue explicitly: $Q(\theta) = \exp\{-\sigma(\theta - \underline{\theta})^2/4\}$ and $\gamma_\infty = 1 - \sigma$. Though, for any value of $\sigma > 0$, $m$ has quadratic growth, $\gamma_\infty$ can be positive or negative depending on $\sigma$.

following:

**Theorem 2.4** (B., Chan, Henderson, Kim [B9]). *Suppose that $\gamma_\infty \le 0$ and $Supp(u_0)$ is compact. Then*

$$\lim_{t\to\infty} \sup_{(x,\theta)\in\mathbb{R}\times\Theta} u(t,x,\theta) = 0.$$

In the case of non-extinction, i.e. when $\gamma_\infty > 0$, one may expect propagation of the initial population. A first attempt in this direction is to look for travelling waves. We obtain the following result.

**Theorem 2.5** (B., Chan, Henderson, Kim [B9]). *Suppose that $\gamma_\infty > 0$ and that $\lim_{\theta\to\infty} m(\theta)/\theta$ is positive. Then (2.7) admits a travelling wave solution $(c^*, \mu)$, with $c^* := \inf_{\lambda>0} c_\lambda$. In other words $u(t,x,\theta) = \mu(x - c^* t, \theta)$ solves (2.7), with $c^* > 0$, and*

$$\liminf_{\xi\to-\infty} \mu(\xi, \underline{\theta}) > 0 \qquad \text{and} \qquad \lim_{\xi\to\infty} \sup_{\theta\in\Theta} \mu(\xi,\theta) = 0.$$

For any given $\lambda > 0$, we apply the Krein-Rutman theorem to solve the following spectral problem and get $c_\lambda$,

$$\begin{cases} Q_\lambda'' + \left[\lambda^2\theta - \lambda c_\lambda + (1 - m(\theta))\right] Q_\lambda = 0, & \theta \in \Theta, \\ Q_\lambda'(\underline{\theta}) = 0, \qquad Q_\lambda > 0 \end{cases}$$

when $m$ increases sufficiently quickly.

Our main interest is in a spreading result for the Cauchy problem (2.7). We thus ask whether the travelling wave constructed in Theorem 2.5 is stable. This is answered by the following theorem.

**Theorem 2.6** (B., Chan, Henderson, Kim [B9]). *Suppose the conditions of Theorem 2.5 hold. Suppose that $u$ solves (2.7) with a compactly supported initial data. Then there exists $\underline{u} > 0$ such that for every $\varepsilon > 0$, we have*

$$\liminf_{t\to\infty} \inf_{|x|\le(c^*-\varepsilon)t} u(t,x,\underline{\theta}) \ge \underline{u}, \qquad \text{and} \qquad \lim_{t\to\infty} \sup_{x\ge(c^*+\varepsilon)t} \sup_{\theta\in\Theta} u(t,x,\theta) = 0.$$

This type of result is standard going back to [13, 14] in the local Fisher-KPP setting. Since the dynamics of the solution are so complicated, it would be interesting to obtain more precise estimates on the propagation speed. We expect a logarithmic delay *a la* Bramson, see [B20] and Chapter 1 for the delay in the cane toads equation.

The condition for the existence of travelling waves may be roughly re-written as $m$ is at least linear. On the other hand, when $m$ is sub-linear and $\gamma_\infty > 0$, we still expect propagation. Since the spectral problem (2.3) is not solvable, we may expect an acceleration phenomenon exactly as for the cane toads equation [B6, B21]. For $\theta \in \overline{\Theta}$, define $\Phi(\theta) = \int_{\underline{\theta}}^{\theta} \sqrt{m(s)}ds$. Define $\eta(t) \in \overline{\Theta}$ to be the unique solution of $\Phi(\eta(t)) = t$. Our main result goes as follows.

**Theorem 2.7** (B., Chan, Henderson, Kim [B9]). *Suppose that $\gamma_\infty > 0$, $m(\theta)/\theta$ tends to zero as $\theta$ tends to $+\infty$. Then there exist positive constants $\underline{u}$, $\underline{c}$, and $\overline{c}$ such that*

$$\liminf_{t\to\infty} \inf_{|x|\le\underline{c}\eta(t)^{3/2}} u(t,x,\underline{\theta}) \ge \underline{u} \qquad \text{and} \qquad \lim_{t\to\infty} \sup_{x\ge\overline{c}\eta(t)^{3/2}} \sup_{\theta\in\Theta} u(t,x,\theta) = 0.$$

To illustrate the result, we discuss a concrete choice of $m$ : if $m(\theta) \sim \theta^p$ for $p \in (0,1)$, one can check that $\eta(t) \sim t^{2/(2+p)}$, hence the front is at $\eta^{3/2}(t) \sim t^{3/(2+p)}$. We point out that this is an interpolation between the cases $p = 1$, when no acceleration occurs (see Theorem 2.6), and $p = 0$, when acceleration is of order $t^{3/2}$ (se Theorem 2.2). This result might be surprising at first glance, since populations with very high traits have a negative growth rate. It turns out that the spatial sorting still gives a strong propagation force to population with high traits at the edge of the invasion.

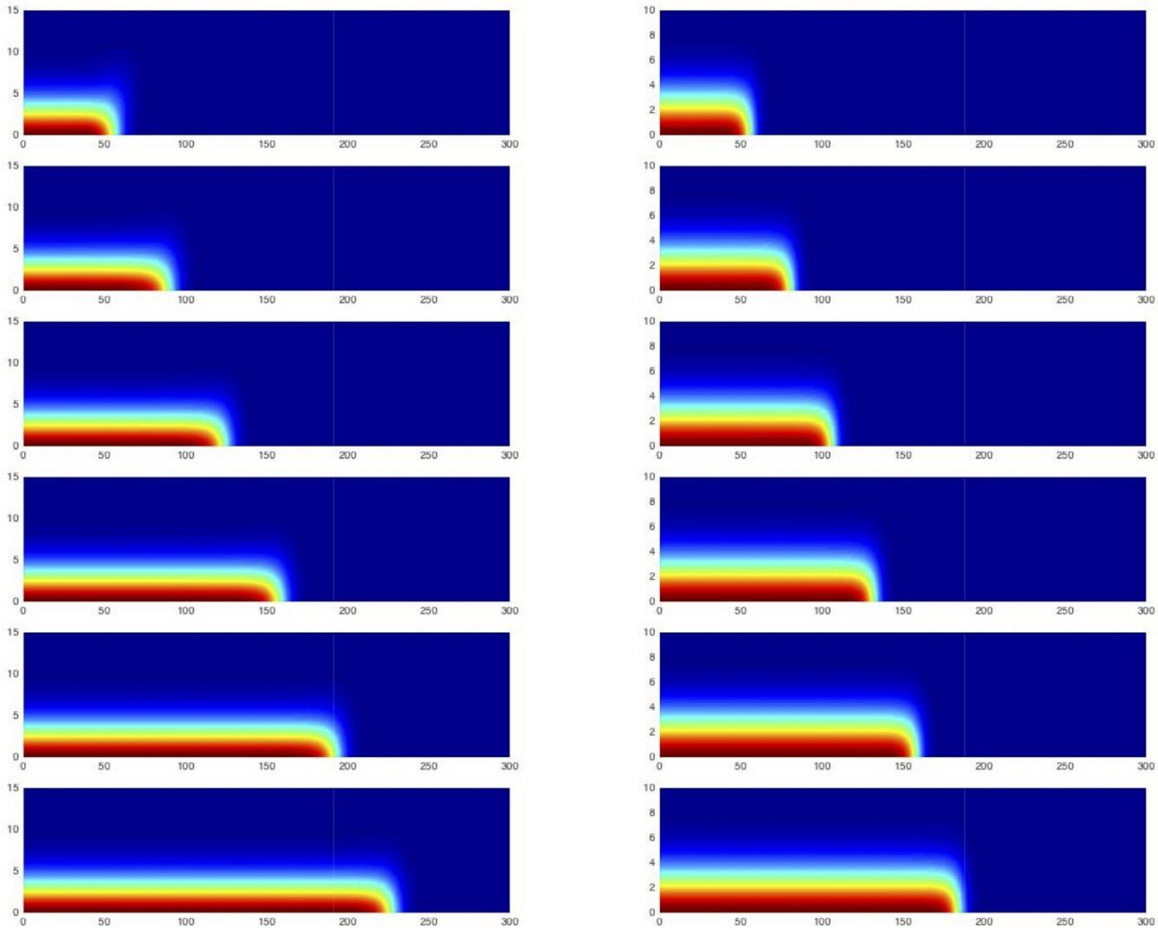


Figure 2.7: Numerical simulations of the Cauchy problem of equation (2.7) at a fixed time, in the phase space $\mathbb{R} \times \Theta$ at times $t = 10$, $t = 20$, $t = 30$, $t = 40$, $t = 50$, $t = 60$ with the choice $\underline{\theta} = .01$. Left column: $p = 1$. Right column: $p = 4/3$. Both exhibit propagation at a linear rate.

### A word about the proofs

The construction of a travelling wave solution with minimal speed of Theorem 2.5 is done by building a solution to an approximate problem on a finite “slab” by a degree theory fixed point

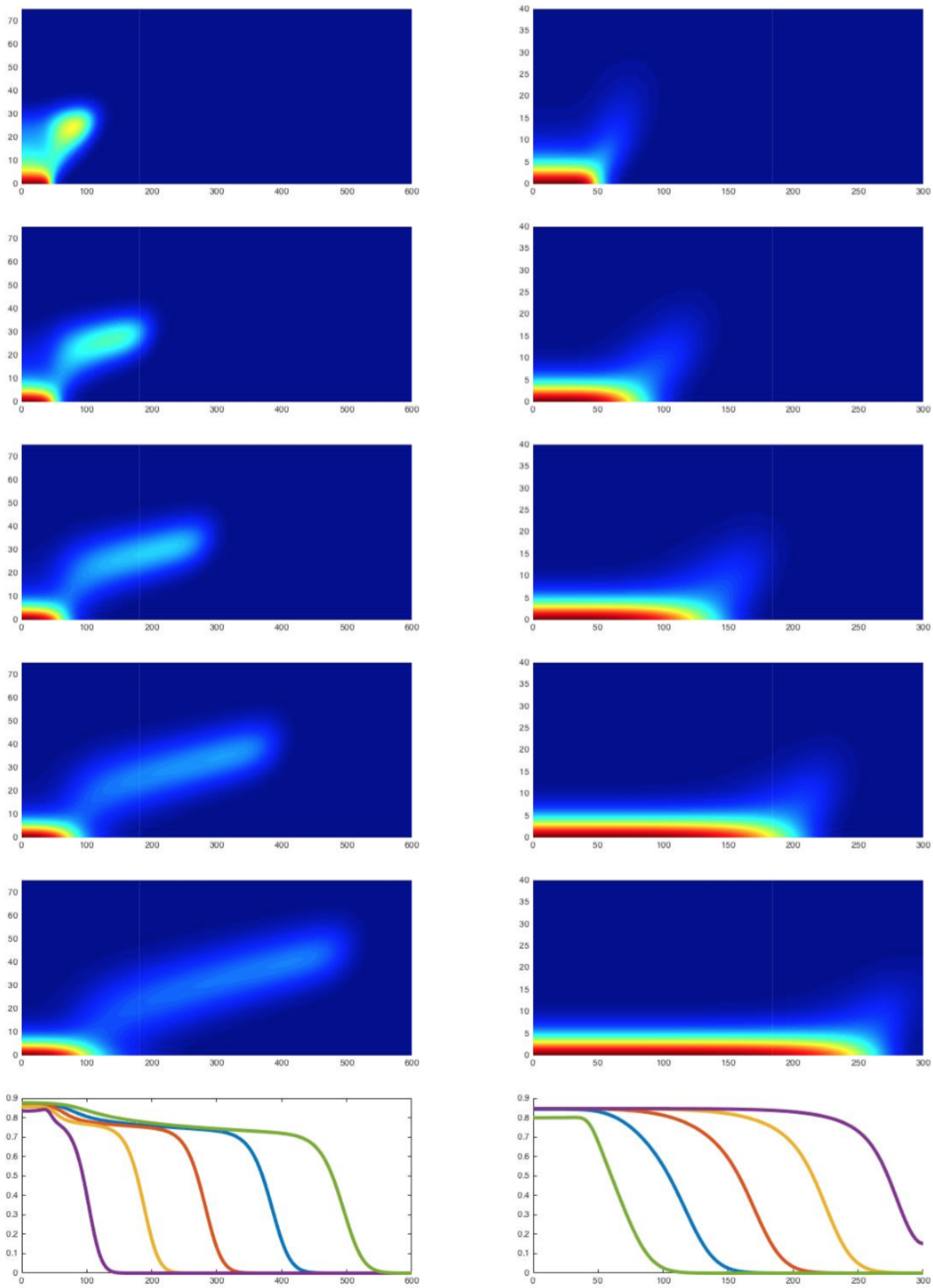


Figure 2.8: Numerical simulations of the Cauchy problem of equation (2.7) at a fixed time, in the phase space $\mathbb{R} \times \Theta$ at times (from top to bottom) $t = 10$, $t = 20$, $t = 30$, $t = 40$, $t = 50$. Left column: $p = 1/3$. Right column: $p = 2/3$. Last line: evolution of $\rho$ at times $t = 10$, $t = 20$, $t = 30$, $t = 40$, $t = 50$. Both exhibit propagation at a super-linear rate. The transient dynamics driving the acceleration are seen in the “head” – the light diagonal line moving and up and to the right of the front.

argument. This construction appears to be a non-trivial extension of the one for the cane toads equation with bounded traits [B4]. In particular, our proof differs from the usual procedure in [B4] because we have both unbounded diffusivity and unbounded growth rates. In this direction, we also point out connections to [5, 7, 26], where travelling waves for structured models were constructed.

The proof of the spreading result in Theorem 2.6 proceeds as follows. We directly construct a super-solution of $u$ using (2.3), which provides the upper bound. The lower bound follows by building a solution to a related problem on a moving ball to using the intermediate steps of the construction of the travelling wave and applying a local-in-time Harnack inequality to compare this to $u$ (see the spirit of Section 1.2 and Proposition 1.4).

Let us now comment on the difficulties of the proof of Theorem 2.7. In order to obtain the lower bound, we build a sub-solution on a moving ball using the principal Dirichlet eigenvalue. There are two main difficulties here. First, the problem is nonlocal and thus does not have a comparison principle. To overcome this, we relate it to a local problem by estimating the nonlocal term $\rho$ using two ingredients: when $\theta$ is small, we may use a local-in-time Harnack inequality and when $\theta$ is large, we may obtain a priori estimates on the tails in trait of the solution $u$. Second, in contrast to [B21] presented in the previous section, the path of this moving "bump" sub-solution cannot be found explicitly since the ODE system for the optimal path given by the Euler-Lagrange equation is not explicitly solvable. Instead, we must optimize over rectangular paths as shown in Figure 2.9. First, we move mass upwards. Of course, this mass reaches a place where the death rate is highly negative due to the strength of the trade-off. This movement is justified by the second part of our trajectory, where we are able to move forward in space with a very high velocity, since the space diffusivity is very high in this zone. However, due to the strong trade-off, the mass at the end of this second step is extremely small. It is thus mandatory to move down to small traits again, to reach a zone where it is possible for the population to grow. Finally, in this region, we grow the population to order one in the last step.

The strategy of the proof of the upper bound is related in spirit to [B21] but is technically completely different. In order to avoid complications with the nonlocal term, we notice that solutions to the linearised equation

$$\overline{u}_t = \theta \overline{u}_{xx} + \overline{u}_{\theta\theta} + \overline{u}(1 - m(\theta)) \tag{2.9}$$

are super-solutions to $u$. As such we seek bounds on $\overline{u}$. As such, we proceed by considering $\mathbb{R} \times \Theta$ as a two dimensional Riemannian manifold with boundary with the appropriate metric $g$. After removing an integrating factor, we may view the linearised operator (2.9) as the Laplace-Beltrami operator $\Delta_g$ with a potential $-m(\theta)$. We may then appeal to the methods of [137] to obtain bounds on the fundamental solution of (2.9). After some careful modification of this fundamental solution and after reinserting the integrating factor, this provides a super-solution to $u$. We note that these heat kernel estimates do not provide the propagation result immediately. Indeed, the results in [137] give heat kernel estimates in terms of a Lagrangian, and this Lagrangian is itself difficult to estimate precisely. The use of the estimates coming from [137] is not common in works investigating propagation, and we believe that this is an important addition to the toolbox for these types of problems.

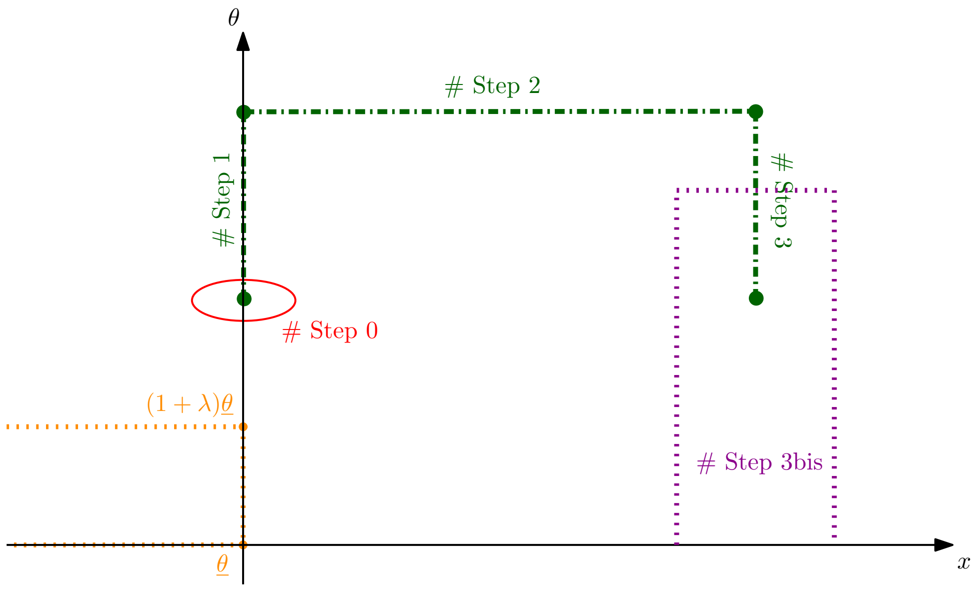



Figure 2.9: The trajectory used in the proof of Theorem 2.7 and the different steps.

## 2.4 Perspectives and related topics

1. *Sharp spreading rate in the bistable model.*

   In the paper with Henderson [B18], we have proved, see Theorem 2.7, that the local bistable cane toads equation exhibits acceleration. We were not able to derive a sharp constant because Lagrangian trajectories are less immediate and this question remains. Concerning the nonlocal bistable cane toads model (2.6), the structure of the nonlinearity being a bit similar to the standard cane toads (a product of $u$ and a function of $\rho$), the approach by Calvez in [46] could be tried.

2. *Bramson type correction in accelerated regimes.*

   We know the rate of spreading in the cane toads equation. It is interesting to wonder which kind of lower order terms may follow. In the local equation, a conjecture is that after the leading term $\frac{4}{3}t^{\frac{3}{2}}$, will come something of the form $\sqrt{t}\ln(t)$. Showing such a result could be done using the BBM probabilistic strategy.

3. *Invasion profiles.*

   A particular feature of accelerated motions (see for example Figure 2.7) is that usually solutions flatten in space during the evolution. This is drastically different from the finite speed propagation case for which travelling waves exist. An interesting issue would be to be more precise on the invasion profiles for cane toads evolutions (see also Chapter 3, Chapter 4 for similar issues on related but different models). This could require to study more precisely thing in self-similar variables. An additional difficulty here is that the behaviour in the direction $\theta$ is very different from the one in direction $x$.

4. *Toads with sexual reproduction.*

A recent topic in the mathematical biology community is to take into account sexual reproduction in models, and then study the influence on invasions. First formal arguments for toads were given in [44] and [74]. Moreover, rigorous mathematical analysis of the infinitesimal operator have started, see *e.g.* [45, 157]. I believe that interesting mathematical questions will arise from there.

5. *Nonlocal dispersion operators.*

   In the models above, mutations are modelled using a diffusion operator. It would be more relevant biologically to take into account large jumps, with kernel operators. This would change the analysis and require to adapt tools.

6. *Invasions with nonlinear gradients.*

   A related topic is to study invasions with nonlinear gradients. This means looking at models of the form
   $$u_t = u_{xx} + u_{\theta\theta} + u(a(x,\theta) - \rho),$$
   with $a(x,\theta) := 1 - A(\theta - \Theta(x))^2$. The fittest trait is such heterogeneous in space, following the fonction $\Theta$. The case of a linear function $\Theta$ has been discussed in [7]. Still with a linear function $\Theta$, the case of superlinear spreading with heavy tails is done in [158]. We believe that some ideas from this chapter (sliding on trajectories, use of Riemannian metrics) could be of help to describe propagation in these cases. Some early simulations are in Figure 2.10.

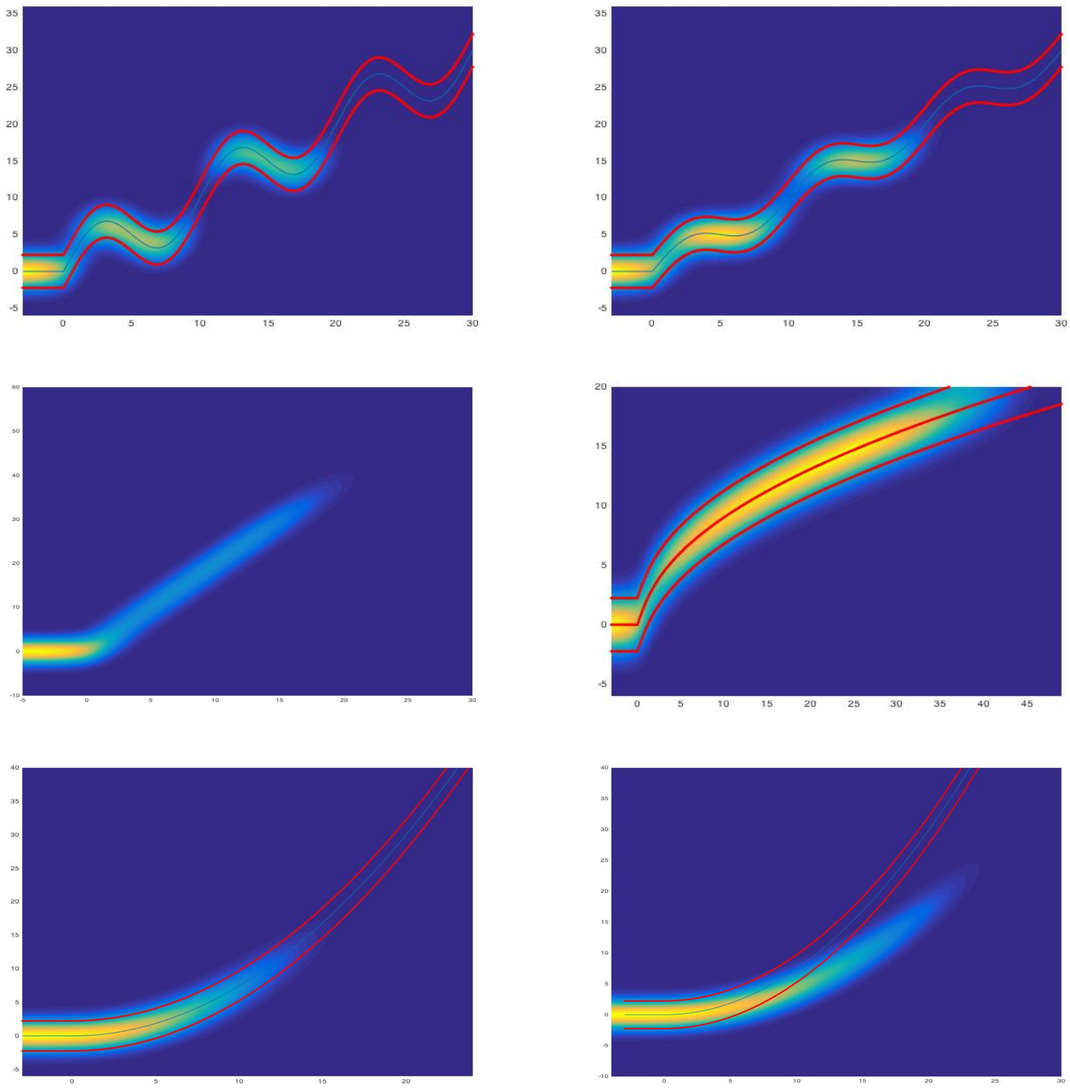


Figure 2.10: Various cases of non linear clines. Observe that when the gradient of the cline is small enough regularly, propagation seems to occur. If it gets too large, the populations get off-road.

# Chapter 3

# Sharp spreading in integro-differential equations

**Contents**


**3.1 More precise rates of expansion when $f$ is of KPP type** . . . . . . . . . . 42
**3.2 The case of a weak Allee effect** . . . . . . . . . . . . . . . . . . . . . . . . . . . 45
**3.3 The lower bound with ignition nonlinearities** . . . . . . . . . . . . . . . . 52
**3.4 Perspectives** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54


In this third chapter, I will mainly discuss acceleration in integro-differential models of the form

$$u_t = \mathcal{D}[u] + f(u), \tag{3.1}$$

where $\mathcal{D}$ is typically a jump operator with measure $J$:

$$\mathcal{D}[u](t,x) := P.V.\left(\int_{\mathbb{R}} [u(t,y) - u(t,x)]J(x-y)\,dy\right)$$

In this context, the unknown function $u$ represents a density of individuals at time $t$ and at position $x$. Here, $J$ could have a singularity at the origin as in fractional laplacians, or could be very regular with an emphasis put on the decay of its tails, as in kernel type operators. In this chapter, the nonlinearity $f$ is of monostable type, but potentially with degeneracies. Indeed, we will consider standard Fisher-KPP, together with Allee effects and ignition type nonlinearities. Precise assumptions will be made later on depending on the different contexts.

This kind of models arise naturally in population dynamics to model systems with non–local effects [90, 144]. One of the most interesting features of this model, compared to the classical Fisher-KPP equation, is that it allows for long range dispersal events. The existence of these events depends critically on the tail of the kernel $J$.

When the kernel is thin–tailed, solutions to (3.1) generally exhibit the same behaviour as solutions to the Fisher–KPP equation in the sense that travelling wave solutions exist and solutions of the Cauchy problem typically propagate at a constant speed. There is an extensive literature about this [171, 185, 50, 65, 64, 140, 187]. See also the work and references contained in the habilitation thesis of Coville [63].

On the other hand, super-linear in time propagation phenomena can occur in ecology, as evidenced and discussed in Chapter 2. Another classical example is Reid's paradox of rapid plant migration [58, 57] that is usually resolved using jump processes with fat–tailed kernels. Note that the mechanism at stage here is different from the one of cane toads equation. In general, proving that propagation is super-linear in nonlocal dispersion models like (3.1) is not extremely difficult. We can readily cite [144, 187, 98] for acceleration when the dispersion kernel $J$ is smooth with heavy tails, or [40, 41] where a group around Cabré and Roquejoffre studied in details the fractional Fisher-KPP equation concluding to an exponential propagation behaviour. However, getting sharp rates of expansion is much harder since it requires to provide very precise estimates. Even though the maximum and comparison principles are available, this is still quantitatively involved in full generality due to the non-local nature of the model.

We will make mathematically precise what we call a *thin–tailed* or a *fat–tailed* kernel in the following sections.

This chapter is devoted to presenting several results I obtained in [B10, B11, B17]. My contributions are around finding sharp rates of acceleration in three configurations. The first one deals with the case of a convolution operator with a rather general fat-tailed kernel and a KPP-type nonlinearity. The second one is about getting a sharp rate of expansion when the reaction takes into account an Allee effect, and the dispersal has a rather general jump measure. Lastly, I will describe the case of an ignition nonlinearity.

## 3.1 More precise rates of expansion when $f$ is of KPP type

We focus on the asymptotic behaviour of solutions to the following integro–differential equation

$$\begin{cases} u_t = J \star u - u + u(1-u), & \text{in } (0,+\infty) \times \mathbb{R}, \\ u(t=0,\cdot) = u_0. \end{cases} \tag{3.2}$$

where

$$(J \star u)(t,x) := \int_{\mathbb{R}} J(x-y)u(t,y)\,dy.$$

The kernel $J$ is a symmetric probability density, that is, for all $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} J(x)\,dx = 1, \qquad J(x) = J(|x|) \qquad \text{and} \qquad J(x) > 0.$$

The decay of $J$ is encoded by the function $k := -\ln(J)$, for which we further assume the following three properties:

1. **Monotonicity and asymptotic convexity of $J$.** The function $k \in \mathcal{C}^2(\mathbb{R})$ is strictly increasing on $(0,+\infty)$ and asymptotically concave, that is, there exists $x_{\text{conc}} > 0$ such that

$$\begin{cases} k(x) > k(y), & \text{if } x > y \geq 0 \qquad \text{and} \\ k''(x) \leq 0 & \text{if } x \geq x_{\text{conc}}. \end{cases}$$

Without loss of generality, we suppose that $k(0) = 0$, or $J(0) = 1$, since otherwise we may re-scale the equation. This implies that $J(\mathbb{R}) = (0,1]$. Moreover, $J$ is invertible on $\mathbb{R}^+$, this inverse from $(0,1]$ to $\mathbb{R}^+$ is what we denote $J^{-1}$ in the sequel. Similarly, $k$ is invertible on $\mathbb{R}^+$, this inverse from $\mathbb{R}^+$ to $\mathbb{R}^+$ is what we denote $k^{-1}$ in the sequel.

2. **Lower bound on the tail of $J$.** The kernel $J$ decays slower than any exponential in the sense that
$$\limsup_{x\to\infty} \frac{x\,k'(x)}{k(x)} < 1.$$
Roughly speaking, this implies that $k$ grows sub-linearly and that $J'(x) = o_{x\to\infty}(J(x))$.

3. **Upper bound on the tail of $J$.** The tail of $J$ is thinner than $|x|^{-1}$, the threshold for integrability, in the sense that
$$\mu := \liminf_{x\to\infty} x\,k'(x) > 1.$$

The main examples of kernels $J$ that satisfy all this are either sub–exponential kernels where $f(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha < 1$ or polynomial kernels where $f(x) = \alpha\ln(1+|x|^2)/2$ with $\alpha > 0$.

Garnier [98] proved that the acceleration propagation of the solution of (3.2) can be measured by tracking the level sets $E_\lambda(t) := \{x \in \mathbb{R} : u(t,x) = \lambda\}$ of the solution $u$, where $\lambda \in (0,1)$. Under the fat–tailed kernel hypothesis, these level sets move super-linearly in time. More precisely, Garnier [98] proved that there exists a constant $\rho > 1$ such that for any $\lambda \in (0,1)$ and any $\varepsilon > 0$, any element $x_\lambda$ of the level set $E_\lambda$ satisfies for all $\varepsilon > 0$ and $t$ large enough:
$$J^{-1}\left(e^{-(1-\varepsilon)t}\right) \leq |x_\lambda(t)| \leq J^{-1}\left(e^{-\rho t}\right),$$
or equivalently,
$$k^{-1}\left((1-\varepsilon)t\right) \leq |x_\lambda(t)| \leq k^{-1}\left(\rho t\right). \quad (3.3)$$

With Garnier, Henderson and Patout, we were interested in refining these estimates. Our starting point is the following double inequality that we obtain by means of sub- and super-solutions.

**Proposition 3.1** (B., Garnier, Henderson, Patout [B17]). *Assume that there exists two positive constants $\underline{C}$ and $\overline{C}$ such that $\underline{C} < 1 < \overline{C}$ and*
$$\underline{C}J \leq u_0 \leq \overline{C}J, \quad (3.4)$$
*and that $u_0$ is symmetric. Let $u$ be a solution of eq.* (3.2) *with initial data $u_0$. Then, there exists a bounded positive function $\theta$, which only depends on $J$, such that $\theta(s) \to 0$ as $s \to +\infty$, and positive constants $\underline{C} < 1 < \overline{C}$, such that, for all $t > 0$ and $x \in \mathbb{R}$,*
$$\frac{\underline{C}\exp\left(-\int_0^t \theta(s)ds\right)}{1+e^{k(x)-t}} \leq u(t,x) \leq \frac{2\overline{C}\exp\left(\int_0^t \theta(s)ds\right)}{1+e^{k(x)-t}}.$$

We show that the left hand side and the right hand side of Proposition 3.1 above are respectively a sub- and a super–solution of eq. (3.2). Observe these are both constructed from solutions to the family of decoupled ODEs:
$$\begin{cases} \dfrac{d\varphi(t,x)}{dt} = \varphi(1-\varphi), \\ \varphi(0,\cdot) = \frac{J}{1+J} \leq J, \end{cases}$$

parametrized by $x \in \mathbb{R}$. The construction of the function $\theta$ appears when showing that one can achieve

$$-\theta\varphi \leq J \star \varphi - \varphi \leq \theta\varphi.$$

The latter step is the most involved and requires some computations, presented in [B17]. As already noticed by Garnier in [98], in this kind of models, dispersion and growth are kind of decoupled. Once the initial data is prepared after a short time to look like $J$, the propagation is only triggered by growth.

The first result that we are able to get is an asymptotic expansion of $x_\lambda$. We can state it in two ways.

The first way is an echo again to the last decade of work on the use of Hamilton-Jacobi equations to describe invasion dynamics. This is a generalisation of [150]. Rescale time and space as follows: $t \mapsto \frac{t}{\varepsilon}$ and $x \mapsto \psi_\varepsilon(x) := \text{sign}\,(x)k^{-1}\left(\frac{k(x)}{\varepsilon}\right)$ and define the solution $u_\varepsilon$ in the new variables: $u_\varepsilon(t,x) = u(\frac{t}{\varepsilon}, \psi_\varepsilon(x))$ where $u$ solves (3.2) with initial condition $u^0$ satisfying (3.4). In the large scale limit $\varepsilon \to 0$ with our change of variables, we expect this propagation to be transformed into dynamics of an interface moving with time. To capture this phenomenon, we use the logarithmic Hopf–Cole transform [85, 95] as follows:

$$\varphi_\varepsilon := -\varepsilon \ln u_\varepsilon.$$

**Theorem 3.2** (B., Garnier, Henderson, Patout [B17])**.** *Let $u_\varepsilon$ be the solution of* (3.2) *with initial condition satisfying* (3.4)*. As $\varepsilon \to 0$, the sequence $\varphi_\varepsilon$ converges locally uniformly on $(0,\infty) \times \mathbb{R}$ to*

$$\varphi(t,x) := \max\{f(x) - t, 0\}.$$

*Moreover,*

*(a) uniformly on compact subsets of $\{\varphi > 0\}$,*

$$\lim_{\varepsilon \to 0} u_\varepsilon = 0;$$

*(b) for every compact subset $K \subset Int\,(\{\varphi(t,x) = 0\})$,*

$$\lim_{\varepsilon \to 0} u_\varepsilon(t,x) = 1,$$

*where the limit is uniform in $K$.*

Since $k$ is a continuous and increasing function of $|x|$, the boundary of $\{\varphi(t,x) = 0\}$ is given by $|x| = k^{-1}(t)$. Hence, as $\varepsilon \to 0$, $u_\varepsilon \sim 1$ if and only if $|x| < k^{-1}(t)$. As such, Theorem 3.3 implies that the location of the front of $u$ is $\sim f^{-1}(t)$. When $k$ is sub–exponential of the form $k(x) = (1 + |x|^2)^{\alpha/2}$ with $\alpha \in (0,1)$, we see that the front is located at $\sim t^{1/\alpha}$. In the thin-tailed limit $\alpha \to 1$ see recover constant speed propagation. On the other hand, when $J$ is a polynomial function such that $k(x) = (1+\alpha)\ln(1 + |x|^2)/2$, with $\alpha > 0$, we see that the front is located at $\sim e^{t/(1+\alpha)}$.

The second way is to recast the double inequality of Proposition 3.1 and combine to [98] to get

**Theorem 3.3** (B., Garnier, Henderson, Patout [B17]). *Let $u$ be the solution of* (4.1) *with initial condition satisfying* (3.4). *Then,*

$$k(|x_\lambda(t)|) = t + o(t),$$

*where the $o(t)$ may depend on the level set $\lambda$.*

Actually, improved estimates can be obtained by having better knowledge on the function $\theta$. Tracking line by line the computations in [B17], one could, in practice, compute $\theta$ and determine for which kernels $J$ the function $\theta$ is integrable on $\mathbb{R}^+$. When this is the case, the estimate given by Proposition 3.1 is more precise since $\int_0^t \theta(s)ds$ could be replaced by a constant on both sides of the equation. One could then quantify and compare more precisely the expansion of the $\lambda$-level lines of the solution $n$ for various values of $\lambda$. The threshold for integrability of $\theta$ appears to be kernels like $\exp\{-\sqrt{|x|}\}$: those which are fatter yield an integrable $\theta$.

**Proposition 3.4** (B., Garnier, Henderson, Patout, [B17]). *Assume that there exists two positive constants $\underline{C}$ and $\overline{C}$ such that $\underline{C} < 1 < \overline{C}$ and*

$$\underline{C}J \leq u_0 \leq \overline{C}J,$$

*and that $u_0$ is symmetric. Assume that $J$ is such that the function $\theta$ is integrable. Then,*

$$k\left(|x_\lambda(t)|\right) = t + O(1).$$

All our results can be generalised to a nonlocal nonlinearity of the form $n(r - \int_{\mathbb{R}} n(x)dx)$, exactly as in [150]. Moreover, we restrict our focus to the effects of the tails of $J$ on the rate of propagation. As a consequence, we do not include potential singularities at the origin, which is the case for a fractional Laplacian operator, for example. We expect however that our results also hold for these cases.

We end this section by presenting some numerical evidence of the acceleration phenomena on various examples in Figure 3.1.

## 3.2 The case of a weak Allee effect

In this section, we elaborate on the case where the nonlinearity $f$ takes into account a possible Allee effect:

$$f(u) = u^\beta(1-u).$$

The parameter $\beta$ above describes the possibility of a weak Allee effect that the population overcomes. A biological description and discussion about the origin and relevance of such an effect may be found in a book by Courchamp *et al.* [59] but also in [10, 75, 25]. In crude terms, the Allee effect means that a too small population will not have enough strength to survive and expand. This effect is said to be weak whenever the growth rate of a very small population is eventually extremely small but still positive as opposed to a strong Allee effect leading to negative growth rates for small populations. In the sequel, and without further notice, we take $\beta > 1$ (again, yielding small growth rates for small densities). Moreover, we could generalise our results to nonlinearities with the same behaviour around zero and

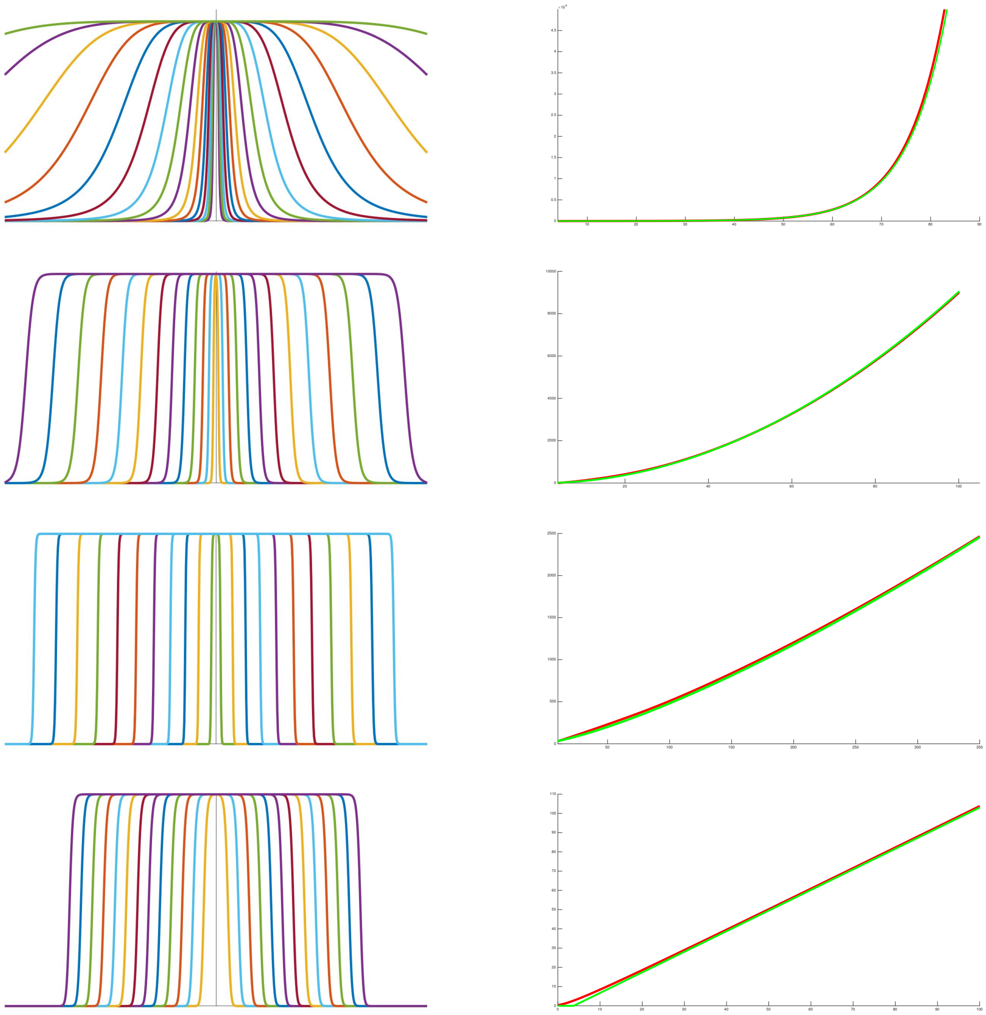


Figure 3.1: Numerical simulations of the Cauchy problem (4.1), for various kernels $J$. The first line corresponds to $J \sim |x|^{-5}$, the second line to $J \sim \exp\left(-|x|^{1/2}\right)$, the third line to $J \sim \exp\left(-|x|^{3/4}\right)$ and the last line to a Gaussian kernel $J \sim \exp\left(-|x|^{2}\right)$. For each kernel, we present in the left column the evolution of the solution by plotting it on the same figure for various successive (linearly chosen) values of time $t$. To quantify this and recover and illustrate Theorem 3.3, we present in the right column the time evolution of the level set $\{x \in \mathbb{R} : u(t,x) = 1/2\}$ for each kernel. The red bold curve is the numerical simulation, starting from an initial condition of the form $J$. The green curve is the expected asymptotic rate of expansion predicted by Theorem 3.3, that is $k^{-1}(t)$, except for the Gaussian kernel, in which case it is a line.

one but there we choose to stick to this exact $u^{\beta}(1-u)$ not to complexify the presentation unnecessarily.

Moreover, $J$ is a nonnegative function satisfying the following properties.

**Hypothesis 3.5.** *Let $s > 0$. The kernel $J \geq 0$ is symmetric and is such that there exists positive* *constants* $\mathcal{J}_0, \mathcal{J}_1$ *and* $R_0 \geq 1$ *such that*

$$\int_{|z|\leq 1} J(z)|z|^2\,dz \leq 2\mathcal{J}_1 \quad \textit{and} \quad \frac{\mathcal{J}_0}{|z|^{1+2s}}\mathbb{1}_{\{|z|\geq 1\}} \geq J(z) \geq \frac{\mathcal{J}_0^{-1}}{|z|^{1+2s}}\mathbb{1}_{\{|z|\geq R_0\}}.$$

and $u_0 \in \mathcal{C}(\mathbb{R}, [0,1])$ is such that

**Hypothesis 3.6.** $\quad 1 \geq u_0 \geq a\mathbb{1}_{(-\infty,b]} \quad$ *for some* $a > 0$ *and* $b \in \mathbb{R}$.

As a matter of fact, the parameter $s$ will thus appear in the rates we obtain later. One may readily notice that our hypothesis on $J$ allows to cover the two broad types of integro-differential operators $\mathcal{D}[\,u\,]$ usually considered in the literature which are the fractional laplacian $(-\Delta)^s u$ and the standard convolution operators with integrable kernels often written $J \star u - u$. This universality is one main contribution here.

When an Allee effect is introduced, the study of propagation is more subtle. Alfaro started the program with a paper about the interplay between heavy tailed initial data and Allee effect in local reaction-diffusion equations [4]. Coville *et. al.* [65, 61, 64] have proved existence of travelling fronts when the dispersal kernel $J$ is exponentially bounded and the Cauchy problem typically does not lead to acceleration [188]. When not, the competition between heavy tails and the Allee effect leads to intense discussions. For algebraic decaying kernels, Alfaro and Coville [6] provide the exact separation between existence and non existence of travelling waves. This in turn provides the exact separation between non acceleration and acceleration in the Cauchy problem. In the same spirit, Gui and collaborators discuss the existence of travelling waves and the possibility of acceleration for a fractional equation with Allee effet in [107]. However, in this latter paper, no precise rates of acceleration were given. Before reviewing the last-to-date results on (3.1), let us also mention that acceleration phenomenon also appears in some porous medium equations [126, 174, 8, 9].

As far as (3.1) is concerned, when $J \propto |\cdot|^{-(1+2s)}$ or integrable with a finite first moment, bounds on the expansion of the level sets of $u$ have been already obtained in [66, 4] showing a delicate interplay between the tails of $J$ and the power $\beta$. Namely, Coville *et al.* obtained a sharp upper bound of acceleration when $J \propto |\cdot|^{-(1+2s)}$ or integrable with a finite first moment and assuming that $\beta < 1 + \frac{1}{2s-1}$: they got an expansion at at most $t^{\frac{\beta}{2s(\beta-1)}}$. However, they were unable to provide a matching lower bound: their lower bound was growing faster than linearly, but not fast enough. The following result clarify the situation.

**Theorem 3.7** (B., Coville, Legendre [B11])**.** *Assume that* $J$ *satisfies Hypothesis 3.5 and that*

$$\beta < 1 + \frac{1}{2s-1}.$$

*Then for any* $\lambda \in (0,1)$*, the level line* $x_\lambda(t)$ *accelerates with the following rate*[9]*,*

$$x_\lambda(t) \asymp_\lambda t^{\frac{\beta}{2s(\beta-1)}}.$$

Up to our knowledge, this is the first, sharp, unified estimate of the level sets in this context. As already explained above, previous papers were able to derive correct upper bounds but getting a precise lower bound was left open. Our contribution is thus a lower bound that

[9]We use the notation $u \asymp_\lambda v$ for the existence of a positive constant $C_\lambda$ such that $C_\lambda v \leq u \leq C_\lambda^{-1} v$.

matches the already known upper bounds. To give the reader a clear panorama of the rates of invasion occurring in integro-differential models, we may summarise our and previous contributions in Figure 3.2. Note that the condition on $\beta$ fits and unifies all related papers [6, 107, 66].

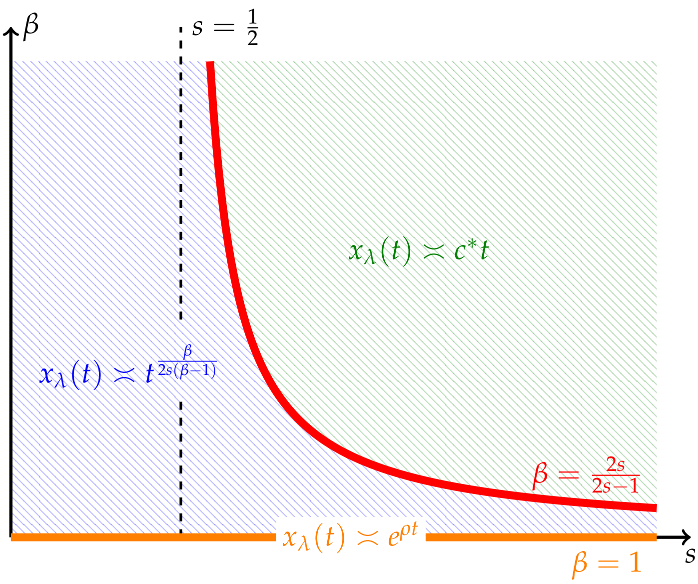



Figure 3.2: In the green zone, the model enjoys linear propagation with existence of travelling fronts [60]: $x_\lambda(t) \sim c^* t$. In the blue zone, we provide the sharp lower bound, upper bounds being given in [4, 66]: $x_\lambda(t) \asymp t^{\frac{\beta}{2s(\beta-1)}}$. The orange zone is a zone of exponential propagation, after *e.g* [41, 98, B17]: $x_\lambda(t) \sim \exp(\rho t)$.

A parallel work is a paper by Zhang and Zlatos [189] on the fractional laplacian version of the equation obtaining similar bounds using a different approach relying strongly on the properties of this operator.

To achieve this lower bound, two ingredients are important. First, we have to show that a front like initial data triggers a decay $x^{-2s}$ at $+\infty$ at time 1. This is enough to start when $s \geq 1$, but actually, when $s \in (0,1)$, we even need to prove a more precise flattening property, which is that *for all* $C > 0$, there exists $t_C$ such that

$$\lim_{x \to +\infty} x^{2s} u(t,x) \geq C \quad \text{for all} \quad t \geq t_C.$$

For the fractional laplacian, for instance, such type of estimates can be obtained easily through of the properties of the solution of the diffusion problem

$$\begin{cases} v_t(t,x) = \mathcal{D}[v](t,x) \\ v(0,x) = \mathbb{1}_{(-\infty,0]} \end{cases}$$

exploiting the particular time and space scaling properties of the associated heat kernel or even explicit sub- and super-solutions. However, although the characterisation of the heat kernel associated to the generator of a Levy process is a well known problem in probability theory and analysis that dates back to the original work of Pólya [163] and Blumenthal and Getoor [30] on $\alpha$-stable processes, up to our knowledge characterisations of the heat kernel that may induce such flattening estimates have only been established for some specific class of Levy processes [31, 72, 104, 124] and do not really exist for a generic Levy process. Our proof relies on a different approach using a subtle construction of a sub-solution of the linear problem that mimics the expected scaling behaviour of the heat kernel. In the regime $s \geq 1$, the flattening of cannot be uniquely explained through the diffusion process and is a consequence of the nonlinear invasion (or at least the Heaviside initial data). We made clear in [B11] the differences of necessary ingredients between $s \in (0,1)$ and $s \geq 1$. See Figure 3.3 for a schematic view of the expected behaviour of the solution given this flattening information.

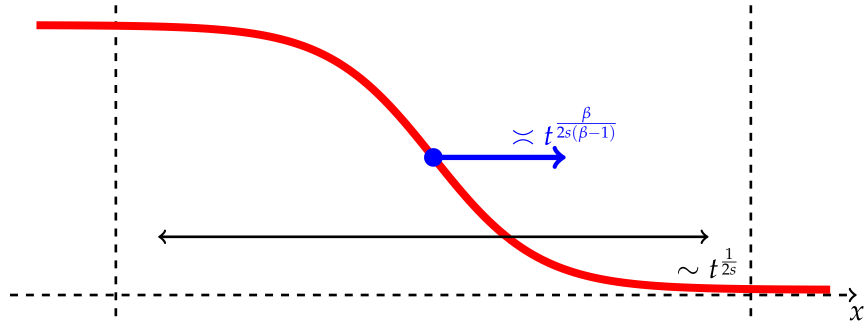



Figure 3.3: Schematic view of the expected behaviour of solution at a given time $t$.

Once the initial data has been well-prepared, our strategy consists in the construction of a new type of sub-solution that captures all the expected dynamics of the solution $u$. It has the form, for $t > 1$,

$$\underline{u}(t,x) := \begin{cases} \varepsilon & \text{for all } x \leq X(t), \\ 3\left(1 - \dfrac{w(t,x)}{\varepsilon} + \dfrac{w^2(t,x)}{3\varepsilon^2}\right) w(t,x) & \text{for all } x > X(t), \end{cases}$$

where,

$$w(t,x) := \left[\left(\frac{\kappa t}{x^{2s}}\right)^{1-\beta} - \gamma(\beta-1)t\right]^{-\frac{1}{\beta-1}}, \quad \text{and} \quad X(t) = (\kappa t)^{\frac{1}{2s}}\left[\varepsilon^{1-\beta} + \gamma(\beta-1)t\right]^{\frac{1}{2s(\beta-1)}}.$$

The parameters $\kappa$, $\gamma$ and $\varepsilon$ are fitted suitably in the proof can one can find in [B11]. As depicted in Figure 3.4, understanding the dynamics asks to split space in different zones, each one corresponding to different features of the equation. The main difficulty, once the candidate $\underline{u}$ is found, is to estimate $\mathcal{D}[\underline{u}]$ properly in all the space-time zones. This is the largest part of [B11].

We conclude by presenting numerical evidence of the acceleration in several cases. We have put a significant energy in getting such illustrative figures, since numerical simulations for acceleration phenomena are not so common, and still very costly due to the flattening of solutions while acceleration happens. Not so many general methods are yet developed to cope with this specific feature.

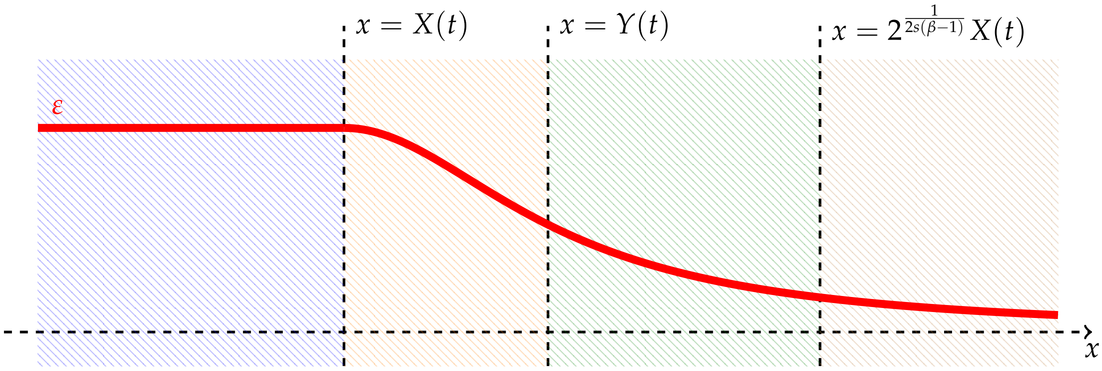
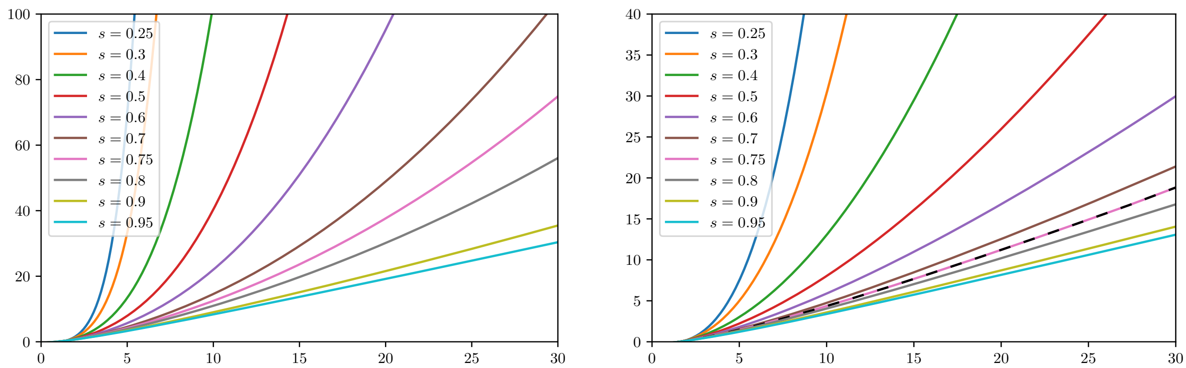



Figure 3.4: Schematic view of the sub-solution at a given time $t$. Several zones have to be considered. The exact expression of $Y(t)$ appears naturally in [B11]. The blue zone is where $\underline{u}$ is constant, making computations easier. In the orange zone, the fact that $\underline{u}$ looks like a solution to an ODE $n' = n^\beta$ is crucial. In the brown (far-field) zone, the decay imitating a fractional Laplace equation gives the right behaviour. The green zone is subtle and needs a mixture between both surrounding zones.

In Figure 3.5, the position of the level line of height $\lambda = 0.5$ is plotted as a function of time for two different values of $\beta$ and several values of the fractional Laplacian exponent $s$. In one of the two configurations, namely for $\beta = 1.5$, the critical value of the exponent $s$ above which there exists a travelling front is strictly greater than 1. As a consequence, the level set accelerates for any of the chosen values for $s$, but this acceleration clearly decreases to zero as $s$ tends to 1, as expected from the existing results with local diffusion. This is no longer the case for $\beta = 3$, as one can observe a switching from an accelerated regime to a travel at constant speed around the critical value $s = 0.75$ (the corresponding curve is plotted with a dashed line).




(a) $\beta = 1.5$.

(b) $\beta = 3$.

Figure 3.5: Position of the level line of height $\frac{1}{2}$ of the solution to the problem with fractional Laplacian operator, plotted as a function of time, for two different values of $\beta$ and several values of $s$ in $(0, 1)$.

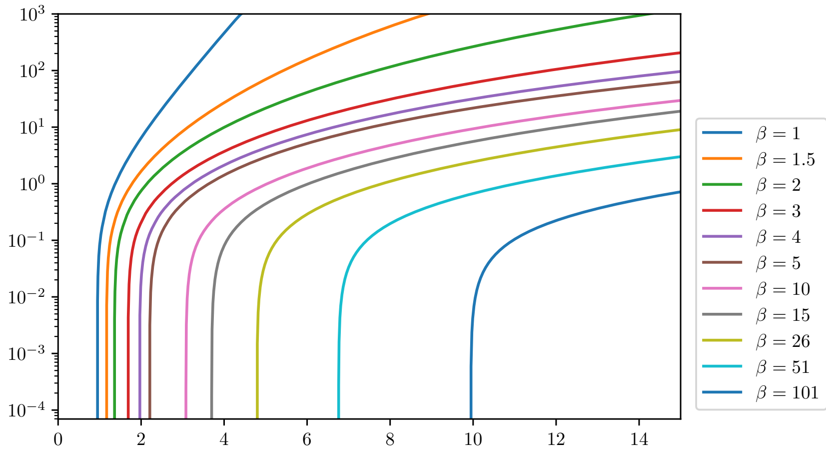



Figure 3.6: Logarithm of the position of the level line of height $\frac{1}{2}$ of the solution to the problem with fractional Laplacian operator, plotted as a function of time, for different values of $\beta$ and $s$ equal to $\frac{1}{2}$. We observe that curves are indeed logarithmic in time, giving polynomial rate of expansion, with a prefactor that converges, as expected (to $\frac{1}{2s}$ theoretically). The translation factor is due to the constant in front of the algebraic rate, that may vary with $\beta$.

On Figure 3.6, we check the polynomial expansion. On Figure 3.7, we illustrate the accelerated expansion and the flattening of the profiles as time goes to infinity.

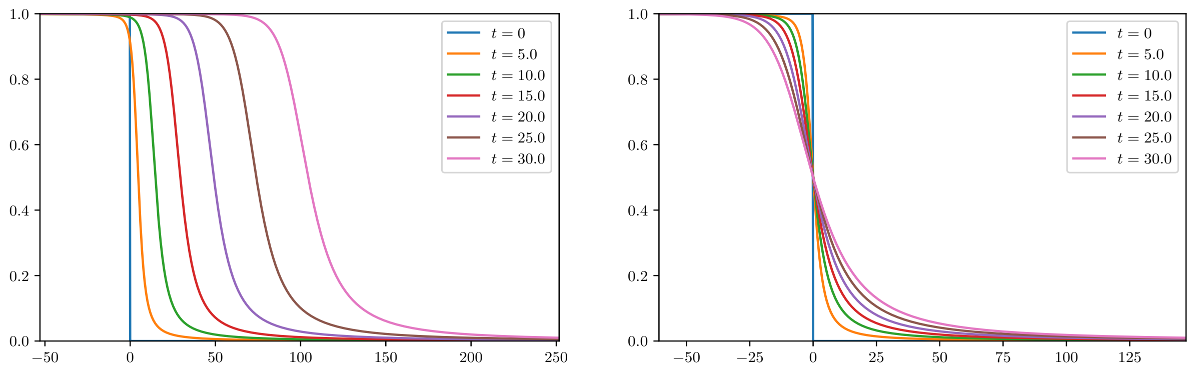
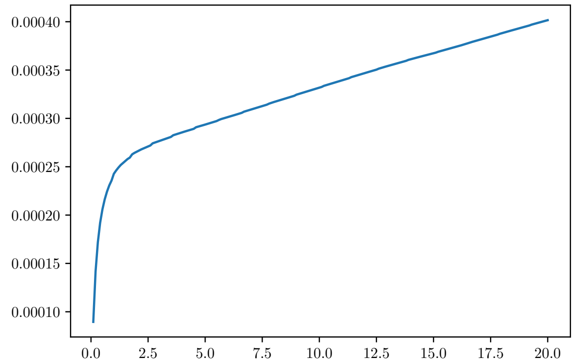
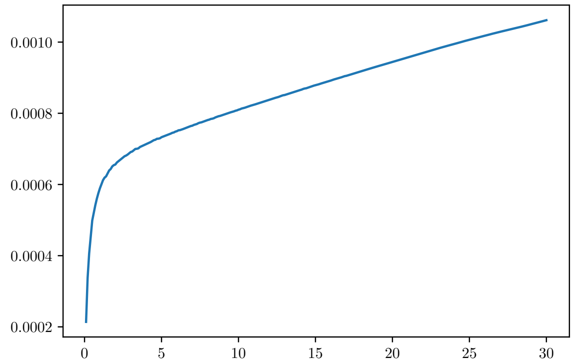



Figure 3.7: Approximations of the solution at different times for $\beta = 1.5$ and $s = 0.7$. On the right, the solutions have been recentered by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes.

To have a more convincing picture of this flattening effect, we may also plot in Figure 3.8 the evolution over time of the best constant $C$ such that the tail of the solution fits with $\frac{C}{x^{2s}}$ in the least square sense. After a rapid transition the graph obtained describes a linear growth of this constant $C$.




(a) $s = 0.4$.




(b) $s = 0.5$.

Figure 3.8: Evolution over time of the fitting constant for the part of the tail of the approximation solution at time $t = 1$ bounded by value $10^{-2}$ on the left and value $10^{-5}$ on the right using the function $\frac{C}{x^{2s}}$ for the solution of the problem with fractional diffusion and $\beta = 1.5$ and $s = 0.4/0.5$.

## 3.3 The lower bound with ignition nonlinearities

We follow the same lines as above but with an ignition nonlinearity.

**Hypothesis 3.8.** *Take $\theta > 0$,*

$$f(1) = 0, \qquad f(u) = 0 \quad \textit{if } u \leq \theta \qquad \textit{and } f(u) > 0 \quad \textit{in } [\theta, 1].$$

Let us review some existing works around this issue. Existence of fronts have been obtained for the fractional laplacian for $s > 1/2$ by Mellet et al in [147] and for the convolution type operator $J \star u - u$ provided $J$ has a first moment by Coville [61, 55]. See also some related works by Shen et al [172, 173]. Here we explore a situation where no first moment at infinity exists, that is $s \leq 1/2$. Thanks to the monostable results [66, 107], we already know that accelerated propagation can only occur in this region of parameter. This is drastic contrast with truly monostable nonlinearities [40, 41].

Let $s \in [0, \frac{1}{2}]$. The kernel $J$ is symmetric and is such that there exists positive constants $\mathcal{J}_0, \mathcal{J}_1$ and $R_0 \geq 1$ such that

$$\int_{|z|\leq 1} J(z)|z|^2\, dz \leq 2\mathcal{J}_1 \quad \text{and} \quad \frac{\mathcal{J}_0}{|z|^{1+2s}} \mathbb{1}_{\{|z|\geq 1\}} \geq J(z) \geq \frac{\mathcal{J}_0^{-1}}{|z|^{1+2s}} \mathbb{1}_{\{|z|\geq R_0\}}.$$

Observe that $\frac{1}{2}$ is exactly consistent with formally taking $\beta$ to $\infty$ in Theorem 3.7 above.

**Theorem 3.9** (B., Coville, Legendre [B10]). *Assume that $J$ satisfies Hypothesis 3.5 with $s < \frac{1}{2}$ and that $f$ is an ignition nonlinearity. For any $\lambda \in (0, 1)$, the level line $x_\lambda(t)$ accelerates with the following rate,*

$$x_\lambda(t) \gtrsim t^{\frac{1}{2s}}.$$

Again, observe that $\frac{1}{2s}$ is exactly consistent with formally taking $\beta$ to $+\infty$ in Theorem 3.7 above.

We briefly present the way that we construct a sub-solution to prove the lower bounds in Theorem 3.9. As previously for the monostable case, we expect $\underline{u}$ to look like a solution of the standard fractional Laplace equation with Heaviside initial data at the far edge. In this situation, a natural candidate would be given by

$$w(t,x) := \left[\frac{x^{2s}}{\kappa t} + \gamma\right]^{-1}. \tag{3.5}$$

with $\kappa, \gamma$ positive free parameter that will be determined later on. Note that this function is well defined for $t \geq 1$ and $x > 0$. The expected decay in space of a solution of the standard fractional Laplace equation with Heaviside initial data being at least of order $tx^{-2s}$, such a $w$ would have the good asymptotics. For $\varepsilon \in (0,1)$, let us define $X(t) > 0$ such that $w(t, X(t)) = \varepsilon$. For such $X(t)$ to be well defined, we need to impose that $\gamma < \frac{1}{\varepsilon}$, and thus for such $\varepsilon$ and $\gamma < \frac{1}{\varepsilon}$, $X(t)$ is then defined by the following formula

$$X(t) = \left[\varepsilon^{-1} - \gamma\right]^{\frac{1}{2s}} (\kappa t)^{\frac{1}{2s}}. \tag{3.6}$$

One may observe that $X(t)$ moves with the speed that we expect in Theorem 3.9. As in the monostable case, to get a $C^2$ function at $x = X(t)$, we complete our construction by taking $\varphi$ such that

$$\underline{u}(t,x) := \begin{cases} \varepsilon & \text{for all } x \leq X(t), \\ 3\left(1 - \dfrac{w(t,x)}{\varepsilon} + \dfrac{w^2(t,x)}{3\varepsilon^2}\right) w(t,x) & \text{for all } x > X(t), \end{cases} \tag{3.7}$$

for $t > 1$. The interested reader may consult [B10] for the proof of the fact that $\underline{u}$ is indeed a sub-solution.

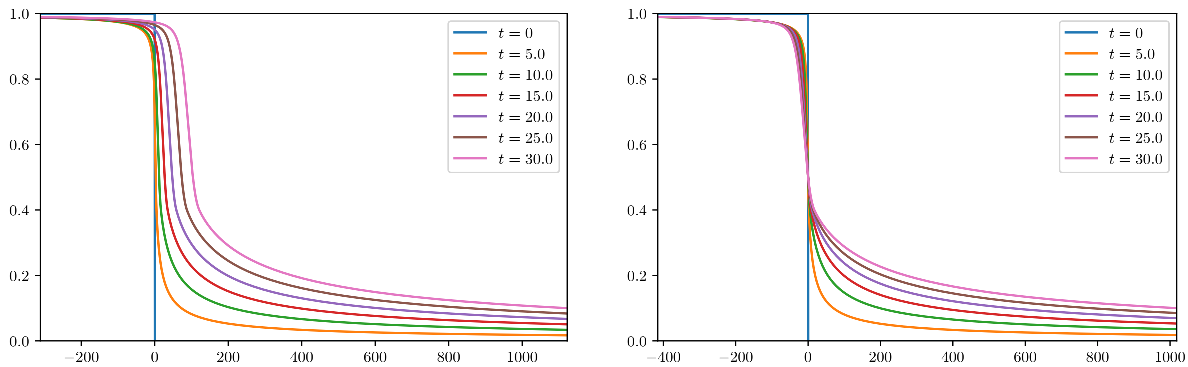



Figure 3.9: Numerical approximations of the solution to the problem with fractional diffusion at different times for $s$ equal to $\frac{1}{3}$ and $\theta = 0.4$. On the right, the graphs have been shifted by setting the position of the level line of value $\frac{1}{2}$ at $x = 0$, for comparison purposes. The latter exhibits more clearly the deformation of the solution.

## 3.4 Perspectives

1. *Symmetrisation of the solutions.*

   An interesting question is to know more about the profiles of invasion. For fractional laplacian invasions, it is known from Roquejoffre and Tarfulea that multi-dimensional invasions symmetrise (that is, become radial) and relaxation to asymptotic profiles are proved *via* gradient estimates. The same issue is of interest here, for example in the situation described in Section 3.1. In particular, when the function $\theta$ is integrable, it is rather conceivable to try to perform the same kind of estimates.

2. *Upper bound and description of the profiles.*

   The upper bounds given by Coville and co-authors in [66] allow to find the acceleration law, be the shape of the profiles used are not likely to be optimal. A refinement of the analysis there could allow a better understanding, or at least a rigorous understanding, of the shape of the invasion profiles. Moreover, nothing is known for the moment about optimal constants in front of the power law in Theorem 3.7 : an asymptotic analysis is of great interest, may be in tentative self-similar variables.

# Part II

# Long time behaviour and scaling limits in kinetic theory

# Chapter 4

# Spreading in kinetic reaction transport equations: local and nonlocal Hamilton-Jacobi equations

## Contents


**4.1 Propagation in higher dimensions : influence of the velocity set . . . . . 58**
**4.2 Large deviations and nonlocal Hamilton-Jacobi equations . . . . . . . . 61**
4.2.1 The notion of viscosity solution and the convergence result . . . . 62
4.2.2 Informal discussion on the limit system . . . . . . . . . . . . . . . . 63
4.2.3 Heuristics on the convergence result . . . . . . . . . . . . . . . . . . 64
4.2.4 Representation formula and consequences . . . . . . . . . . . . . . 65
4.2.5 Qualitative behaviour (in the long term) . . . . . . . . . . . . . . . . 66
4.2.6 Getting the acceleration constant . . . . . . . . . . . . . . . . . . . . 66
**4.3 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 68**


In this chapter, we are interested in propagation phenomena occurring in the following reaction-transport equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{M}\rho - f + r\rho\left(\mathcal{M} - f\right), & \text{in } \mathbb{R}_+ \times \mathbb{R}^n \times V\,, \\ f(0,\cdot,\cdot) = f_0\,, & \text{in } \mathbb{R}^n \times V\,, \end{cases} \tag{4.1}$$

where $r > 0$. The mesoscopic density $f$ depends on time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^n$ and velocity $v \in V$ and describes a population of individuals. The macroscopic density is $\rho(t,x) = \int_V f(t,x,v)\,dv$. The subset $V \subset \mathbb{R}^n$ is the set of all possible velocities.

Individuals move following a so-called velocity-jump process. That is, they alternate successively a run phase, with velocity $v \in V$, and a change of velocity at rate 1, which we call the tumbling. The new velocity is chosen according to the probability distribution $\mathcal{M}$. This is the typical motion of bacteria E. Coli for instance. As such, the underlying velocity-jump process belongs to the class of so-called Piecewise Deterministic Markov Processes (PDMP). The reproduction of individuals is taken into account through a reaction term of monostable type. The constant $r > 0$ is the growth rate in absence of any saturation. New individuals start with a velocity chosen at random with the same probability distribution $\mathcal{M}$. The quadratic saturation term accounts for local competition between individuals, regardless of their speed. Note that this model is nonlocal, but interestingly enough enjoys maximum and comparison principles.

It is relatively natural to address the question of spreading for (4.1) since there is a strong link between (4.1) and the classical Fisher-KPP equation [91, 131]. Indeed, a suitable parabolic rescaling leads to the Fisher-KPP equation (see [71] for example).

In this chapter, I will summarise my contributions (papers [B5, B2]) in this context.

## 4.1 Propagation in higher dimensions : influence of the velocity set

In the paper [B8] with Calvez and Nadin, propagation for the full kinetic model (4.1) has been investigated. In one dimension of velocities, and when the velocities are bounded, we obtained the existence and stability of travelling waves solutions to (4.1). The minimal speed of propagation of the waves is determined by the resolution of a spectral problem in the velocity variable. In particular, it is not related with the KPP speed, except that the speeds coincide in the diffusive regime.

With Caillerie, we focused in [B2] on the case of bounded velocities having dimension higher than one. For any given direction $e \in \mathbb{S}^{n-1}$ and $p \in \mathbb{R}^n$, we define

$$\overline{v}(e) = \max\left\{v \cdot e, v \in V\right\}, \qquad \mu(0) = 0, \qquad \mu(p) = |p|\overline{v}\left(\frac{p}{|p|}\right), \quad \text{if } p \neq 0,$$

and

$$\operatorname{Arg}\mu(p) = \left\{v \in V \mid v \cdot p = \mu(p)\right\}.$$

We set

$$v_{max} := \sup_{v \in V} |v|, \qquad |V| := \int_V dv.$$

We perform the hyperbolic scaling $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ in (4.1) and the *kinetic Hopf-Cole transformation* (already used in [B1, B3, 42]),

$$f^\varepsilon = \mathcal{M}e^{-\frac{u^\varepsilon}{\varepsilon}}, \text{ or equivalently } u^\varepsilon = -\varepsilon \ln\left(\frac{f^\varepsilon}{\mathcal{M}}\right). \tag{4.2}$$

The convergence result for the sequence of functions $u^\varepsilon$ is as follows.

**Theorem 4.1** (B., Caillerie, [B2])**.** *Suppose that the initial data satisfies*

$$\forall (x, v) \in \mathbb{R}^n \times V, \qquad u^\varepsilon(0, x, v) = u_0(x, v).$$

*Then, $(u^\varepsilon)_\varepsilon$ converges uniformly on all compacts of $\mathbb{R}_+^* \times \mathbb{R}^n \times V$ towards $u^0$, where $u^0$ does not depend on $v$. Moreover $u^0$ is the unique viscosity solution of the following Hamilton-Jacobi equation:*

$$\begin{cases} \min\left\{\partial_t u^0 + (1+r)\mathcal{H}\left(\frac{\nabla_x u^0}{1+r}\right) + r, u^0\right\} = 0, & (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ u^0(0, x) = \min\limits_{v \in V} u_0(x, v), & x \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

The Hamiltonian involved in this result is defined as follows.

**Definition 4.2.** *We define, for $e \in \mathbb{S}^{n-1}$,*

$$l(e) = \int_V \frac{\mathcal{M}(v)}{\overline{v}(e) - v \cdot e} dv.$$

*The so-called singular set is defined by*

$$\mathrm{Sing}\,(\mathcal{M}) := \left\{ p \in \mathbb{R}^n, \int_V \frac{\mathcal{M}(v)}{\mu(p) - v \cdot p} dv \leq 1 \right\} = \left\{ p \in \mathbb{R}^n, l\left(\frac{p}{|p|}\right) \leq |p| \right\}. \tag{4.4}$$

*Then, the Hamiltonian $\mathcal{H}$ involved in this paper is given as follows:*

- *If $p \notin \mathrm{Sing}\,(\mathcal{M})$, then $\mathcal{H}$ is uniquely defined by the following implicit relation :*

$$\int_V \frac{\mathcal{M}(v)}{1 + \mathcal{H}(p) - v \cdot p} dv = 1, \tag{4.5}$$

- *else, $\mathcal{H}(p) = \mu\,(p) - 1$.*

We represent in Figure 4.1 the shape of $\mathrm{Sing}(\mathcal{M})$ when $V = [-1,1]^2$. It is worth mentioning that the set $\mathrm{Sing}(\mathcal{M})^c$ is not bounded here.

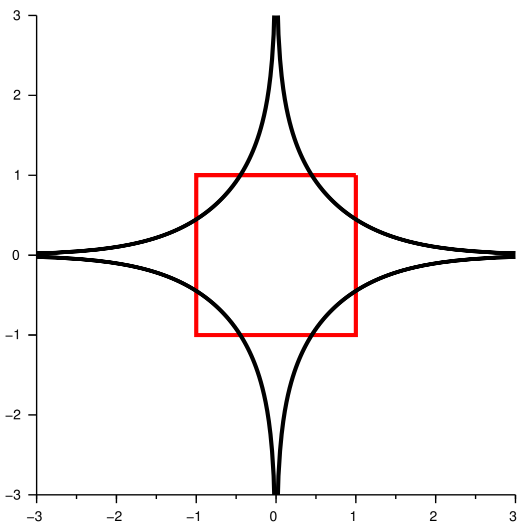



Figure 4.1: The set $\mathrm{Sing}(\mathcal{M})$ when $V = [-1,1]^2$. Red plain line: $\partial V$. Black plain line: $\partial\mathrm{Sing}(\mathcal{M})$. The set $\mathrm{Sing}(\mathcal{M})^c$ is the connected component of $\mathbb{R}^n$ that contains $(0,0)$. The lines $\{y = 0\}$ and $\{x = 0\}$ are included in $\mathrm{Sing}(\mathcal{M})^c$.

This Hamiltonian is the same as the one from [42]. Note that one interest of our paper is also that the proof we have designed for Theorem 4.1 is simpler and more adaptable since we manage to use the half-relaxed limits of Barles and Perthame [22] in the spirit of [B1].

Let us focus on the existence of travelling waves and the influence of the geometry on the speed of propagation. One can define

$$c(\lambda, e) = (1+r)\mathcal{H}\left(\frac{\lambda e}{1+r}\right) + r,$$

and the minimal speed in the direction $e \in \mathbb{S}^{n-1}$.

$$c^*(e) = \inf_{\lambda>0} c(\lambda, e)\,.$$

We obtain the following existence result.

**Theorem 4.3** (B., Caillerie, [B2]). *Let $e \in \mathbb{S}^{n-1}$. For all $c \in [c^*(e), \overline{v}(e))$, there exists a travelling wave solution of* (4.1) *with speed $c$ and direction $e$. Moreover, there exists no positive travelling wave solution of speed $c \in [0, c^*(e))$.*

The main difference with standard strategies is the way we prove the minimality of the speed $c^*(e)$. Indeed, it might happen that $c(\lambda, e)$ is singular at its minimum $\lambda^*$ so that one can not reproduce the same argument as for the mono-dimensional case used in [B8], that was based on the Rouché theorem.

We finally obtain two spreading results using the Hamilton-Jacobi framework in a similar fashion as in [27]. The first one is for travelling bands.

**Proposition 4.4.** *Let $f_0$ be a non-zero initial data, compactly supported in some direction $e_0$, that is to say there exists $\gamma < 1$ such that*

$$\gamma\mathcal{M}(v)\mathbf{1}_{[-x_m,x_m]\cdot e_0+e_0^\perp}(x) \le f_0(x,v) \le \mathcal{M}(v)\mathbf{1}_{[-x_M,x_M]\cdot e_0+e_0^\perp}(x),$$

*for all $(x,v) \in \mathbb{R}^n \times V$, where $e_0^\perp$ denotes the orthogonal complement of* $\mathrm{Span}(e_0)$. *Let $f$ be the solution of the Cauchy problem* (4.1) *associated to this initial data. Then we have*

$$\lim_{t\to+\infty} \sup_{x\cdot e_0>ct} \rho(t,x) = 0, \text{ if } c > c^*(e_0),$$

$$\lim_{t\to+\infty} f(t, e_0^\perp + cte_0, v) = \mathcal{M}(v)\,, \text{ if } c < c^*(e_0),$$

*uniformly in $v \in V$.*

The second one is for compactly supported initial data. It involves a Freidlin-Gärtner type formula (see [96] for its first derivation). Define, for any direction $e_0 \in \mathbb{S}^{n-1}$,

$$w^*(e_0) = \min_{\substack{e\in\mathbb{S}^{n-1}\\ e_0\cdot e>0}} \left(\frac{c^*(e)}{e_0\cdot e}\right).$$

We obtain the following result.

**Proposition 4.5.** *Let $f^0$ be a non-zero compactly supported initial data such that $0 \le f_0(x,v) \le \mathcal{M}(v)$ for all $(x,v) \in \mathbb{R}^n \times V$. Let $f$ be the solution of the Cauchy problem* (4.1) *associated to this initial data. Then for any $e_0 \in \mathbb{S}^{n-1}$ and all $x \in \mathbb{R}^n$, we have*

$$\lim_{t\to\infty} f(t, x+cte_0, v) = 0, \qquad \text{if } c > w^*(e_0),$$

*pointwise and*

$$\lim_{t\to\infty} f(t, cte_0, v) = \mathcal{M}(v), \qquad \text{if } 0 \le c < w^*(e_0),$$

*for all $v \in V$.*

This result is interesting since contrary to the case of the usual Fisher-KPP equation for which the Freidlin-Gärtner formula holds when coefficients vary in space, this Freidlin-Gärtner formula in space is induced by an heterogeneity of the velocity set.

## 4.2 Large deviations and nonlocal Hamilton-Jacobi equations

There is a strong link between the KPP type propagation phenomena presented in the previous section and large deviations for the underlying velocity-jump process. Indeed, it is well known that fronts in standard Fisher-KPP equations are so-called *pulled fronts*, that is, are triggered by very small populations at the edge that are able to reproduce almost exponentially. Our work can be viewed as a contribution to the theory of large deviations for velocity jump processes, which is an interesting problem in itself. The previous section gave some answers when the velocity set is compact.

In the one-dimensional case $n = 1$, it was established in [B8] that when velocities are not bounded, solutions to (4.1) behave in the long-time asymptotics as accelerating fronts due to the (rare) occurence of high velocities that send particles far from the bulk. Furthermore, the location of the front is of the order of $t^{3/2}$, in accordance with the scaling limit of the linear problem performed in (4.1). The location of the front $X(t)$ (such that $\rho(t, X(t)) = \frac{1}{2}$) was determined via the construction of sub- and super-solutions, with some room in between. More precisely, it was estimated that

$$\left(\frac{r}{r+2}\right)^{3/2} \leq \frac{X(t)}{t^{3/2}} \leq \sqrt{2r}\,,$$

in a weak sense (see [B8, Theorem 1.11] for details). With the work below, we are able to refine the estimate (4.2) to get that the front is located around

$$X(t) = \frac{\left((2/3)r\right)^{3/2}}{1+r}\, t^{3/2}\,, \tag{4.6}$$

in a weak sense.

To get to this quantitative result, we start by investigating the asymptotic limit of the following linear kinetic transport equation as $\varepsilon \to 0$,

$$\partial_t f^\varepsilon(t,x,v) + v \cdot \nabla_x f^\varepsilon(t,x,v) = \frac{1}{\varepsilon}\left(\mathcal{M}_\varepsilon(v)\rho^\varepsilon(t,x) - f^\varepsilon(t,x,v)\right), \quad t > 0\,,\, x \in \mathbb{R}^n\,,\, v \in \mathbb{R}^n\,. \tag{4.7}$$

The velocity distribution $\mathcal{M}_\varepsilon$ of reorientation events is given. We opt here for the Gaussian distribution with variance $\varepsilon$:

$$\mathcal{M}_\varepsilon(v) = \frac{1}{(2\pi\varepsilon)^{n/2}} \exp\left(-\frac{|v|^2}{2\varepsilon}\right).$$

However, we believe that our methodology could be applied to a broader range of distributions. The equation (4.7) is obtained from the unscaled problem ($\varepsilon = 1$) in the scaling regime

$$\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{3/2}}, \frac{v}{\varepsilon^{1/2}}\right)$$

which is crucial for the analysis and appropriate to establish a Large Deviation Principle. Define the rate function $u^\varepsilon = -\varepsilon \log f^\varepsilon$. About the initial data, we assume for simplicity that it is well-prepared, in the form $f^\varepsilon(0, x, v) = \exp\left(-u_0(x, v)/\varepsilon\right)$. Note that here we choose not divide by $\mathcal{M}$: it could be done but at the expense of slightly less nice expressions later on.

### 4.2.1 The notion of viscosity solution and the convergence result

Our convergence result is the following.

**Theorem 4.6** (B., Calvez, Grenier, Nadin [B5]). *Assume that $u_0$ is continuous and satisfies the following property:*

$$u_0 - \frac{|v|^2}{2} \in L^\infty(\mathbb{R}^{2n}), \tag{4.8}$$

*Let $u^\varepsilon$ be the solution of* (4.2.3)*, with the initial data $u^\varepsilon(0, \cdot) = u_0$. Then, $u^\varepsilon$ converges locally uniformly towards $u$ as $\varepsilon \to 0$, where $u$ is the unique viscosity solution of*

$$\begin{cases} \max\left(\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v) - 1, u(t,x,v) - \min\limits_{v' \in \mathbb{R}^n} u(t,x,v') - \dfrac{|v|^2}{2}\right) = 0\,, \\ \partial_t \left(\min\limits_{v' \in \mathbb{R}^n} u(t,x,v')\right) \leq 0\,, \quad \textit{and} \quad \partial_t \left(\min\limits_{v' \in \mathbb{R}^n} u(t,x,v')\right) = 0 \;\; \textit{if} \;\; \operatorname*{argmin}\limits_{v' \in \mathbb{R}^n} u(t,x,v') = \{0\}\,. \end{cases} \tag{4.9}$$

*with initial data* $\min\left(u_0, \min' u_0 + \frac{|v|^2}{2}\right)$.

The averaging phenomenon which is central in the case of bounded velocities [86, B3, B1, 42] does not occur in the case of unbounded velocities. One immediate consequence is that the rate function $u$ depends on the velocity variable $v$. More profound consequences are the seemingly new structure of the nonlocal Hamilton-Jacobi problem (4.9) and its consequences. Alternatively speaking, the nature of the PDMP persists in the regime of large deviations.

The main issue in this result is to understand what "viscosity solution" there means. To the best of our knowledge, system (4.9) is entirely original. Actually, no results for large deviations of such a PDMP with unbounded velocities did exist earlier. We refer to it as a Hamilton-Jacobi problem by analogy with the usual procedures in large deviations theory. We define what sub- and super-solutions mean, which gives the notion of solution after proving a strong comparison principle.

Equation (4.9) can be viewed as a coupled system of Hamilton-Jacobi equations on $u$ and $\min' u := \min_{v'} u(\cdot, \cdot, v')$. Accordingly, we define viscosity solutions to (4.9) using a pair of test functions as *e.g.* in [136, 83].

**Definition 4.7** (Sub-solution). *Let $\underline{u}_0$ be a continuous function, and $T > 0$. An upper semi-continuous function $\underline{u}$ is a **viscosity sub-solution** of* (4.9) *on $(0, T) \times \mathbb{R}^{2n}$ with initial data $\underline{u}_0$ if the following conditions are fulfilled:*

(i) $\underline{u}(0+, \cdot, \cdot) \leq \underline{u}_0$.

(ii) *It satisfies the constraint*

$$\forall (t, x, v) \in (0, T) \times \mathbb{R}^{2n} \quad \underline{u}(t, x, v) - \min{}' \underline{u}(t, x) - \frac{|v|^2}{2} \leq 0\,.$$

(iii) *For all pair of test functions* $(\varphi, \psi) \in \mathcal{C}^1\left((0,T)\times\mathbb{R}^{2n}\right)\times\mathcal{C}^1\left((0,T)\times\mathbb{R}^n\right)$, *if* $(t_0, x_0, v_0)$ *is such that both* $\underline{u}(\cdot,\cdot,v_0) - \varphi(\cdot,\cdot,v_0)$ *and* $\min' \underline{u} - \psi$ *have a local maximum at* $(t_0, x_0)$ *with* $t_0 > 0$, *then*

$$\begin{cases} \partial_t\varphi(t_0,x_0,v_0) + v_0\cdot\nabla_x\varphi(t_0,x_0,v_0) - 1 \le 0, \\ \partial_t\psi(t_0,x_0) \le 0. \end{cases}$$

**Definition 4.8** (Super-solution)**.** *Let* $\overline{u}_0$ *be a continuous function, and* $T > 0$. *A lower semi-continuous function* $\overline{u}$ *is a* ***viscosity super-solution*** *of* (4.9) *on* $(0,T)\times\mathbb{R}^{2n}$ *with initial data* $\overline{u}_0$ *if the following conditions are fulfilled:*

(i) $\overline{u}(0+,\cdot,\cdot) \ge \overline{u}_0$.

(ii) *For all pair of test functions* $(\varphi, \psi) \in \mathcal{C}^1\left((0,T)\times\mathbb{R}^{2n}\right)\times\mathcal{C}^1\left((0,T)\times\mathbb{R}^n\right)$, *if* $(t_0, x_0, v_0)$ *is such that both* $\overline{u}(\cdot,\cdot,v_0) - \varphi(\cdot,\cdot,v_0)$ *and* $\min' u - \psi$ *have a local minimum at* $(t_0, x_0)$ *with* $t_0 > 0$, *then*

$$\begin{cases} \partial_t\varphi(t_0,x_0,v_0) + v_0\cdot\nabla_x\varphi(t_0,x_0,v_0) - 1 \ge 0 & \text{if} \quad \overline{u}(t_0,x_0,v_0) - \min'\overline{u}(t_0,x_0) - \dfrac{|v_0|^2}{2} < 0, \\ \partial_t\psi(t_0,x_0) \ge 0, & \text{if} \quad \operatorname{argmin}'\overline{u}(t_0,x_0) = \{0\}\,. \end{cases}$$

Let us mention that the mimimality (*resp.* maximality) condition in the definition of the super- (*resp.* sub-) solution arises with respect to variables $(t,x)$ only. This is consistent with the fact that there is no derivative in the velocity variable in (4.9).

**Definition 4.9** (Solution)**.** *Let* $u_0$ *be a continuous function, and* $T > 0$. *A function* $u$ *is a* ***viscosity solution*** *of* (4.9) *on* $(0,T)\times\mathbb{R}^{2n}$ *with initial data* $u_0$ *if its upper (resp. lower) semi-continuous envelope is a sub- (resp. super-) solution in the sense of definitions Definition 4.7 and Definition 4.8.*

The following theorem states a comparison principle for viscosity (sub/super-)solutions of the system (4.9). This establishes uniqueness of viscosity solutions as a corollary.

**Theorem 4.10** (Comparison principle)**.** *Let* $\underline{u}$ *(resp.* $\overline{u}$*) be a viscosity sub-solution (resp. super-solution) of* (4.9) *on* $(0,T)\times\mathbb{R}^{2n}$ *with continuous initial data* $\underline{u}_0 \le \overline{u}_0$. *Assume that* $\underline{u}$ *and* $\overline{u}$ *are such that*

$$\overline{u} - \frac{|v|^2}{2} \in L^\infty\left((0,T)\times\mathbb{R}^{2n}\right), \quad \underline{u} - \frac{|v|^2}{2} \in L^\infty\left((0,T)\times\mathbb{R}^{2n}\right).$$

*Then* $\underline{u} \le \overline{u}$ *on* $(0,T)\times\mathbb{R}^{2n}$.

### 4.2.2 Informal discussion on the limit system

The system (4.9) is not a standard Hamilton-Jacobi equation. The first equation of (4.9) does not contain enough information due to the occurrence of $\min u$ for which extra dynamics are required. Although it seems somehow sparse, the two additional (in)equations $\partial_t(\min u) \le 0$ $(= 0)$ are sufficient to determine a unique solution of the Cauchy problem.

In order to get some insight about the well-posedness of (4.9), we propose the following description of the typical dynamics of its solution $u$. The first condition in (4.9) guarantees that the following constraint must be satisfied everywhere:

$$u(t,x,v) \le \min' u(t,x) + \frac{|v|^2}{2}\,. \tag{4.10}$$

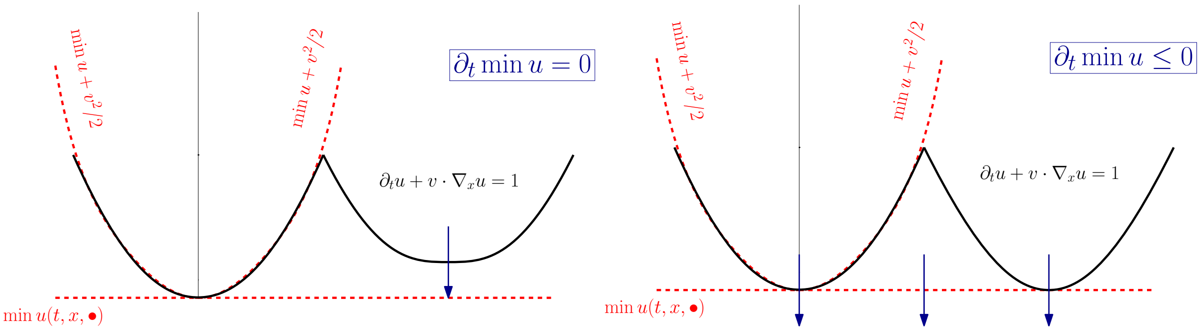



Figure 4.2: Typical dynamics of solutions to (4.9). Left: the case $\operatorname{argmin}' u(t,x) = \{0\}$. Right: the case $\operatorname{argmin}' u(t,x) \neq \{0\}$.

Consequently, the solution reaches its global minimum with respect to the velocity variable at $v = 0$. Furthermore, the following dichotomy holds:

(i) either the constraint is saturated: $u = \min' u + \frac{|v|^2}{2}$,

(ii) or the solution is driven by free transport: $\partial_t u + v \cdot \nabla_x u = 1$.

Then, two more cases must be distinguished: if $v = 0$ is the only global minimal point with respect to velocity ($\operatorname{argmin}' u(t,x) = \{0\}$), then the minimal value does not change, see Figure 4.2. Hence, the parabolic constraint (4.10) does not change as well. Nevertheless, the solution in the unsaturated area can still evolve by free transport and decay. If it touches the minimal value somewhere else, then the condition $\operatorname{argmin}' u(t,x) = \{0\}$ is not satisfied anymore, and the minimal value can possibly decrease, together with the parabolic constraint, see Figure 4.2. The decay in the free zone drives the global decay of the solution.

### 4.2.3 Heuristics on the convergence result

Let us provide some heuristics to describe the link between (4.1) and (4.9). Equation (4.1) is equivalent to the following equation on $u^\varepsilon$:

$$\partial_t u^\varepsilon(t,x,v) + v \cdot \nabla_x u^\varepsilon(t,x,v) - 1 = -\frac{1}{(2\pi\varepsilon)^{n/2}} \int_{\mathbb{R}^n} \exp\left( \frac{u^\varepsilon(t,x,v) - u^\varepsilon(t,x,v') - |v|^2/2}{\varepsilon} \right) dv' .$$

On the one hand, it is immediate that the constraint (4.10) is fulfilled in the limit $\varepsilon \to 0$ provided that the left-hand-side is locally uniformly bounded. On the other hand, the continuity equation $\partial_t \int f^\varepsilon \, dv + \nabla_x \cdot \int v f^\varepsilon \, dv = 0$ is equivalent to

$$\int_{\mathbb{R}^n} (\partial_t u^\varepsilon + v \cdot \nabla_x u^\varepsilon) \, d\mu^\varepsilon(v) = 0 , \quad d\mu^\varepsilon(v) = \frac{f^\varepsilon(t,x,v)}{\rho^\varepsilon(t,x)} \, dv . \tag{4.11}$$

The probability measure $d\mu^\varepsilon$ is expected to concentrate on the minimum points of $u$ with respect to $v$ as $\varepsilon \to 0$. Let assume that we do have in some sense,

$$d\mu^\varepsilon \rightharpoonup \sum_{w \in \operatorname{argmin}' u(t,x)} p_w \delta(v - w) \; = \; p_0 \delta(v) + \sum_{w' \in \operatorname{argmin}' u(t,x) \setminus \{0\}} p_{w'} \delta(v - w') , \tag{4.12}$$

where the weights satisfy $\sum p_w = 1$. The constraint (4.10) at each $w' \in \operatorname{argmin}' u(t,x) \setminus \{0\}$ is clearly unsaturated, in the sense that $u(t,x,w') < \min' u + |w'|^2/2$. There, we expect to see the right-hand-side contribution of (4.2.3) vanish. This would lead to $\partial_t u(t,x,w') + w' \cdot \nabla_x u(t,x,w') = 1$ for each $w' \in \operatorname{argmin}' u(t,x) \setminus \{0\}$. Plugging this into (4.11), and using (4.12), we obtain successively,

$$
\begin{aligned}
0 = \sum_{w \in \operatorname{argmin}' u(t,x)} p_w \left(\partial_t u + w \cdot \nabla_x u\right) = p_0 \partial_t u(t,x,0) + \sum_{w' \in \operatorname{argmin}' u(t,x) \setminus \{0\}} p_{w'} \\
= p_0 \partial_t u(t,x,0) + 1 - p_0 \, .
\end{aligned}
$$

As we have formally $\partial_t u(t,x,0) = \partial_t \left(\min' u\right)(t,x)$ by the chain rule, we expect eventually that $\partial_t(\min' u) \le 0$ and even $\partial_t(\min' u) = 0$ if $p_0 = 1$, that is, somehow $\operatorname{argmin}' u(t,x) = \{0\}$.

### 4.2.4 Representation formula and consequences

The limit system being unusual, we are interested in deriving formulas that help computing, or at least understanding, its solutions. First, we derive the representation formula for (4.9).

$$
u(t,x,v) = \inf_{\{\gamma \,:\, \gamma(t) = x,\, \dot\gamma(t) = v\}} \left\{ \mathcal{A}_0^t[\dot\gamma] + u_0(\gamma(0), \dot\gamma(0)) \right\} \tag{4.13}
$$

where the action of a piecewise linear curve $\gamma$ over the time interval $(0,t]$ is given by:

$$
\mathcal{A}_0^t[\dot\gamma] = \frac{1}{2} \sum_{\sigma \in \dot\Gamma_*} |\sigma|^2 + \operatorname{Leb}\left\{ s \in (0,t] : \dot\gamma(s) \neq 0 \right\} ,
$$

where $\dot\Gamma_*$ denotes the finite list of velocities $(\dot\gamma(s))_{s \in (0,t]}$ but the initial one. Alternatively speaking, each non-zero velocity $\sigma$ after the first velocity jump contributes to a single cost of $\frac{1}{2}|\sigma|^2$ and a running cost of one per unit of time.

Second, we also establish the variational formulation (4.13) of the viscosity solution. Let $\Sigma_s^t$ be the space of piecewise constant, *cádlág* functions defined over the time interval $(s,t]$ taking values in $\mathbb{R}^n$. For any $y \in \mathbb{R}^n$ and $\sigma \in \Sigma_s^t$, we define the piecewise linear curve $\gamma$ as

$$
\gamma(\tau) = y + \int_s^\tau \sigma(\tau') d\tau' \, .
$$

Denoting by $(t_i)_{1 \le i \le N}$ the times of discontinuity of $\sigma$ in $(s,t]$, such that

$$
\sigma = \sigma_0 \mathbf{1}_{(s,t_1)} + \sum_{i=1}^{N-1} \sigma_i \mathbf{1}_{[t_i,t_{i+1})} + \sigma_N \mathbf{1}_{[t_N,t]},
$$

we define the action of $\sigma$ on $(s,t]$ as follows:

$$
\mathcal{A}_s^t[\sigma] = \frac{1}{2} \sum_{i=1}^N |\sigma_i|^2 + \sum_{i=0}^N (t_{i+1} - t_i) \mathbf{1}_{\sigma_i \neq 0} \, ,
$$

with the convention $t_0 = s$ and $t_{N+1} = t$. The main result is the following kinetic Hopf-Lax formula.

**Theorem 4.11** (B., Calvez, Grenier, Nadin [B5]). *Let $u_0$ be a continuous function verifying* (4.8). *Then, the following representation formula*

$$U(t,x,v) = \inf_{\substack{\{(y,\sigma) \in \mathbb{R}^n \times \Sigma_0^t : \\ \gamma(t) = x, \sigma_N = v\}}} \left\{ \mathcal{A}_0^t[\sigma] + u_0(y,\sigma_0) \right\} \tag{4.14}$$

*is the viscosity solution of* (4.9) *with initial data* $\min\left(u_0, \min' u_0 + \frac{|v|^2}{2}\right)$.

The structure of the representation formula is similar to usual Hopf-Lax formulas for Hamilton-Jacobi equations. Nevertheless, one important difference is the nature of the action. While large deviations of random paths are measured typically by $\int_0^t L(\gamma(s), \dot\gamma(s))\,ds$ or related functionals for diffusions and Lévy processes (see the original theorem by Schilder [170] for the Brownian motion, and the introduction in [89] for further examples), here the action is of a different nature, and is supported by piecewise linear trajectories instead of smooth curves (4.2.4).

### 4.2.5 Qualitative behaviour (in the long term)

Our first interest is to get qualitative insight on the long time behaviour of (4.1). It is thus rather important to be able to understand more quantitatively the previous systems we derived. In the one-dimensional case, it is possible to find more precise values for the action $\mathcal{A}_0^t[\sigma]$ (and the minimisers) with prescribed endpoints $(y, \sigma_0)$ and $(x, \sigma_N)$ (we can set $y = 0$ without loss of generality by translation invariance). We refer to [B5] for detailed calculations, but to conclude this presentation, let us just say that the action between a position $x$ and the origin, with both initial and final velocities at rest is:

$$\mathfrak{A}_0^t(x,0,0) = \begin{cases} \dfrac{3}{2}|x|^{2/3} & \text{if } |x| \le t^{3/2}, \\ \dfrac{|x|^2}{2t^2} + t & \text{if } |x| \ge t^{3/2}. \end{cases} \tag{4.15}$$

This spatial behaviour is illustrated in Figure 4.3 in comparison to the kernel $\frac{x^2}{2t}$ associated with the Hamilton-Jacobi equation coming from the heat equation with vanishing viscosity. In the latter case, the kernel is a family of parabola converging to zero as $t \to \infty$, uniformly on compact intervals. It means that, despite the rarity of finding a Brownian particle far from its origin, the small probability is not uniformly exponentially small. Contrarily, the kernel (4.15) converges towards its envelope $(3/2)|x|^{2/3}$, which is obviously uniformly positive on closed intervals that do not contain the origin. Alternatively speaking, the probability of finding a particle far from its origin remains uniformly exponentially small in the velocity-jump process under study.

### 4.2.6 Getting the acceleration constant

We restrict to the one-dimensional case $n = 1$ for simplicity. We shall quickly comment on the way we get the acceleration constant (4.6). First, we identify the limit problem for the full nonlinear equation (4.1), which is an obstacle version of (4.9) due to the saturation term (as

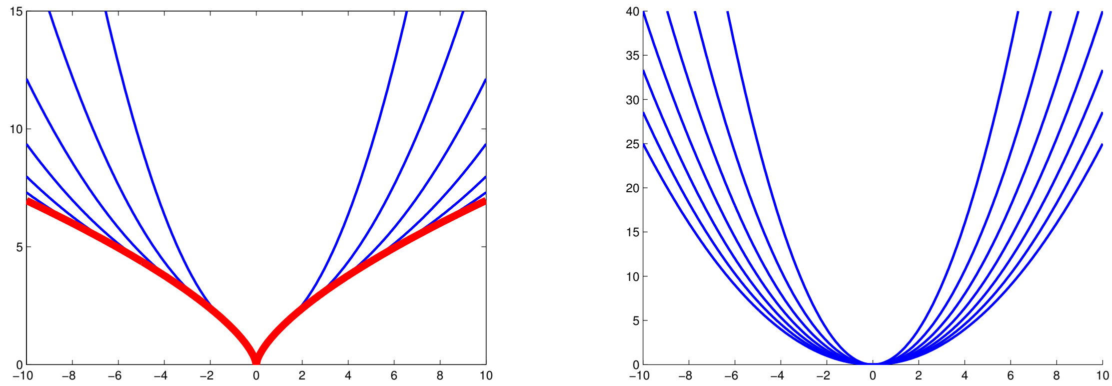



Figure 4.3: Comparison of different qualitative behaviours between the velocity-jump process and the Brownian motion. Left : plot of the kernel (4.15) with initial and final velocities at rest. The red curve is the envelope as $t \to +\infty$. Right : plot of the kernel $\frac{x^2}{2t}$ associated with the heat equation with vanishing viscosity.

usual in Evans-Souganidis type arguments [85], see also Freidlin [95] and Barles, Evans and Souganidis [21]). This requires to redefine notions of viscosity solutions again, study an action as above. I briefly describe the way of thinking here but this is actually not so easy! See [B5] for all the arguments in full details. Once all this is done, we get what we show in Figure 4.4: a cartoon of the tails of the spatial density resulting from the approximation of geometric optics. There is a first zone far ahead where the density is uniformly exponentially small, independent of $r$. It is followed by a region where the density is approximately of separable variables: a sub-exponential anomalous spatial profile multiplied by a growing exponential. It is where the front is actually emerging at $X(t)$ (roughly).

Observe interestingly that this is a structure that was depicted in Chapter 3 in another context of acceleration: two different zones ahead of the dynamics.

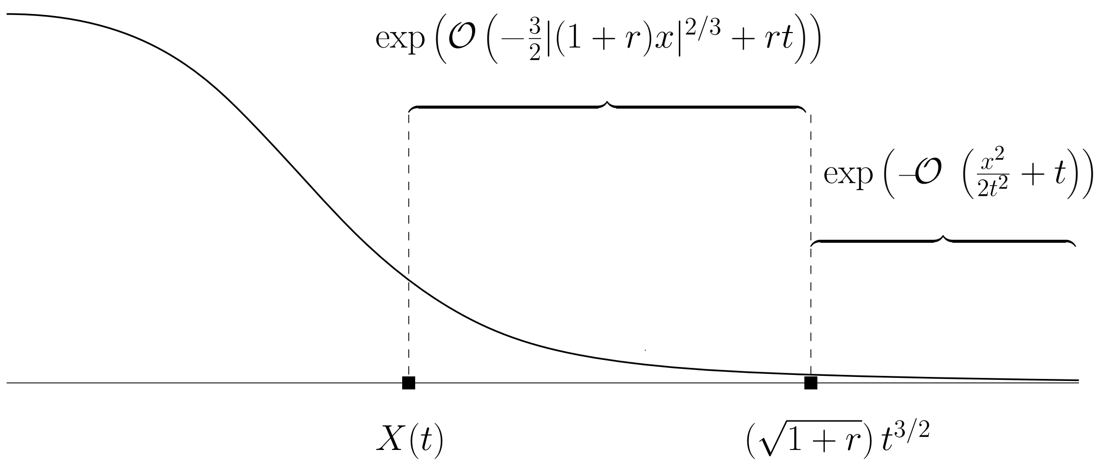



Figure 4.4: Loose description of the dynamics of front acceleration.

## 4.3 Perspectives

1. *Improvements in the kinetic Hamilton-Jacobi method.*

   With Guerand, with are interested in refining the knowledge on the solution $u^0$ found in Theorem 4.1. Namely, a formal expansion has been presented by Luo and Payne in [138, 139] for numerical purposes (note that Hivert proposed a more accurate scheme in [116]) and it would be nice to get this expansion rigorously. This is related to knowing more precisely the corrector in the test function method used to prove the Hamilton-Jacobi limit in Theorem 4.1.

2. *Bramson correction in kinetic reaction transport waves.*

   During my PhD thesis, we proved with Calvez and Nadin in [B8] the existence of travelling waves for a one-dimensional kinetic reaction transport problem. In Section 4.1, an extension to the multi-dimensional case has been presented. Spreading results giving finite speed propagation were also derived. An interesting issue is the existence of a Bramson correction (see Chapter 1) for the Cauchy problem in situations where these waves exist. This would be interesting since the underlying microscopic movement of the particules is purely hyperbolic. Actually, this question is particularly relevant is the context of this memoir, since it makes a bridge between the first and the second part. Indeed, Chapter 1 explains that a core ingredient is to estimate Dirichlet problems and the way to do this in the kinetic framework would be to use hypocoercive estimates on the half line. This is under investigation with Mouhot and Mischler, see also Chapter 6.

3. *Large deviations for jump processes with more general reorientation densities.*

   The paper [B5] considers only $\mathcal{M}_\varepsilon$ being a Gaussian density. The structure of the mathematical treatment in that case makes us think that other densities could be considered, for example $\mathcal{M}_\varepsilon \propto \exp\left(-\gamma^{-1}|v|^\gamma\right)$. This of course implies scaling changes and deserves some attention. To enlarge results to polynomially decaying kernels, however, one would need to change quite a lot of things, since one expects at least exponentially fast propagation, which implies changes of variables of a different type (see [B17, 150]).

4. *Sub-exponential travelling waves in kinetic models.*

   We have shown above that the kinetic reaction-transport equation (4.1) with a KPP type monostable nonlinearity and unbounded velocities exhibits a super-linear propagation phenomena. This is linked to the special behaviour of the linear BGK equation (4.7) when velocities are unbounded. It is yet possible to find situations where there is finite speed propagation for kinetic models with unbounded velocities and it seems that interesting features may arise on the profiles. This seems to be the case for example for kinetic models for conservation laws [70],

   $$\partial_t f + v \cdot \nabla_x f = M(\rho, v) - f, \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},$$

   where

   $$\int_V M(\rho,v)dv = \rho, \quad \int_V vM(\rho,v)dv = a(\rho),$$

   or a bistable kinetic model

   $$\partial_t f + v \cdot \nabla_x f = \mathcal{M}(v)\rho - f + r\rho(\rho-\alpha)\left(\mathcal{M}(v) - f\right), \qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \quad (4.16)$$

for which it is natural that such nonlinearity will stop the acceleration (see the flavour of [56]). Another example would be kinetic models for chemotaxis [47, 84]. For all these examples, the conjecture is that the profiles would decay sub-exponentially in space. We show a simulation in Figure 4.5 on the bistable equation (4.16) that corroborate this possibility.

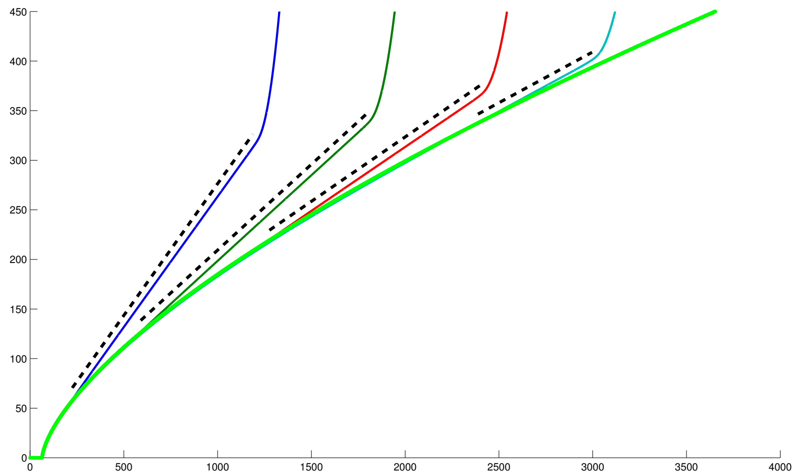



Figure 4.5: Coloured thin lines: Plot of $-\ln(\rho)$, where $\rho$ is the macroscopic density, when $f$ solves (4.16) with a Gaussian $\mathcal{M}$. The curves are ordered by increasing maximal velocity from left to right. For each truncation, the front decays exponentially (dashed line). The envelop of the curves (bold green curve) shows a lighter decay (here, $x \mapsto x^{\frac{2}{3}}$).

# Chapter 5

# A unified approach to fluid approximations of linear kinetic models

**Contents**


**5.1 The question at hand** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 71

**5.2 The history of the mathematical treatment of the problem** . . . . . . . . 72

**5.3 The spectral approach and the result** . . . . . . . . . . . . . . . . . . . . . . 74

**5.4 Application to concrete models** . . . . . . . . . . . . . . . . . . . . . . . . . . 80

**5.5 A word about the proofs** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 82

**5.6 Perspectives** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 83


This chapter is about my recent paper with Clément Mouhot [B26].

## 5.1 The question at hand

The main mathematical object of study in *transport theory* is the linear equation

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \tag{5.1}$$

on the time-dependent density of particles $f = f(t,x,v) \geq 0$ over $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, for $t \geq 0$. The left hand side accounts for free motion and the right hand side accounts for the interaction with a background, for instance scatterers, with an operator $\mathsf{L}$ that only acts on the kinetic variable $v$. Several forms are possible. In nuclear reactor, radiative transfer and semi-conductor theories it is common to consider *scattering operators*, sometimes also called *linear Boltzmann operators*, which have the form

$$\mathsf{L} f(v) = \left( \int_{\mathbb{R}^d} b(v,v') f(v') \, \mathrm{d}v' \right) \mathcal{M}(v) - \nu(v) f(v) \tag{BGK}$$

$$\text{given the } \textit{collision frequency} \quad \nu(v) := \int_{\mathbb{R}^d} b(v,v') \mathcal{M}(v') \, \mathrm{d}v',$$

some *collisional kernel* $b = b(v, v')$ and an *equilibrium distribution* $\mathcal{M}(v)$. In astrophysics and sometimes in semi-conductor theory, one also considers *Fokker-Planck operators* which may be written

$$\mathsf{L}f := \nabla_v \cdot \left( \mathcal{M} \nabla_v \left( \frac{f}{\mathcal{M}} \right) \right). \tag{FP}$$

Finally, as a simplified model of long-range collisional interactions in a gas of charged particles, we also consider *Lévy-Fokker-Planck operators* (given $s \in (0,1)$):

$$\begin{cases} \mathsf{L}(f) = \Delta_v^s f + \nabla_v \cdot (U f) \quad \text{with } U(v) = U(|v|) \text{ radially symmetric so that} \\ \Delta_v^s \mathcal{M} + \nabla_v \cdot (U \mathcal{M}) = 0. \end{cases} \tag{LFP}$$

Denoting $\mathcal{F}$ the Fourier transform, the fractional Laplacian is defined as

$$\Delta_v^s f(v) := -\mathcal{F}^{-1} \left[ |\iota|^{2s} \mathcal{F} f(\iota) \right] (v).$$

The equation (5.1) is too intricate for many applications. When the relevant time and space scales of observation are much larger than the mean free time and mean free path, it is thus natural to search for a simplified regime. The so-called *diffusion theory* was born out of this endeavour, and in the words of Wigner [186], 'this [diffusion] theory gives the spatial variation of the [neutron transport] flux quite accurately in regions well removed from interfaces'. We also refer to [184, Chap. IX] for the diffusion theory of monoenergetic neutrons, to [164, Chap. III.2] for the so-called *Eddington approximation* in radiative transfer theory, and to [34, Chap. 2] for a modern mathematical review. Note important that some *anomalous* diffusions and Levy flights are also observed by biologists and physicists [11, 176, 3, 143, 168]. Anomalous diffusion and power law heavy-tails occur in many contexts: astrophysics, granular gases, collective behaviour ....

The question is thus the following. Given a solution $f$ in $L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ to equation (5.1) we denote

$$f_\varepsilon(t,x,v) := f\left( \frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v \right) \in L^\infty_t \left( [0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}) \right),$$

where $\varepsilon > 0$, and $\theta(\varepsilon)$ is an unknown scaling function. The equation satisfied by $f_\varepsilon$ is

$$\theta(\varepsilon) \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \mathsf{L} f_\varepsilon. \tag{5.2}$$

Can one find a suitable scaling in time $\theta(\varepsilon)$ such that $\frac{f_\varepsilon}{\mathcal{M}}$ converges in some explicit sense and with explicit rate to a macroscopic function $r$, where $r$ solves a known standard or anomalous diffusion type equation?

## 5.2 The history of the mathematical treatment of the problem

Let us start by reviewing the literature on this kind of problems. It has been growing on scattering and Fokker-Planck collision operators. As said earlier, the first investigation was by Wigner for neutron transport [186]. Larsen and Keller [133] and then later on Frisch *et al* [97], Bensoussan *et al* [24] and Bardos *et al* [20] have shown that when $\mathcal{M}$ is given by a Maxwellian distribution function, the mean square displacement of particle is a linear function of time.

The correct time scaling is then $\theta(\varepsilon) = \varepsilon^2$ and leads to a diffusion equation for the density of particles

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v \xrightarrow[\varepsilon\to 0]{} \rho(t,x)$$

where $\rho$ solves

$$\rho_t = \nabla_x \cdot (A\nabla_x \rho) \quad \text{with} \quad A := \int_{\mathbb{R}^d} v \otimes \mathbf{G}(v)\,\mathrm{d}v \quad \text{with} \quad \mathsf{L}(\mathbf{G}) = -v\mathcal{M}.$$

Later on, Degond *et al* obtained the same kind of results in a larger context [73]. As long as the diffusion matrix exists, the limit is a standard diffusion. However when the diffusion matrix $A$ is infinite, these methods, in particular the *Hilbert expansion* do not apply and thus scaling limits were unclear at that stage. This actually happens, especially when $\mathcal{M}$ is a power law.

A first answer was given simultaneously by Mellet, Mouhot and Mischler in [146] by analytic arguments and by Jara, Komorowski and Olla with probabilistic arguments [123]. The right scaling is then "anomalous" and the limit is a fractional diffusion equation. Flights are Levy walks rather than Brownian motions. Before these, very few results of anomalous diffusion limits for kinetic models were derived. Things appear in Frisch *et al.* [97] in the physics literature for radiative transfer ($\Delta^{\frac{1}{2}}$ in the limit), and in the mathematics literature [39, 101, 77] for a gas confined between two plates when the distance between the plates goes to zero (standard diffusive limit but anomalous time scaling $\theta(\varepsilon) = \varepsilon^2|\ln\varepsilon|$). More precisely, [146] considered (BGK) with $b(v,v') = \lfloor v\rceil^{-\beta}\lfloor v'\rceil^{-\beta}$ and an algebraically decaying $\mathcal{M}$,

$$\mathcal{M}(v) := c_0\lfloor v\rceil^{-d-\alpha}, \text{where } \lfloor v\rceil = (1+|v|^2)^{\frac{1}{2}},$$

with $\beta > 0$ and $\alpha \in (0, 2+\beta)$ and showed that the relevant time scaling is $\theta(\varepsilon) = \varepsilon^{\frac{\alpha+\beta}{1+\beta}}$ and that $\rho_\varepsilon$ converges towards $\rho$ solving

$$\rho_t = \kappa\Delta_x^{\frac{1}{2}\frac{\alpha+\beta}{1+\beta}}\rho,$$

where $\kappa$ is computed through a simple scaled integral of $\mathcal{M}$. The topology of the convergence is weak-$*$ in $L^\infty(0,T,L^2_{x,v}(\mathcal{M}^{-1}))$ but no rate was provided, even though the limit equation was characterised. The original proof in [146] relies on Laplace-Fourier calculation in time-space, which makes it quite rigid. This first result was then reproved with different methods: a moment method by Mellet [145] and a modified Hilbert expansion approach by Mellet *et al.* [165].

For the Fokker-Planck equation with an algebraically decaying $\mathcal{M}$, there has been a recent activity. Let us review what is known. When $\alpha > 4$, a standard diffusion limit is shown with PDE arguments by Nasreddine *et al.* [153]. The diffusion coefficient is characterised but no convergence rates are given. The critical case $\alpha = 4$ leads to taking an anomalous scaling $\theta(\varepsilon) = \varepsilon^2|\ln(\varepsilon)|$ but the limit is still a standard diffusion, as proved by probabilistic arguments by Cattiaux *et al.* in [51]. The more involved case $\alpha \in (0,4)$ originates intense discussions. In dimension $d = 1$, Lebeau and Puel obtain in [135] by PDE methods that the relevant scaling function is $\theta(\varepsilon) = \varepsilon^{\frac{\alpha+2}{3}}$ and that the macroscopic limit is fractional. Their method of proof is by studying a spectral problem reminiscent of Ellis–Pinsky seminal work [82]. We will come back to such a method later on. Fournier and Tardif recover this result with probabilistic arguments in the paper [94]. The multidimensional case is more recently treated in [93] by probabilistic methods, sometimes without characterisation of the coefficients of the limit equation and without rate.

## 5.3 The spectral approach and the result

Our aim is to provide a framework that can handle all three types of collision operators (BGK), (FP), (LFP) in one go, with explicit coefficients in the limit equation and explicit convergence rates. We can view this approach as a approach *à la* Ellis and Pinsky for Boltzmann [82] (see also Nicolaenko [154]). The paper by Lebeau and Puel [135] has been of crucial inspiration on some aspects.

We rewrite the equation (5.1) by changing the unknown to $h := \frac{f}{\mathcal{M}}$:

$$\partial_t h + v \cdot \nabla_v h = Lh \quad \text{where} \quad Lh := \mathcal{M}^{-1} \mathsf{L}\left(\mathcal{M}h\right). \tag{5.3}$$

This change of unknown is convenient since asymptotic estimates compare $f$ with the equilibrium $\mathcal{M}$.

Consider the complex Hilbert spaces $L^2(\mathbb{R}^d; \mathcal{M}\,\mathrm{d}v) =: L^2_v(\mathcal{M})$ and $L^2(\mathbb{R}^d \times \mathbb{R}^d; \mathcal{M}\,\mathrm{d}x\,\mathrm{d}v) =: L^2_{x,v}(\mathcal{M})$ and denote $\|h\|_k := \|(1+|\cdot|^2)^{\frac{k}{2}} h\|_{L^2(\mathcal{M})}$ (the integration variable(s) will be emphasized when there is ambiguity). We omit the index when $k = 0$. The scalar product $\langle \cdot, \cdot \rangle$ refers to $L^2_v(\mathcal{M})$ or $L^2_{x,v}(\mathcal{M})$ depending on the context.

We assume, for some $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\lambda \in \mathbb{R}^*_+$, the following,

**Hypothesis 5.1** (Equilibria)**.** *The equilibrium $\mathcal{M}$ takes one of the following two forms.*

*(i) Either it is given by*

$$\mathcal{M}(v) = c_{\alpha,\beta} \lfloor v \rceil^{-(d+\alpha)} \textit{ with } c_{\alpha,\beta} := \left( \int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha-\beta} \,\mathrm{d}v \right)^{-1} \textit{ and } \lfloor v \rceil := (1+|v|^2)^{\frac{1}{2}}. \tag{5.4}$$

*(ii) Or it is a smooth positive radially symmetric function decaying faster than any polynomial. The latter case is denoted by '$\alpha = +\infty$' in the sequel.*

Note that the normalisation implies the following **generalised mass condition**

$$\int_{\mathbb{R}^d} \mathcal{M}_\beta(v)\,\mathrm{d}v = 1 \quad \text{with} \quad \mathcal{M}_\beta := \lfloor \cdot \rceil^{-\beta} \mathcal{M}. \tag{5.5}$$

**Hypothesis 5.2** (Weighted coercivity)**.** *The operator $L$ is linear, independent of time $t$ and space $x$, commutes with rotations in $v$, is closed densely defined on* $\mathrm{Dom}(L) \subset L^2_v(\mathcal{M})$ *and satisfies $L(1) = L^*(1) = 0$, where $L^*$ is the $L^2_v(\mathcal{M})$-adjoint. Finally $\tilde{L} := \lfloor \cdot \rceil^{\frac{\beta}{2}} L(\lfloor \cdot \rceil^{\frac{\beta}{2}} \cdot)$ is closed densely defined on* $\mathrm{Dom}(\tilde{L}) \subset L^2_v(\mathcal{M})$*, with the spectral gap estimate*

$$\forall\, g \in \mathrm{Dom}(\tilde{L}), \quad g \perp \lfloor \cdot \rceil^{-\frac{\beta}{2}}, \quad -\mathrm{Re}\, \langle \tilde{L} g, g \rangle \geq \lambda\, \|g\|^2.$$

*The latter means, translating back to $L$,*

$$\forall\, h \in \mathrm{Dom}(L), \quad -\mathrm{Re}\, \langle Lh, h \rangle \geq \lambda\, \|h - \mathcal{P}h\|^2_{-\beta} \quad \textit{with} \quad \mathcal{P}h := \left( \int_{\mathbb{R}^d} h(v') \mathcal{M}_\beta(v')\,\mathrm{d}v' \right).$$

**Hypothesis 5.3** (Amplitude of collisions at large velocities)**.** *Given* $0 \leq \chi \leq 1$ *a smooth function that is* 1 *on $B(0,1)$ and* 0 *outside $B(0,2)$, and $\chi_R = \chi(\frac{\cdot}{R})$ for $R \geq 1$, one has*

$$\|L(\chi_R)\|_\beta \lesssim R^{-\frac{\alpha+\beta}{2}}.$$

Our first result, on the basis of the three previous hypothesis, is a quantitative construction of a branch of 'fluid eigenmode' in the asymptotic of large time and small spatial frequencies, i.e. a unique eigenvalue branching from zero for $\tilde{L}^* + i\eta\lfloor v\rceil^{\beta}(v\cdot\sigma)$ for small $\eta$ (see Figure 5.1):

**Lemma 5.4** (Construction of the fluid mode)**.** *Given Hypothesis 5.1–5.2–5.3, there are $\eta_0 > 0$ and $r_0 \in (0,\lambda)$, explicit in terms of the constants in these hypothesis, such that for any $\eta \in (0,\eta_0)$ and any $\sigma \in \mathbb{S}^{d-1}$, there is a unique solution $\varphi_\eta = \varphi_\eta(v) \in L^2_v(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and $\mu(\eta) \in B(0,r_0)$ to*

$$-L^*\varphi_\eta - i\eta(v\cdot\sigma)\varphi_\eta = \mu(\eta)\lfloor v\rceil^{-\beta}\varphi_\eta \quad \textit{with} \quad \int_{\mathbb{R}^d}\varphi_\eta(v)\,\mathcal{M}_\beta(v)\,\mathrm{d}v = 1.$$

*Moreover, the branch $(\varphi_\eta,\mu(\eta))$ connects to $(1,0)$ as $\eta\to 0$, with $\mu(\eta)>0$ and the asymptotics*

$$\|\varphi_\eta - 1\|_{-\beta} \lesssim \mu(\eta)^{\frac{1}{2}} \quad \textit{and} \quad \mu(\eta) \in (\mathrm{R}_0\Theta(\eta), \mathrm{R}_1\Theta(\eta)) \tag{5.6}$$

*for some $0 < \mathrm{R}_0 < \mathrm{R}_1$, where the function $\Theta$ is defined by*

$$\Theta(\eta) := \begin{cases} \eta^2 & \textit{when } \alpha > 2+\beta, \\ \eta^2|\ln(\eta)| & \textit{when } \alpha = 2+\beta, \\ \eta^{\frac{\alpha+\beta}{1+\beta}} & \textit{when } -\beta < \alpha < 2+\beta. \end{cases} \tag{5.7}$$

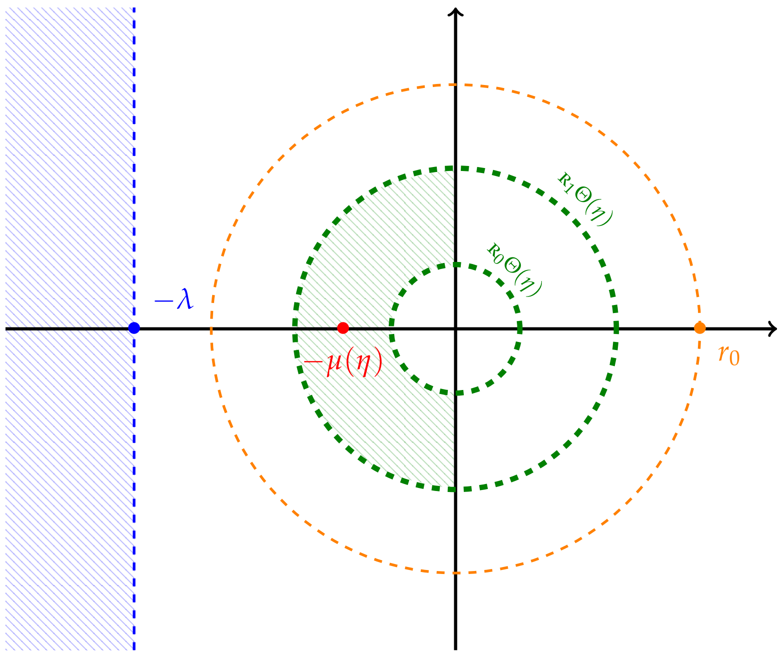



Figure 5.1: The blue dashed zone on the left of $\mathrm{Re}\, z = -\lambda$ corresponds to the spectral gap estimates on $\tilde{L}^* + i\eta\lfloor v\rceil^{\beta}(v\cdot\sigma)$ for $g \perp \lfloor\cdot\rceil^{-\frac{\beta}{2}}$ (Hypothesis 5.2). The green dashed zone is where Lemma 5.4 construct a unique real eigenvalue $-\mu(\eta)$ of the latter operator, that goes to zero as $\eta \to 0$.

Note that $\Theta$ is well-defined in the case $\alpha \in (-\beta, 2+\beta)$ since $(1+\beta) > (\alpha+\beta)/2 > 0$. To identify the macroscopic limit with quantitative rates and constants, it is necessary to estimate the leading order of $\mu(\eta)$, and this requires estimates on the eigenvector, which is our last hypothesis. We denote $|u|_\eta := (\eta^{\frac{2}{1+\beta}} + |u|^2)^{\frac{1}{2}}$.

**Hypothesis 5.5** (Scaling of the fluid mode)**.** *We make different assumptions depending on $\alpha$:*

*(i)* Case $\alpha > 2+\beta$: *The fluid mode $\varphi_\eta$ constructed in Lemma 5.4 satisfies*

$$\forall\, \ell < \alpha, \quad \left\|\varphi_\eta\right\|_\ell \lesssim_\ell 1.$$

*(ii)* Case $\alpha \in (-\beta, 2+\beta]$: *The rescaled fluid mode $\Phi_\eta := \varphi_\eta(\eta^{-\frac{1}{1+\beta}}\cdot)$ is converging in $L^2_{\mathrm{loc}}(\mathbb{R}^d \backslash 0)$ as $\eta \to 0$ to a limit $\Phi$ and satisfies the pointwise controls*

$$\forall\, \eta \in (0, \eta_1),\ \forall\, u \in \mathbb{R}^d, \quad \begin{cases} \left|\Phi_\eta(u)\right| \lesssim |u|_\eta^{C\mu(\eta)}, \\ \left|\operatorname{Im}\Phi_\eta(u)\right| \lesssim |u|_\eta^{\beta+\min(\alpha,1)-\delta} \end{cases} \tag{5.8}$$

*for some $\eta_1 \in (0, \eta_0)$ and $C > 0$ and $\delta < 1 + \min(\alpha, 1) + \beta - \alpha$. We also make the following additional assumptions depending in the two following subcases:*

*(ii)-(a)* Case $\alpha = 2+\beta$: *There are $\mathfrak{a}(\eta) \to 0$ and $\Omega : \mathbb{R}^d \to \mathbb{R}$ locally integrable such that*

$$\begin{cases} \left|\displaystyle\int_{1 \geq |u| \geq \eta^{\frac{1}{1+\beta}}} (u\cdot\sigma)\Big[\operatorname{Im}\Phi_\eta(u) - \operatorname{Im}\Phi(u)\Big]|u|_\eta^{-d-\alpha}\,\mathrm{d}u\right| \leq \mathfrak{a}(\eta)|\ln(\eta)|, \\ \forall\, \sigma' \in \mathbb{S}^{d-1}, \quad \dfrac{\operatorname{Im}\Phi(\lambda\sigma')}{\lambda^{1+\beta}} \xrightarrow[\lambda\to 0]{\lambda\neq 0} \Omega(\sigma') \quad in \quad L^1(\mathbb{S}^{d-1}). \end{cases}$$

*(ii)-(b)* Case $\alpha \in [0, \beta]$: *The additional following integral control holds:*

$$\int_{|u|\geq 1} \left|\Phi_\eta(u)\right|^2 |u|_\eta^{-d-\alpha+\beta}\,\mathrm{d}u \lesssim 1. \tag{5.9}$$

*(ii)-(c)* Case $\alpha \in (-\beta, 0)$: *The integral control* (5.9) *holds, and furthermore the the limit rescaled mass is positive:*

$$\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u > 0. \tag{5.10}$$

Note that in (5.8), $|u|_\eta^{C\mu(\eta)} \sim 1$ as $\eta \to 0$ in the region $|u| \lesssim \eta^{\frac{1}{1+\beta}}$. The second part of point (ii) above is subtle and made necessary by the fact that the case $\alpha = 2+\beta$ is borderline between two different regimes (standard diffusion vs. fractional diffusion) as well as borderline between two different scalings for obtaining the diffusion coefficient (fluid mode in variable $v$ vs. fluid mode in the rescaled variable $u = \eta^{-\frac{1}{1+\beta}}v$).

With these four hypothesis we can characterise the precise scaling of the fluid eigenvalue:

**Lemma 5.6** (Rescaled limit of the fluid eigenvalue)**.** *Assume Hypothesis 5.1–5.2–5.3–5.5. The eigenvalue $\mu(\eta)$ constructed in Lemma 5.4 satisfies (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypothesis)*

$$\mu(\eta) \sim_{\eta\to 0} \mu_0\Theta(\eta), \tag{5.11}$$

*where the constant $\mu_0 > 0$ is positive and determined as follows:*

$$
\begin{cases}
\mu_0 := \int_{\mathbb{R}^d} (v \cdot \sigma)\, F(v) \mathcal{M}(v)\, \mathrm{d}v \quad \text{when } \alpha > 2+\beta, \\
\qquad \text{where } F = \lim_{\eta \to 0} \frac{\operatorname{Im} \varphi_\eta}{\eta} \text{ is solution to } LF = -(v \cdot \sigma) \text{ and } \int_{\mathbb{R}^d} F(v)\, \mathcal{M}_\beta(v)\, \mathrm{d}v = 0, \\
\mu_0 := \frac{c_{2+\beta,\beta}}{1+\beta} \int_{\mathbb{S}^{d-1}} (\sigma \cdot \sigma') \Omega(\sigma')\, \mathrm{d}\sigma' \quad \text{when } \alpha = 2+\beta, \\
\qquad \text{where } \Omega(u) = \lim_{\lambda \to 0,\, \lambda \neq 0} \frac{\operatorname{Im} \Phi(\lambda u)}{\lambda^{1+\beta}} \text{ and } \Phi = \lim_{\eta \to 0} \Phi_\eta = \lim_{\eta \to 0} \varphi_\eta \left( \eta^{-\frac{1}{1+\beta}} \cdot \right), \\
\mu_0 := c_{\alpha,\beta} \int_{\mathbb{R}^d} (u \cdot \sigma) \operatorname{Im} \Phi(u) |u|^{-d-\alpha}\, \mathrm{d}u \quad \text{when } \alpha \in (-\beta, 2+\beta).
\end{cases}
$$

Note how in the previous statement, when $\alpha > 2+\beta$, the function $F$ used in the previous works on standard diffusive limit (usually with $\beta = 0$) is recovered here as a limit of our fluid mode; this allows our proof to track the convergence rate. Define the *diffusion exponent*

$$
\zeta = \zeta(\alpha, \beta) := \begin{cases} 2 & \text{when } \alpha \in [2+\beta, +\infty] \\ \dfrac{\alpha_+ + \beta}{1+\beta} & \text{when } \alpha \in (-\beta, 2+\beta), \end{cases} \tag{5.12}
$$

with $\alpha_+ := \max(\alpha, 0)$, and the *scaling function*

$$
\theta(\varepsilon) := \begin{cases} \varepsilon^\zeta & \text{when } \alpha \in (-\beta, +\infty] \setminus \{0, 2+\beta\}, \\ \varepsilon^2 |\ln \varepsilon| & \text{when } \alpha = 2+\beta, \\ \dfrac{\varepsilon^{\frac{\beta}{1+\beta}}}{|\ln \varepsilon|} & \text{when } \alpha = 0. \end{cases} \tag{5.13}
$$

Note that the threshold $\alpha = 2+\beta$ between standard and fractional diffusion corresponds to whether or not $\mathcal{M}_\beta$ has finite variance. We finally derive the *diffusion coefficient*:

**Lemma 5.7** (Diffusion coefficient)**.** *Assume Hypothesis 5.1–5.2–5.3–5.5. Then the following limit holds (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypothesis)*

$$
\kappa := \lim_{\eta \to 0} \frac{\mu(\eta) |\xi|^{-\zeta}}{\theta(\varepsilon) \langle 1, \varphi_\eta \rangle} = \mu_0 \times \begin{cases} \|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} & \text{when } \alpha > 0, \\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} & \text{when } \alpha = 0, \\ \left[ c_{\alpha,\beta} \displaystyle\int_{\mathbb{R}^d} \Phi(u) |u|^{-d-\alpha}\, \mathrm{d}u \right]^{-1} & \text{when } \alpha \in (-\beta, 0). \end{cases} \tag{5.14}
$$

Note that the last line is well-defined thanks to Hypothesis 5.5-(v) (equation (5.10)). The diffusion coefficient thus emerges from ratios between (rescaled) integrals as follows:

$$\kappa := \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} (v\cdot\sigma)\, F(v)\mathcal{M}(v)\,\mathrm{d}v}{\|\mathcal{M}\|_{L^1(\mathbb{R}^d)}} & \boxed{\text{when } \alpha > 2+\beta} \\[2ex] \dfrac{1}{1+\beta}\dfrac{\displaystyle\int_{\mathbb{S}^{d-1}} (\sigma\cdot\sigma')\Omega(\sigma')\,\mathrm{d}\sigma'}{\displaystyle\int_{\mathbb{R}^d} \lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha = 2+\beta} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d} (u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u}{\displaystyle\int_{\mathbb{R}^d} \lfloor v\rceil^{-d-\alpha}\,\mathrm{d}v} & \boxed{\text{when } \alpha \in (0,2+\beta)} \\[2ex] \dfrac{1+\beta}{|\mathbb{S}^{d-1}|}\dfrac{\displaystyle\int_{\mathbb{R}^d} (u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u}{\displaystyle\int_{\mathbb{R}^d} \lfloor v\rceil^{-d-\alpha-\beta}\,\mathrm{d}v} & \boxed{\text{when } \alpha = 0} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d} (u\cdot\sigma)\,\mathrm{Im}\,\Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u}{\displaystyle\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u} & \boxed{\text{when } \alpha \in (-\beta,0)} \end{cases} \tag{5.15}$$

where we recall, for the legibility of this catalogue of formula:

$$F = \lim_{\eta\to 0}\frac{\mathrm{Im}\,\varphi_\eta}{\eta}, \qquad \Phi = \lim_{\eta\to 0}\Phi_\eta = \lim_{\eta\to 0}\varphi_\eta\left(\eta^{-\frac{1}{1+\beta}}\cdot\right), \qquad \Omega(u) = \lim_{\lambda\to 0,\ \lambda\neq 0}\frac{\mathrm{Im}\,\Phi(\lambda u)}{\lambda^{1+\beta}},$$

and (note that $\alpha > 2+\beta$ in this case) $F$ is also the unique solution to $LF = -(v\cdot\sigma)$ with $\int_{\mathbb{R}^d} F(v)\lfloor v\rceil^{-d-\alpha-\beta}\,\mathrm{d}v = 0$. For legibility again, we wrote, in the cases $\alpha \in (-\beta, 2+\beta]$, the formula for $\kappa$ with $\mathcal{M}$ given by (5.4), and we refer to [B26, Section 9 (Remarks and extensions)] for more general $\mathcal{M}$.

**Theorem 5.8** (B., Mouhot [B26] - Unified second fluid approximation, see Figure 5.2)**.** *Assume Hypothesis 5.1–5.2–5.3–5.5, and consider $f_\varepsilon \in L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ solving* (5.1) *in the weak sense with initially*

$$r_\varepsilon(0,\cdot) \xrightarrow[\varepsilon\to 0]{H^{-\zeta}(\mathbb{R}^d)} r(0,\cdot), \tag{5.16}$$

*and additional conditions on $f_\varepsilon$ detailed in [B26]. Then for any $T > 0$ (and recalling the definition of $\zeta$ in* (5.12)*)*

$$\left\|\frac{f_\varepsilon}{\mathcal{M}} - r\right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

*when $\alpha > \beta$ and*

$$\left\|\left|\ln\frac{2|\nabla_x|}{1+|\nabla_x|}\right|\left(\frac{f_\varepsilon}{\mathcal{M}} - r\right)\right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

*when $\alpha = \beta$ and*

$$\left\||\nabla_x|^{\frac{\beta-|\alpha|}{2(1+\beta)}}\lfloor\nabla_x\rceil^{-\frac{\beta-|\alpha|}{2(1+\beta)}}\left(\frac{f_\varepsilon}{\mathcal{M}} - r\right)\right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

*when $\alpha \in (-\beta, \beta)$, where $r = r(t,x)$ solves*

$$\partial_t r = \kappa \, \Delta_x^{\frac{\zeta}{2}} r, \quad t > 0, \quad \textit{with initial data } r(0,\cdot) \textit{ defined in } (5.16).$$

*The rates of convergence are estimated in terms of $T$, the constants, error terms and convergence rates in Hypothesis 5.1–5.2–5.3–5.5, and the initial convergence rate on $r_\varepsilon$ in* (5.16)*. Apart from the latter (that depends on the initial data), the errors we obtain are polynomial in $\varepsilon$ for $\alpha \in (-\beta, +\infty) \setminus \{0, 2+\beta\}$ and logarithmic for $\alpha \in \{0, 2+\beta\}$.*

This theorem is the core contribution of [B26], and is used to obtain results on concrete models in the corollaries below. Together with Lemmas 5.4–5.6–5.7, it reveals the relevant macroscopic scales for a large class of operators in any dimension and provides a unified theoretical framework to answer questions of the last decades on the topic. The diffusive limit is reduced to a spectral problem –the construction of the fluid mode– that we solve in a general setting. The proof is constructive and the key constants governing the macroscopic behaviour are derived.

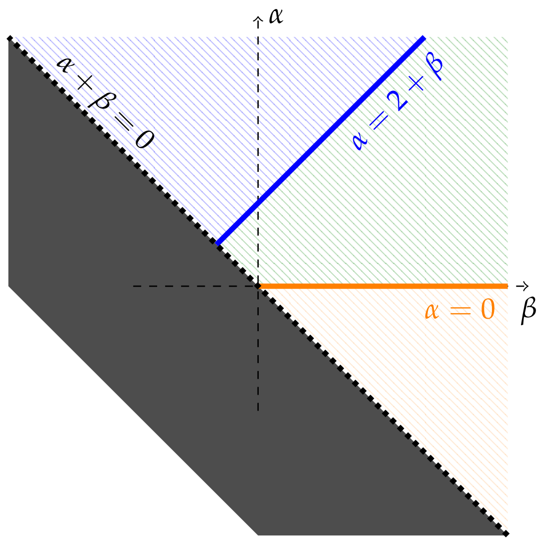



Figure 5.2: Summary of the results in the $(\alpha, \beta)$ plane. Admissible parameters are in half-plane $\alpha + \beta > 0$. The blue hatched area leads to $\theta(\varepsilon) = \varepsilon^2$ and a standard diffusive limit with symbol $\kappa|\xi|^2$. The blue line is the set of parameters yielding the anomalous scaling $\theta(\varepsilon) = \varepsilon^2|\ln(\varepsilon)|$ but still a standard diffusive limit with symbol $\kappa|\xi|^2$. The green hatched area results into the fractional scaling $\theta(\varepsilon) = \varepsilon^{\frac{\alpha+\beta}{1+\beta}}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\alpha+\beta}{1+\beta}}$. The orange bold line yields the fractional scaling $\theta(\varepsilon) = \varepsilon^{\frac{\beta}{1+\beta}}|\ln(\varepsilon)|^{-1}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\beta}{1+\beta}}$. Finally, the orange hatched area yields the fractional scaling $\theta(\varepsilon) = \varepsilon^{\frac{\beta}{1+\beta}}$ and a fractional diffusive limit with symbol $\kappa|\xi|^{\frac{\beta}{1+\beta}}$.

Note that $r(t,x)$ is the limit (in the topology of the above theorem) of the weighted velocity average

$$r_\varepsilon(t,x) = \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) \lfloor v \rceil^{-\beta} \, dv.$$

When $\alpha > 0$, the density $\rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f\left(\frac{t}{\theta(\varepsilon)}, \frac{x}{\varepsilon}, v\right) \mathrm{d}v$ exists and also converges to $r(t,x)$.

## 5.4 Application to concrete models

We now apply the previous abstract theorem to particular models.

**Corollary 5.9** (Scattering equation)**.** *Assume that* $\mathsf{L}$ *is the scattering operator* (BGK) *with* $b \in \mathcal{C}^1$ *and* $\mathcal{M}$ *satisfying Hypothesis 5.1 and that, for some constant* $\nu_0 > 0$ *and* $\beta > -\alpha$

$$\begin{cases} \forall\, v \in \mathbb{R}^d, \quad \lfloor v \rceil^{-\beta} \lesssim \nu(v) \lesssim \lfloor v \rceil^{-\beta} \\ \forall\, v \in \mathbb{R}^d \setminus \{0\}, \quad \lambda^\beta \nu(\lambda v) \sim_{\lambda \to \infty} \nu_0 |v|^{-\beta} \\ \forall\, v \in \mathbb{R}^d, \quad \|b(v,\cdot)\|_\beta + \|b(\cdot,v)\|_\beta \lesssim \lfloor v \rceil^{-\beta}. \end{cases} \tag{5.17}$$

*This includes* $b(v,v') = \lfloor v \rceil^{-\beta} \lfloor v' \rceil^{-\beta}$ *for any* $\alpha + \beta > 0$*, and* $b(v,v') = \lfloor v - v' \rceil^{-\beta}$ *when* $\beta < 0$ *and* $\alpha + \beta > 0$ *or when* $\beta \geq 0$ *and* $\alpha > 3\beta$*. Then Theorem 5.8 applies with* $\alpha, \beta$ *given in Hypothesis 5.1 and* (5.17)*. This proves the diffusive limit for solutions to* (5.2) *with quantitative rate, diffusion exponent* $\zeta = \frac{\alpha_+ + \beta}{1+\beta}$*, scaling function* (5.13) *and diffusion coefficient* (5.15)*.*

This recovers and unifies all results from [165, 73, 145, 146] (except for the case of space-dependent collision kernels in [73]) and extend them to new cases such as $\alpha \in (-\beta, 0)$ (infinite mass). The convergence rate is also new. Our approach bears partial similarities with, but differs from, the Hilbert expansions in [165] and [73], the moment method in [145] and the Fourier-Laplace calculation in [146].

In fact the constants can be computed explicitly since then

$$\begin{cases} F(u) = \nu(v)^{-1}(v \cdot \sigma) & \text{when } \alpha > 2 + \beta, \\ \Omega(u) = \nu_0^{-1} |u|^\beta (u \cdot \sigma) & \text{when } \alpha = 2 + \beta, \\ \Phi(u) = \dfrac{\nu_0}{\nu_0 - i|u|^\beta (u \cdot \sigma)} & \text{when } \alpha \in (-\beta, 2+\beta), \end{cases}$$

resulting in the diffusion coefficient

$$\kappa := \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} (v \cdot \sigma)^2 \, \nu(v)^{-1} \lfloor v \rceil^{-d-\alpha} \, \mathrm{d}v}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha} \, \mathrm{d}v} & \boxed{\text{when } \alpha \in (2+\beta, +\infty)} \\[2ex] \dfrac{1}{\nu_0(1+\beta)} \dfrac{\int_{\mathbb{S}^{d-1}} (\sigma \cdot \sigma')^2 \, \mathrm{d}\sigma'}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha} \, \mathrm{d}v} & \boxed{\text{when } \alpha = 2+\beta} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d} \frac{\nu_0 |u|^\beta (u \cdot \sigma)^2}{\nu_0^2 + |u|^{2\beta}(u \cdot \sigma)^2} \frac{\mathrm{d}u}{|u|^{d+\alpha}}}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha} \, \mathrm{d}v} & \boxed{\text{when } \alpha \in (0, 2+\beta)} \\[2ex] \dfrac{(1+\beta)}{|\mathbb{S}^{d-1}|} \dfrac{\displaystyle\int_{\mathbb{R}^d} \frac{\nu_0 |u|^\beta (u \cdot \sigma)^2}{\nu_0^2 + |u|^{2\beta}(u \cdot \sigma)^2} \frac{\mathrm{d}u}{|u|^{d}}}{\displaystyle\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\beta} \, \mathrm{d}v} & \boxed{\text{when } \alpha = 0} \\[2ex] \dfrac{\displaystyle\int_{\mathbb{R}^d} \frac{\nu_0 |u|^\beta (u \cdot \sigma)^2}{\nu_0^2 + |u|^{2\beta}(u \cdot \sigma)^2} \frac{\mathrm{d}u}{|u|^{d+\alpha}}}{\displaystyle\int_{\mathbb{R}^d} \frac{\nu_0^2}{\nu_0^2 + |u|^{2\beta}(u \cdot \sigma)^2} \frac{\mathrm{d}u}{|u|^{d+\alpha}}} & \boxed{\text{when } \alpha \in (-\beta, 0)} \end{cases}$$

as well as $\kappa := \|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} (v \cdot \sigma)^2 \nu^{-1} \mathcal{M}(v) \, \mathrm{d}v$ is the case "$\alpha = +\infty$".

**Corollary 5.10** (Kinetic Fokker-Planck equation)**.** *Assume that* $\mathsf{L}$ *is the Fokker-Planck operator* (FP) *with* $\mathcal{M}$ *satisfying Hypothesis 5.1 with* $\alpha > 0$*. Then Theorem 5.8 applies with* $\alpha$ *given in Hypothesis 5.1 and* $\beta = 2$*. This proves the diffusive limit for solutions to* (5.2) *with quantitative rate, diffusion exponent* $\zeta = \min\left(2, \frac{\alpha+2}{3}\right)$*, scaling function* (5.13) *and diffusion coefficient* (5.15)*.*

Note that the constants may be precised using that $\Phi$ solves the Schrödinger-type equation

$$-|u|^2 \Delta_u \Phi + (d+\alpha) u \cdot \nabla_u \Phi - i(u \cdot \sigma)|u|^2 \Phi = 0 \quad \text{with the normalisation } \Phi(0) = 1.$$

In particular in the case $\alpha = 2 + \beta = 4$, the function $\Omega$ solves

$$-|u|^2 \Delta_u \Omega + (d+\alpha) u \cdot \nabla_u \Omega = (u \cdot \sigma)|u|^2 \quad \text{with } \Omega(0) = 0 \quad \Longrightarrow \quad \Omega(u) := \frac{|u|^2 (u \cdot \sigma)}{d+8}.$$

This recovers and unifies all results from [51, 153, 94, 93, 135] and obtains the first derivation of the diffusion coefficient in dimension higher than 1. The convergence rate is also new.

**Corollary 5.11** (Kinetic Lévy-Fokker-Planck equation)**.** *Assume that* $\mathsf{L}$ *is the Lévy-Fokker-Planck operator* (LFP) *with parameter* $s \in \left(\frac{1}{2}, 1\right)$ *and with* $\mathcal{M}$ *satisfying Hypothesis 5.1 with* $\alpha > s$*. Then Theorem 5.8 applies with* $\beta := 2s - \alpha$*. This proves the diffusive limit for solutions to* (5.2) *with quantitative rate and diffusion exponent*

$$\zeta = \begin{cases} 2 & \textit{when } \alpha \geq 1+s \\[1ex] \dfrac{2s}{1+2s-\alpha} & \textit{when } \alpha \in (s, 1+s), \end{cases}$$

*and scaling function* (5.13) *and diffusion coefficient* (5.15).

The formula (5.15) for the diffusion coefficient may be precised with

$$\Phi(u) := \exp\left(i\frac{2sc_{\alpha,0}}{c_{\alpha,\beta}}\frac{|u|^{\beta}(u\cdot\sigma)}{1+\beta}\right), \qquad \Omega(u) := \frac{2sc_{\alpha,0}}{c_{\alpha,\beta}}\frac{|u|^{\beta}(u\cdot\sigma)}{1+\beta}.$$

This gives, in particular,

$$\kappa := \begin{cases} \dfrac{2sc_{\alpha,0}^2}{c_{\alpha,\beta}(1+\beta)^2}\displaystyle\int_{\mathbb{S}^{d-1}}(\sigma'\cdot\sigma)^2\,\mathrm{d}\sigma' & \boxed{\text{when } \alpha = 1+s} \\[2ex] \dfrac{c_{\alpha,0}}{1+\beta}\left(\dfrac{2sc_{\alpha,0}}{c_{\alpha,\beta}(1+\beta)}\right)^{\frac{\alpha-1}{1+\beta}}\displaystyle\int_{\mathbb{R}^d}(w\cdot\sigma)\sin(w\cdot\sigma)\frac{dw}{|w|^{d+\frac{\alpha+\beta}{1+\beta}}} & \boxed{\text{when } \alpha \in (s, 1+s)} \end{cases}$$

This recovers and extends the qualitative results in [1, 54] to general equilibria, with quantitative error estimates and characterizations of the diffusion coefficient. In the latter papers, the moment method initiated by Mellet is used to derive a fractional limit in the case $\beta = 0$.

Let us summarise our contributions. Theorem 5.8 and Corollaries 5.9–5.10–5.11 recover the results of [1, 165, 51, 73, 94, 93, 135, 145, 146, 153] with a shorter and unified constructive method and prove new results for (1) Lévy-Fokker-Planck operators, (2) scattering operators with decaying collision kernel and infinite mass equilibria and importantly (3) Fokker-Planck operators in any dimension (for which the characterization of the diffusion coefficient was not known). The quantitative error in this fluid approximation seems to also be novel for all equations considered. For detailed comparison with previous works, *e.g.* [94, 93], we refer to the original paper [B26].

## 5.5 A word about the proofs

As explained above, the method of the present paper extends to the fractional diffusive limit the approach pioneered in [154, 82] of constructing exact dispersion laws in the regime of parabolic time-space scaling and small eigenvalues; this extension is inspired by the recent one-dimensional result [135] and in particular we use and generalise the idea of rescaling velocities to obtain a non-trivial dispersion law in the latter paper. In comparison with [135], the main novelty of the present paper is a quantitative spectral method for constructing the branch of fluid eigenvalue: in [135] it was done by a one-dimensional argument connecting two infinite series on $\mathbb{R}_-$ and $\mathbb{R}_+$ (and it was done by fixed points in the simpler case of classical diffusive limit in the older works [154, 82]).

The proof of Theorem 5.8 goes as follows. We start by constructing the fluid mode, by carefully showing the existence of the resolvent of $\tilde{L}^* + i\eta\lfloor v\rceil^{\beta}(v\cdot\sigma)$ near zero ($|z| < \mathrm{R}_0\Theta(\eta)$) and away from zero ($|z| > \mathrm{R}_1\Theta(\eta)$). Then, estimating the spectral projector on the larger circle and showing that is it close to the one of $L$ when $\eta$ is small, we get the existence and uniqueness of a real $\mu$. This part of the proof is exactly where relevant scalings are revealed, by very neat and careful estimates. Then, it is necessary to prove the convergence of $\frac{\mu(\eta)}{\Theta(\eta)}$ when $\eta$ goes to zero (Lemma 5.6). Once this is done, one can test the equation written in Fourier variables on the eigenvector built right before, to see what the diffusion coefficient should be. Separate estimates are mandatory to show that the natural quantity is indeed well defined, this is Lemma 5.7.

Naturally, the application to concrete cases needs to prove Hypothesis 5.5 for all of the three cases (BGK), (FP) and (LFP), which is quite an amount of work.

## 5.6 Perspectives

1. *Some reasonably reachable extensions.*

   It would be interesting to try and apply this method in other settings such as [19, 97] (radiative transfer theory), [39, 77, 101] (rarefied gas in a region between two parallel plates), [2] (scattering with external acceleration field), [161] (models for chemotaxis) among others. These are some cases that I feel reachable.

2. *Fokker-Planck with infinite mass.*

   Amusingly, the case $\alpha < 0$ for the Fokker-Planck equation is open, since we have shown that $\int_{\mathbb{R}^d} \Phi(u)|u|^{-d-\alpha}\,\mathrm{d}u = 0$ in that case!

3. *The influence of space.*

   Now comes a more involved issue. Our approach uses crucially the fact that the space variable is in the full space $\mathbb{R}^d$ since we perform Fourier transforms in $x$. At that stage, using our approach in bounded domains (even half-space domains, depending on boundary conditions), or even in the full space with space dependant coefficients, or with space dependant confinements, is impossible. Note several contributions around Cesbron that carry out limits that could eventually be non-trivial fractional type operators [52, 1, 53]. I will be very interested in the next future to see how the method could be adapted or transformed to tackle non-standard cases, in order to identify key physical quantities and mathematical objects.

4. *First fluid approximation (more invariants).*

   What about the first (instead of second) fluid approximation when the collision operator also preserves momentum and energy? Partial answer to first fluid approximation in a particular case has been given in [115] (derivation of fractional Stokes and Fourier-Stokes systems from momentum-conserving scattering type operators). Extending our approach to diffusion limits with several invariants is interesting and seems reachable at that stage.

5. *Some fractional cases.*

   Our contribution concerning the Levy-Fokker-Planck raises several interesting questions: (1) can our approach be extended to $s \in (0, \frac{1}{2})$? (this seems to be a technical difficulty), (2) is the fractional diffusive limit having any meaning and is possible for infinite mass equilibria? (i.e. $\alpha < 0$) In this case, early computations seem to say that the fractional Laplace would not disappear in the spectral problem, (3) can the connexion between the kinetic Lévy-Fokker-Planck equation with $\alpha = 2s$ (for which the $\mathsf{L}$ is the generator of a Lévy process) and the standard kinetic Fokker-Planck equation with Gaussian equilibrium be clarified as $s \to 1$? (our diffusion constant $\kappa$ above diverges as $s \to 1$ so the two limits in $\varepsilon \to 0$ and $s \to 1$ do not commute which calls for further investigation).

6. *Fractional Hilbert expansion.*

   One major difficulty arising from the series of works by Mellet (see [165] among others) is the fact that using Hilbert expansions is more involved and difficult when the macroscopic limit is of fractional type. An interesting question would be to develop a fractional Hilbert expansion based on the apparent multi-scale structure of the problem. We have started discussions around this.

7. *Asymptotic Preserving schemes.*

   The numerical simulation of macroscopic limits of fractional (or anomalous) type has attracted a lot of attention in the last decades. The numerical difficulty is due to the fact that large velocities are responsible for the fractional behaviour as opposed to the fact that usually kinetic schemes involve a Courant–Friedrichs–Lewy condition that limitates the largest velocities that one can bear. Some works on scattering models by Crouseilles *et al.* [67, 69, 68] and Hivert [117] have taken advantage of a multi-scale strategy and a Fourier approach. It could be interesting to know whether the spectral approach of this chapter could give a systematic numerical treatment of the fluid approximation of linear kinetic models.

# Chapter 6

# Contributions to hypocoercivity

**Contents**


**6.1 Hypocoercivity without confinement . . . . . . . . . . . . . . . . . . . . . . 87**

6.1.1 When $\mathcal{M}$ is thin-tailed . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 88

6.1.2 When $\mathcal{M}$ has a sub-exponential decay . . . . . . . . . . . . . . . . . . 90

6.1.3 When $\mathcal{M}$ has algebraic decay . . . . . . . . . . . . . . . . . . . . . . . . 91

**6.2 The influence of a weak confinement . . . . . . . . . . . . . . . . . . . . . . 95**

**6.3 Hypocoercivity when the steady state is not known . . . . . . . . . . . . 98**

**6.4 Perspectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 100**


In this last chapter of the memoir, we present contributions in the field of hypocoercivity of linear kinetic equations. The general framework is the study of large time behaviour of solutions to Cauchy problems of the type

$$\begin{cases} \partial_t f + \mathsf{T} f = \mathsf{L} f\,, \\ f(0,\cdot,\cdot) = f_0. \end{cases} \tag{6.1}$$

Here, $\mathsf{T}$ is a linear transport operator and $\mathsf{L}$ is a linear collision operator. Specific forms of them will be described later on. In this chapter, the distribution function is $f(t,x,v)$, with *position* variable $x \in \mathbb{R}^d$, *velocity* variable $v \in \mathbb{R}^d$, and with *time* $t \geq 0$.

The word *hypocoercivity* was coined by T. Gallay and widely disseminated in the context of kinetic theory by C. Villani. In [149, 181, 182], the method deals with large time properties solutions to such kinetic equations by considering $\mathrm{H}^1$-norms (in $x$ and $v$ variables) and taking into account cross-terms. This is very well explained in [181, Section 3], but was already present in earlier works like [120]. Hypocoercivity theory is inspired by and related to the earlier *hypoellipticity* theory. The latter has a long history in the context of the kinetic Fokker-Planck equation. One can refer for instance to [81, 120] and much earlier to Hörmander's theory [121]. The seed for such an approach can even be traced back to Kolmogorov's computation of Green's kernel for the kinetic Fokker-Planck equation in [130], which has been reconsidered in [122] and successfully applied, for instance, to the study of the Vlasov-Poisson-Fokker-Planck system in [179, 33].

As in previous chapters, the *microscopic equilibrium* is denoted by $\mathcal{M}$, and is such that

$$\mathrm{Ker}(\mathsf{L}) = \mathrm{Span}(\mathcal{M}).$$

We normalise $\mathcal{M}$ to be a probability density : $\int_{\mathbb{R}^d} \mathcal{M}(v)\,dv = 1$. The decay of $\mathcal{M}$ will be important, and will be alternatively thin-tailed (at least exponential) or fat-tailed (sub-exponential or algebraic). Getting quantitative decays for a wide range of these decays is part of our program in this chapter. We are also interested in covering most of the collision operators appearing in the literature, that is

(a) Fokker-Planck operators,

$$\mathsf{L}f = \nabla_v \cdot \left(\mathcal{M}\,\nabla_v \left(\mathcal{M}^{-1} f\right)\right). \tag{FP}$$

(b) Scattering collision operators,

$$\mathsf{L}f = \int_{\mathbb{R}^d} \mathsf{b}(\cdot, v') \left(f(v')\,\mathcal{M}(\cdot) - f(\cdot)\,\mathcal{M}(v')\right) dv' , \tag{BGK}$$

with

$$\int_{\mathbb{R}^d} \left(\mathsf{b}(v,v') - \mathsf{b}(v',v)\right) \mathcal{M}(v')\,dv' = 0, \quad \forall\, v \in \mathbb{R}^d.$$

(c) Fractional Fokker-Planck operators,

$$\mathsf{L}f := \Delta_v^{\frac{s}{2}} f + \nabla_v \cdot (E\,f) \tag{LFP}$$

with $0 < s < 2$ and a radial *friction force* $E = E(v)$ as a solution of

$$\mathsf{L}\mathcal{M} = \Delta_v^{\frac{s}{2}} \mathcal{M} + \nabla_v \cdot (E\,\mathcal{M}) = 0\,.$$

Unless otherwise stated, the transport operator will be of the general form

$$\mathsf{T}f = v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f \tag{T}$$

where $V$ will be a confinement potential. Its form will be made precise case by case later. Notably, we will devote a lot of attention situations where $V$ is weak, that is having a slow growth rate at infinity, and even $V = 0$. All operators $\mathsf{L}$ will satisfy

$$\int_{\mathbb{R}^d} \mathsf{L}f \,\mathrm{d}v = 0\,,$$

so that all our linear kinetic equations enjoy mass conservation.

Most results of this chapter are based on the $L^2$ approach to hypocoercivity (that is, hypocoercivity without regularity) developed by Dolbeault, Mouhot and Schmeiser in [80, 79]. Let us recall the basic observation leading to the birth of the hypocoercivity theory. The basic $L^2(\mathcal{M}^{-1} dx\,dv)$ energy estimate is

$$\frac{1}{2}\frac{d}{dt}\,\|f\|^2_{L^2(\mathcal{M}^{-1} dx\,dv)} = \langle \mathsf{L}f, f\rangle.$$

Observe that since $\mathsf{L}$ has a nontrivial kernel, the latter energy estimate will not give any time decay. To recover a decay, their framework is the following. Denote by $\Pi$ the orthogonal projection onto $\mathrm{Ker}(\mathsf{L})$. With the following crucial ingredients,

1. *Microscopic coercivity*, that is a Poincaré type inequality in $v$:

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \,\|(1-\Pi)F\|^2\,,$$

2. *Macroscopic coercivity*, that is a Poincaré type inequality in $x$:

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \,\|\Pi F\|^2\,,$$

3. *Parabolic macroscopic dynamics:*

$$\Pi\mathsf{T}\Pi \; = 0\,.$$

and a few other hypothesis including high order moments for $\mathcal{M}$ (see [80] for details), Dolbeault, Mouhot and Schmeiser obtain quantitative exponential convergence to equilibrium:

$$\left\| e^{(\mathsf{L}-\,\mathsf{T})\,t}\right\|^2 \leq 3\,e^{-\lambda\,t}$$

where $\lambda$ is explicit.

The idea of [79, 80] (reminiscent to [120]) is to define an equivalent Lyapunov functional $\mathsf{H}$ by,

$$\mathsf{A} := \big(1+(\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*, \qquad \mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle,$$

for which they prove exponential decay.

One main objective of this chapter is to explain how to modify and extend this approach to situations for which some or even all ingredients listed above are not available. This happens for example when the space confinement is weak, killing the Poincaré inequality in space, or when $\mathcal{M}$ has fat tails, leading to the impossibility to use its moments, or any Poincaré inequality with this measure. Finally, it is clear that the strategy requires knowledge on $\mathcal{M}$, and this is not necessarily available.

The rest of the chapter is organised as follows. In Section 6.1, we discuss the case of no space confinement but with emphasis on the influence of the decay of the microscopic equilibrium $\mathcal{M}$. We first report the paper [B14] where $\mathcal{M}$ is a least exponentially decaying. When it is fat-tailed, with sub-exponential decay, we recast the Dolbeault-Mouhot-Schmeiser method with Nash and weighted Poincaré inequalities to derive the trend to zero [B13]. When it is fat-tailed, with algebraic decay, the situation is much more involved a requires a lot of adaptations and corresponds to the paper [B12]. Next, in Section 6.2, we discuss results with Dolbeault and Schmeiser [B15] giving rates of decay when the space confinement is too weak to prevent convergence to zero but prevents the use of Fourier techniques: deriving good functional inequalities is necessary. Section 6.3 is slightly different in terms of topic: we discuss a situation where transport and collision operators are such that the macroscopic equilibrium is not known to exist and is *a fortiori* not explicit, and show how to use hypocoercivity to deduce its existence in a perturbative regime and get a rate of convergence.

## 6.1 Hypocoercivity without confinement

In this section, the transport operator (T) is $\mathsf{T} = v\cdot\nabla_x$, that is, the confinement potential is $V \equiv 0$.

Observe crucially that due to total mass conservation, an initial data with finite total mass will necessary go to zero as $t$ goes to infinity since there is no global equilibrium state with finite mass except from zero. The aim of this section is to derive rates of decay to zero depending on the tails of $\mathcal{M}$. It is expected that such tails have a huge influence on the rates, since as we have shown in Chapter 5, they influence drastically the macroscopic limit of the kinetic model (6.1). Since hypocoercivity functionals coming from the Dolbeault-Mouhot-Schmeiser approach have a strong link with the underlying macroscopic dynamics, the necessary adaptations will be important.

### 6.1.1 When $\mathcal{M}$ is thin-tailed

Let us start by presenting the paper [B14]. We consider a kinetic equation without confinement ($V \equiv 0$ in (T)) and with either a Fokker-Planck (FP) or a scattering collision operator (BGK). We shall make the following assumptions on $\mathcal{M}$ and on the *scattering rate* $\mathrm{b}(v, v')$:

$$\begin{aligned}
&\nabla_v \sqrt{\mathcal{M}} \in \mathrm{L}^2(\mathbb{R}^d)\,, \quad \mathcal{M} \in C(\mathbb{R}^d)\,,\\
&\mathcal{M} = \mathcal{M}(|v|)\,, \quad 0 < \mathcal{M}(v) \le c_1 e^{-c_2|v|}\,, \quad \forall\, v \in \mathbb{R}^d\,, \quad \text{for some } c_1,\ c_2 > 0\,.\\
&1 \le \mathrm{b}(v, v') \le \overline{\mathrm{b}}\,, \quad \forall v, v' \in \mathbb{R}^d\,, \quad \text{for some } \overline{\mathrm{b}} \ge 1\,.
\end{aligned}$$

Rotational symmetry is not important, but assumed for computational convenience. However the property

$$\int_{\mathbb{R}^d} v M(v) dv = 0\,,$$

*i.e., zero flux in local equilibrium*, is essential.

Let us consider the measures

$$d\gamma_k := \gamma_k(v)\, dv \quad \text{where} \quad \gamma_k(v) = \lfloor v \rceil^k = (1 + |v|^2)^{\frac{k}{2}} \quad \text{and} \quad k > d\,,$$

such that $1/\gamma_k \in \mathrm{L}^1(\mathbb{R}^d)$. The condition $k \in (d, \infty]$ then covers the case of weights with a growth of the order of $|v|^k$, when $k$ is finite, and we denote $k = \infty$ the case when the weight $\gamma_\infty = \mathcal{M}^{-1}$ grows at least exponentially fast.

By replacing the Poincaré inequality by Nash's inequality or using direct estimates in Fourier variables, we adapt the $\mathrm{L}^2$ hypocoercivity method presented in the above short introduction and prove that an appropriate norm of the solution decays at a rate which is the rate of the heat equation. This observation is compatible with diffusion limits, which have been a source of inspiration for building Lyapunov functionals and establishing the $\mathrm{L}^2$ hypocoercivity method of [79, 80]. We use the *factorization* method of [105] and obtain estimates in large functional spaces (that is, to go from the exponential weight $\gamma_\infty$ to larger spaces corresponding to the algebraic weights $\gamma_k$ with $k \in (d, \infty)$). Note that the method based on the use of the Nash inequality would be applicable to problems with non-constant coefficients like scattering operators with $x$-dependent scattering rates $\sigma$, or Fokker-Planck operators with $x$-dependent diffusion constants like $\nabla_v \cdot \big(\mathcal{D}(x)\, \mathcal{M}\, \nabla_v(\mathcal{M}^{-1} f)\big)$. Our result is the following.

**Theorem 6.1** (B., Dolbeault, Mischler, Mouhot, Schmeiser [B14]). *There exists a constant $C > 0$ such that solutions $f$ of* (6.1) *with either a Fokker-Planck* (FP) *or a scattering collision operator* (BGK), *with initial datum $f_0 \in \mathrm{L}^2(dx\, d\gamma_k) \cap \mathrm{L}^2(d\gamma_k;\, \mathrm{L}^1(dx))$ satisfy, for all $t \ge 0$,*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\, d\gamma_k)} \le C\, \frac{\|f_0\|^2_{\mathrm{L}^2(dx\, d\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\, \mathrm{L}^1(dx))}}{(1+t)^{d/2}}\,,$$

*where* $k \in (d, \infty]$.

In the Fourier style proof, we slightly strengthen the abstract hypocoercivity result of [80] by allowing complex Hilbert spaces and by providing explicit formulas for the coefficients in the decay rate. This result is applied for fixed $\xi$ to the Fourier transformed problem (with respect to $x$), where integrals are computed with respect to the measure $d\gamma_\infty$ in the velocity variable $v$. Since the frequency $\xi$ can be considered as a parameter, we shall speak of a *mode-by-mode hypocoercivity* result. It provides exponential decay, however with a rate deteriorating as $\xi \to 0$, resulting in a polynomial rate when turning back to physical variables.

In a second phase, we focus on improved decay rates when Fourier modes with slowest decay are eliminated from the initial data. For the heat equation, improved decay rates can be shown by Fourier techniques. The following two results are in this spirit, but for the full kinetic model.

**Theorem 6.2** (B., Dolbeault, Mischler, Mouhot, Schmeiser [B14])**.** *Let the assumptions of Theorem 6.1 hold, and let*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0\,dx\,dv = 0\,.$$

*Then there exists* $C > 0$ *such that solutions* $f$ *of* (6.1) *with initial datum* $f_0$ *satisfy, for all* $t \geq 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \leq C\,\frac{\|f_0\|^2_{\mathrm{L}^2(d\gamma_{k+2};\,\mathrm{L}^1(dx))} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))} + \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)}}{(1+t)^{d/2+1}}\,,$$

*with* $k \in (d, \infty)$.

For the formulation of a result corresponding to the cancellation of higher order moments, we introduce the set $\mathbb{R}_\ell[X,V]$ of polynomials of order at most $\ell$ in the variables $X$, $V \in \mathbb{R}^d$ (the sum of the degrees in $X$ and in $V$ is at most $\ell$). We also need that the kernel of the collision operator is spanned by a Gaussian function in order to keep polynomial spaces invariant. This means that for any $P \in \mathbb{R}_\ell[X,V]$, one has $(\mathsf{L}-\mathsf{T})\,(PM) \in \mathbb{R}_\ell[X,V]M$. Since the transport operator mixes both variables $x$ and $v$, one needs moments with respect to both $x$ and $v$ variables.

**Theorem 6.3** (B., Dolbeault, Mischler, Mouhot, Schmeiser [B14])**.** *Let* $\mathsf{L}$ *be with either a Fokker-Planck* (FP) *with* $\mathcal{M}$ *a normalized Gaussian or a scattering collision operator* (BGK) *with* $\overline{\mathrm{b}} = 1$. *Let* $k \in (d, \infty]$, $\ell \in \mathbb{N}$ *and assume that the initial datum* $f_0 \in \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ *is such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\,P(x,v)\,dx\,dv = 0$$

*for all* $P \in \mathbb{R}_\ell[X,V]$. *Then there exists a constant* $c_k > 0$ *such that any solution* $f$ *of* (6.1) *with initial datum* $f_0$ *satisfies, for all* $t \geq 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \leq c_k\,\frac{\|f_0\|^2_{\mathrm{L}^2(d\gamma_{k+2};\,\mathrm{L}^1(dx))} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))} + \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)}}{(1+t)^{d/2+1+\ell}}\,.$$

### 6.1.2 When $\mathcal{M}$ has a sub-exponential decay

We now start reviewing cases where $\mathcal{M}$ has fat tails, which prevents from having a microscopic coercivity of Poincaré type. In this paragraph, let $\mathcal{M}$ be of the form

$$\mathcal{M}(v) = C_\alpha\, e^{-\lfloor v \rceil^\alpha}, \quad v \in \mathbb{R}^d, \qquad \text{with } C_\alpha^{-1} = \int_{\mathbb{R}^d} e^{-\lfloor v \rceil^\alpha}\, \mathrm{d}v\,,$$

with $0 < \alpha < 1$. The results can easily be extended to more general distributions $\mathcal{M}$, satisfying

$$\alpha := \lim_{|v|\to+\infty} \frac{\log(-\log \mathcal{M}(v))}{\log|v|} \in (0,1)\,.$$

We shall consider two types of collision operators, either the Fokker-Planck operator (FP) or the scattering operator (BGK). In the latter case, we assume the existence of constants $\beta$, $\overline{\mathrm{b}}$, $\underline{\mathrm{b}} > 0$, $\gamma \geq 0$, with $\gamma \leq \beta$, $\gamma < d$, such that

$$\underline{\mathrm{b}}\lfloor v \rceil^{-\beta}\lfloor v' \rceil^{-\beta} \leq \mathrm{b}(v,v') \leq \overline{\mathrm{b}} \min\left\{|v-v'|^{-\beta}, |v-v'|^{-\gamma}\right\}.$$

The upper bound with the restriction on the exponent $\gamma$ is a local integrability assumption. This typically allows for the choice $\mathrm{b}(v,v') = \lfloor v \rceil^{-\beta}\lfloor v' \rceil^{-\beta}$ with arbitrary $\beta > 0$, as well as Boltzmann kernels $\mathrm{b}(v,v') = |v-v'|^{-\beta}$ with $0 < \beta < d$. As a consequence, the collision frequency

$$\nu(v) = \int_{\mathbb{R}^d} \mathrm{b}(v,v')\, \mathcal{M}(v')\, \mathrm{d}v'$$

satisfies $\nu(v) \asymp \lfloor v \rceil^{-\beta}$. For the formulation of our results, we introduce the norms $\|f\|_k$ *via*

$$\|f\|_k^2 := \int_{\mathbb{R}^d\times\mathbb{R}^d} f^2\, \lfloor v \rceil^k\, \mathrm{d}x\, \mathrm{d}\mu\,, \qquad k \in \mathbb{R},$$

as well as the scalar product $\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d\times\mathbb{R}^d} f_1\, f_2\, \mathrm{d}x\, \mathrm{d}\mu$ on $\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$ with the induced norm $\|f\|^2 := \|f\|_0^2 = \langle f, f \rangle$.

Actually, the parameter $\beta$ as a common origin for both types of operators. One way of seeing it is through the dissipation property of $\mathsf{L}$. When $\mathcal{M}$ is sub-exponential, the standard Poincaré inequality does not hold. We need to replace it with a *weighted Poincaré* inequality.

**Lemma 6.4.** *Let* $\mathsf{L}$ *be either the Fokker-Planck operator* (FP) *with* $\beta = 2(1-\alpha)$ *or the scattering operator* (BGK) *with* $\beta$ *defined above. Then, with all hypothesis above, there exists* $\mathsf{C} > 0$ *such that*

$$\forall\, f \in \mathcal{D}(\mathbb{R}^d\times\mathbb{R}^d) \qquad -\langle \mathsf{L}f, f \rangle \geq \mathsf{C}\, \|(1-\Pi)f\|_{-\beta}^2.$$

The parameter $\beta$ is thus a structural loss of weight. Note that it is formally possible to guess $\beta$ directly from the expression of $\mathsf{L}$ by writing $\mathsf{L}$ on the form $\mathsf{B}[f] - \nu(v)\, f$ and defining $-\beta$ as the exponent at infinity of the function $\nu$.

With this in mind, we can state the main result.

**Theorem 6.5** (B., Dolbeault, Lafleche, Schmeiser [B13])**.** *Let* $\alpha \in (0,1)$*,* $\beta > 0$*,* $k > 0$*. Let* $\mathsf{L}$ *be either the Fokker-Planck operator* (FP) *with* $\beta = 2(1-\alpha)$ *or the scattering operator* (BGK) *with* $\beta$ *defined above. Then there exists a constant* $C > 0$ *such that any solution* $f$ *of* (4.1) *with initial datum* $f^{\mathrm{in}} \in \mathrm{L}^2(\lfloor v \rceil^k \mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$ *satisfies*

$$\forall\, t \geq 0, \qquad \|f(t,\cdot,\cdot)\|^2 \leq C\, \frac{\|f^{\mathrm{in}}\|^2}{(1+\kappa\, t)^{\zeta}},$$

*with rate* $\zeta = \min\{d/2, k/\beta\}$ *and with* $\kappa > 0$*, which is an explicit function of the two quotients* $\|f^{\rm in}\| \,/\, \|f^{\rm in}\|_k$ *and* $\|f^{\rm in}\| \,/\, \|f^{\rm in}\|_{{\rm L}^1({\rm d}x\,{\rm d}v)}$*.*

The loss of information due to the weight $\lfloor v \rceil^{-2(1-\alpha)}$ has to be compensated by a $\mathrm{L}^2$-bound for the initial datum with a weight $\lfloor v \rceil^k$. We thus need to show propagation of weighted norms with weights $\lfloor v \rceil^k$ of arbitrary positive order $k \in \mathbb{R}^+$.

**Proposition 6.6.** *Let* $k > 0$ *and* $f$ *be solution of* (4.1) *with* $f^{\rm in} \in \mathrm{L}^2(\lfloor v \rceil^k\,{\rm d}x\,{\rm d}\mu)$*. Then there exists a constant* $\mathcal{K}_k > 1$ *such that*

$$\forall\, t \geq 0 \quad \|f(t,\cdot,\cdot)\|_k \leq \mathcal{K}_k\, \|f^{\rm in}\|_k\,.$$

For this problem, estimates based on *weak Poincaré inequalities* are also very popular in the scientific community of semi-group theory and Markov processes (see [167, 125], [18, Proposition 7.5.10]). Estimates based on weak Poincaré inequalities rely on a uniform bound for the initial data for $\alpha < 1$ which is not needed for $\alpha \geq 1$, while the approach developed in [B13] provides a continuous transition from the range $0 < \alpha < 1$ to the range $\alpha \geq 1$ since we may choose $k \searrow 0$ as $\alpha \nearrow 1$. Note that for $\alpha = 1$, the weighted Poincaré inequality of Lemma 6.4 reduces to the standard Poincaré inequality. The advantage of weighted Poincaré inequalities compared to the more classical weak Poincaré inequalities is that the description of the convergence rates to the local equilibrium does not require extra regularity assumptions to cover the transition from super-exponential and exponential local equilibria to sub-exponential local equilibria.

The proof of Theorem 6.5 goes along the lines of the hypocoercivity approach (with $\alpha \geq 1$) of [79, 80] and its extension to cases without confinement as in [B14, B15]. It combines information on the *microscopic* and the *macroscopic* dissipation properties. The major difficulty is to understand how to take into account the macroscopic degenerate dissipation. Thus, the core of the microscopic part is given in Lemma 6.4. Since the macroscopic limit of (4.1) is the heat equation on the whole space, it is natural that for the estimation of the macroscopic dissipation we use Nash's inequality,

$$\|u\|^2_{\mathrm{L}^2(\,{\rm d}x)} \leq \mathcal{C}_{\rm Nash}\, \|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(\,{\rm d}x)}\, \|\nabla u\|^{\frac{2d}{d+2}}_{\mathrm{L}^2(\,{\rm d}x)}\,,$$

a tool which has been developed for this purpose. The result of Theorem 6.5 can be interpreted as giving the weaker of the microscopic decay rate $t^{-k/\beta}$ and the macroscopic decay rate $t^{-d/2}$. Only for $k \geq \beta\, d/2$, the decay rate of the macroscopic diffusion limit is recovered.

### 6.1.3 When $\mathcal{M}$ has algebraic decay

In this last paragraph, we shall now focus on even fatter tails. The local equilibrium $\mathcal{M}$ will now have an algebraic tail given for some $\alpha > 0$ by

$$\forall\, v \in \mathbb{R}^d, \quad \mathcal{M}(v) = \frac{c_\alpha}{\lfloor v \rceil^{d+\alpha}}.$$

The normalization constant is $c_\alpha = \pi^{-d/2}\,\Gamma\big((d+\alpha)/2\big)/\Gamma(\alpha/2)$ and associated to the measure

$${\rm d}\mu = \mathcal{M}^{-1}(v)\,{\rm d}v\,,$$

We consider in one go all three examples of linear collision operators L, (FP), (BGK) and (LFP), that are all compatible with algebraically decaying local equilibria. For (BGK), assume

$$\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v,v')\,\mathcal{M}(v')\,\mathrm{d}v' \underset{|v|\to+\infty}{\sim} |v|^{-\beta},$$

for a given $\beta \in \mathbb{R}$. Some extra structural hypothesis on b are needed but to simplify the presentation of this memoir, let us only say that they are verified when

$$\begin{aligned} &\text{either} && \mathrm{b}(v,v') \propto \lfloor v' \rceil^{-\beta} \lfloor v \rceil^{-\beta} && \text{with} \quad |\beta| \le \alpha\,,\\ &\text{or} && \mathrm{b}(v,v') = |v'-v|^{-\beta} && \text{with} \quad \beta \in \left[0, \frac{d}{2}\right)\,. \end{aligned}$$

For (LFP) with such equilibrium $\mathcal{M}$, the friction force $E$ behaves like $\lfloor v \rceil^{-(s-\zeta)}\, v$ at infinity, see [B12].

Exactly for the same reasons as in the previous subsection, each of them has an associated parameter $\beta$ which represents a loss of weight in the following weighted Poincaré inequality.

$$-\operatorname{Re}\langle \mathsf{L}\varphi, \varphi\rangle_{L^2(\mathcal{M}^{-1})} \gtrsim \left\| \varphi - \left( \int_{\mathbb{R}^d} \varphi(v') \lfloor v' \rceil^{-\beta}\, dv' \right) \mathcal{M} \right\|^2_{L^2(\lfloor v \rceil^{-\beta} \mathcal{M}^{-1})}.$$

For scattering operators (BGK), the parameter $\beta$ above defined is named on purpose. We find $\beta = 2$ for the Fokker-Planck operator (FP) and $\beta = s - \alpha$ for Levy-Fokker-Planck operators (LFP).

For any $k \in \mathbb{R}$, we define

$$|||f|||_k := \|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap \mathrm{L}^2(\lfloor v \rceil^k\,\mathrm{d}x\,\mathrm{d}\mu)} := \left( \|f\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)} + \|f\|^2_{\mathrm{L}^2(\lfloor v \rceil^k\,\mathrm{d}x\,\mathrm{d}\mu)} \right)^{1/2},$$

and

$$\zeta := \begin{cases} \frac{\alpha+\beta}{1+\beta} \in (0,2) & \text{if} \quad \alpha < 2+\beta\,,\\ 2 & \text{if} \quad \alpha \ge 2+\beta\,. \end{cases}$$

The main result of this paragraph, obtained with Dolbeault and Lafleche, is the following.

**Theorem 6.7** (B., Dolbeault, Lafleche [B12]). *Let $d \ge 2$, $\alpha > \max\{0, -\beta\}$ and $k \in [0,\alpha)$. Let us consider a solution $f$ to (6.1) with initial condition $f^{\mathrm{in}} \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2\big(\lfloor v \rceil^k \mathrm{d}x\,\mathrm{d}\mu\big)$.*

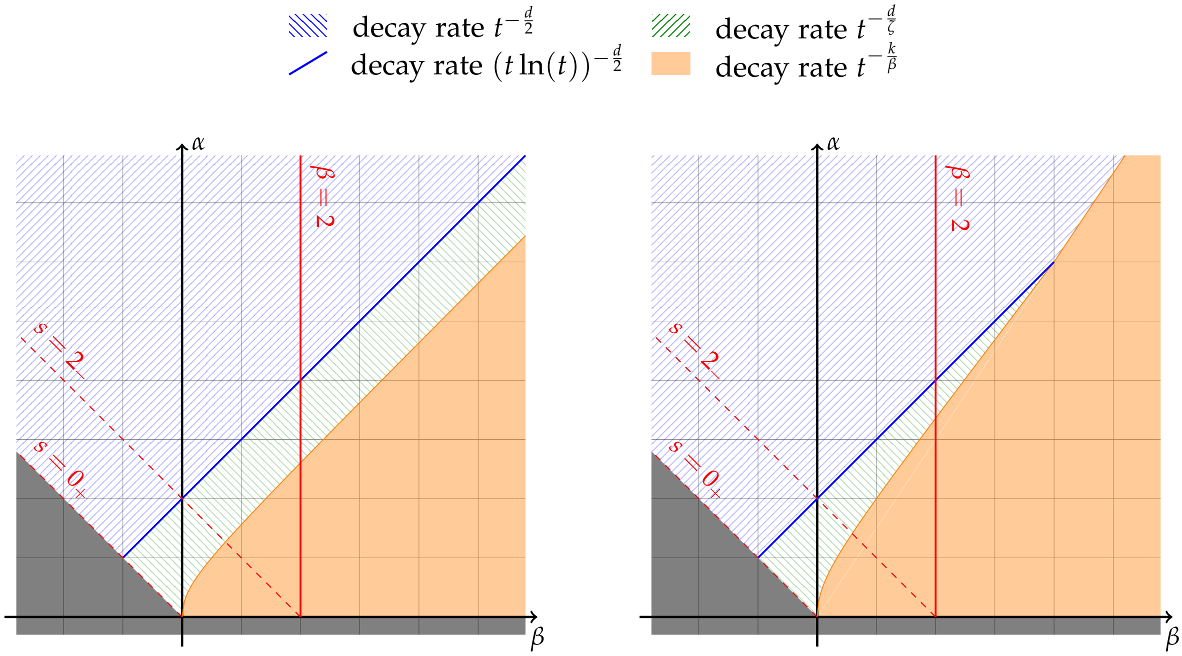
*If $\alpha \ne 2+\beta$ or if $\alpha = 2+\beta$ and $\frac{k}{\beta_+} > \frac{d}{2}$, then*

$$\forall\, t \ge 0\,, \quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \frac{\mathrm{C}}{(1+t)^\tau}\, |||f^{\mathrm{in}}|||^2_k \quad \textit{with} \quad \tau = \min\left\{\tfrac{d}{\zeta}, \tfrac{k}{\beta_+}\right\}.$$

*In the critical case $\alpha = 2+\beta$, and with either $k = 0$ if $\beta < 0$, or $k > 0$ if $\beta \ge 0$, and under the additional condition $\frac{k}{\beta_+} \le \frac{d}{2}$ if $d \ge 3$,*

$$\forall\, t \ge 2\,, \quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \frac{C}{(t\,\log t)^{d/2}}\, |||f^{\mathrm{in}}|||^2_k\,.$$

*In the above estimates, $\mathrm{C} > 0$ is a constant which does not depend on $f^{\mathrm{in}}$.*




(a) The case $d = 2$. (b) The case $d = 3$.

Figure 6.1: The rates of decay to 0 in the $(\beta, \alpha)$ plane.

The case $d = 1$ is also treated in the paper [B12] but we omit it here not to overload this memoir (a little subtlety appears in the statement).

The results are depicted on Figure 6.1. It is worth comparing the rates we obtain with the one of the macroscopic limit of the model, which is a (fractional/standard) heat equation, as described in Chapter 5 and [B26]. The macroscopic limit is the fractional heat equation,

$$\widehat{r}_t + \kappa\, |\xi|^{\zeta}\, \widehat{r} = 0\,.$$

If $\widehat{\rho}_0$ solves (6.1.3), then using the *fractional Nash inequality* and Plancherel's identity, we obtain $\|\rho(t,\cdot)\|_{\mathrm{L}^2(\mathrm{d}x)} = O\big(t^{-d/\zeta}\big)$ as $t \to +\infty$. Observe that this inequality is responsible for the $d/\zeta$ in Theorem 6.7, all other exponents being due to the microscopic part, *i.e.* the decay of $\mathcal{M}$.

Let us outline the method of proof, to highlight how we palliate the lack of Poincaré inequality and moments of $\mathcal{M}$ that are necessary to carry out the Dolbeault-Mouhot-Schmeiser method [80]. If $f$ solves (6.1), then the equation satisfied by $\widehat{f}$ is

$$\partial_t \widehat{f} + \mathsf{T}\widehat{f} = \mathsf{L}\widehat{f}\,, \quad \widehat{f}(0,\xi,v) = \widehat{f}^{\mathrm{in}}(\xi,v)$$

where $\mathsf{T}$ is the *transport operator* (recall that $V \equiv 0$) in Fourier variables given by

$$\mathsf{T}\widehat{f} = i\, v \cdot \xi\, \widehat{f}\,,$$

and $\xi \in \mathbb{R}^d$ can be seen as a parameter, so that for each Fourier mode $\xi$, $\mathsf{T}$ is a multiplication operator and we can study the decay of $t \mapsto \widehat{f}(t,\xi,\cdot)$. For this reason, we call it a *mode-by-mode analysis*, as in [B13]. Let us define the operator

$$\mathsf{A}_\xi := \frac{1}{\lfloor v \rceil^2}\, \Pi\, \frac{(-i\, v\cdot\xi)\, \lfloor v \rceil^{\beta}}{1 + \lfloor v \rceil^{2\,|1+\beta|}\, |\xi|^2}\,,$$

and the *entropy functional* by

$$\mathsf{H}_\xi[f] := \left\|\widehat{f}\right\|^2 + \delta\ \mathrm{Re}\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle\,, \qquad \mathsf{H}[f] := \int_{\mathbb{R}^d} \mathsf{H}_\xi[\widehat{f}]\,\mathrm{d}\xi.$$

This is one main contribution of this work: finding a new workable operator A. This is done mode-by-mode. The flavour is the same as in macroscopic limits for scattering operators [146] and in the original Dolbeault-Mouhot-Schmeiser method [80] but with very serious adaptations.

We observe that if $f$ solves (6.1), then

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] = -2\int_{\mathbb{R}^d} \langle \mathsf{L}\widehat{f}, \widehat{f}\rangle\,\mathrm{d}\xi + \delta\int_{\mathbb{R}^d} \mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi$$

where $\mathsf{R}_\xi[\widehat{f}] = -\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Re}\,\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle$. The difficult part of the proof is to estimate properly both terms of the latter r.h.s..

The main steps of our method are as follows. We only sketch the main elements here, sending to [B12] for explicit computations. First, to estimate the last integral, we prove the following dissipation inequalities.

**Proposition 6.8.** *Let $\alpha > \max\{0, -\beta\}$ and $\eta \in (-\alpha, \alpha)$ such that $\eta \geq -\beta$. Then*

$$\int_{\mathbb{R}^d} \mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi \ \gtrsim\ \|\Pi f\|^{2\,(1+\frac{\zeta}{d})}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} - |||(1-\Pi)f|||^2_{-\beta} \quad \text{if} \quad \alpha \neq 2+\beta\,,$$

$$\int_{\mathbb{R}^d} \mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi \quad \gtrsim \quad \|\Pi f\|^{2\,(1+\frac{\zeta}{d})}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \log\left(\frac{\|\Pi f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}}{\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu)}}\right) - |||(1-\Pi)f|||^2_{-\beta} \quad \text{if} \quad \alpha = 2+\beta\,.$$

Second, we show in [B12] how to tackle the microscopic part. As already done a bit in [B13], we use conservation of moments.

**Proposition 6.9.** *Let $d \geq 1$, $\alpha > 0$, $\alpha+\beta \geq 0$, $k \in (0,\alpha)$ and $f$ be a solution of* (6.1) *with initial condition $f^{\mathrm{in}} \in \mathrm{L}^2(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu)$. Then, there exists a positive constant $\mathcal{C}_k$ depending on $d$, $\alpha$, $\beta$ and $k$ such that*

$$\forall\, t \geq 0\,, \quad \|f(t,\cdot,\cdot)\|_{\mathrm{L}^2(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu)} \leq \mathcal{C}_k\, \left\|f^{\mathrm{in}}\right\|_{\mathrm{L}^2(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu)}\,.$$

This is crucial to be able to use the following interpolated weighted Poincaré inequality, that we also need to derive for all operators.

**Proposition 6.10.** *Let $d \geq 1$, $\alpha > 0$, $\alpha+\beta \geq 0$, $\eta \in [-\beta, \alpha)$ and $k \in (0,\alpha)$. Then there exists a positive constant $\mathcal{C}$ depending on $\|f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}$ such that for any $f \in \mathrm{L}^2(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu)$,*

$$\mathcal{C}\, \left\|(1-\Pi_\eta)f\right\|^{2\,\frac{k+\beta}{k-\eta}}_{\mathrm{L}^2(\lfloor v\rceil^\eta \mathrm{d}x\,\mathrm{d}\mu)}\, \|f\|^{-2\,\frac{\eta+\beta}{k-\eta}}_{\mathrm{L}^2(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu)} \leq -\langle \mathsf{L}f, f\rangle.$$

We shall use Proposition 6.10 with $\eta = -\beta$ if $\alpha > \beta$ and for some $\eta \in (-\alpha, 0)$ if $\alpha \leq \beta$. The latter case is actually more involved, we refer to [B12].

Once all previous estimates are combined, we conclude the proof by using Grönwall type strategies, as usual. This is a little bit more involved here but tractable.

## 6.2 The influence of a weak confinement

We are now interested in a situation where particles overcome a confinement in space, but this confinement is weak. The potential has an at most logarithmic growth and is such that $e^{-V}$ is not integrable. Again, the only integrable equilibrium state is 0. Thus, if the initial datum is integrable, we expect that the solution to our different models converge to 0 as $t \to +\infty$.

When there was no confinement potential, it was possible (see Section 6.1) to rely on Fourier analysis and mode-by-mode estimates to tackle kinetic equations. Here, this is not possible anymore and we develop an alternative approach based on moment estimates and Caffarelli-Kohn-Nirenberg inequalities of Nash type for diffusion and kinetic equations.

Since results were not known for the associated (macroscopic) Fokker-Planck equation, we start by that analysis. Then, we turn to kinetic equations with this weak confinement.

### The macroscopic Fokker-Planck equation

Start with

$$u_t = \Delta_x u + \nabla_x \cdot (\nabla_x V\, u) = \nabla_x \left(e^{-V}\, \nabla_x \left(e^{V}\, u\right)\right) \tag{6.2}$$

where $x \in \mathbb{R}^d$, $d \geq 3$, and $V$ is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$, that is, $e^{-V}\, dx$ is an *unbounded invariant measure*.

We shall investigate the two following examples

$$V_1(x) = \gamma \, \log|x| \quad \text{and} \quad V_2(x) = \gamma \, \log\lfloor x \rceil ,$$

with $\gamma < d$ and $\lfloor x \rceil := \sqrt{1+|x|^2}$ for any $x \in \mathbb{R}^d$. These two potentials share the same asymptotic behaviour as $|x| \to \infty$. The potential $V_1$ is invariant under scalings, whereas $V_2$ is smooth at the origin.

When $\gamma > 0$, the potential $V$ is *very weakly confining* in the sense that, even if it eventually slows down the decay rate, it is not strong enough to produce a stationary state of finite mass: the diffusion wins over the drift. Our goal to establish the rate of convergence in suitable norms. We shall use the notation $\|\cdot\|_p := \|\cdot\|_{\mathrm{L}^p(dx)}$ in case of Lebesgue's measure and specify the measure otherwise.

**Theorem 6.11** (B., Dolbeault, Schmeiser [B15]). *Assume that either $d \geq 3$, $\gamma < (d-2)/2$ and $V = V_1$ or $V = V_2$, or $d = 2$, $\gamma \leq 0$ and $V = V_2$. Then any solution $u$ of* (6.2) *with initial datum $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(\mathbb{R}^d)$ satisfies, for all $t \geq 0$,*

$$\|u(t,\cdot)\|_2^2 \leq \frac{\|u_0\|_2^2}{(1+c\,t)^{\frac{d}{2}}} \quad \textit{with} \quad c := \frac{4}{d}\, \min\left\{1, 1-\tfrac{2\,\gamma}{d-2}\right\} \mathcal{C}_{\mathrm{Nash}}^{-1}\, \frac{\|u_0\|_2^{4/d}}{\|u_0\|_1^{4/d}} \,.$$

Here $\mathcal{C}_{\mathrm{Nash}}$ denotes the optimal constant in Nash's inequality [152, 49, B16]

$$\|u\|_2^{2+\frac{4}{d}} \leq \mathcal{C}_{\mathrm{Nash}}\, \|u\|_1^{\frac{4}{d}}\, \|\nabla u\|_2^2 \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)\,.$$

Note that the rate of decay is independent of $\gamma$ and we recover the classical estimate due to J. Nash when $V = 0$ (here $\gamma = 0$).

Our method involves the computation of $\Delta V$. In dimension $d=2$, $V=V_1$ would produce a singularity (which could be handled by an appropriate regularization procedure). In dimension $d=1$, $V_2''(x)=(1-x^2)/(1+x^2)^2$ has no definite sign and would require new estimates, which are not covered by our result.

Theorem 6.11 does not cover the interval $(d-2)/2<\gamma<d$. This range is covered by employing the natural setting of $\mathrm{L}^2(e^V)$ and by requiring additional moment bounds.

**Theorem 6.12** (B., Dolbeault, Schmeiser [B15]). *Let $d\geq 1$, $\gamma<d$, $V=V_1$ or $V=V_2$, and $u_0\in\mathrm{L}^1_+\cap\mathrm{L}^2(e^V)$. If $\gamma>0$, let us assume that $\big\||x|^k u_0\big\|_1<\infty$ for some $k\geq\max\{2,\frac{\gamma}{2}\}$. Then any solution of* (6.2) *with initial datum $u_0$ satisfies*

$$\forall\,t\geq 0\,,\quad \|u(t,\cdot)\|^2_{\mathrm{L}^2(e^V dx)}\leq\|u_0\|^2_{\mathrm{L}^2(e^V dx)}\,(1+c\,t)^{-\frac{d-\gamma}{2}}\,.$$

*The constant $c$ depends on $d$, $\gamma$, $k$, $\|u_0\|_{\mathrm{L}^2(e^V dx)}$, $\|u_0\|_1$, and $\big\||x|^k u_0\big\|_1$.*

Our proofs are actually based on using suitable functional inequalities. Recall that when potentials $V$ have a sufficient growth at infinity: typically, if $V(x)=|x|^\alpha$, with $\alpha\geq 1$, then a *Poincaré inequality* exists and the rate of convergence to a unique stationary solution is then exponential, when measured in the appropriate norms; see [18] for a general overview. An interesting family of *weakly confining* potentials is made of functions $V$ with an intermediate growth, such that $e^{-V}$ is integrable but $\lim_{|x|\to\infty}V(x)/|x|=0$: all solutions of (6.2) are attracted by a unique stationary solution, but the rate is expected to be algebraic rather than exponential. A typical example is $V(x)=|x|^\alpha$ with $\alpha\in(0,1)$. The underlying functional inequality is a *weak Poincaré inequality*: see [167, 125], and [17] for related Lyapunov type methods *à la* Meyn and Tweedie or [23] for recent spectral considerations. We refer to [16] and [183] for further considerations on, respectively, *weighted Nash inequalities* and spectral properties of the diffusion operator. This problem has also attracted attention in the physics literature (see [3] and the references therein for a list of interesting examples).

To achieve this theorem, it is a bit different. We make use of Nash or Hardy-Nash type inequalities, usual Caffarelli-Kohn-Nirenberg inequalities when $V=V_1$ but also inhomogeneous Caffarelli-Kohn-Nirenberg inequalities when $V=V_2$, that we have been proving (see [B15, Appendix B, Theorem B.1.]) and are of independent interest, and that have the form

**Proposition 6.13.** *If $d\geq 3$, $\gamma\in(0,d)$ and $k\geq\gamma/2$, then there exists a function $\mathcal{H}$ such that, for any $w\in\mathrm{H}^1(\mathbb{R}^d,\lfloor x\rceil^{-\gamma}\,dx)$ such that $\lfloor x\rceil^{k-\gamma}\,v\in\mathrm{L}^1(\mathbb{R}^d,dx)$, we have*

$$\int_{\mathbb{R}^d}w^2\,\lfloor x\rceil^{-\gamma}\,dx\leq\left(\int_{\mathbb{R}^d}\lfloor x\rceil^{k-\gamma}\,|w|\,dx\right)^2\,\mathcal{H}\left(\frac{\int_{\mathbb{R}^d}|\nabla w|^2\,\lfloor x\rceil^{-\gamma}\,dx}{\left(\int_{\mathbb{R}^d}\lfloor x\rceil^{k-\gamma}\,|w|\,dx\right)^2}\right)\,.$$

*The function $\mathcal{H}$ is such that $\mathcal{H}(X)\leq\mathcal{K}\,X^a$ for some optimal constant $\mathcal{K}>0$, where $a=\frac{d+2k-\gamma}{d+2+2k-\gamma}$ if $d\geq 3$ and $\gamma\leq 2\,(d-2)$. Otherwise, if $2\,(d-2)<\gamma<d$ (which is possible only if $d=3$) or $d\leq 2$, then $\mathcal{H}(X)\leq\mathcal{K}\,\big(X^a+X^b\big)$ where $b=1-\frac{4}{\gamma+2}\,(d+2\,k+2-\gamma)^{-1}$.*

To be used in our proofs, these suitable Caffarelli-Kohn-Nirenberg inequalities require moment estimates. Additionally to the fact that no Fourier technique is available in this framework, the main difference with [B14] is to rely on the moments of the solutions, that have to be controlled independently.

Although this is a side result, let us notice that the case in which the potential contributes to the decay, *i.e.*, when $\gamma < 0$, is also covered in Theorem 6.12.

The scale invariance of (6.2) with $V = V_1$ can be exploited to obtain intermediate asymptotics in self-similar variables. Let us define

$$u_\star(t,x) = \frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}}\,|x|^{-\gamma}\exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)\,, \tag{6.3}$$

The following result on *intermediate asymptotics* allows us to identify the leading order term of the solution of (6.2) as $t \to +\infty$. It is the strongest of our results on (6.2) but initial data need to have a sufficient decay as $|x| \to \infty$.

**Theorem 6.14** (B., Dolbeault, Schmeiser [B15]). *Let $d \geq 1$, $\gamma \in (0,d)$ and $V = V_1$. If for some constant $K > 1$, the function $u_0$ is such that*

$$\forall\, x \in \mathbb{R}^d\,,\quad 0 \leq u_0(x) \leq K\,u_\star(0,x)$$

*where $c_\star$ is chosen such that $\|u_\star\|_1 = \|u_0\|_1$ then the solution $u$ of* (6.2) *with initial datum $u_0$ satisfies*

$$\forall\, t \geq 0\,,\quad \|u(t,\cdot) - u_\star(t,\cdot)\|_p \leq K\,c_\star^{1-\frac{1}{p}}\,\|u_0\|_1^{\frac{1}{p}}\left(\frac{e}{2\,|\gamma|}\right)^{\frac{\gamma}{2}\left(1-\frac{1}{p}\right)}\,(1+2\,t)^{-\zeta_p}$$

*for any $p \in [1,+\infty)$, where $\zeta_p := \frac{d}{2}\left(1-\frac{1}{p}\right) + \frac{1}{2\,p}\,\min\left\{2, \frac{d}{d-\gamma}\right\}$.*

### The kinetic counterpart

The second part of the paper [B15] is devoted to *kinetic equations* involving a degenerate diffusion operator acting only on the velocity variable, for *very weak potentials* like $V_1$ or $V_2$.

The study of decay to equilibrium needs hypocoercivity methods, as earlier in this chapter. Note importantly that the *sub-exponential* regime or the regime with *weak confinement*, *i.e.*, of a potential $V$ such that a weak Poincaré inequality holds, has also been studied in [48, 118]. We now focus on the kinetic counterpart to Theorems 6.11, 6.12 and 6.14. As in the case of (6.2) when $\gamma \geq 0$, the drift is opposed to the diffusion, but it is not strong enough to prevent that the solution locally vanishes.

We consider the kinetic model (6.1), with for $\mathsf{L}$ either a Fokker-Planck operator (FP) of a scattering operator (BGK), where the *scattering rate* $\mathrm{b}$ satisfies $1 \leq \mathrm{b} \leq \overline{\mathrm{b}}$ for some $\overline{\mathrm{b}} \geq 1$ (in addition). This is the kinetic equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f = \mathsf{L} f\,.$$

Observe that the global equilibrium is of the form

$$\forall\,(x,v) \in \mathbb{R}^d\times\mathbb{R}^d\,,\quad G(x,v) = \mathcal{M}(v)\,e^{-V(x)}\quad\text{where}\quad M(v) = (2\pi)^{-\frac{d}{2}}\,e^{-\frac{1}{2}\,|v|^2}\,.$$

Our main result is a decay rate in the presence of a very weak potential. It is an extension of the results of Theorem 6.12 to the framework of kinetic equations.

**Theorem 6.15** (B., Dolbeault, Schmeiser [B15]). *Let $d \geq 1$, $V = V_2$ with $\gamma \in [0,d)$ and $k > \max\{2,\gamma/2\}$. We assume that* (H1)–(H2) *hold and consider a solution $f$ of* (6.1) *with initial datum $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)$ such that $\int_{\mathbb{R}^d\times\mathbb{R}^d}\lfloor x\rceil^k\,f_0\,dx\,dv + \int_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k\,f_0\,dx\,dv < +\infty$. Then there exists $\mathcal{C} > 0$ such that*

$$\forall\, t \geq 0\,,\quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)} \leq \mathcal{C}\,(1+\,t)^{-\frac{d-\gamma}{2}}\,.$$

The expression of the constant $C$ is explicit. However, due to the method, we cannot claim optimality in the estimate of Theorem 6.15, but at least the asymptotic rate is expected to be optimal by consistency with the diffusion limit, as it is the case when $V = 0$, studied in [B14].

We have now reached the end of the discussion on hypocoercivity without confinement. A summary will be given in the perspectives below.

## 6.3 Hypocoercivity when the steady state is not known

In this last section, we change a little bit the spirit and we describe a situation where the construction of hypocoercivity functionals can allow to prove the existence of a global equilibrium when it is not known *a priori*. This is a result with Dolbeault and Hoffmann [B23].

The mathematical analysis of the fibre lay-down process in the production of non-woven textiles has seen a lot of interest in recent years [141, 142, 109, 127, 128, 78, 129]. We now describe the model we are interested in, which comes from [109]. It consists in the following Fokker-Planck equation with forced speed:

$$\partial_t f + (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha \left(\tau^\perp \cdot \nabla_x V f\right) = \partial_{\alpha\alpha} f, \qquad x \in \mathbb{R}^2, \alpha \in \mathbb{S}^1. \tag{6.4}$$

The speed of the moving belt is constant, and denoted by $\kappa$. The confining potential satisfies,

**(H1)** *Regularity and symmetry*: $V \in C^2(\mathbb{R}^2)$ and $V$ is spherically symmetric outside some ball $B(0, R_V)$.

**(H2)** *Normalisation*: $\int_{\mathbb{R}^2} e^{-V(x)}\,\mathrm{d}x = 1$.

**(H3)** *Spectral gap condition* (Poincaré inequality): there exists a positive constant $\Lambda$ such that for any $u \in H^1(e^{-V}\mathrm{d}x)$ with $\int_{\mathbb{R}^2} u e^{-V}\mathrm{d}x = 0$,

$$\int_{\mathbb{R}^2} |\nabla_x u|^2 e^{-V}\,\mathrm{d}x \geq \Lambda \int_{\mathbb{R}^2} u^2 e^{-V}\,\mathrm{d}x.$$

**(H4)** *Pointwise regularity condition on the potential*: there exists $c_1 > 0$ such that for any $x \in \mathbb{R}^2$, the Hessian $\nabla_x^2 V$ of $V$ satisfies

$$|\nabla_x^2 V(x)| \leq c_1 (1 + |\nabla_x V(x)|).$$

**(H5)** *Behaviour at infinity*:

$$\lim_{|x|\to\infty} \frac{|\nabla_x V(x)|}{V(x)} = 0, \qquad \lim_{|x|\to\infty} \frac{|\nabla_x^2 V(x)|}{|\nabla_x V(x)|} = 0\,.$$

Observe that this model falls down in the type of models we are interested in all along this chapter. Indeed, rewrite (6.4) as

$$\partial_t f = \mathsf{L} f - \mathsf{T} f + \mathsf{T}_\kappa f\,, \tag{6.5}$$

where the collision operator $\mathsf{L} := \partial_{\alpha\alpha}$ acts as a multiplicator in the space variable $x$, $\mathsf{T}_\kappa$ is the perturbation introduced by the moving belt.

$$\mathsf{T}_\kappa f := -\kappa e_1 \cdot \nabla_x f\,,$$

and the transport operator $\mathsf{T}$ is given by

$$\mathsf{T}f := \tau \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right).$$

We consider solutions to (6.4) in the space $L^2(\mathrm{d}\mu_\kappa) := L^2(\mathbb{R}^2 \times \mathbb{S}^1, \mathrm{d}\mu_\kappa)$ with measure

$$\mathrm{d}\mu_\kappa(x,\alpha) = \left( e^{V(x)} + \zeta \kappa g(x,\alpha) \right) \frac{\mathrm{d}x \, \mathrm{d}\alpha}{2\pi}.$$

We denote by $\langle \cdot, \cdot \rangle_\kappa$ the corresponding scalar product and by $\| \cdot \|_\kappa$ the associated norm. Here, $\zeta > 0$ is a free parameter to be chosen later. The construction of the weight $g$ depends on the boundedness of $\nabla_x V$. When it is bounded, no additional weight is needed to control the effect of the perturbation $\mathsf{T}_\kappa$, and so we simply set $g \equiv 0$ in that case. When the gradient is unbounded, the weight is constructed thanks to the following proposition:

**Proposition 6.16.** *Assume that $V$ satisfies **(H1)** and **(H5)** and that*

$$\lim_{|x| \to \infty} |\nabla_x V| = +\infty.$$

*If $\kappa < 1/3$ holds true, then there exists a function $g(x,\alpha)$, a constant $c = c(\kappa) > 0$ and a finite radius $R = R(\kappa, V) > 0$ such that*

$$\forall\, |x| > R, \forall\, \alpha \in \mathbb{S}^1, \quad \mathcal{L}_\kappa(g)(x,\alpha) \leq -c\, |\nabla_x V(x)| g(x,\alpha),$$

*where $\mathcal{L}_\kappa$ is defined by*

$$\mathcal{L}_\kappa(h) := \partial_{\alpha\alpha} h + (\tau + \kappa e_1) \cdot \nabla_x h - \left( \tau^\perp \cdot \nabla_x V \right) \partial_\alpha h - (\tau \cdot \nabla_x V)\, h.$$

*The weight $g$ is of the form*

$$g(x,\alpha) := \exp\left( \beta V(x) + |\nabla_x V(x)| \Gamma \left( \tau(\alpha) \cdot \frac{\nabla_x V(x)}{|\nabla_x V(x)|} \right) \right),$$

*where the parameter $\beta > 1$ and the function $\Gamma \in C^1([-1,1])$, $\Gamma > 0$ are determined along the proof and only depend on $\kappa$.*

When $\kappa = 0$, the steady state (global) is $\frac{1}{2\pi} e^{-V}$. When $\kappa > 0$, it is not explicitly computable and actually it is not even known to exist. Our aim is to show existence and uniqueness of a stationary state. For this, we derive a more general hypocoercivity estimate from which existence, uniqueness and exponential convergence can be derived. The main result is the following.

**Theorem 6.17** (B., Hoffmann, Mouhot [B23])**.** *Let $f_{\mathrm{in}} \in L^2(\mathrm{d}\mu_\kappa)$ and let **(H1-2-3-4-5)** hold. For $0 < \kappa < 1$ small enough (with a quantitative estimate) and $\zeta > 0$ large enough (with a quantitative estimate), there exists a unique non-negative stationary state $F_\kappa \in L^2(\mathrm{d}\mu_\kappa)$ with unit mass $M_{F_\kappa} = 1$. In addition, for any solution $f$ of (6.4) in $L^2(\mathrm{d}\mu_\kappa)$ with mass $M_f$ and subject to the initial condition $f(t=0) = f_{\mathrm{in}}$, we have*

$$\left\| f(t,\cdot) - M_f F_\kappa \right\|_\kappa \leq C \left\| f_{\mathrm{in}} - M_f F_\kappa \right\|_\kappa e^{-\lambda_\kappa t},$$

*where the rate of convergence $\lambda_\kappa > 0$ depends only on $\kappa$ and $V$, and the constant $C > 0$ depends only on and $V$.*

Adding the movement of the conveyor belt, Theorem 6.17 shows that $\operatorname{Ker} \mathsf{L}_\kappa = \langle F_\kappa \rangle$ and the exponential decay to equilibrium with rate $\lambda_\kappa$ corresponds to a spectral gap of size at least $[-\lambda_\kappa, 0]$. Further, it allows to recover an explicit expression for the rate of convergence $\lambda_0$ for $\kappa = 0$. In general, we are not able to compute the stationary state $F_\kappa$ for $\kappa > 0$ explicitly, but $F_\kappa$ converges to $F_0 = e^{-V}$ weakly as $\kappa \to 0$. Let us finally emphasise that a specific contribution of this work is to introduce two modifications of the entropy: 1) we first modify the *space itself* with the coercivity weight $g$, then 2) we change the norm with an auxiliary operator following the Dolbeault-Mouhot-Schmeiser hypocoercivity approach.

The proof of the existence is by a fixed point argument on a convex that is stable by the flow. This stability is obtained by the following inequality.

**Proposition 6.18.** *Assume that hypothesis* ***(H1-2-3-4-5)*** *hold and that* $0 < \kappa < 1$ *is small enough (with a quantitative estimate). Let* $f_{\text{in}} \in L^2(\mathrm{d}\mu_\kappa)$ *and* $f = f(t,x,\alpha)$ *be a solution of* (6.4) *in* $L^2(\mathrm{d}\mu_\kappa)$ *subject to the initial condition* $f(t=0) = f_{\text{in}}$*. Then* $f$ *satisfies the following Grönwall type estimate:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f(t,\cdot)] \leq -\gamma_1 \mathsf{H}[f(t,\cdot)] + \gamma_2 M_f^2, \tag{6.6}$$

*where* $\gamma_1 > 0, \gamma_2 > 0$ *are explicit constants only depending on* $\kappa$ *and* $V$*.*

When applied to the difference of two solutions with the same mass, (6.6) gives an estimate on the exponential decay rate towards equilibrium.

## 6.4 Perspectives

1. *Clean the table.*

   The conclusions of Section 6.1 can be summed up in the following table.

| $\mathcal{M}$ \ $V$ | $V \equiv 0$ | $V \sim \gamma \ln(\lvert x \rvert)$ | $V \sim \lvert x \rvert^\alpha$, $\alpha \in (0,1)$ | $V \sim \lvert x \rvert^\alpha$, $\alpha \geq 1$ |
|---|---|---|---|---|
| $\mathcal{M} \lesssim e^{-\lfloor v \rceil}$ | [B14] : $t^{-\frac{d}{2}}$ | [B15] : $t^{-\frac{d-\gamma}{2}}$ | [48] : $e^{-\lambda t^b}$ | [80] : $e^{-\lambda t}$ |
| $\mathcal{M} \asymp e^{-\lfloor v \rceil^\alpha}$, $\alpha \in (0,1)$ | [B13] : $t^{-\min\left(\frac{d}{2}, \frac{k}{\beta}\right)}$ | Open | Open | Open |
| $\mathcal{M} \asymp \lfloor v \rceil^{-d-\alpha}$, $\alpha > 0$ | [B12] : $t^{-\min\left(\frac{d}{\alpha}, \frac{k}{\beta_+}\right)}$ *e.g.* | Open | Open | Open |

   We observe that some cases are officially still open. However, we believe that the strategies designed in this chapter can provide answers. One first thing would be to rewrite the Fourier approach using only operators in the original space variable: this would allow to consider cases with weak confinement potential and fat-tailed $\mathcal{M}$. In cases where an integrable equilibrium exists, the point is that the global equilibrium $G$ is not $\mathcal{M}e^{-V}$ but associated to an energy. This was already present in the original method in [80] so one can really imagine to take this into account.

2. *Half-space hypocoercivity.*

   A question of great interest is to be able to work in bounded or semi bounded domains. In such geometries, using Fourier approaches is not possible anymore. A first interesting scenario is the half line $\mathbb{R}^+$, on which one could for example look at the polynomial relaxation rate of

$$\partial_t f + v \cdot \nabla_x f = \mathcal{M}\rho_f - f, \quad \rho[f] := \int_V f(v)\,\mathrm{d}v, \quad M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{d}{2}}} \tag{6.7}$$

and $x \in \mathbb{R}_+$, $v \in \mathbb{R}$ with boundary conditions at $\{x = 0\}$ and $f_{\text{in}} \in L^2(\mathcal{M}^{-1})$. The norm $\|\cdot\| := \|\cdot\|_{L^2(\mathcal{M}^{-1})}$. The conjecture is that the solution to (6.7) with specular conditions at the boundary $\{x = 0\}$ satisfies the optimal decay $\|f\| = O(t^{-\frac{d}{4}})$ (similar to that in the whole space), whereas the solution to (6.7) with the "no inflow" condition satisfies the improved decay $\|f\| = O(t^{-\frac{d+2}{4}})$. This is actually interesting for applications, one being the Bramson correction in kinetic reaction transport waves, see Chapter 4. This is a bridge that I would like to make soon, and is a work in progress with Mouhot and Mischler.

3. *Numerical schemes for rates of convergence.*

   Another interesting perspective is the development of robust schemes that are able to provide precise rates of decay to zero (or to equilibrium). Exactly for the same reasons as for the conception of numerical schemes for anomalous macroscopic limits (see the perspectives in Chapter 5), getting rates is numerically difficult in cases where the microscopic part is supposed to have a non-trivial influence. An attempt on the fractional Fokker-Planck equation has been made by Ayi *et al.* in [15].

4. *Nonlinear models.*

   Developing strategies to understand very well linear semigroups is fundamental, since it is usually one important ingredient in the understanding of nonlinear problems. As an example of nonlinear problems that I got interested in recently with Amic Frouvelle, for which linear hypocoercivity is a first step, let me say a few words about a model of alignement of self-propelled particles,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \Delta_v f - \nabla_v \cdot \left(P_{v^\perp} J_f f\right), \\ J_f(t,x) := \int_{\mathbb{S}} v f(t,x,v)\,\mathrm{d}v \end{cases}$$

   Here, the velocities are on the unit sphere of $\mathbb{R}^d$ and the space variable in the torus or the full space. The operator $P_{v^\perp}$ is a projection on $v^\perp$. Hypocoercive techniques are useful to obtain for example uniform bounds for the linear semi-group, that would potentially yield nonlinear bounds by perturbation. The fact that velocities are on the sphere adds quite a lot of technical and structural difficulties. Actually, getting precise qualitative and quantitative results on this model is still open from a lot of points of view (long time behaviour, profiles, travelling wave solutions ...).

# References


[1] P. Aceves-Sanchez and L. Cesbron. Fractional Diffusion Limit for a Fractional Vlasov–Fokker–Planck Equation. *SIAM Journal on Mathematical Analysis*, 51(1):469–488, Jan. 2019. Publisher: Society for Industrial & Applied Mathematics (SIAM).

[2] P. Aceves-Sanchez and A. Mellet. Anomalous diffusion limit for a linear Boltzmann equation with external force field. *Mathematical Models and Methods in Applied Sciences*, 27(5):845–878, 2017.

[3] E. Aghion, D. A. Kessler, and E. Barkai. From Non-Normalizable Boltzmann-Gibbs Statistics to Infinite-Ergodic Theory. *Phys. Rev. Lett.*, 122(1):010601, Jan. 2019. Publisher: American Physical Society.

[4] M. Alfaro. Slowing Allee effect versus accelerating heavy tails in monostable reaction diffusion equations. *Nonlinearity*, 30(2):687, 2017. Publisher: IOP Publishing.

[5] M. Alfaro, H. Berestycki, and G. Raoul. The Effect of Climate Shift on a Species Submitted to Dispersion, Evolution, Growth, and Nonlocal Competition. *SIAM Journal on Mathematical Analysis*, 49(1):562–596, 2017. Publisher: SIAM.

[6] M. Alfaro and J. Coville. Propagation phenomena in monostable integro-differential equations: Acceleration or not? *Journal of Differential Equations*, 263(9):5727 – 5758, 2017.

[7] M. Alfaro, J. Coville, and G. Raoul. Travelling waves in a nonlocal reaction-diffusion equation as a model for a population structured by a space variable and a phenotypic trait. *Communications in Partial Differential Equations*, 38(12):2126–2154, 2013.

[8] M. Alfaro and T. Giletti. Interplay of nonlinear diffusion, initial tails and Allee effect on the speed of invasions. *arXiv preprint arXiv:1711.10364*, 2017.

[9] M. Alfaro and T. Giletti. When fast diffusion and reactive growth both induce accelerating invasions. *Communications on Pure & Applied Analysis*, 18(6), 2019.

[10] W. C. Allee. *The Social Life of Animals*. Norton, 1938.

[11] G. Ariel, A. Rabani, S. Benisty, J. D. Partridge, R. M. Harshey, and A. Be'er. Swarming bacteria migrate by Lévy Walk. *Nature Communications*, 6(1):8396, Sept. 2015.

[12] A. Arnold, L. Desvillettes, and C. Prévost. Existence of nontrivial steady states for populations structured with respect to space and a continuous trait. *Communications on Pure and Applied Analysis*, 11(1):83–96, 2012.

[13] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974)*, pages 5–49. Lecture Notes in Math., Vol. 446. Springer, Berlin, 1975.

[14] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics*, 30(1):33–76, 1978.

[15] N. Ayi, M. Herda, H. Hivert, and I. Tristani. On a structure-preserving numerical method for fractional fokker-planck equations, 2021.

[16] D. Bakry, F. Bolley, I. Gentil, and P. Maheux. Weighted Nash inequalities. *Revista Matemática Iberoamericana*, 28(3):879–906, 2012.

[17] D. Bakry, P. Cattiaux, and A. Guillin. Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *Journal of Functional Analysis*, 254(3):727–759, 2008.

[18] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

[19] C. Bardos, F. Golse, and I. Moyano. Linear Boltzmann equation and fractional diffusion. *Kinetic and Related Models*, 11(4):1011–1036, 2018.

[20] C. Bardos, R. Santos, and R. Sentis. Diffusion approximation and computation of the critical size. *Trans. AMS*, 284(2):617–649, 1984.

[21] G. Barles, L. C. Evans, and P. E. Souganidis. Wavefront propagation for reaction-diffusion systems of PDE. *Duke Mathematical Journal*, 61(3):835–858, 1990.

[22] G. Barles and B. Perthame. Exit Time Problems in Optimal Control and Vanishing Viscosity Method. *SIAM Journal on Control and Optimization*, 26(5):1133–1148, Sept. 1988.

[23] J. Ben-Artzi and A. Einav. Weak Poincaré Inequalities in the Absence of Spectral Gaps. *Annales Henri Poincaré. A Journal of Theoretical and Mathematical Physics*, 21(2):359–375, 2020.

[24] A. Bensoussan, J. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. Contributions to Economic Analysis. North-Holland Publishing Company, 1978.

[25] L. Berec, E. Angulo, and F. Courchamp. Multiple Allee effects and population management. *Trends Ecol. Evol.*, 22:185–191, 2007.

[26] H. Berestycki, T. Jin, and L. Silvestre. Propagation in a non local reaction diffusion equation with spatial and genetic trait structure. *Nonlinearity*, 29(4):1434–1466, 2016. Publisher: IOP Publishing.

[27] H. Berestycki and G. Nadin. Spreading speeds for one-dimensional monostable reaction-diffusion equations. *Journal of Mathematical Physics*, 53(11):115619, Nov. 2012.

[28] H. Berestycki, G. Nadin, B. Perthame, and L. Ryzhik. The non-local Fisher-KPP equation: travelling waves and steady states. *Nonlinearity*, 22(12):2813–2844, 2009.

[29] N. Berestycki, C. Mouhot, and G. Raoul. Existence of self-accelerating fronts for a non-local reaction-diffusion equation, 2015.

[30] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. *Transactions of the American Mathematical Society*, 95(2):263–273, 1960. Publisher: JSTOR.

[31] K. Bogdan, T. Grzywny, and M. Ryznar. Density and tails of unimodal convolution semigroups. *Journal of Functional Analysis*, 266(6):3543–3571, 2014.

[32] D. Bonte and others. Costs of dispersal. *Biological Reviews*, 87(2):290–312, 2012. Publisher: Blackwell Publishing Ltd.

[33] F. Bouchut. Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions. *Journal of Functional Analysis*, 111(1):239–258, 1993.

[34] F. Bouchut, F. Golse, and M. Pulvirenti. *Kinetic equations and asymptotic theory*, volume 4 of *Series in Applied Mathematics (Paris)*. Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, 2000.

[35] M. Bramson. Maximal displacement of branching Brownian motion. *Communications on Pure and Applied Mathematics*, 31(5):531–581, 1978.

[36] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Memoirs of the American Mathematical Society*, 44(285):iv+190, 1983.

[37] N. F. Britton. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. *SIAM Journal on Applied Mathematics*, 50(6):1663–1688, 1990.

[38] O. Bénichou, V. Calvez, N. Meunier, and R. Voituriez. Front acceleration by dynamic selection in Fisher population waves. *Physical Review E*, 86:041908, 2012.

[39] C. Börgers, C. Greengard, and E. Thomann. The diffusion limit of free molecular flow in thin plane channels. *SIAM Journal on Applied Mathematics*, 52(4):1057–1075, 1992. Publisher: SIAM.

[40] X. Cabré and J.-M. Roquejoffre. Propagation de fronts dans les équations de Fisher-KPP avec diffusion fractionnaire. *C. R. Math. Acad. Sci. Paris*, 347(23–24):1361–1366, 2009.

[41] X. Cabré and J.-M. Roquejoffre. The Influence of Fractional Diffusion in Fisher-KPP Equations. *Communications in Mathematical Physics*, 320(3):679–722, 2013.

[42] N. Caillerie. Large deviations of a velocity jump process with a Hamilton–Jacobi approach. *Comptes Rendus Mathematique*, 355(2):170–175, Feb. 2017.

[43] V. Calvez. Chemotactic waves of bacteria at the mesoscale. *Journal of the European Mathematical Society*, 22(2):593–668, 2019.

[44] V. Calvez, J. Crevat, L. Dekens, B. Fabrèges, F. Kuczma, F. Lavigne, and G. Raoul. Influence of the mode of reproduction on dispersal evolution during species invasion. *ESAIM: ProcS*, 67:120–134, 2020.

[45] V. Calvez, J. Garnier, and F. Patout. Asymptotic analysis of a quantitative genetics model with nonlinear integral operator. *Journal de l'École polytechnique — Mathématiques*, 6:537–579, 2019.

[46] V. Calvez, C. Henderson, S. Mirrahimi, O. Turanova, and T. Dumont. Non-local competition slows down front acceleration during dispersal evolution. *arXiv:1810.07634 [math]*, Oct. 2018.

[47] V. Calvez, G. Raoul, and C. Schmeiser. Confinement by biased velocity jumps: aggregation of ıt Escherichia coli. *Kinetic and Related Models*, 8(4):651–666, 2015.

[48] C. Cao. The kinetic Fokker-Planck equation with weak confinement force. *Communications in Mathematical Sciences*, 17(8):2281–2308, 2019. Publisher: International Press of Boston.

[49] E. A. Carlen and M. Loss. Sharp constant in Nash's inequality. *International Mathematics Research Notices*, 1993(7):213–215, 04 1993.

[50] J. Carr and A. Chmaj. Uniqueness of travelling waves for nonlocal monostable equations. *Proceedings of the American Mathematical Society*, 132(8):2433–2439 (electronic), 2004.

[51] P. Cattiaux, E. Nasreddine, and M. Puel. Diffusion limit for kinetic Fokker-Planck equation with heavy tails equilibria: The critical case. *Kinetic & Related Models*, 12(4):727–748, 2019.

[52] L. Cesbron. Anomalous Diffusion Limit of Kinetic Equations in Spatially Bounded Domains. *Communications in Mathematical Physics*, 364(1):233–286, Nov. 2018.

[53] L. Cesbron, A. Mellet, and M. Puel. Fractional diffusion limit of a kinetic equation with diffusive boundary conditions in the upper-half space. *Archive for Rational Mechanics and Analysis*, 235(2):1245–1288, 2020.

[54] L. Cesbron, A. Mellet, and K. Trivisa. Anomalous transport of particles in plasma physics. *Applied Mathematics Letters*, 25(12):2344–2348, Dec. 2012.

[55] X. Chen. Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations. *Advances in Differential Equations*, 2(1):125–160, 1997.

[56] A. Chmaj. Existence of traveling waves in the fractional bistable equation. *Archiv der Mathematik*, 100(5):473–480, May 2013.

[57] J. S. Clark. Why trees migrate so fast: confronting theory with dispersal biology and the paleorecord. *The American Naturalist*, 152(2):204–224, 1998. Publisher: JSTOR.

[58] J. S. Clark, C. Fastie, G. Hurtt, S. T. Jackson, C. Johnson, G. A. King, M. Lewis, J. Lynch, S. Pacala, C. Prentice, and others. Reid's Paradox of Rapid Plant Migration Dispersal theory and interpretation of paleoecological records. *BioScience*, 48(1):13–24, 1998. Publisher: Oxford University Press.

[59] F. Courchamp, L. Berec, and J. Gascoigne. *Allee Effects in Ecology and Conservation*. Oxford University Press, Oxford, 2008.

[60] J. Coville. On uniqueness and monotonicity of solutions of non-local reaction diffusion equation. *Annali di Matematica Pura ed Applicata. Series IV*, 185(3):461–485, 2006.

[61] J. Coville. Travelling fronts in asymmetric nonlocal reaction diffusion equations: The bistable and ignition cases. *CCSD-Hal e-print*, pages –, May 2007.

[62] J. Coville. Harnack type inequality for positive solution of some integral equation. *Annali di Matematica Pura ed Applicata*, 191(3):503–528, Sept. 2012.

[63] J. Coville. *Contribution à l'étude d'équations non locales en dynamique des populations*. Habilitation à diriger des recherches, Aix Marseille Université, CNRS, I2M UMR 7373,, Nov. 2015.

[64] J. Coville, J. Davila, and S. Martinez. Nonlocal anisotropic dispersal with monostable nonlinearity. *Journal of Differential Equations*, 244(12):3080–3118, 2008.

[65] J. Coville and L. Dupaigne. On a non-local equation arising in population dynamics. *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, 137(4):727–755, 2007.

[66] J. Coville, C. Gui, and M. Zhao. Propagation acceleration in reaction diffusion equations with anomalous diffusions. *Nonlinearity*, 34(3):1544–1576, Mar. 2021.

[67] N. Crouseilles, H. Hivert, and M. Lemou. Multiscale numerical schemes for kinetic equations in the anomalous diffusion limit. *Comptes Rendus Mathematique*, 353(8):755–760, 2015.

[68] N. Crouseilles, H. Hivert, and M. Lemou. Numerical schemes for kinetic equations in the anomalous diffusion limit. Part I: The case of heavy-tailed equilibrium. *SIAM Journal on Scientific Computing*, 38(2):A737–A764, 2016.

[69] N. Crouseilles, H. Hivert, and M. Lemou. Numerical schemes for kinetic equations in the anomalous diffusion limit. Part II: Degenerate collision frequency. *SIAM Journal on Scientific Computing*, 38(4):A2464–A2491, 2016.

[70] C. Cuesta and C. Schmeiser. Weak shocks for a one-dimensional BGK kinetic model for conservation laws. *SIAM Journal on Mathematical Analysis*, 38:637–656, 2006.

[71] C. M. Cuesta, S. Hittmeir, and C. Schmeiser. Traveling Waves of a Kinetic Transport Model for the KPP-Fisher Equation. *SIAM J. Math. Anal.*, 44(6):4128–4146, Dec. 2012.

[72] W. Cygan, T. Grzywny, and B. Trojan. Asymptotic behavior of densities of unimodal convolution semigroups. *Transactions of the American Mathematical Society*, 369(8):5623–5644, 2017.

[73] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana University Mathematics Journal*, 49(3):1175–1198, 2000.

[74] L. Dekens and F. Lavigne. Front propagation of a sexual population with evolution of dispersion: A formal analysis. *SIAM Journal on Applied Mathematics*, 81(4):1441–1460, 2021.

[75] B. Dennis. Allee effects: population growth, critical density, and the chance of extinction. *Nat. Resour. Model.*, 3:481–538, 1989.

[76] O. Diekmann, P.-E. Jabin, S. Mischler, and B. Perthame. The dynamics of adaptation: an illuminating example and a Hamilton-Jacobi approach. *Theoretical Population Biology*, 67(4):257–271, 2005. Publisher: Elsevier.

[77] C. Dogbé. Anomalous diffusion limit induced on a kinetic equation. *J. Statist. Phys.*, 100(3-4):603–632, 2000.

[78] J. Dolbeault, A. Klar, C. Mouhot, and C. Schmeiser. Exponential rate of convergence to equilibrium for a model describing fiber lay-down processes. *Applied Mathematics Research Express. AMRX*, 2013(2):165–175, 2013.

[79] J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for Kinetic Equations with Linear Relaxation Terms. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 347(9-10):511–516, May 2009.

[80] J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for Linear Kinetic Equations Conserving Mass. *Transactions of the American Mathematical Society*, 367(6):3807–3828, Feb. 2015.

[81] J.-P. Eckmann and M. Hairer. Spectral properties of hypoelliptic operators. *Communications in Mathematical Physics*, 235(2):233–253, 2003.

[82] R. S. Ellis and M. A. Pinsky. The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. Pures Appl. (9)*, 54:125–156, 1975.

[83] H. Engler and S. M. Lenhart. Viscosity Solutions for Weakly Coupled Systems of Hamilton-Jacobi Equations. *Proceedings of the London Mathematical Society*, s3-63(1):212–240, 1991.

[84] J. Evans and H. Yoldaş. On the asymptotic behaviour of a run and tumble equation for bacterial chemotaxis. *arXiv preprint arXiv:2103.16524*, 2021.

[85] L. C. Evans and P. E. Souganidis. A pde approach to geometric optics for certain semilinear parabolic equations. *Indiana University Mathematics Journal*, 38(1):141–172, 1989.

[86] A. Faggionato, D. Gabrielli, and M. R. Crivellari. Averaging and large deviation principles for fully-coupled piecewise deterministic Markov processes and applications to molecular motors. *arXiv:0808.1910 [math-ph]*, Aug. 2008.

[87] J. Fang and X.-Q. Zhao. Monotone wavefronts of the nonlocal Fisher-KPP equation. *Nonlinearity*, 24(11):3043–3054, 2011.

[88] G. Faye and M. Holzer. Modulated traveling fronts for a nonlocal Fisher-KPP equation: a dynamical systems approach. *Journal of Differential Equations*, 258(7):2257–2289, 2015.

[89] J. Feng and T. G. Kurtz, editors. *Large deviations for stochastic processes*. Number v. 131 in Mathematical surveys and monographs. American Mathematical Society, Providence, R.I, 2006.

[90] P. C. Fife. *Mathematical aspects of reacting and diffusing systems*, volume 28. Springer Science & Business Media, 2013.

[91] R. A. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:335–369, 1937.

[92] W. H. Fleming and P. E. Souganidis. PDE-viscosity solution approach to some problems of large deviations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 4:171–192, 1986.

[93] N. Fournier and C. Tardif. Anomalous diffusion for multi-dimensional critical kinetic Fokker–Planck equations. *The Annals of Probability*, 48(5):2359–2403, 2020.

[94] N. Fournier and C. Tardif. One Dimensional Critical Kinetic Fokker-Planck Equations, Bessel and Stable Processes. *Communications in Mathematical Physics*, 381(1):143–173, Jan. 2021. Publisher: Springer Science and Business Media LLC.

[95] M. Freidlin. Limit theorems for large deviations and reaction-diffusion equations. *The Annals of Probability*, pages 639–675, 1985. Publisher: JSTOR.

[96] M. I. Freidlin and J. Gärtner. On the propagation of concentration waves in periodic and random media. *Sov. Math. Dokl.*, 20:1282–1286, 1979.

[97] U. Frisch and H. Frisch. Non-LTE transfer–III. Asymptotic expansion for small $\varepsilon$. *Monthly Notices of the Royal Astronomical Society*, 181(2):273–280, 1977. Publisher: Oxford University Press.

[98] J. Garnier. Accelerating solutions in integro-differential equations. *SIAM J. Math. Anal.*, 43:1955–1974, 2011.

[99] S. Genieys, V. Volpert, and P. Auger. Pattern and waves for a model in population dynamics with nonlocal consumption of resources. *Mathematical Modelling of Natural Phenomena*, 1(1):65–82, 2006.

[100] T. Giletti. Monostable pulled fronts and logarithmic drifts. 2021.

[101] F. Golse. Anomalous diffusion limit for the Knudsen gas. *Asymptotic Analysis*, 17(1):1–12, 1998.

[102] S. A. Gourley. Travelling front solutions of a nonlocal Fisher equation. *Journal of Mathematical Biology*, 41(3):272–284, 2000.

[103] C. Graham. The bramson correction for fisher–kpp equations with nonlocal diffusion, 2020.

[104] T. Grzywny, M. Ryznar, and B. Trojan. Asymptotic behaviour and estimates of slowly varying convolution semigroups. *International Mathematics Research Notices*, 2019(23):7193–7258, 2019.

[105] M. P. Gualdani, S. Mischler, and C. Mouhot. Factorization of Non-Symmetric Operators and Exponential H-Theorem. *Mémoires de la Société Mathématique de France. Nouvelle Série*, 153(153):1–137, June 2017.

[106] P. A. Guerra. Evaluating the life-history trade-off between dispersal capability and reproduction in wing dimorphic insects: a meta-analysis. *Biological Reviews*, 86(4):813–835, 2011. Publisher: Blackwell Publishing Ltd.

[107] C. Gui and T. Huan. Traveling wave solutions to some reaction diffusion equations with fractional Laplacians. *Calculus of Variations and Partial Differential Equations*, 54(1):251–273, 2015. Publisher: Springer.

[108] C. Gui and M. Zhao. Traveling wave solutions of Allen–Cahn equation with a fractional Laplacian. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 32(4):785–812, 2015. Publisher: Elsevier.

[109] T. Götz, A. Klar, N. Marheineke, and R. Wegener. A stochastic model and associated Fokker-Planck equation for the fiber lay-down process in nonwoven production processes. *SIAM Journal on Applied Mathematics*, 67(6):1704–1717, 2007.

[110] F. Hamel. Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 89(4):355–399, 2008.

[111] F. Hamel, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. A short proof of the logarithmic Bramson correction in Fisher-KPP equations. *Networks and Heterogeneous Media*, 8(1):275–289, 2013.

[112] F. Hamel, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. The logarithmic delay of kpp fronts in a periodic medium. *Journal of the European Mathematical Society*, 018(3):465–505, 2016.

[113] F. Hamel and L. Roques. Fast propagation for KPP equations with slowly decaying initial conditions. *J. Diff. Equations*, 249:1726–1745, 2010.

[114] F. Hamel and L. Ryzhik. On the nonlocal Fisher-KPP equation: steady states, spreading speed and global bounds. *Nonlinearity*, 27(11):2735–2753, 2014.

[115] S. Hittmeir and S. Merino-Aceituno. Kinetic derivation of fractional Stokes and Stokes-Fourier systems. *Kinetic and Related Models*, 9(1):105–129, 2016.

[116] H. Hivert. A first-order asymptotic preserving scheme for front propagation in a one-dimensional kinetic reaction–transport equation. *Journal of Computational Physics*, 367:253–278, 2018.

[117] H. Hivert. Numerical schemes for kinetic equation with diffusion limit and anomalous time scale. *Kinetic & Related Models*, 11(2):409–439, 2018.

[118] S. Hu and X. Wang. Subexponential decay in kinetic Fokker–Planck equation: Weak hypocoercivity. *Bernoulli*, 25(1):174–188, Feb. 2019.

[119] C. L. Hughes, C. Dytham, and J. K. Hill. Modelling and analysing evolution of dispersal in populations at expanding range boundaries. *Ecological Entomology*, 32(5):437–445, 2007. Publisher: Blackwell Publishing Ltd.

[120] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Archive for Rational Mechanics and Analysis*, 171(2):151–218, 2004.

[121] L. Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119:147–171, 1967.

[122] A. M. Il'in and R. Z. Khas'minskii. On the equations of Brownian motion. *Akademija Nauk SSSR. Teorija Verojatnoste\textbackslashuı \textbackslash i ee Primenenija*, 9:466–491, 1964.

[123] M. Jara, T. Komorowski, and S. Olla. Limit theorems for additive functionals of a Markov chain. *The Annals of Applied Probability*, 19(6):2270–2300, 2009.

[124] K. Kaleta and P. Sztonyk. Spatial asymptotics at infinity for heat kernels of integro-differential operators. *Transactions of the American Mathematical Society*, 371(9):6627–6663, 2019.

[125] O. Kavian, S. Mischler, and M. Ndao. The Fokker-Planck equation with subcritical confinement force, 2020. _eprint: 1512.07005.

[126] J. R. King and P. M. McCabe. On the Fisher-KPP equation with fast nonlinear diffusion. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 459(2038):2529–2546, 2003. Publisher: The Royal Society _eprint: http://rspa.royalsocietypublishing.org/content/459/2038/2529.full.pdf.

[127] A. Klar, J. Maringer, and R. Wegener. A 3D model for fiber lay-down in nonwoven production processes. *Mathematical Models and Methods in Applied Sciences*, 22(9):1250020, 18, 2012.

[128] M. Kolb, M. Savov, and A. Wübker. Geometric Ergodicity of a Hypoelliptic Diffusion Modelling The Melt-Spinning Process of Nonwoven Materials. *arXiv/1112.6159*, 2011.

[129] M. Kolb, M. Savov, and A. Wübker. (Non-)ergodicity of a degenerate diffusion modeling the fiber lay down process. *SIAM Journal on Mathematical Analysis*, 45(1):1–13, 2013.

[130] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934.

[131] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bulletin Université d'État à Moscow (Bjul. Moskowskogo Gos. Univ)*, Série Internationale(Section A):1–26, 1937.

[132] G. A. Langellotto, R. F. Denno, and J. R. Ott. A Trade-Off between Flight Capability and Reproduction in Males of a Wing-Dimorphic Insect. *Ecology*, 81(3):865–875, 2000. Publisher: Ecological Society of America.

[133] E. W. Larsen and J. B. Keller. Asymptotic solution of neutron transport problems for small mean free paths. *Journal of Mathematical Physics*, 15:75–81, 1974.

[134] K.-S. Lau. On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov. *Journal of Differential Equations*, 59(1):44–70, 1985.

[135] G. Lebeau and M. Puel. Diffusion approximation for Fokker Planck with heavy tail equilibria: a spectral method in dimension 1. *Communications in Mathematical Physics*, 366(2):709–735, 2019.

[136] S. M. Lenhart. Viscosity solutions for weakly coupled systems of first-order partial differential equations. *Journal of Mathematical Analysis and Applications*, 131(1):180–193, Apr. 1988.

[137] P. Li and S.-T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156(3-4):153–201, 1986.

[138] S. Luo and N. Payne. An asymptotic method based on a hopf–cole transformation for a kinetic bgk equation in the hyperbolic limit. *Journal of Computational Physics*, 341:295–312, 2017.

[139] S. Luo and N. Payne. Properties-preserving high order numerical methods for a kinetic eikonal equation. *J. Comput. Phys.*, 331(C):73–89, Feb. 2017.

[140] F. Lutscher, E. Pachepsky, and M. A. Lewis. The effect of dispersal patterns on stream populations. *SIAM Rev.*, 47(4):749–772, 2005.

[141] N. Marheineke and R. Wegener. Fiber dynamics in turbulent flows: general modeling framework. *SIAM Journal on Applied Mathematics*, 66(5):1703–1726 (electronic), 2006.

[142] N. Marheineke and R. Wegener. Fiber dynamics in turbulent flows: specific Taylor drag. *SIAM Journal on Applied Mathematics*, 68(1):1–23 (electronic), 2007.

[143] S. Marksteiner, K. Ellinger, and P. Zoller. Anomalous diffusion and Lévy walks in optical lattices. *Phys. Rev. A*, 53(5):3409–3430, May 1996. Publisher: American Physical Society.

[144] J. Medlock and M. Kot. Spreading disease: integro-differential equations old and new. *Mathematical Biosciences*, 184(2):201–222, 2003.

[145] A. Mellet. Fractional Diffusion Limit for Collisional Kinetic Equations: a Moments Method. *Indiana University Mathematics Journal*, 59(4):1333–1360, 2010.

[146] A. Mellet, S. Mischler, and C. Mouhot. Fractional Diffusion Limit for Collisional Kinetic Equations. *Archive for Rational Mechanics and Analysis*, 199(2):493–525, Feb. 2011.

[147] A. Mellet, J.-M. Roquejoffre, and Y. Sire. Existence and asymptotics of fronts in non local combustion models. *Commun. Math. Sci.*, 12(1):1–11, 2014.

[148] S. Mole and A. J. Zera. Differential allocation of resources underlies the dispersal-reproduction trade-off in the wing-dimorphic cricket, Gryllus rubens. *Oecologia*, 93(1):121–127, 1993.

[149] C. Mouhot and L. Neumann. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity*, 19(4):969–998, 2006.

[150] S. Méléard and S. Mirrahimi. Singular limits for reaction-diffusion equations with fractional Laplacian and local or nonlocal nonlinearity. *Communications in Partial Differential Equations*, 40(5):957–993, 2015. Publisher: Taylor & Francis.

[151] G. Nadin, L. Rossi, L. Ryzhik, and B. Perthame. Wave-like solutions for nonlocal reaction-diffusion equations: a toy model. *Mathematical Modelling of Natural Phenomena*, 8(3):33–41, 2013.

[152] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80:931–954, 1958.

[153] E. Nasreddine and M. Puel. Diffusion limit of Fokker-Planck equation with heavy tail equilibria. *ESAIM. Mathematical Modelling and Numerical Analysis*, 49(1):1–17, 2015.

[154] B. Nicolaenko. Dispersion Laws for plane wave propagation. In A. Grünbaum, editor, *The Boltzmann Equation*. Courant Institute, 1971.

[155] J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. Convergence to a single wave in the Fisher-KPP equation. *Chinese Annals of Mathematics. Series B*, 38(2):629–646, 2017.

[156] J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. Refined long-time asymptotics for Fisher-KPP fronts. *Communications in Contemporary Mathematics*, 21(7):1850072, 25, 2019.

[157] F. Patout. The Cauchy problem for the infinitesimal model in the regime of small variance. working paper or preprint, Dec. 2019.

[158] G. Peltier. Accelerating invasions along an environmental gradient. *Journal of Differential Equations*, 268(7):3299–3331, 2020.

[159] S. Penington. The spreading speed of solutions of the non-local Fisher-KPP equation. *Journal of Functional Analysis*, 275(12):3259–3302, 2018.

[160] B. Perthame and G. Barles. Dirac concentrations in Lotka-Volterra parabolic PDEs. *Indiana University Mathematics Journal*, 57(7):3275–3301, 2008.

[161] B. Perthame, W. Sun, and M. Tang. The fractional diffusion limit of a kinetic model with biochemical pathway. *Zeitschrift für angewandte Mathematik und Physik*, 69(3):67, May 2018.

[162] B. Phillips, G. Brown, J. Webb, and R. Shine. Invasion and the evolution of speed in toads. *Nature*, 439(7078):803–803, 2006.

[163] G. Pólya. On the zeros of an integral function represented by fourier's integral. *Messenger of Math*, 52:185–188, 1923.

[164] G. C. Pomraning. *The Equations of Radiation Hydrodynamics*. Pergamon Press, 1973.

[165] M. Puel, A. Mellet, and N. Ben Abdallah. Fractional diffusion limit for collisional kinetic equations: a Hilbert expansion approach. *Kinetic and Related Models*, 4(4):873–900, Nov. 2011. Publisher: American Institute of Mathematical Sciences (AIMS).

[166] M. I. Roberts. A simple path to asymptotics for the frontier of a branching Brownian motion. *The Annals of Probability*, 41(5):3518–3541, 2013.

[167] M. Röckner and F.-Y. Wang. Weak Poincaré Inequalities and $\mathrm L^2$-Convergence Rates of Markov Semigroups. *Journal of Functional Analysis*, 185(2):564–603, Oct. 2001.

[168] Y. Sagi, M. Brook, I. Almog, and N. Davidson. Observation of Anomalous Diffusion and Fractional Self-Similarity in One Dimension. *Phys. Rev. Lett.*, 108(9):093002, Mar. 2012. Publisher: American Physical Society.

[169] J. Saragosti, V. Calvez, N. Bournaveas, B. Perthame, A. Buguin, and P. Silberzan. Directional persistence of chemotactic bacteria in a traveling concentration wave. *Proceedings of the National Academy of Sciences of the United States of America*, 108(39):16235–16240, Sept. 2011.

[170] M. Schilder. Some Asymptotic Formulas for Wiener Integrals. *Transactions of the American Mathematical Society*, 125(1):63–85, 1966.

[171] K. Schumacher. Travelling-front solutions for integro-differential equations. I. *Journal für die Reine und Angewandte Mathematik*, 316:54–70, 1980.

[172] W. Shen and Z. Shen. Regularity and stability of transition fronts in nonlocal equations with time heterogeneous ignition nonlinearity. *Journal of Differential Equations*, 262(5):3390–3430, 2017. Publisher: Elsevier.

[173] W. Shen and Z. Shen. Transition fronts in nonlocal equations with time heterogeneous ignition nonlinearity. *Discrete & Continuous Dynamical Systems-A*, 37(2):1013–1037, 2017.

[174] D. Stan and J. L. Vazquez. The Fisher-KPP Equation with Nonlinear Fractional Diffusion. *SIAM Journal on Mathematical Analysis*, 46(5):3241–3276, 2014. _eprint: http://dx.doi.org/10.1137/130918289.

[175] C. D. Thomas, E. J. Bodsworth, R. J. Wilson, A. D. Simmons, Z. G. Davis, M. Musche, and L. Conradt. Ecological and evolutionary processes at expanding range margins. *Nature*, 411:577 – 581, 2001.

[176] Y. Tu and G. Grinstein. How White Noise Generates Power-Law Switching in Bacterial Flagellar Motors. *Phys. Rev. Lett.*, 94(20):208101, May 2005. Publisher: American Physical Society.

[177] O. Turanova. On a model of a population with variable motility. *Mathematical Models and Methods in Applied Sciences*, 25(10):1961–2014, 2015.

[178] K. Uchiyama. The behavior of solutions of some nonlinear diffusion equations for large time. *Journal of Mathematics of Kyoto University*, 18(3):453–508, 1978.

[179] H. D. Victory, Jr. and B. P. O'Dwyer. On classical solutions of Vlasov-Poisson Fokker-Planck systems. *Indiana University Mathematics Journal*, 39(1):105–156, 1990.

[180] C. Villani. Cercignani's conjecture is sometimes true and always almost true. *Communications in Mathematical Physics*, 234(3):455–490, 2003.

[181] C. Villani. Hypocoercive diffusion operators. *Bollettino della Unione Matematica Italiana. Serie VIII. Sezione B. Articoli di Ricerca Matematica*, 10(2):257–275, 2007.

[182] C. Villani. Hypocoercivity. *Memoirs of the American Mathematical Society*, 202(950):iv+141, 2009. ISBN: 978-0-8218-4498-4.

[183] F.-Y. Wang. Functional inequalities and spectrum estimates: the infinite measure case. *Journal of Functional Analysis*, 194(2):288–310, 2002.

[184] A. M. Weinberg and E. P. Wigner. *The physical theory of neutron chain reactors*. The University of Chicago Press, Chicago, Ill., 1958.

[185] H. F. Weinberger. Long-time behavior of a class of biological models. *SIAM Journal on Mathematical Analysis*, 13(3):353–396, 1982.

[186] E. P. Wigner. Mathematical problems of nuclear reactor theory. In *Proc. Sympos. Appl. Math., Vol. XI*, pages 89–104. American Mathematical Society, Providence, R.I., 1961.

[187] H. Yagisita. Existence and nonexistence of travelling waves for a nonlocal monostable equation. *Publ. RIMS, Kyoto Univ.*, 45:925–953, 2009.

[188] G.-B. Zhang, W.-T. Li, and Z.-C. Wang. Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity. *Journal of Differential Equations*, 252(9):5096 – 5124, 2012.

[189] Y. P. Zhang and A. Zlatoš. Optimal estimates on the propagation of reactions with fractional diffusion. arXiv:2105.12800 [math.AP], 2021.

# Abstract

This memoir presents the research work I have conducted after my doctoral thesis. It is applied analysis, dealing with qualitative and quantitative study of nonlocal models arising from physics and biology. This manuscript is structured in two parts. The first part is about spreading in nonlocal reaction-diffusion models (roughly). We study Bramson corrections and accelerated propagation. The accent is put on getting sharp rates of invasion as much as possible. The second part is about kinetic theory, we study several scaling limits that have importance and interpretation in physics, and expand the theory of hypocoercivity to have a better knowledge on long time behaviour of linear kinetic semigroups.



**Keywords:** Kinetic equations, reaction-diffusion equations, Hamilton-Jacobi equations, front propagation, modelling, convergence to equilibrium, scaling limits.

## Résumé

Ce mémoire est une synthèse des travaux de recherche que j'ai conduits depuis ma soutenance de thèse. Il s'agit majoritairement de contributions en analyse appliquée, autour de l'étude qualitative et quantitative de modèles venant de la physique et de la biologie. L'accent est mis sur les modèles non locaux en réaction-diffusion (au sens large) et en théorie cinétique. Le manuscript décrit en particulier l'extension de méthodes d'analyse pour l'étude de la correction de Bramson, l'étude quantitative de phénomènes d'accélération dans des équations non locales de type "crapauds buffles" ou intégro-différentielles, puis dans un second temps quelques développements de la théorie de l'hypocoercivité pour comprendre le retour à l'équilibre d'équations cinétiques linéaires, ainsi que quelques limites d'échelles macroscopiques ou de grandes déviations pour ce dernier type de modèles.

**Mots-clés:** Equations cinétiques, équations de reaction-diffusion, équations de Hamilton-Jacobi, propagation, modélisation, convergence vers l'équilibre, limites d'échelle.